A Mathematical Model and Numerical Simulations of Traffic Flow Problems

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Abstract

This report consists of three main parts. The first part explores how the density of traffic on a road can be mathematically modeled as a partial differential equation: a relation between a function and its derivatives. The second part describes an algorithm which can find a numerical solution for the differential equation. This algorithm is used in the third part of the report where simulations are run in order to study the effects of traffic lights on traffic.

Populärvetenskaplig sammanfattning

Fenomenet trafikflöde är en integrerad del av det moderna samhället. Varje dag åker folk bil eller buss dit de vill åka och varor transporteras mellan producenter och säljare. Dessutom har ett hinder i trafikflödet en negativ effekt på ekonomin och vår vardag. Därför är det viktigt att förstå hur detta fenomen fungerar.

I denna rapport framställs trafikflödesfenomen matematiskt genom en så kallad differentialekvation, och en algoritm beskrivs och används för att hitta approximationer av lösningarna till ekvationen. Detta görs eftersom det ofta är omöjligt att hitta exakta lösningar.

Denna algoritm kan användas för att simulera trafik och erhålla information för att förbättra trafikflöde. Till exempel i den sista sektionen av rapporten presenteras ett kriterium för att optimera trafikfködet. Detta kriterium innehåller förhållandet mellan grönt och rött ljus vilket måste överkrida en konstant. Om kriteriet uppfylls, slipper fordon vänta för länge framför trafikljuset.

1 Introduction

One's ability to commute between places and the ability to transport goods from manufacturers to vendors are both crucial to modern society since any impediment of either of them might lead to serious repercussions for the economy and one's quality of life. Traffic flow is thus an integral part of modern day life; to better understand it, this phenomenon is studied throughout this report.

The traffic flow problem is finding the density of cars on a road after a certain time, given that the density of cars on the road at said time is known. In section one of this report, it is seen that the traffic flow problem can be restated as an initial value problem for a partial differential equation (PDE). A graphical solution can then be found using the method of characteristics.

Unfortunately traffic flow problems can get very complicated, which makes finding exact solutions very difficult. This problem is addressed in section two of the report where a numerical method is found that solves the PDE.

In the third section of the report, this numerical method is used to find solutions for traffic lights problems; a traffic lights problem is a traffic flow problem with traffic lights somewhere on the road. This allows for simulating the movement of traffic on a road with traffic lights, which in turn allows for finding a criterion, which when followed, reduces the waiting time of cars at a red light.

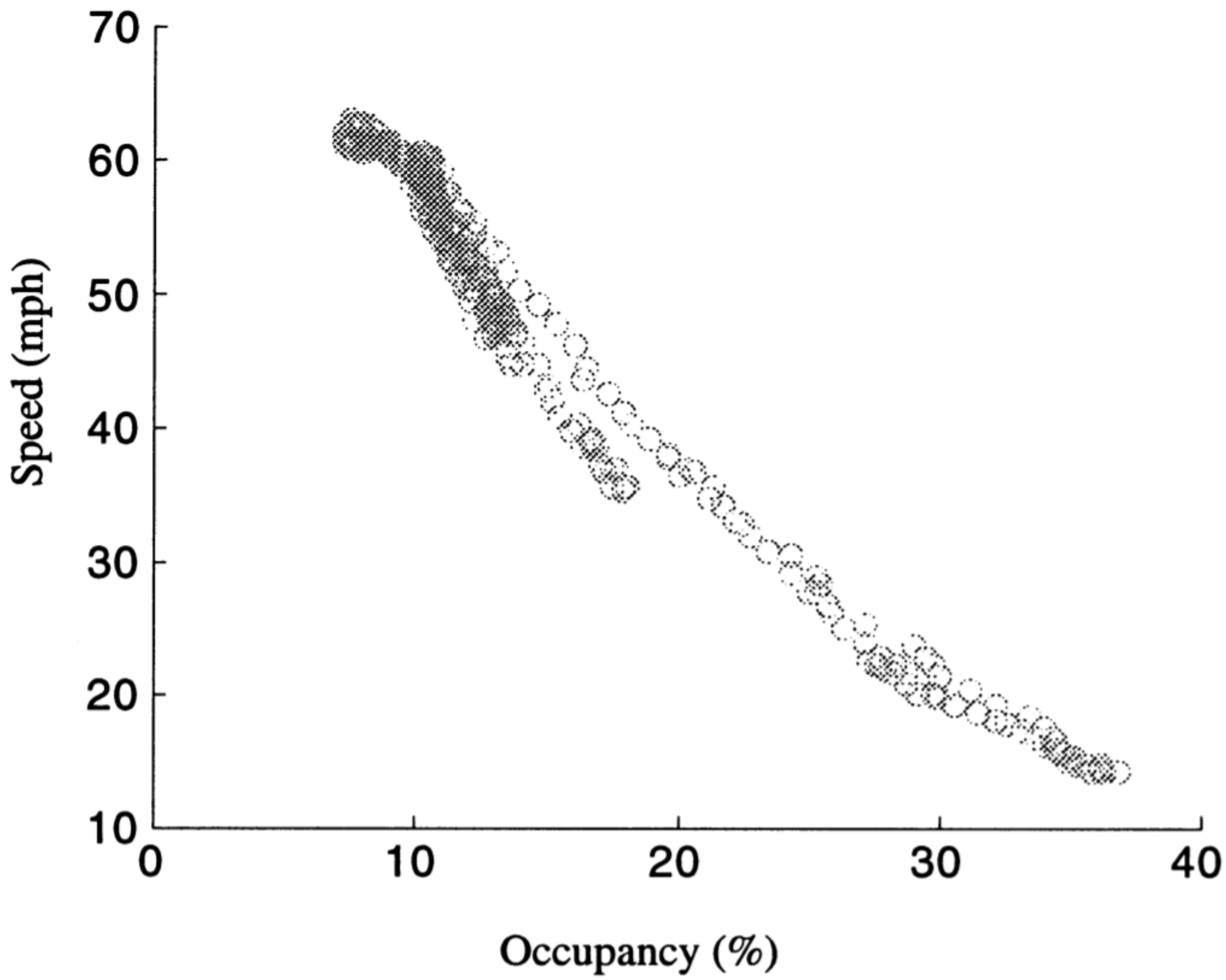

Figure 1: The velocity-occupancy plot[1]

2 The Model

This section is loosely based on the lecture notes by Stefan Diehl[3].

2.1 Framework

Cars flowing down a one-way infinite road can be modeled as follows:

- The road is modeled as $X = \mathbb{R}$ where the origin, $x = 0$, is some chosen point on the road. The positive direction is chosen as that of the cars on the road.
- Time is modeled as $T = \mathbb{R}^+$.
- On $X \times T$ define the non-negative real valued density function $u(x, t)$, i.e. the number of cars per unit of length at point x at time t.
- On $X \times T$ define the non-negative real valued velocity function $v(x, t)$, i.e. the velocity of the cars at point x at time t. Consider Figure 1 which shows the speed-occupancy plot based on experimental data measured by detectors on a freeway in San Francisco [1]. It suggests that the speed of cars decreases as the occupancy of the road increases. The occupancy of a road is what percentage of that road is occupied, for example the occupancy is 100% if the density of cars on that road is $u_{\max}$. Thus the velocity function can be redefined as $v(u)$, a non-decreasing function with $v(0) = v_{\max}$ and $v(u_{\max}) = 0$.

- On $X \times T$ define the non-negative real valued flux function $f(x,t)$, i.e. the number of cars traversing the point x at time t. However, $f(x,t) = v(x,t)u(x,t)$. Therefore, it can be redefined as $f(u) = v(u)u$. For this model, the choices of v are restricted to those such that $f(u) = v(u)u$ is a strictly concave function.

Since f is a strictly concave function, it has the following properties:

1. $f''(u) < 0, \forall u \in [0, u_{\max}] \quad \iff \forall u_1, u_2 \in [0, u_{\max}], \quad u_1 < u_2 \implies f'(u_1) > f'(u_2).$
2. $\exists \hat{u} \in [0, u_{\max}]$ such that $f(\hat{u}) = \max\limits_{u \in [0, u_{\max}]} f(u)$ and $f'(\hat{u}) = 0$.
3. $\forall u \in [0, \hat{u}), \quad u < \hat{u} \implies f'(u) > f'(\hat{u}) > 0 \implies f$ increasing on $[0, \hat{u})$.
4. $\forall u \in (\hat{u}, u_{\max}], \quad u > \hat{u} \implies f'(u) < f'(\hat{u}) = 0 \implies f$ decreasing on $(\hat{u}, u_{\max}]$.
5. $\forall u_1, u_2 \in [0, u_{\max}], \quad \forall \lambda \in [0,1], \quad \lambda f(u_1) + (1-\lambda) f(u_2) < f(\lambda u_1 + (1-\lambda) u_2).$

2.2 The conservation law

Consider a section of the road $[x_{\text{start}}, x_{\text{end}}]$, and assume $u(x,t)$ and $f(x,t)$ are $C^1([x_{\text{start}}, x_{\text{end}}] \times [0, t_{end}])$ with $t_{\text{end}} > 0$. Let $a < b$ be two arbitrary points of $[x_{\text{start}}, x_{\text{end}}]$. Cars are neither created nor destroyed on the road, so the change in the number of cars on $[a,b]$ at a time $t \in [0, t_{\text{end}}]$ is the difference between the number of cars entering a at t and the number of cars leaving b at t, i.e.

$$\frac{d}{dt} \int_a^b u(x,t) dx = f(a,t) - f(b,t). \tag{1}$$

Since $u(x,t)$ and $f(x,t)$ are C^1, then the above equations can be restated as:

$$\int_a^b \frac{\partial}{\partial t} u(x,t) dx = -\int_a^b \frac{\partial}{\partial x} f(x,t) dx, \tag{2}$$

and therefore:

$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) dx = 0. \tag{3}$$

Since $u_t + f_x$ is continuous, and a and b where arbitrarily chosen, then:

$$u_t + f_x = 0 \tag{4}$$

almost everywhere on $[x_{start}, x_{end}] \times [0, t_{end}]$. Since f is a function of u, the above partial differential equation can be restated as:

$$u_t + f'(u) u_x = 0. \tag{5}$$

2.3 Characteristics

Assume one knows that the solution u of the following initial-value problem:

$$\begin{aligned} u_t + f'(u) u_x &= 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

is in fact C^1. Let $x(t)$ be a level curve of $u(x,t)$ in $X \times T$, i.e.

$$\begin{aligned} & \exists U_0 \in \mathbb{R} : U_0 = u(x(t), t) \\ \iff & 0 = \frac{\partial u}{\partial t}(x(t), t) \\ \iff & 0 = u_x x'(t) + u_t \\ \iff & 0 = u_x x'(t) - f'(u) u_x \\ \iff & 0 = u_x (x'(t) - f'(U_0)). \end{aligned}$$

If $x'(t) = f'(U_0)$ then $x(t)$ is straight line with slope $\frac{1}{f'(U_0)}$ in $X \times T$. The level curve x(t) is called a signal, and $x'(t) = f'(U_0)$ is its speed.

If $u_x = 0$ in some region, then $u_t = 0$ by the conservation law. Therefore, u is constant in this region, and any curve in it is a level curve, particularly $x(t)$ the straight line with slope $\frac{1}{f'(U_0)}$.

These lines are called characteristics and they define an implicit solution

$$\begin{cases} x = f'(u_0(x_0))t + x_0 \\ u(x,t) = u(x_0). \end{cases}$$

This allows for the construction of a geometric solution, by drawing the characteristics at each point of the initial data.

2.4 Example 1

Consider the following traffic flow problem:

$$\begin{aligned} & f(u) = v(u)u = (1-u)u \\ & u_t + f(u)_x = 0 \\ & u_0 = u(x,0) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geqslant 0. \end{cases} \end{aligned} \tag{6}$$

A geometric solution can be constructed as follows:

$$f'(u_0) = 1 - 2u_0 = \begin{cases} 1 & \text{if} \quad x < 0 \\ -1 & \text{if} \quad x \geqslant 0. \end{cases}$$

By the implicit solution

$$u(x,t) = \begin{cases} 0 & \text{if} \quad \exists x_0 < 0 \quad \text{such that} \quad x = t + x_0 \\ 1 & \text{if} \quad \exists x_0 \geqslant 0 \quad \text{such that} \quad x = -t + x_0. \end{cases}$$

The characteristics passing through $\{(x,0) : x \in \mathbb{R}\}$ are drawn to get Figure 2, which gives the following solution:

$$u(x,t) = \begin{cases} 0 & \text{if} \quad x < 0 \quad \text{and} \quad t < -x \\ 1 & \text{if} \quad x \geqslant 0 \quad \text{and} \quad t < x \\ \text{undetermined} & \text{if} \quad -t \leqslant x \leqslant t. \end{cases}$$

Note that the method of characteristics requires the solution to be C^1 on the region of interest. It is therefore no surprise that the method fails near the discontinuity $(0,0)$.

2.5 A shock wave and its speed

If a discontinuity appears at (x_d, t_d), it might lead to a shock wave i.e. a jump from $u^- = u(x_d^-, t_d)$ to $u^+ = u(x_d^+, t_d)$. If the shock wave does occur, its speed s can be computed. This allows for the plotting of the shock wave in the $X \times T$ plane. Moreover, u will have the value u^- to the left of the shock wave and u^+ to the right of it.

Let s be the speed of the shock wave. Consider a new inertial system moving with speed s with respect to the original system such that at $t = 0$ the coordinates of the origin of the new system with respect to the original system are (x_d, t_d), i.e. the discontinuity is always at point $(0,0)$ with respect to the new inertial system.

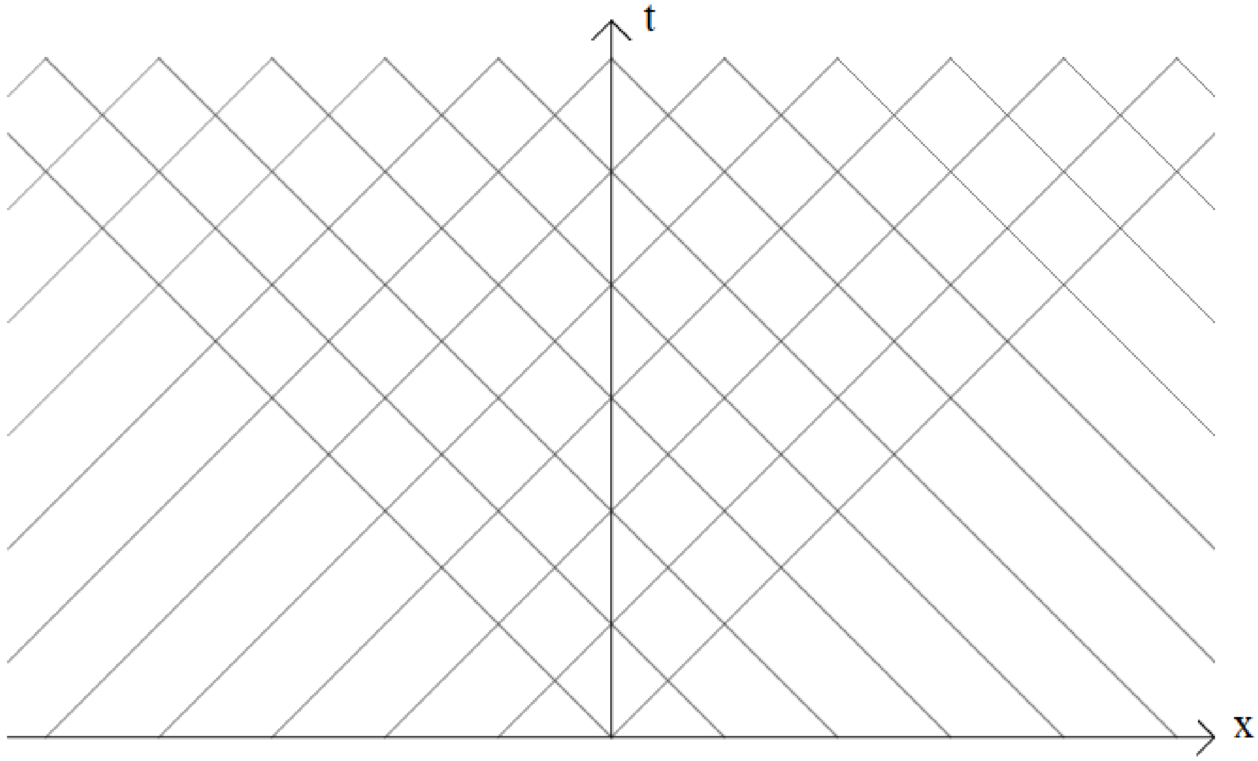

Figure 2: Characteristics of (6)

The flux with respect to the new system is $f_{\text{new}}(u) = uv_{\text{new}}(u) = u(v(u) - s)$. Since the number of cars passing the point $x = 0^-$ is equal to the number of cars passing the point $x = 0^+$ at time $t = 0$, then:

$$\begin{aligned}
& f_{\text{new}}(u(0^-, 0)) = f_{\text{new}}(u(0^+, 0)) \\
\Longleftrightarrow\ & f_{\text{new}}(u^-) = f_{\text{new}}(u^+) \\
\Longleftrightarrow\ & u^-(v(u^-) - s) = u^+(v(u^+) - s) \\
\Longleftrightarrow\ & s = \frac{u^+ v(u^+) - u^- v(u^-)}{u^+ - u^-} \\
\Longleftrightarrow\ & s = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.
\end{aligned} \tag{7}$$

2.6 Example 1 (continued)

If a shock wave was created at $(0, 0)$, a complete solution of example 1 can be found by computing its speed which is $s = \frac{f(1)-f(0)}{1-0} = 0$, then plotting the characteristics as in Figure 3. This figure gives the following solution:

$$\forall t > 0, \quad u(x,t) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geqslant 0. \end{cases}$$

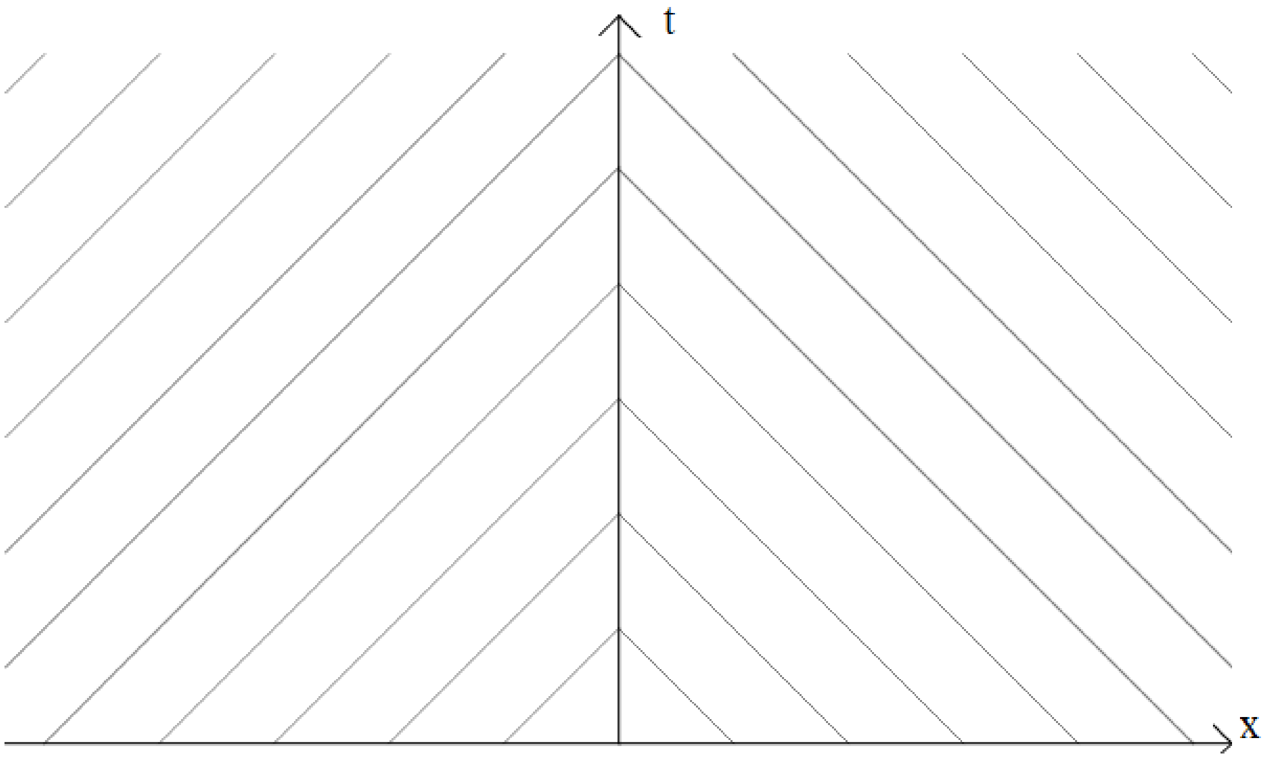

Figure 3: Characteristics (6) and the shock wave

2.7 Example 2

Consider the following traffic flow problem:

$$\begin{aligned} & f(u) = v(u)u = (1-u)u \\ & u_t + f(u)_x = 0 \\ & u_0 = u(x,0) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geqslant 0. \end{cases} \end{aligned} \tag{8}$$

A geometric solution can be constructed as follows:

$$f'(u_0) = 1 - 2u_0 = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geqslant 0. \end{cases}$$

By the implicit solution

$$u(x,t) = \begin{cases} 1 & \text{if} \quad \exists x_0 < 0 \quad \text{such that} \quad x = -t + x_0 \\ 0 & \text{if} \quad \exists x_0 \geqslant 0 \quad \text{such that} \quad x = t + x_0. \end{cases}$$

The characteristics passing through $\{(x,0) : x \in \mathbb{R}\}$ are drawn to get Figure 4, which gives the following solution:

$$u(x,t) = \begin{cases} 1 & \text{if} \quad x < 0 \quad \text{and} \quad t < -x \\ 0 & \text{if} \quad x \geqslant 0 \quad \text{and} \quad t < x \\ \text{undetermined} & \text{if} \quad -t \leqslant x \leqslant t. \end{cases}$$

In Figure 4, it is once again seen how the method of characteristics fails near discontinuities. This time, instead of getting a region with intersecting characteristics, one gets a region without any characteristics at all.

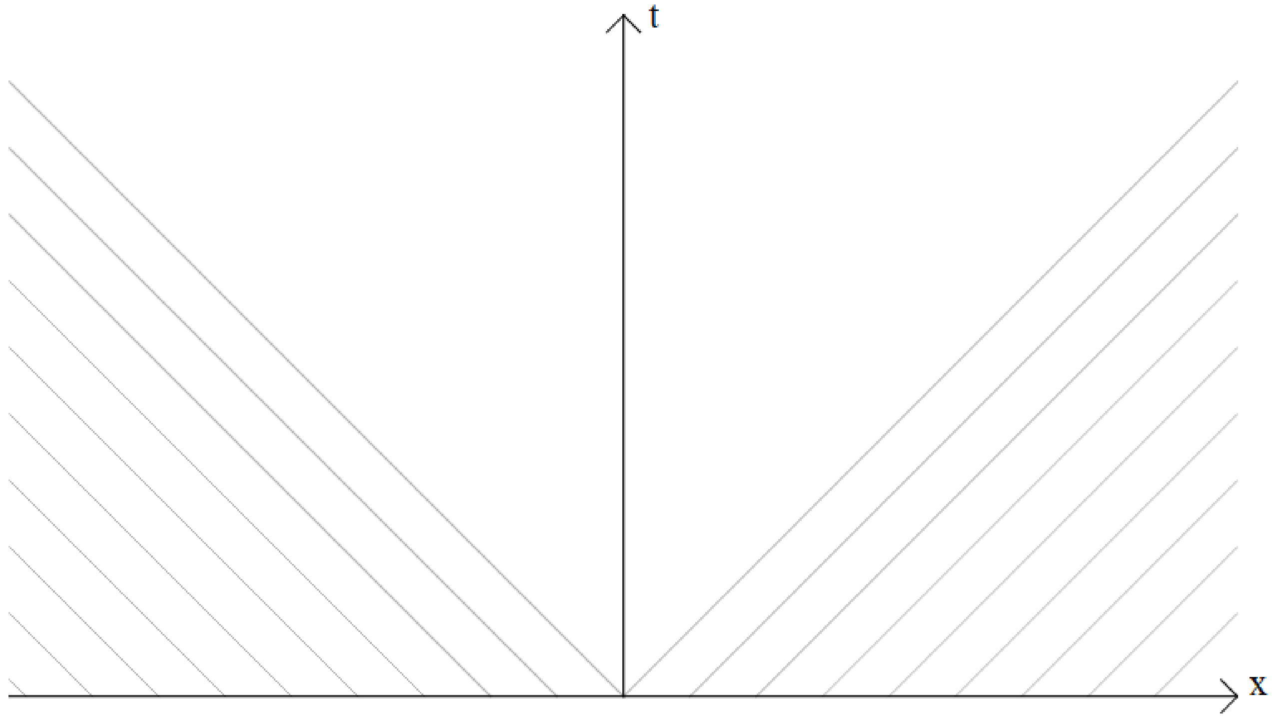

Figure 4: Characteristics of (8)

2.8 The entropy condition

Suppose there is a discontinuity at a point (x_d, t_d). For a shock wave to arise, its speed s must satisfy the entropy condition[2]:

$$s < \frac{f(v) - f(u^-)}{v - u^-} \quad \forall v \in (\min(u^-, u^+), \max(u^-, u^+)). \tag{9}$$

However, for strictly concave functions this condition can be simplified as follows:

$$\frac{f(u^+) - f(u^-)}{u^+ - u^-} - \frac{f(v) - f(u^-)}{v - u^-} < 0$$

$$\iff \qquad \frac{(v - u^-)f(u^+) + (u^+ - v)f(u^-) - (u^+ - u^-)f(v)}{(u^+ - u^-)(v - u^-)} < 0$$

$$\iff \qquad \frac{(v - u^-)}{(u^+ - u^-)(v - u^-)} f(u^+) + \frac{(u^+ - v)}{(u^+ - u^-)(v - u^-)} f(u^-) < \frac{1}{v - u^-} f(v).$$

Since v is in $(\min(u^-,u^+),\max(u^-,u^+))$, then there exists a λ in $(0,1)$ such that $v=\lambda u^-+(1-\lambda)u^+$. Therefore:

$$\frac{(1-\lambda)(u^+-u^-)}{(u^+-u^-)(v-u^-)}f(u^+)+\frac{\lambda(u^+-u^-)}{(u^+-u^-)(v-u^-)}f(u^-)<\frac{1}{v-u^-}f(\lambda u^-+(1-\lambda)u^+)$$

$$\iff \frac{(1-\lambda)}{(v-u^-)}f(u^+)+\frac{\lambda}{(v-u^-)}f(u^-)<\frac{1}{v-u^-}f(\lambda u^-+(1-\lambda)u^+)$$

$$\begin{cases}\text{if} \quad u^+>v>u^-, \quad (1-\lambda)f(u^+)+\lambda f(u^-)<f(\lambda u^-+(1-\lambda)u^+)\\[2ex] \text{if} \quad u^+<v<u^-, \quad (1-\lambda)f(u^+)+\lambda f(u^-)>f(\lambda u^-+(1-\lambda)u^+).\end{cases}$$

By property 5 of subsection 2.1, the first condition is true for all v strictly between u^- and u^+, and the second is false for all such v. The entropy condition is reduced to

$$u^-<u^+, \tag{10}$$

for strictly concave f. In example 1, the jump was from $(u^-=0)$ to $(u^+=1)$. Since (10) holds, a shock wave is created and the solution given in subsection 2.6 is correct. If there is a discontinuity at (x_d,t_d) and (10) does not hold, an expansion wave is created which continuously decreases u from u^- to u^+. This is illustrated in subsection 2.9.

2.9 Example 2 (continued)

At $t=0$ there is a discontinuity in x at 0, where $(u^-=1)>(u^+=0)$. According to the entropy condition, a shock wave will not be created at this point. Instead, an expansion wave is created at $(0,0)$. The solution is:

$$u(x,t)=\begin{cases}1 \quad \text{if} \quad x<0 \quad \text{and} \quad t<-x\\ 0 \quad \text{if} \quad x\geqslant 0 \quad \text{and} \quad t<x\\ \frac{1}{2}-\frac{x}{2t} \quad \text{if} \quad -t\leqslant x\leqslant t.\end{cases}$$

The characteristics of the complete solution can be seen in Figure 5.

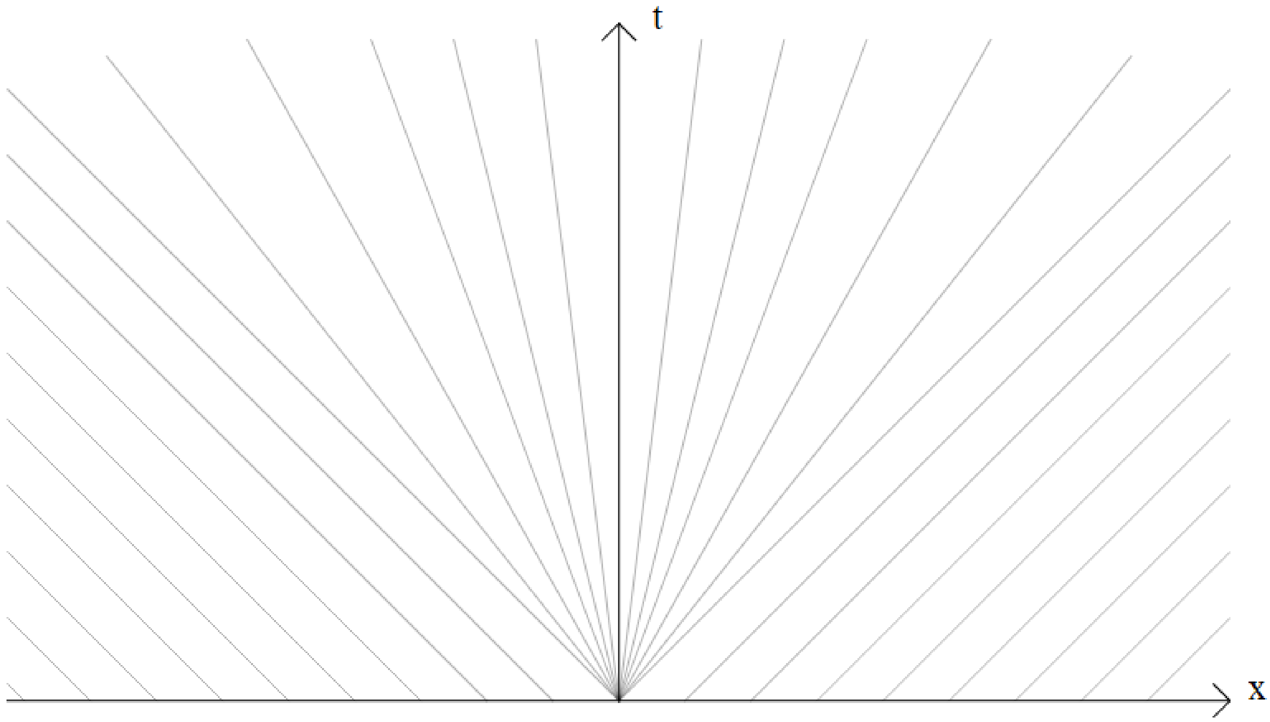

Figure 5: Characteristics of (8) and the expansion wave

3 The Finite Volume Method

This section is loosely based on the book "Front Tracking for Hyperbolic Conservation Laws" [2].

3.1 The numerical density

Consider the traffic flow problem:

$$\begin{aligned} &u_t + f(u)_x = 0 \\ &u_0 = u(x, 0). \end{aligned} \tag{11}$$

The solution to (11) is the density function $u(x,t)$. Moreover, the velocity function $v(u)$, the flux function $f(u) = v(u)u$ and the initial density function $u_0(x) = u(x, 0)$ are all known. The goal is to find a numerical solution U for (11). To do that, step sizes $\Delta x > 0$ and $\Delta t > 0$ are picked according to a criterion discussed in subsection 3.2. Let $x_j = j\Delta x$ and $t_n = n\Delta t$ for all $j \in \mathbb{Z}$ and all $n \in \mathbb{N}$.

Define the average of u on $x \in [x_{j-1}, x_j]$ at time t_n as:

$$U_j^n = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u(x, t_n)dx, \quad \forall j \in \mathbb{Z}, \quad \forall n \in \mathbb{N}. \tag{12}$$

The conservation law states that:

$$\begin{aligned} &\frac{d}{dt}\int_{x_{j-1}}^{x_j} u(x,t)dx = f(u(x_{j-1},t)) - f(u(x_j,t)), \quad \forall j \in \mathbb{Z} \\ \implies &\frac{d}{dt}\int_{x_{j-1}}^{x_j} u(x,t)dx = f_{j-1}(t) - f_j(t) \\ \implies &\frac{d}{dt}\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u(x,t)dx = \frac{1}{\Delta x}(f_{j-1}(t) - f_j(t)) \end{aligned}$$

Integrating both sides between $t_n$and t_{n+1} gives:

$$\begin{aligned}
&\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u(x,t_{n+1})dx - \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u(x,t_n)dx = \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} (f_{j-1}(t) - f_j(t))dt \\
&\implies U_j(t_{n+1}) - U_j(t_n) = \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} (f_{j-1}(t) - f_j(t))dt \\
&\implies U_j^{n+1} = U_j^n + \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} (f_{j-1}(t) - f_j(t))dt. \qquad (13)
\end{aligned}$$

Since $u_0(x)$ is known for all $x \in \mathbb{R}$, then U_j^0 can be computed using (12) for all $j \in \mathbb{Z}$. If the integral on the right hand side of (13) can be computed exactly for all $n \in \mathbb{N}$, then exact averages U_j^n can be computed for all $(j,n) \in \mathbb{Z} \times \mathbb{N}$. Then the numerical density function can be defined as a piecewise constant function:

$$U(x,t) = U_j^n \quad \text{if} \quad (x,t) \in \Omega_{j,n}, \qquad (14)$$

where:

$$\Omega_{j,n} = \{(x,t) : x_{j-1} < x \leqslant x_j \quad \text{and} \quad t_n - \frac{1}{2}\Delta t \leqslant t < t_n + \frac{1}{2}\Delta t\}. \qquad (15)$$

It is easy to see that $U(x,t)$ converges point-wise to $u(x,t)$ as Δx and Δt tend to 0 since the U_j^n's are the exact averages. Unfortunately, for all $j \in \mathbb{Z}$, $f_j(t) = f(u(x_j,t))$ cannot be computed without prior knowledge of the unknown solution u of (11) at the point (x_j,t). However $f_j(t)$ can be approximated with the Godunov flux $F_j(t)$ which does not require any prior knowledge of u. An approximation of right hand side of (13) can be computed using $F_j(t)$ to obtain approximations of averages U_j^n. These approximations U_j^n allow for a construction of U as in (14) and (15) which converges to $u(x,t)$ as Δx and Δt tend to 0[2].

3.2 The Courant–Friedrichs–Lewy condition

Consider traffic flow problem (11). For some arbitrary $n \in \mathbb{N}$, U_j^n is known $\forall j \in \mathbb{Z}$. If $f(u(x_j,t))$ is known for all $t \in [t_n, t_{n+1}]$ and all $j \in \mathbb{Z}$ then U_j^{n+1} can be computed for all $j \in \mathbb{Z}$ by (13). Since $f(u)$ is a known function, $f(u(x_j,t))$ is totally determined by $u(x_j,t)$. Moreover, for all $x \in (x_{j-1}, x_j]$, $u(x,t_n)$ can be approximated with U_j^n, the average of these points.

A problem with piecewise constant initial density function is called a Cauchy problem. Consider the following Cauchy problem:

$$\begin{aligned}
&\tilde{u}_t + f(\tilde{u})_x = 0 \qquad (16)\\
&\tilde{u}_{t_n}(x) = \tilde{u}(x,t_n) = U_j^n, \quad \text{if} \quad x_{j-1} < x \leqslant x_j.
\end{aligned}$$

Note that in (16) the initial density is given at time t_n instead of 0, and the solution to (16) is $\tilde{u}(x,t)$ for all $t \geqslant t_n$. Since the density at time t_n for traffic flow problem (16) is a piecewise constant approximation of the density at time t_n for traffic flow problem (11), the solution $\tilde{u}$ of (16) can be used to approximate the solution u of (11) for times close to t_n, and especially for times $t \in [t_n, t_{n+1}]$. Thus $f_j(t) = f(u(x_j,t)) \approx f(\tilde{u}(x_j,t))$ for all $t \in [t_n, t_{n+1}]$. Assume the solution to the following traffic flow problem:

$$\begin{aligned}
&\tilde{\tilde{u}}_t + f(\tilde{\tilde{u}})_x = 0 \qquad (17)\\
&\tilde{\tilde{u}}_{t_n}(x) = \tilde{\tilde{u}}(x,t_n) = \begin{cases} U_j^n & \text{if} \quad x \leqslant x_j \\ U_{j+1}^n & \text{if} \quad x > x_j \end{cases}
\end{aligned}$$

is known for all $t > t_n$. A solution for $\tilde{u}$ can be constructed along the line $x = x_j$ using $\tilde{\tilde{u}}$. Let $D_1 = (x_{j-1}, x_{j+1}]$ and $D_2 = (-\infty, x_{j-1}] \cup (x_{j+1}, \infty)$. If for all $x_2 \in D_2$ the signal that starts at (x_2, t_n) never reaches the point (x_j, t^*) with $t^* > t_n$, then the density at (x_j, t) is totally determined by the signals that start at (x_1, t_n) with $x_1 \in D_1$ for all $t \in [t_n, t^*]$. However, for all $x_1 \in D_1$, $\tilde{u}(x_1, t_n) = \tilde{\tilde{u}}(x_1, t_n)$, so $\tilde{u}$ and $\tilde{\tilde{u}}$ have the same signals at (x_1, t_n), for all $x_1 \in D_1$. Therefore, $\tilde{u}(x_j, t) = \tilde{\tilde{u}}(x_j, t)$.

Consider an arbitrary $x_2 \in D_2$. The distance between x_j and x_2 is $\Delta x_2 \geqslant \Delta x$. The time needed for the signal that starts at (x_2, t_n) to move a distance of Δx_2 regardless of orientation is

$$T_2 = \frac{\Delta x_2}{|f'(\tilde{u}(x_2, t_n))|} \geqslant \frac{\Delta x}{|f'(\tilde{u}(x_2, t_n))|} \geqslant \frac{\Delta x}{\max\limits_{0 \leqslant \tilde{u} \leqslant u_{\max}} |f'(\tilde{u})|} = \frac{\Delta x}{\max\limits_{0 \leqslant u \leqslant u_{\max}} |f'(u)|}$$

The following is called the Courant–Friedrichs–Lewy (CFL) condition

$$\frac{\Delta x}{\max\limits_{u} |f'(u)|} > \Delta t. \tag{18}$$

If (18) holds, then for all $x_2 \in D_2$, if $T_2 > \Delta t$ then the signal that starts at (x_2, t_n) never reaches the point (x_j, t), $\forall t \in [t_n, t_n + \Delta t] = [t_n, t_{n+1}]$. This implies that $\tilde{u}(x_j, t)) = \tilde{\tilde{u}}(x_j, t)$ for all $t \in [t_n, t_{n+1}]$. Under this condition, the numerical (Godunov) flux is defined as $F_j(t) = f(\tilde{u}(x_j, t)) = f(\tilde{\tilde{u}}(x_j, t)) \approx f(u(x_j, t)) = f_j(t)$ for all $t \in [t_n, t_{n+1}]$ and $j \in \mathbb{Z}$.

3.3 The Riemann Problem

Traffic flow problem (17) is a Riemann problem, a Cauchy problem where there is only one discontinuity in the initial density. For simplicity, (17) can be rewritten as:

$$\begin{aligned} &\tilde{\tilde{u}}_t + f(\tilde{\tilde{u}})_x = 0 \\ &\tilde{\tilde{u}}_0(x) = \tilde{\tilde{u}}(x, 0) = \begin{cases} \tilde{\tilde{u}}^+ = U_{j+1}^n & \text{if} \quad x > 0 \\ \tilde{\tilde{u}}^- = U_j^n & \text{if} \quad x \leqslant 0, \end{cases} \end{aligned} \tag{19}$$

and the corresponding Godunov flux of (11) is $F_j(t + t_n) = f((\tilde{\tilde{u}}(0, t))$ for all $t \in [0, \Delta t]$. Through out this subsection, the value of $f(\tilde{\tilde{u}}(0, t))$ will be investigated for all Riemann problems.

- If $\tilde{\tilde{u}}^-, \tilde{\tilde{u}}^+ \in [0, \hat{u}]$
  - If $\tilde{\tilde{u}}^- < \tilde{\tilde{u}}^+$, then the entropy condition is satisfied and a shock wave is created with speed $s = \frac{f(\tilde{\tilde{u}}^+) - f(\tilde{\tilde{u}}^-)}{\tilde{\tilde{u}}^+ - \tilde{\tilde{u}}^-}$. The denominator is positive since $\tilde{\tilde{u}}^- < \tilde{\tilde{u}}^+$, and the numerator is positive since f is increasing on $[0, \hat{u})$ by property 3 of a strictly concave function f. Therefore, s is positive, and the shock wave is moving to the right. The characteristics and shock wave are plotted in Figure 6.

    For all $t \in [0, \Delta t]$, $F_j(t + t_n) = f(\tilde{\tilde{u}}(0, t)) = f(\tilde{\tilde{u}}^-) = \min\limits_{u \in [U_j^n, U_{j+1}^n]} f(u) = \text{constant}.$
  - If $\tilde{\tilde{u}}^- > \tilde{\tilde{u}}^+$, then the entropy condition is not satisfied and a shock wave cannot exist. Instead an expansion wave is created. Moreover, $\hat{u} \geqslant \tilde{\tilde{u}}^- > \tilde{\tilde{u}}^+$ which implies $0 < f'(\tilde{\tilde{u}}^-) < f'(\tilde{\tilde{u}}^+)$ by property 1 of a strictly concave function f, and therefore, $0 < \frac{1}{f'(\tilde{\tilde{u}}^+)} < \frac{1}{f'(\tilde{\tilde{u}}^-)} \leqslant \infty$. The characteristics and expansion wave are plotted in Figure 7.

    For all $t \in [0, \Delta t]$, $F_j(t + t_n) = f(\tilde{\tilde{u}}(0, t)) = f(\tilde{\tilde{u}}^-) = \max\limits_{u \in [U_{j+1}^n, U_j^n]} f(u) = \text{constant}.$
- If $\tilde{\tilde{u}}^-, \tilde{\tilde{u}}^+ \in (\hat{u}, u_{\max}]$
  - If $\tilde{\tilde{u}}^- < \tilde{\tilde{u}}^+$, then the entropy condition is satisfied and a shock wave is created with speed $s = \frac{f(\tilde{\tilde{u}}^+) - f(\tilde{\tilde{u}}^-)}{\tilde{\tilde{u}}^+ - \tilde{\tilde{u}}^-}$. The denominator is positive since $\tilde{\tilde{u}}^- < \tilde{\tilde{u}}^+$, and the numerator is negative since f is decreasing on $(\hat{u}, u_{\max}]$ by property 4 of a strictly concave function. Therefore, s

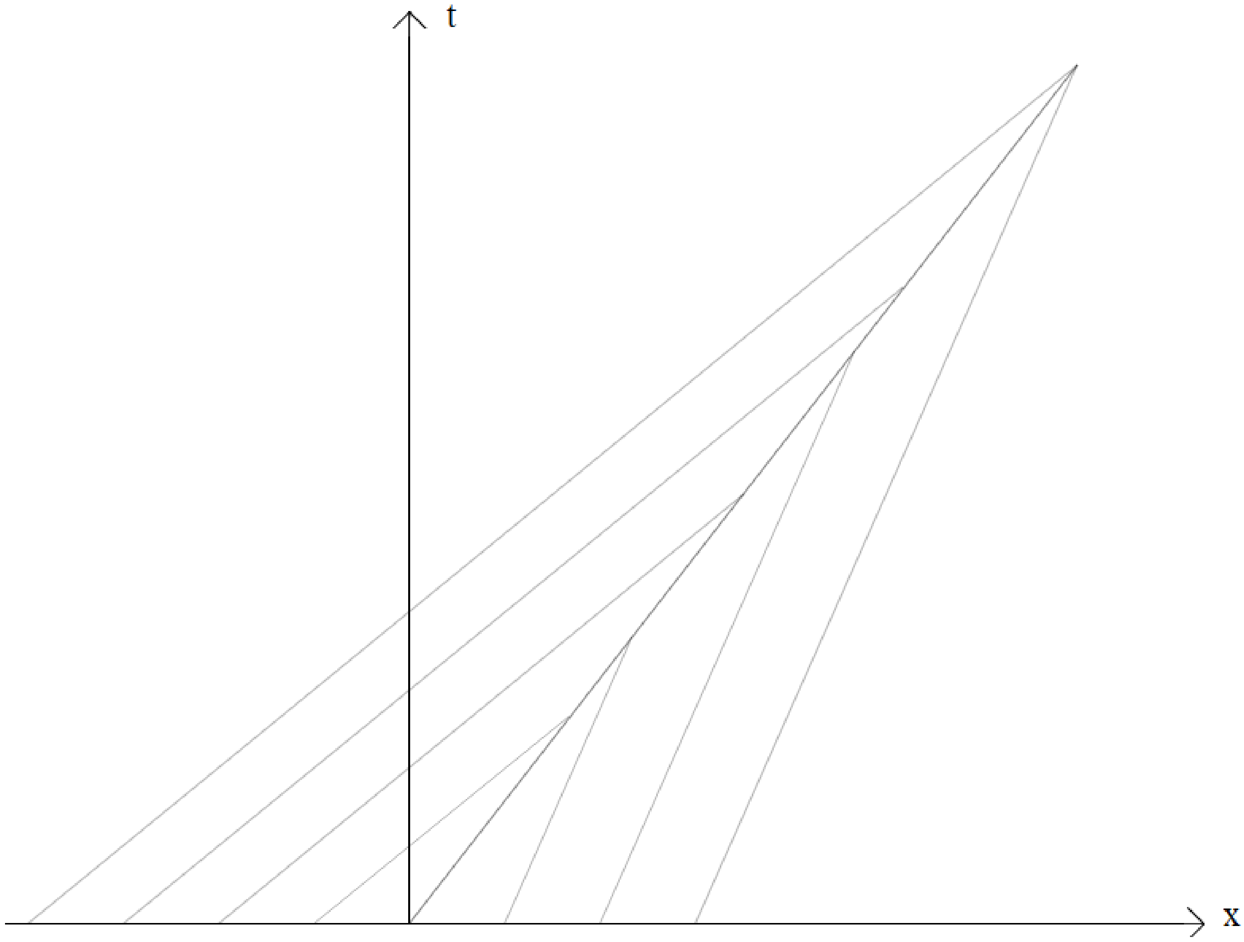

Figure 6: Characteristics of (19) with $\tilde{\tilde{u}}^- < \tilde{\tilde{u}}^+ \leqslant \hat{u}$, and the shock wave moving to the right

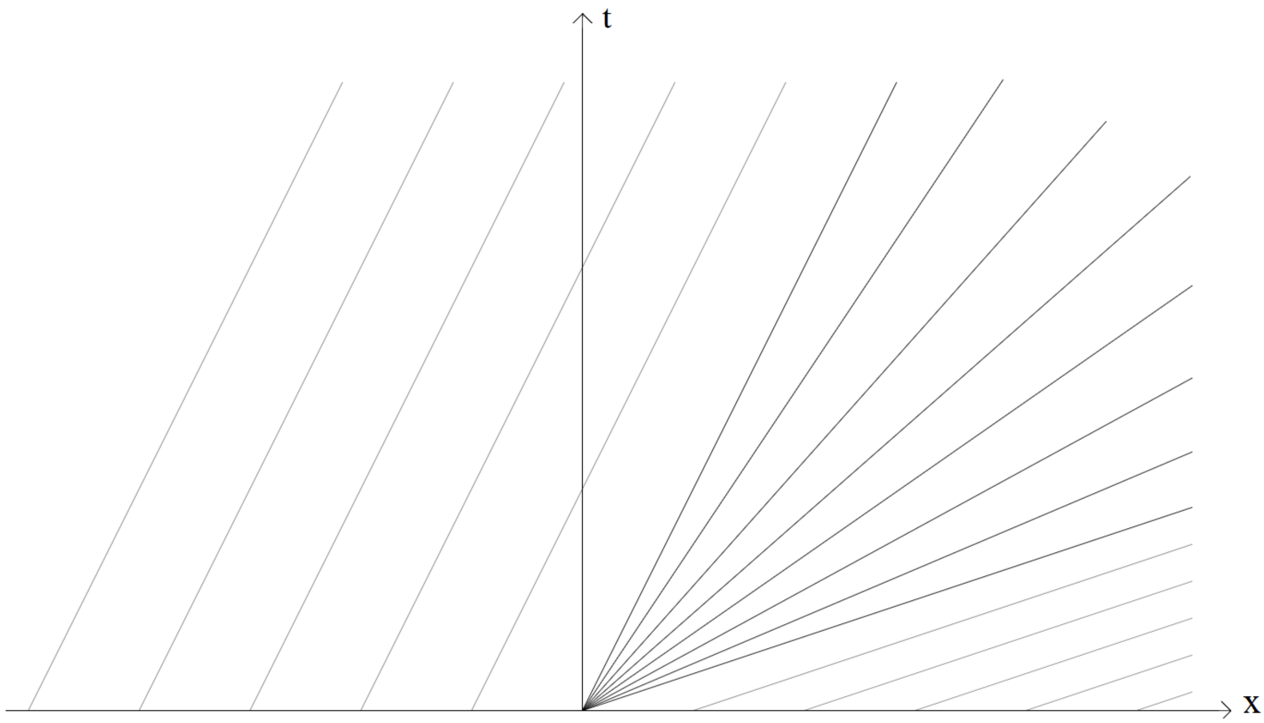

Figure 7: Characteristics of (19) with $\tilde{\tilde{u}}^+ < \tilde{\tilde{u}}^- \leqslant \hat{u}$, and the expansion wave moving to the right

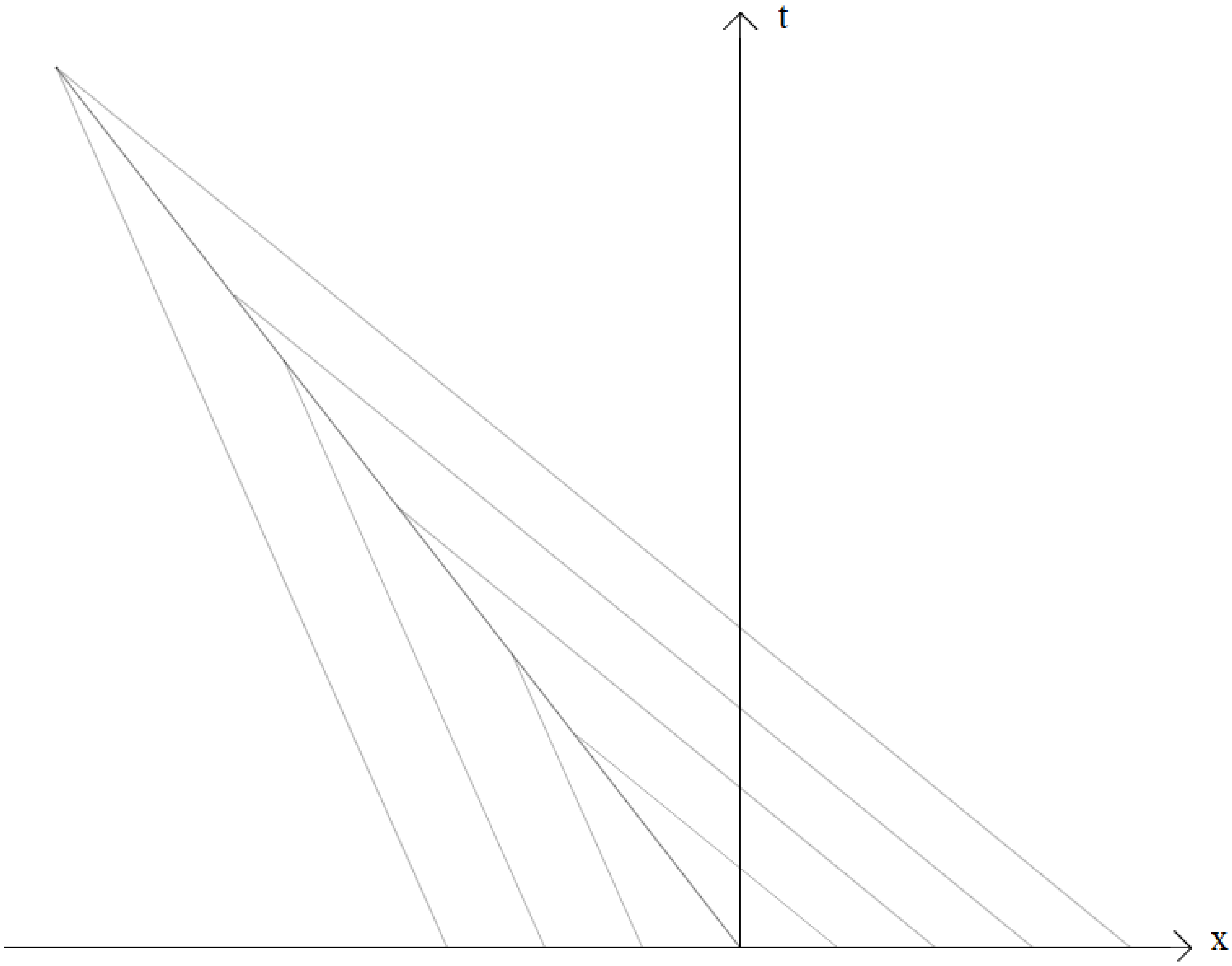

Figure 8: Characteristics of (19) with $\hat{u} < \tilde{\tilde{u}}^{-} < \tilde{\tilde{u}}^{+}$, and the shock wave moving to the left

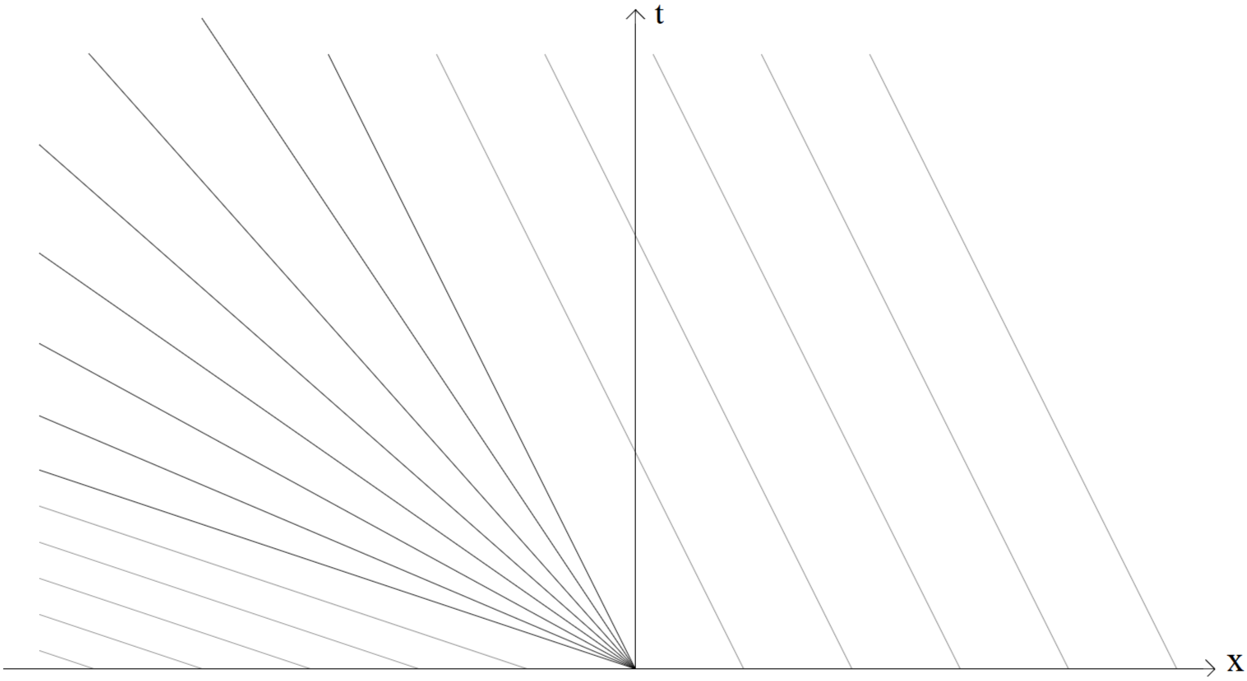

Figure 9: Characteristics of (19) with $\hat{u} < \tilde{\tilde{u}}^{+} < \tilde{\tilde{u}}^{-}$, and the expansion wave moving to the left

is negative, and the shock wave is moving to the left. The characteristics and shock wave are plotted as seen in Figure 8.

For all $t \in [0, \Delta t]$, $F_j(t + t_n) = f(\tilde{\tilde{u}}(0, t)) = f(\tilde{\tilde{u}}^{+}) = \min\limits_{u \in [U_j^n, U_{j+1}^n]} f(u) = \text{constant}.$

- If $\tilde{\tilde{u}}^{-} > \tilde{\tilde{u}}^{+}$, then the entropy condition is not satisfied and a shock wave cannot exist. Instead an expansion wave is created. Moreover, $\hat{u} < \tilde{\tilde{u}}^{-} < \tilde{\tilde{u}}^{+}$ which implies $0 > f'(\tilde{\tilde{u}}^{-}) > f'(\tilde{\tilde{u}}^{+})$ by property 1 of a strictly concave function f, and therefore, $0 < \frac{1}{f'(\tilde{\tilde{u}}^{-})} < \frac{1}{f'(\tilde{\tilde{u}}^{+})}$. The characteristics and expansion wave can are plotted in Figure 9.

  For all $t \in [0, \Delta t]$, $F_j(t + t_n) = f(\tilde{\tilde{u}}(0, t)) = f(\tilde{\tilde{u}}^{+}) = \max\limits_{u \in [U_{j+1}^n, U_j^n]} f(u) = \text{constant}.$

- If $\tilde{\tilde{u}}^{-} \in [0, \hat{u}]$ and $\tilde{\tilde{u}}^{+} \in (\hat{u}, u_{\max}]$ then a shock wave is admitted since $\tilde{\tilde{u}}^{-} < \tilde{\tilde{u}}^{+}$. The shock wave speed is $s = \frac{f(\tilde{\tilde{u}}^{+}) - f(\tilde{\tilde{u}}^{-})}{\tilde{\tilde{u}}^{+} - \tilde{\tilde{u}}^{-}}$. The denominator is positive so the sign of s is that of the numerator. There are two cases to consider:

  - If $f(\tilde{\tilde{u}}^{+}) \geqslant f(\tilde{\tilde{u}}^{-})$, then $s \geqslant 0$ and the shock wave is moving to the right (or is stationary if the equality holds). The characteristics and the shock wave are plotted in Figure 10.

    For all $t \in [0, \Delta t]$, $F_j(t + t_n) = f(\tilde{\tilde{u}}(0, t)) = f(\tilde{\tilde{u}}^{-}) = \min\limits_{u \in [U_j^n, U_{j+1}^n]} f(u) = \text{constant}.$

  - If $f(\tilde{\tilde{u}}^{+}) < f(\tilde{\tilde{u}}^{-})$, then $s < 0$ and the shock wave is moving to the left. The characteristics and the shock wave are plotted in Figure 11.

    For all $t \in [0, \Delta t]$, $F_j(t + t_n) = f(\tilde{\tilde{u}}(0, t)) = f(\tilde{\tilde{u}}^{+}) = \min\limits_{u \in [U_j^n, U_{j+1}^n]} f(u) = \text{constant}.$

- If $\tilde{\tilde{u}}^{+} \in [0, \hat{u}]$ and $\tilde{\tilde{u}}^{-} \in (\hat{u}, u_{\max}]$, then the entropy condition is not satisfied and a shock wave cannot exist. Instead an expansion wave is created. Moreover, $f'(\tilde{\tilde{u}}^{-}) < 0$ and $f'(\tilde{\tilde{u}}^{+}) > 0$ which implies that $\frac{1}{f'(\tilde{\tilde{u}}^{-})} < 0$ and $\frac{1}{f'(\tilde{\tilde{u}}^{+})} > 0$. The characteristics and the expansion wave are plotted in Figure 12.

  For all $t > 0$, $(0, t)$ belongs to the same characteristic line with equation $x = 0$. This means that

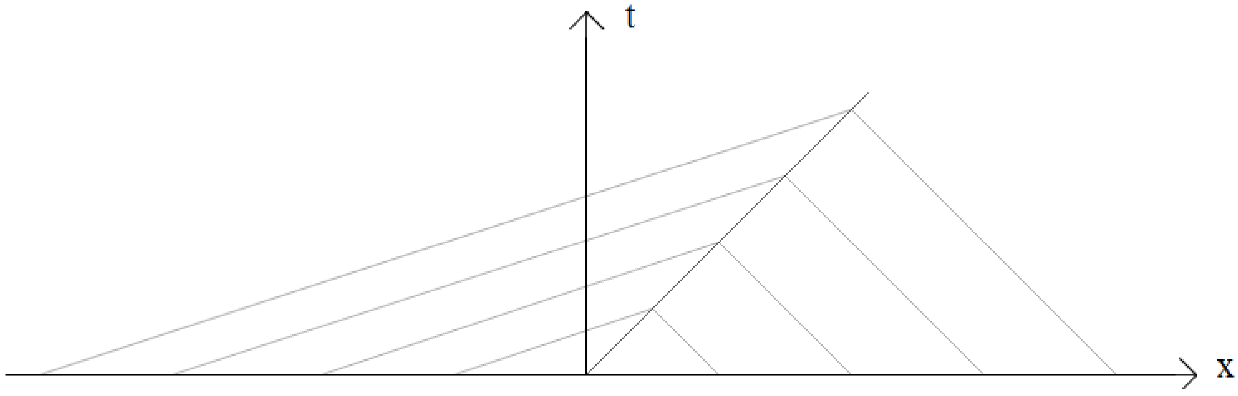

Figure 10: Characteristics of (19) with $\tilde{\tilde{u}}^- \leqslant \hat{u} < \tilde{\tilde{u}}^+$ and $f(\tilde{\tilde{u}}^+) \geqslant f(\tilde{\tilde{u}}^-)$, and the shock wave moving to the right

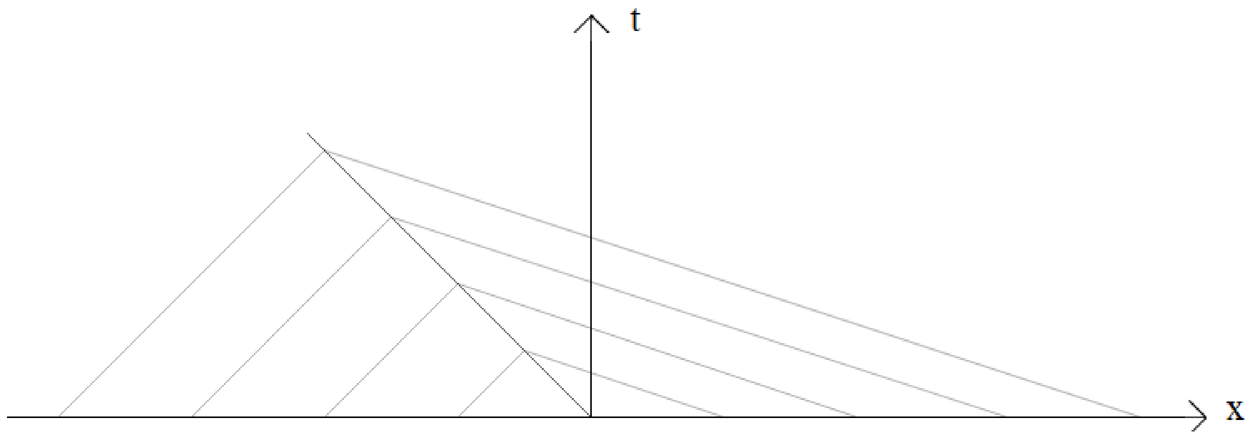

Figure 11: Characteristics of (19) with $\tilde{\tilde{u}}^- \leqslant \hat{u} < \tilde{\tilde{u}}^+$ and $f(\tilde{\tilde{u}}^+) < f(\tilde{\tilde{u}}^-)$, and the shock wave moving to the left

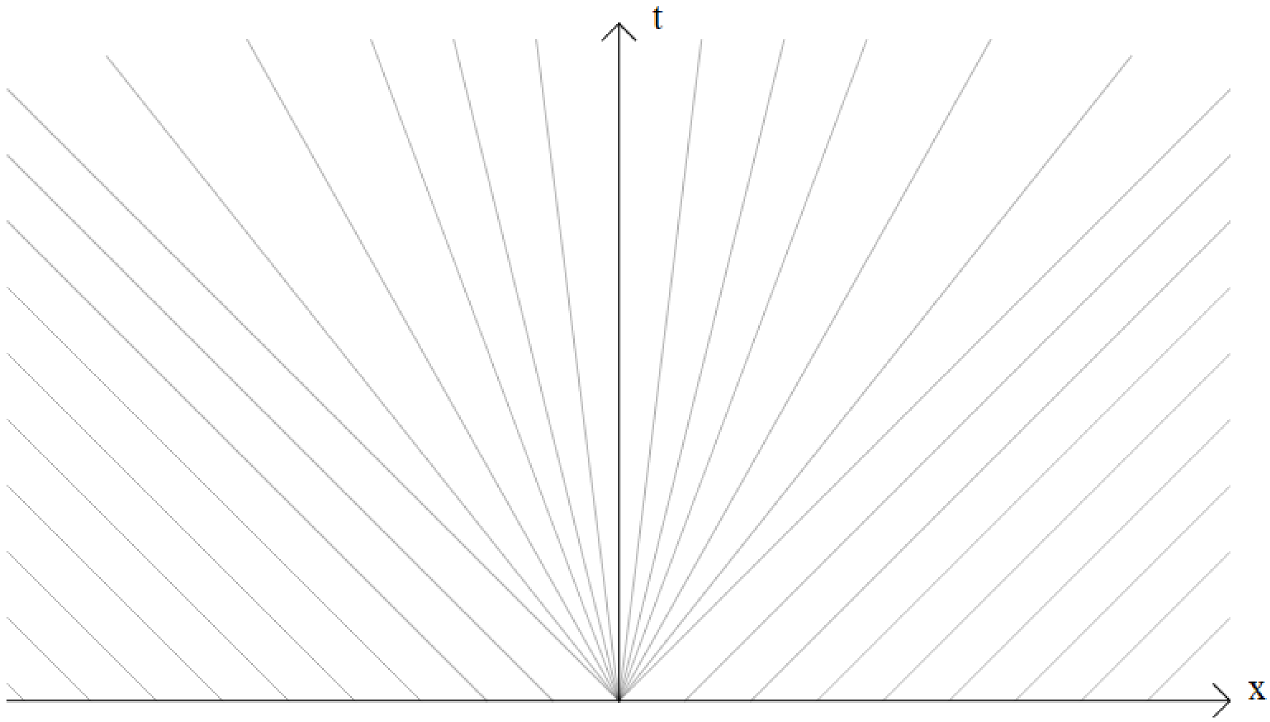

Figure 12: Characteristics of (19) with $\tilde{\tilde{u}}^+ \leqslant \hat{u} < \tilde{\tilde{u}}^-$, and the expansion wave expanding in both directions

the value of $\tilde{\tilde{u}}$ is a constant $\tilde{\tilde{u}}^*$ at these points, and $f'(\tilde{\tilde{u}}^*) = 0$. Therefore, $\tilde{\tilde{u}}^* = \hat{u}$, and for all $t \in [0, \Delta t]$, $F_j(t + t_n) = f(\tilde{\tilde{u}}(0,t)) = f(\hat{u}) = \max_{u \in [U_{j+1}^n, U_j^n]} f(u) = \text{constant}$.

Thus for all $t \in [t_n, t_{n+1}]$, $F_j(t)$ is a constant F_j^n, where

$$F_j^n = \begin{cases} \min_{u \in [U_j^n, U_{j+1}^n]} f(u) & \text{if} \quad U_j^n \leqslant U_{j+1}^n \\ \max_{u \in [U_{j+1}^n, U_j^n]} f(u) & \text{if} \quad U_j^n > U_{j+1}^n. \end{cases} \tag{20}$$

3.4 The algorithm

Recall that the exact averages U_j^n's cannot be found without prior knowledge u. What is computed instead are $\tilde{U}_j^n$'s, the approximations of the averages. This can be done as follows:

1. Δx and Δt are chosen so that they satisfy the CFL condition (18).

2. For all $j \in \mathbb{Z}$, U_j^0 is constructed from $u_0(x)$ by equation (12). Set $\tilde{U}_j^0 = U_j^0$.

3. For all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, $F_j^n = \begin{cases} \min_{u \in [\tilde{U}_j^n, \tilde{U}_{j+1}^n]} f(u) & \text{if} \quad \tilde{U}_j^n \leqslant \tilde{U}_{j+1}^n \\ \max_{u \in [\tilde{U}_{j+1}^n, \tilde{U}_j^n]} f(u) & \text{if} \quad \tilde{U}_j^n > \tilde{U}_{j+1}^n \end{cases}$.

4. For all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{U}_j^{n+1} &= \tilde{U}_j^n + \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} (F_{j-1}(t) - F_j(t))dt \\ &= \tilde{U}_j^n + \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} (F_{j-1}^n - F_j^n)dt \\ &= \tilde{U}_j^n + \frac{\Delta t}{\Delta x}(F_{j-1}^n - F_j^n) \quad \forall j \in \mathbb{Z}, \forall n \in \mathbb{N}. \end{aligned} \tag{21}$$

A piecewise constant numerical density U can be constructed from the $\tilde{U}_j^n$'s according to (14).

3.5 Trajectory of a car

Consider a car c whose position at time $t = 0$ is $x = x_0$. Let $x_c(t)$ and $v_c(t)$ be its position and velocity respectively, at any $t \geqslant 0$. Then:

$$\forall t \geqslant 0, \quad x_c(t) = \int_0^t v_c(z)dz + x_0 = \int_0^t v(u(x_c(z), z))dz + x_0.$$

After differentiating both sides with respect to t, one gets:

$$\begin{cases} x_c'(t) = v(u(x_c(t), t)), & \forall t \geqslant 0 \\ x_c(0) = x_0. \end{cases}$$

A numerical solution X_c to the initial value problem above can be constructed using Euler's method as follows

$$\begin{aligned} T^n &= n\Delta t, \quad \forall n \in \mathbb{N} \\ X_c^0 &= x_0 \\ V_c^n &= v(U(X_c^n, T^n)), \quad \forall n \in \mathbb{N} \\ X_c^{n+1} &= X_c^n + \Delta t V_c^n, \quad \forall n \in \mathbb{N} \end{aligned}$$

. Note the X_c^n is the numerically computed position of c at time T^n.

4 Simulating Traffic Lights

4.1 Introduction

Traffic flow problems can get very complicated making finding a solution by hand through the method of characteristics very cumbersome. However, one can use the algorithm described in subsection 3.5 to get numerical solutions to traffic flow problems. Throughout the rest of this section, the algorithm will be used to run simulations in order to understand the effects of traffic lights on the behavior of traffic.

4.2 The traffic lights problem

A traffic lights cycle consists of two parts. The first part lasts for a period of t_r during which the traffic lights are red, and the second part lasts for a period of t_g during which the traffic lights are green.

Consider a road with traffic lights at $x = 0$. The velocity function $v(u)$ is known; so are the flux function $f(u) = v(u)u = v_{\max}(1 - u/u_{\max})u$ and the initial density $u_0(x) = u(x, 0) = u_0$, which is constant for all $x \in \mathbb{R}$. The road can be divided into two parts: the one to the left of the traffic lights and the one to the right of it. During the first part of a cycle, the traffic lights are red. This means that traffic cannot flow from the left part of the road to the right part. Two things can be inferred from this. The first is that the two parts of the road can be treated separately. The second is that the flux at the traffic lights is $f(u(0,t)) = 0$. Since $f(u) = v(u)u$, either $u = 0$ or $v(u) = 0$. During the second part of a cycle, the traffic lights are green, and traffic can flow from the left part of the road to the right part. Therefore, the two parts of the road have to be treated as one system.

The first cycle begins at time $t = 0$ and ends at time $t = t_r + t_g$. What follows is a method for finding the solution to the traffic lights problem during the first cycle. Consider the left part of the road. The initial density is u_0 to the left of the traffic lights, i.e. when $x < 0$. Since $f(u_0(0)) = 0$, either $u_0(0) = 0$ or $v(u_0(0)) = 0$. Moreover, cars to the left of a red traffic light stand stationary, so $v(u_0(0)) = 0$. This implies that $u_0(0) = u_{\max}$. The density of cars to the left of the traffic lights can be found up to a time t_r by solving the following Riemann problem:

$$\tilde{u}_t + f(\tilde{u})_x = 0$$
$$\tilde{u}(x,0) = \begin{cases} u_0 & \text{if} \quad x < 0 \\ u_{\max} & \text{if} \quad x \geqslant 0. \end{cases}$$

Then $u(x,t) = \tilde{u}(x,t)$ for all $x \leqslant 0$ and $t \leqslant t_r$. Now consider the right part of the road. The initial density is u_0 to the right of the traffic lights, i.e. when $x > 0$. Since $f(u_0(0)) = 0$, either $u_0(0) = 0$ or $v(u_0(0)) = 0$. Moreover, there are no cars directly to the right of a red traffic light, as cars to the left of it are not allowed to pass. This implies that $u_0(0) = 0$. The density of cars to the right of the traffic lights can be found up to a time t_r by solving the following Riemann problem:

$$\tilde{u}_t + f(\tilde{u})_x = 0$$
$$\tilde{u}(x,0) = \begin{cases} 0 & \text{if} \quad x \leqslant 0 \\ u_0 & \text{if} \quad x > 0. \end{cases}$$

Then $u(x,t) = \tilde{u}(x,t)$ for all $x > 0$ and $t \leqslant t_r$. After a time t_r has passed the traffic lights turn and remain green for a period of time t_g. During this time, cars are allowed to flow from the left part of the road to the right part of the road. The solution during $t \in (t_r, t_g]$ can be found by solving the following traffic flow problem problem:

$$\tilde{u}_t + f(\tilde{u})_x = 0$$
$$\tilde{u}(x,0) = u(x,t_r) \quad \text{for} \quad x \in \mathbb{R}.$$

Then $u(x,t) = \tilde{u}(x, t - t_r)$ for all x and $t \in (t_r, t_r + t_g]$.

Now that $u(x,t)$ is known for all x and all $t \in [0, t_r + t_g]$, the solution of u for the next cycle can be found by repeating the same method described above with one key difference: a new initial density $u_0(x) = u(x, t_r + t_g)$, is used. In general, if $n \in \mathbb{Z}_+$ and the solution for the n^{th} cycle is known, the solution of u can be found for the $(n+1)^{\text{th}}$ cycle by repeating the same method with $u_0(x) = u(x, n(t_r + t_g))$.

4.3 The solution over one traffic lights cycle with low initial constant density

The problems described above can be solved using the algorithm described in subsection 3.4, to obtain a numerical solution U. This solution can then be graphically visualized.

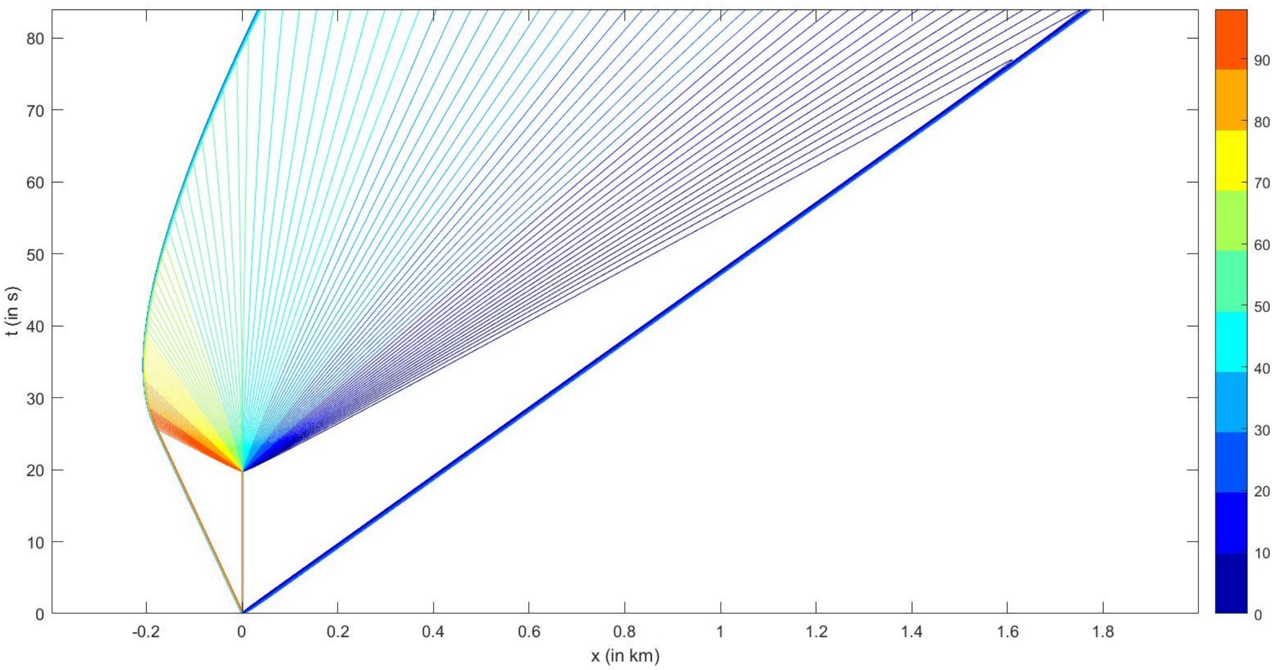

Figure 13: The characteristics of a traffic lights problem with low initial constant density during one traffic lights cycle

For example, the characteristics of the traffic lights problem with the following parameter values:

$$v_{\max} = 100\,\text{km/h}$$
$$u_{\max} = 100\,\text{cars/km}$$
$$u_0 = 25\,\text{cars/km}$$
$$t_r = 20\,\text{s}$$
$$t_g = 65\,\text{s}$$

can be seen in Figure 13.

The program with which Figure 13 was generated plots the contours (or the level curves) of U where U is changing. Therefore the lines seen in Figure 13 are the characteristics. Moreover, the density is constant in the four parts of the figure where there are no contour lines. Another way to visualize U in a two-dimensional plot is to visualize the values of U with colors as in Figure 14. The characteristics can also be seen in Figure 14, as they are the lines made up of points of the same color; these are points that have the same value and therefore belong to the same level curve. It also provides the constant values of U which Figure 13 did not.

Consider Figure 14. At $t = 0\,\text{s}$ the traffic lights turn red. Two shock waves emerge from $x = 0\,\text{km}$. The one on the left is due to a jump from $u^- = u_0 = 25\,\text{cars/km}$ (in light blue) to $u^+ = u_{\max} = 100\,\text{cars/km}$ (in dark red). The one on the right is due to a jump from $u^- = 0\,\text{cars/km}$ (in dark blue) to $u^+ = u_0 = 25\,\text{cars/km}$ (in light blue). This shock wave can be seen in Figure 15, a three-dimensional plot of U.

At $t_r = 20\,\text{s}$ the traffic lights turn green. An expansion wave starts at $x = 0\,\text{km}$ to continuously decrease the density from $u^- = u_{\max} = 100\,\text{cars/km}$ to $u^+ = 0\,\text{cars/km}$. This expansion wave can be divided into a left part which includes the signals with densities in $[u_0, u_{\max}]$ and a right part which includes the signals with densities in $[0, u_0)$.

The shock wave on the left first intersects paths with the expansion wave when it meets the first signal of the expansion wave from the left, which has density $u_{\max}$. This is the top left point of the dark red triangle in Figure 14. The discontinuity is between $u^- = u_0$ and $u^+ = u_{\max}$. After this point the shock wave meets signals with densities continuously decreasing from $u_{\max}$ to u_0.

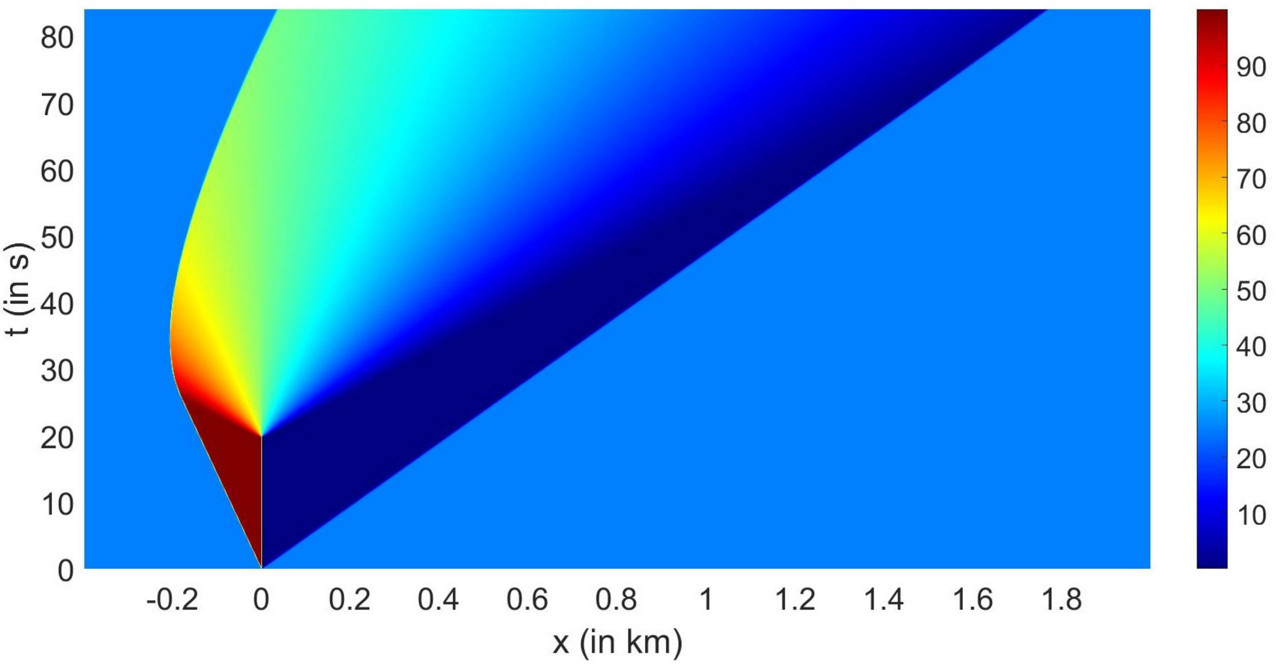

Figure 14: The characteristics of a traffic lights problem with low initial constant density during one traffic lights cycle

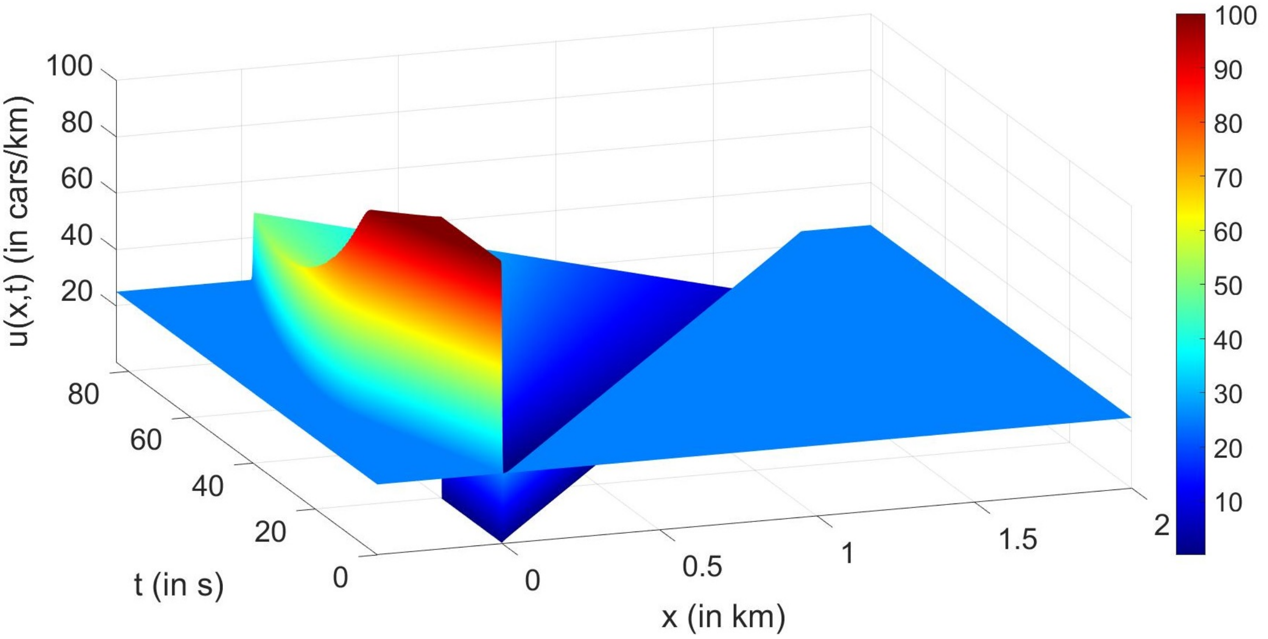

Figure 15: The shock wave of a traffic lights problem with low initial constant density during one traffic lights cycle

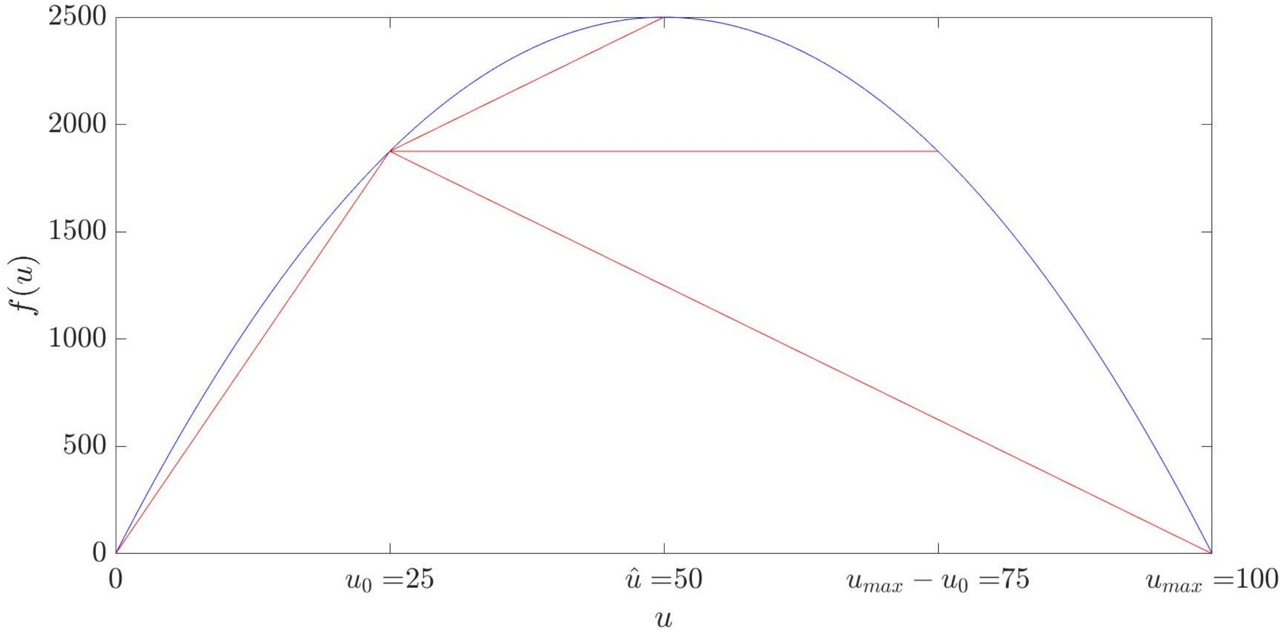

Figure 16: The secants of f between u_0 (low density) and $u \in [0, u_{\max}]$

The speed of the shock wave between u^- and u^+ is $s = \frac{f(u^+)-f(u^-)}{u^+-u^-}$ which is the slope of the secant of f between u^- and u^+ (given that $u^- < u^+$). Therefore the change of the speed of the left shock wave can be seen in Figure 16.

Note that:

$$\begin{aligned} f(u_{\max} - u_0) &= v_{\max}(1 - \frac{u_{\max} - u_0}{u_{\max}})(u_{\max} - u_0) \\ &= v_{\max}(\frac{u_0}{u_{\max}})(u_{\max} - u_0) \\ &= v_{\max}(1 - \frac{u_0}{u_{\max}})(u_0) \\ &= f(u_0). \end{aligned}$$

As the shock wave meets signals continuously decreasing from $u_{max} = 100$ cars/km to $u_{\max} - u_0 = 75$cars/km, the speed of the shock wave continuously increases from the negative value of $s = -25$km/h (the shock wave is traveling to the left) to $s = 0$ km/h (the shock wave is stationary). After that the shock wave meets signals continuously decreasing from 75 cars/km to $\hat{u} = 50$ cars/km. The speed of the shock wave increases continuously from $s = 0$ km/h to $s = 25$ km/h (the shock wave is moving to the right). Finally, as the shock wave meets signals continuously decreasing from $\hat{u}$ to u_0, its speed continues to increase until it reaches a maximum value of $f'(u_0) = 50$ km/h. This happens asymptotically as t_g tends to ∞.

The shock wave on the right first intersects paths with the expansion wave when it meets the first signal of the expansion wave from the right, which has density 0 cars/km. This is the top right point of the dark blue triangle in the Figure 14. The discontinuity is between $u^- = 0$ cars/km and $u^+ = u_0 = 25$ cars/km. After this point the shock wave meets signals with densities continuously increasing from 0 cars/km to 25 cars/km. At this point the speed of the shock wave is $s = 75$ km/h. As the densities of the signals intersecting with the right shock wave increases, the speed of the shock wave decreases until it reaches a minimum of $f'(u_0) = 50$ km/h. This also happens asymptotically as t_g tends to ∞. The intersections of the left and right shock waves with the expansion wave can also be seen in Figure 17.

If a car was in position x_c and time t_c the trajectory of this car can be traced for all $t \geqslant t_c$ using the algorithm described in subsection 3.5. Plotting the trajectories of cars that are evenly spaced with

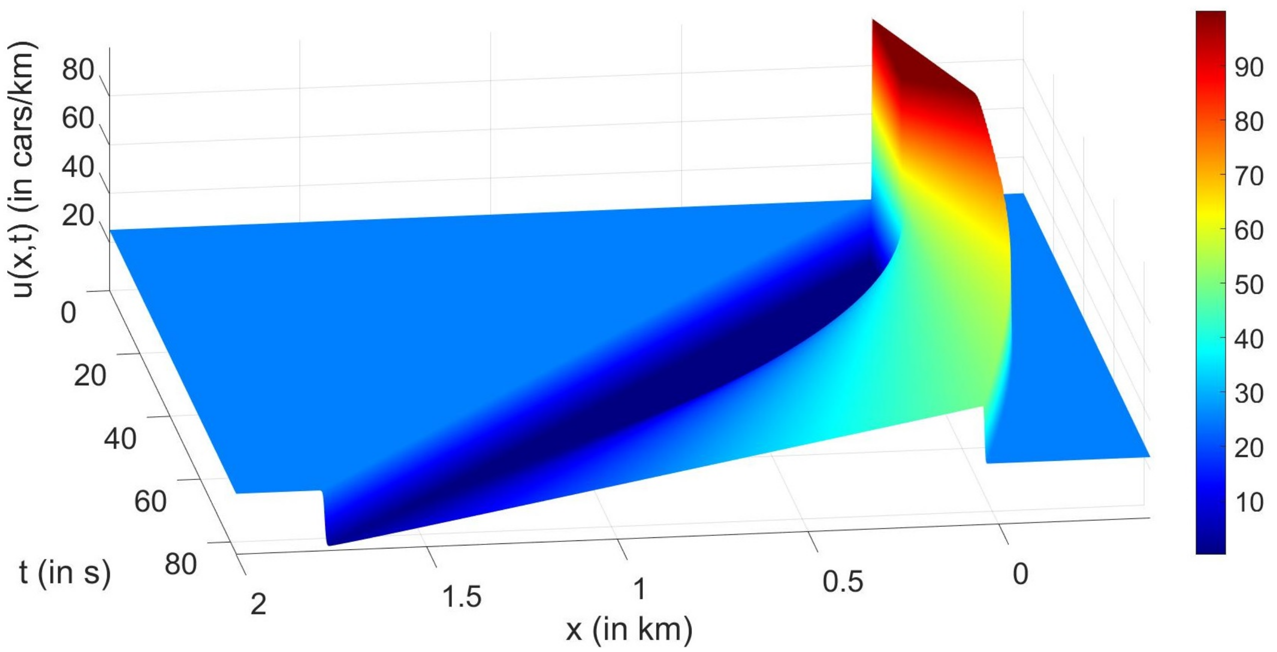

Figure 17: The expansion wave of a traffic lights problem with low initial constant density during one traffic lights cycle

density u_0 at $t = 0$ allows for the visualization of the trajectory of every car on the road. For a clearer figure the trajectory of every second car on the road was plotted in Figure 18.

4.4 Solution over several traffic lights cycles with low initial constant density

In the previous example, it was found that when $u_0 \in [0, \hat{u})$ (i.e for a low initial constant density), the solution over the first cycle can be characterized by two shock waves and an expansion wave. The shock wave on the left was initially moving to the left with constant speed. After the traffic lights turn green, it starts decelerating until it becomes stationary. Then it starts moving to the right with an increasing speed. The second cycle starts at time $t = t_r + t_g$. At this time, the shock wave on the left might still be to the left of the traffic lights, or it might have moved to the right of it. To investigate the effect of this on traffic, two simulations are ran.

For both simulations, $v_{\max} = 100\,\text{km/h}$, $u_{\max} = 100\,\text{cars/km}$, $u_0 = 25\,\text{cars/km}$ and $t_r = 20\,\text{s}$. The only difference is that t_g is chosen to be 65 s for first simulation and 25 s for the second simulation. The solutions to both simulations were found numerically as described in subsection 4.2. The characteristics for simulation 1 can be seen in the Figure 19.

At time $t_r + t_g = 85\,\text{s}$, the shock wave can be seen in the figure above to be to the right of the traffic lights (the time axis). Therefore, $u(x, t_r + t_g) = u_0$ for all $x < 0$. Then the initial density for the left part of the road is the same for both cycles and $u(x,t) = u(x, t + t_r + t_g)$ for all $t \in [0, t_r]$ and for all $x \leqslant 0$. Moreover, when the traffic lights turn green during the second cycle, the same expansion wave as in cycle one will be created to continuously decrease u from $u_{\max} = 100\,\text{cars/km}$ to $0\,\text{cars/km}$. Therefore, $u(x,t) = u(x, t + t_r + t_g)$ for all $t \in [0, t_r + t_g]$ and for all $x \leqslant 0$. By a similar argument this property can be shown for any two consecutive cycles therefore, $u(x,t) = u(x, t + n(t_r + t_g))$ for all $t \in [0, t_r + t_g]$, $x \leqslant 0$ and $n \in \mathbb{Z}_+$. This leads to the following conclusion. If the last car waiting in the line of stationary cars formed during the first part of cycle one passes the traffic lights during the second part of cycle one, then the last car waiting in the line of stationary cars formed during the first part of any cycle passes the traffic lights during the second part of that cycle.

This can be seen in Figure 20 showing the trajectory of every fourth car on the road in simulation 1. Note that the last car waiting in the line of stationary cars in any cycle, passes the traffic lights in

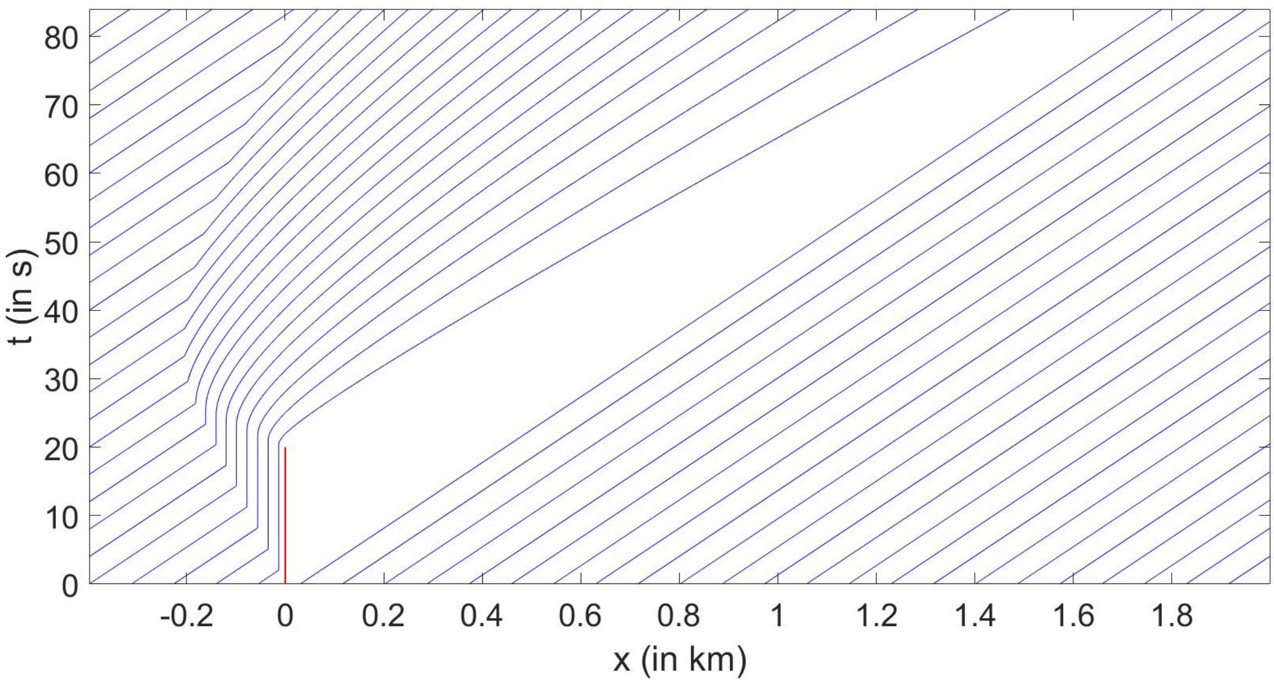

Figure 18: The trajectory of every second car on the road for a traffic lights problem with low initial constant density during one traffic lights cycle

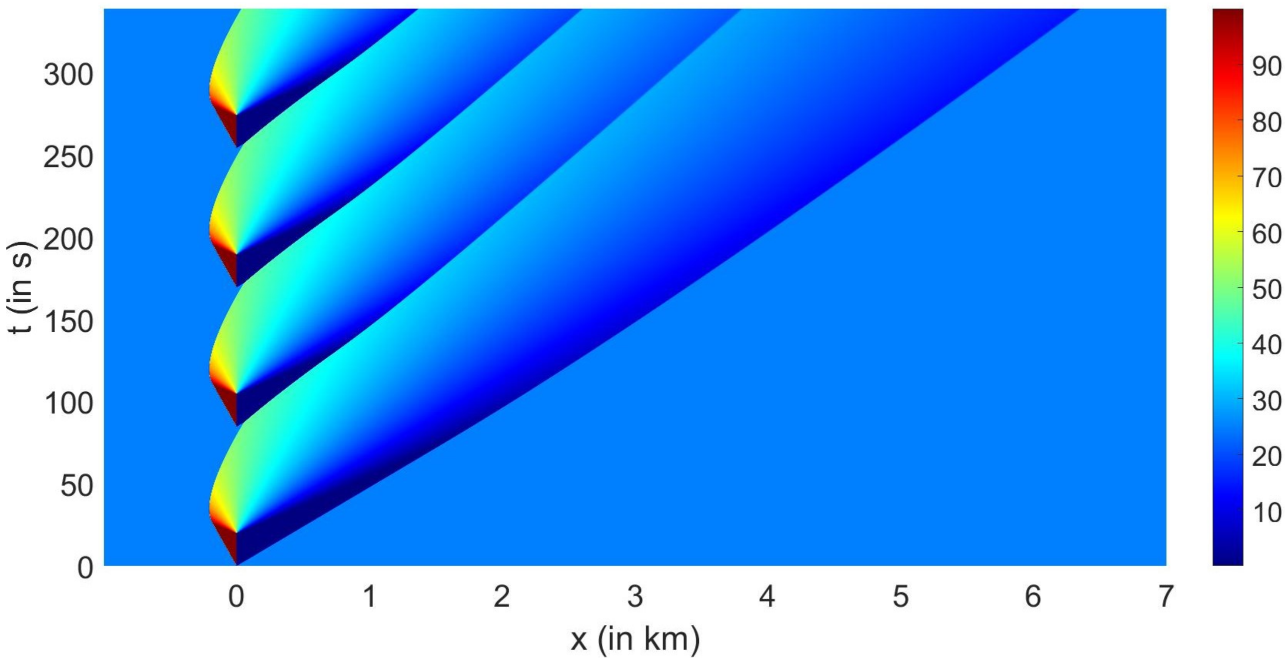

Figure 19: The characteristics of simulation 1 during several traffic lights cycles

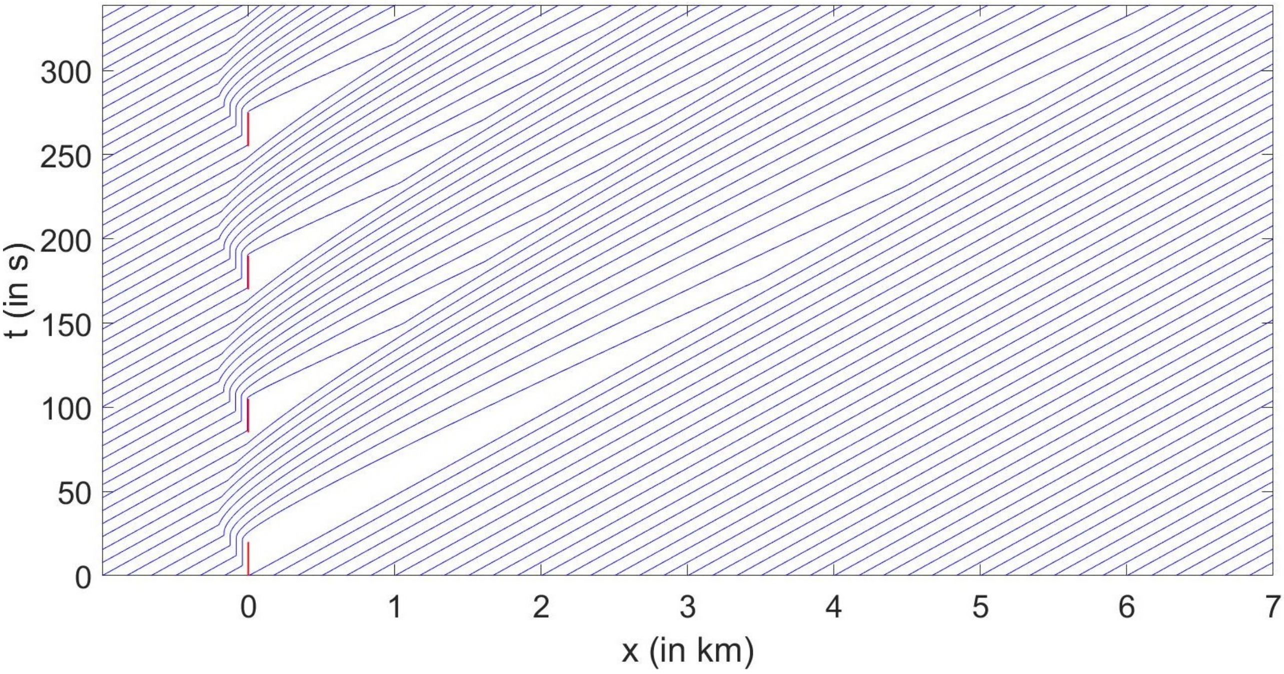

Figure 20: The trajectory of every fourth car on the road in simulation 1 during several traffic lights cycles

that same cycle and there is no traffic build up.

The characteristics for simulation 2 during several traffic lights cycles can be seen in Figure 21. Unlike in simulation 1, at the end of the first cycle ($t = t_r + t_g = 45$ s), the shock wave remains to the left of the traffic lights (the time axis). The initial density for the second cycle is different than that of the first cycle. The difference is over an interval $[\tilde{x}, 0]$ where $\tilde{x}$ is a real negative number. Over this interval, the initial density for the second cycle is a set of continuous high densities (i.e they belong to $[\hat{u}, u_{\max}]$). It can be seen from Figure 16 that jumps from high densities to $u_{\max}$ lead to shock waves moving faster to the left than jumps from u_0 (low density) to $u_{\max}$. Note that the length of the line of stationary cars is the x-coordinate of the intersection of the shock wave on the left, and the first signal of the expansion wave from the left (which has density $u_{\max}$). If the shock wave on the left is moving faster to the left than in the previous cycle, its intersection with that signal of the expansion wave will also be further to the left. Therefore, it leads to longer traffic lines. This can be seen in the Figure 21, where every cycle has a longer traffic line than the one before it. The trajectory of every fourth car on the road can be seen in Figure 22. This figure shows how some cars have to wait at the same traffic lights for two cycles.

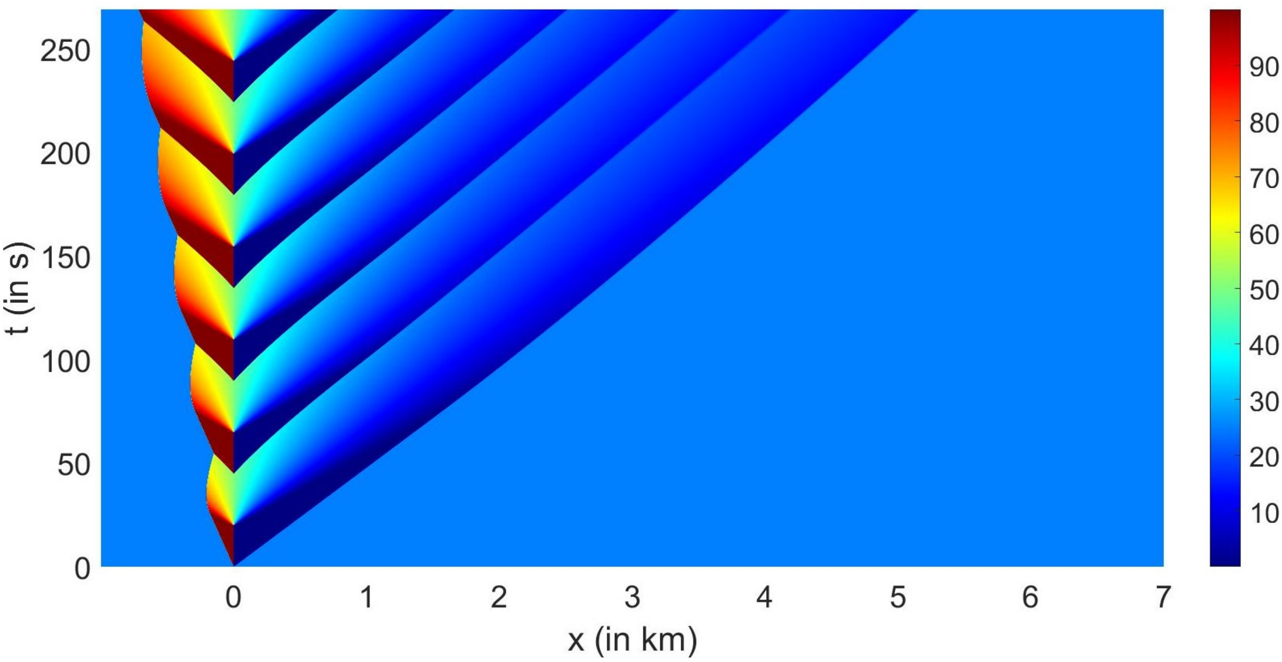

Figure 21: The characteristics of simulation 2 during several traffic lights cycles

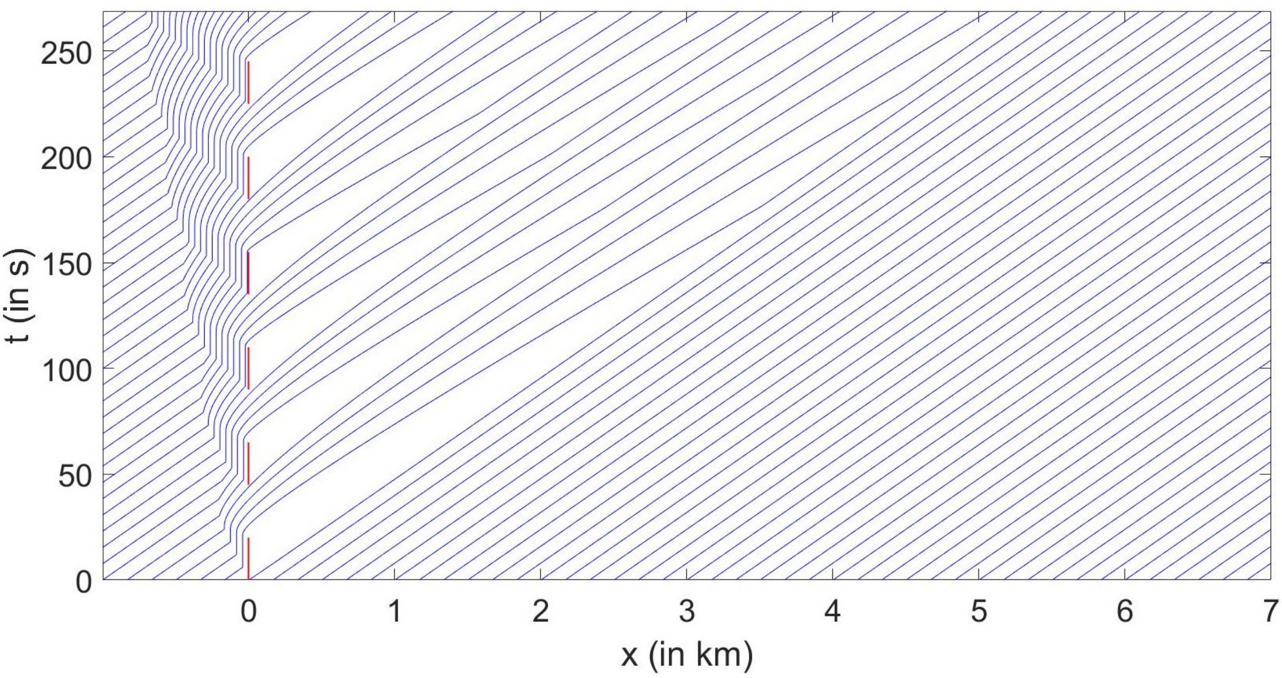

Figure 22: The trajectory of every fourth car on the road in simulation 2 during several traffic lights cycles

4.5 Criterion 1: choice of t_g given t_r and $u_0 \in [0, \hat{u})$

In subsection 4.4, it was seen that for a given time t_r and constant initial density of $u_0 \in [0, \hat{u})$, the choice of t_g had a clear effect on traffic. Therefore, a set of criteria can be set to choose a value for t_g in order to achieve certain desirable effects.

One such desirable effect is that any car that has to stop for the traffic lights during the first part of a traffic lights cycle, must pass the traffic lights during the second part of the same cycle.

During the first cycle, this effect is achieved when the number of cars that pass the traffic lights between the times t_r and $t_r + t_g$ is greater than or equal to the number of cars that had to stop for the red traffic lights during the first part of the cycle.

The number of cars that pass the traffic lights between the times t_r and $t_r + t_g$ is:

$$\int_{t_r}^{t_r+t_g} f(u(0,t))dt.$$

However, the value of $f(u(0,t))$ is determined by the expansion wave which started at $(0, t_r)$ for all $t \in [t_r, t_r + t_g]$. Along the time axis there is only one signal. Therefore $f(u(0,t))$ is constant. Since the speed of the signal is 0, this is the signal with density $\hat{u}$. Thus, $f(u(0,t)) = f(\hat{u})$ for all $t \in [t_r, t_r + t_g]$.

The number of cars that pass the traffic lights between the times t_r and $t_r + t_g$ can be reformulated as:

$$\int_{t_r}^{t_r+t_g} f(\hat{u})dt = f(\hat{u})t_g.$$

The number of cars that had to stop for the red traffic lights is the product of the length of the line of stationary cars and their density $u_{\max}$. However, by referring to Figure 14, it can be seen that the length of the line of stationary cars is the absolute value of the x-coordinate of the intersection point of the shock wave on the left and the first signal from the left of the expansion wave which has density $u_{\max}$.

The shock wave on the left is described by this equation:

$$x = \frac{f(u_{\max}) - f(u_0)}{u_{\max} - u_0}t = \frac{f(u_0)}{u_0 - u_{\max}}t. \tag{22}$$

The first signal from the left of the expansion wave is described by this equation:

$$x = f'(u_{\max})(t - t_r).$$

The x-coordinate of their intersection point is

$$x^* = \frac{t_r f(u_0) f'(u_{\max})}{(u_0 - u_{\max})f'(u_{\max}) - f(u_0)}.$$

The number of cars that had to stop for the red traffic light is

$$-x^* u_{\max} = \frac{u_{\max} t_r f(u_0) f'(u_{\max})}{f(u_0) - (u_0 - u_{\max})f'(u_{\max})}.$$

All the cars that stopped for the red traffic light during the first part of the first traffic lights cycle will pass the traffic lights during the second part of the first cycle if:

$$\begin{aligned} f(\hat{u})t_g &\geqslant \frac{u_{\max} t_r f(u_0) f'(u_{\max})}{f(u_0) - (u_0 - u_{\max})f'(u_{\max})} \\ \iff t_g &\geqslant \frac{u_{\max} t_r f(u_0) f'(u_{\max})}{f(\hat{u})(f(u_0) - (u_0 - u_{\max})f'(u_{\max}))}. \end{aligned} \tag{23}$$

Assume the shock wave on the left reaches the traffic lights at $x = 0$ at time $t = t_{\text{shock}}$. It was seen in the previous section that if $t_r + t_g \geqslant t_{\text{shock}}$, u mimics itself every cycle on the left part of the road. Therefore it is enough to study the behavior of traffic on the left part of the road during the first cycle. If

$$t_g = \max\left(t_{\text{shock}} - t_r, \frac{u_{\max} t_r f(u_0) f'(u_{\max})}{f(\hat{u})(f(u_0) - (u_0 - u_{\max}) f'(u_{\max}))}\right), \tag{24}$$

then any car that is stopped by the traffic lights during a cycle, will pass the traffic lights during that same cycle.

4.6 A numerical method for finding t_{shock}

The time at which the shock wave on the left is in position x^* can be found by plugging in x^* in equation (22) to get

$$t^* = \frac{(u_0 - u_{\max}) t_r f'(u_{\max})}{(u_0 - u_{\max}) f'(u_{\max}) - f(u_0)}.$$

At this point the discontinuity is between $u^- = u_0$ and u^+ which is determined by the expansion wave.

The value of u inside the expansion wave which started at $(0, t_r)$ can be found by the following relation:

$$\begin{aligned} f'(u(x,t)) &= \frac{x - 0}{t - t_r} \\ \implies v_{\max}\left(1 - \frac{2u(x,t)}{u_{\max}}\right) &= \frac{x}{t - t_r} \\ \implies u(x,t) &= \frac{u_{\max}}{2} - \frac{u_{\max} x}{2 v_{\max}(t - t_r)}. \end{aligned}$$

The speed of the shock wave at a time t is can then be found by

$$x'_{\text{shock}}(t) = s(x_{\text{shock}}(t), t) = \frac{f(u(x_{\text{shock}}(t), t)) - f(u_0)}{u(x_{\text{shock}}(t), t) - u_0},$$

where $x_{\text{shock}}(t)$ is the position of the shock wave at time t. This is an ordinary differential equation with an initial value of $x_{\text{shock}}(t^*) = x^*$.

The trajectory of the shock wave can be found using Euler's method:

$$\begin{cases} x_0 = x^* \\ t_0 = t^* \\ s_n = s(x_n, t_n) \quad \forall n \in \mathbb{N} \\ x_{n+1} = x_n + s_n \Delta t \quad \forall n \in \mathbb{N}\backslash\{0\} \\ t_{n+1} = t_n + \Delta t \quad \forall n \in \mathbb{N}\backslash\{0\} \end{cases}$$

where $\Delta t > 0$ is predetermined step size.

This algorithm can be ran while $x_n < 0$. When it terminates the last t_n will be t_{shock}.

4.7 Example

Consider a traffic lights problem with the following parameter values:

$$\begin{aligned} v_{\max} &= 100\ \text{km/h} \\ u_{\max} &= 100\ \text{cars/km} \\ u_0 &= 30\ \text{cars/km} \\ t_r &= 20\ \text{s}. \end{aligned}$$

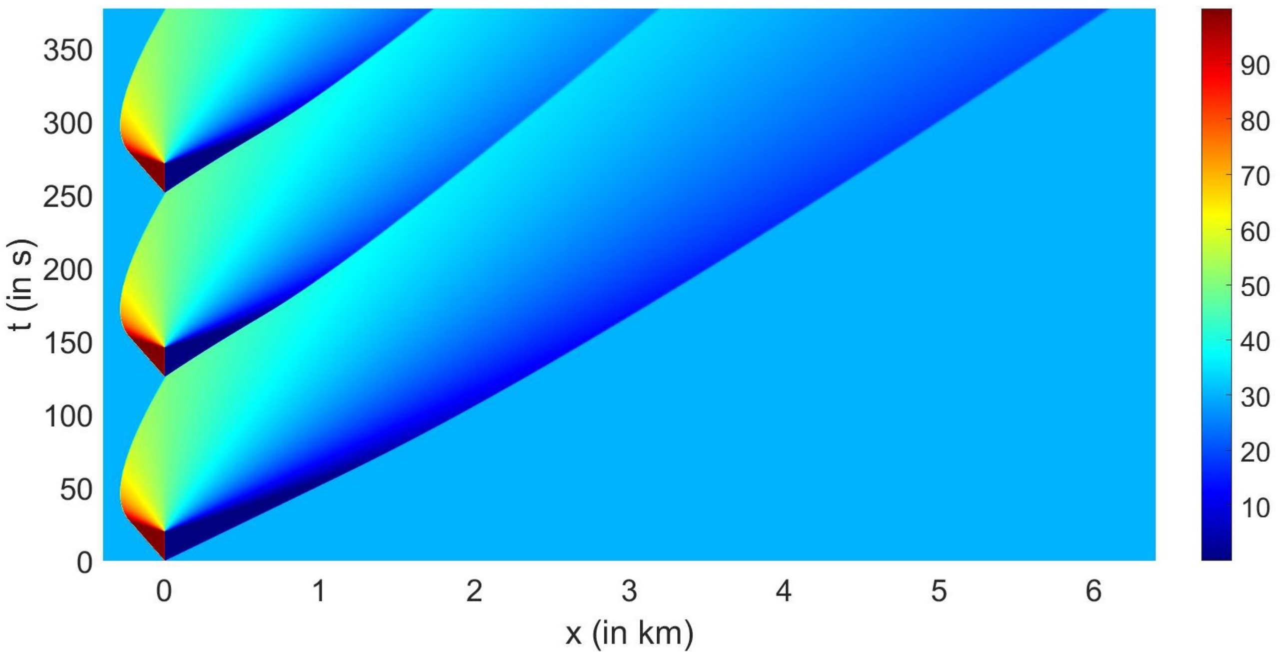

Figure 23: The characteristics for the traffic lights problem with $t_g = 105\,\mathrm{s}$

Using the algorithm described in subsection 4.6, t_{shock} is found to be 125s. Then:

$$\max\left(t_{\text{shock}} - t_r, \frac{u_{\max} t_r f(u_0) f'(u_{\max})}{f(\hat{u})(f(u_0) - (u_0 - u_{\max}) f'(u_{\max}))}\right) = (105, 35) = 105\text{s}.$$

The characteristics for the traffic lights problem with $t_g = 105\,\mathrm{s}$ can be seen in Figure 23. As expected the solution of u mimics itself on the left part of the road during each cycle.

The trajectory of every third car on the road can be seen in Figure 24. As expected any car that is stopped by the red traffic lights in a given cycle, passes the traffic lights during that same cycle.

The characteristics for the traffic lights problem with $t_g = 35\text{s}$ can be seen in Figure 25. The solution does not mimic itself on the left part of the road as expected given that $t_g = 35\text{s} < t_{\text{shock}} - t_r = 105\text{s}$.

The trajectory of every third car on the road can be seen in Figure 26. In the first cycle, any car that is stopped by the red traffic lights, passes the traffic lights before the start of the second cycle. This is expected since $t_g = 35\text{s}$ satisfies (23). However, this is not true for later cycles.

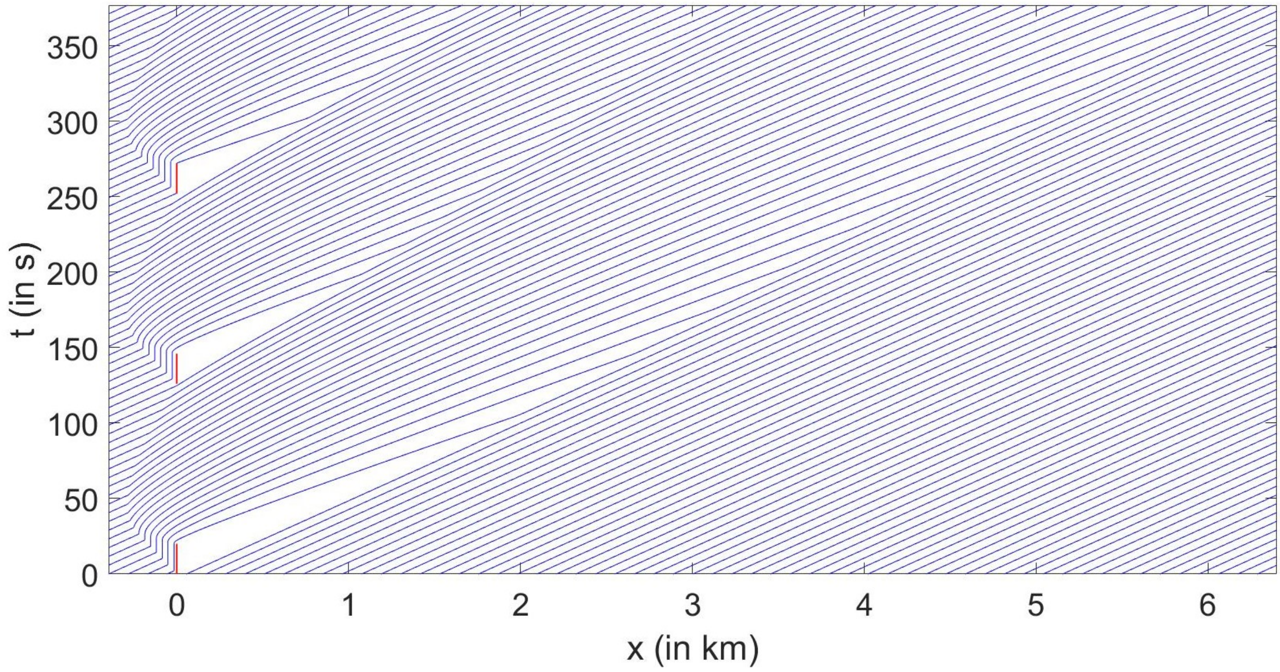

Figure 24: The trajectory of every third car on the road for the traffic lights problem with $t_g = 105\,\mathrm{s}$

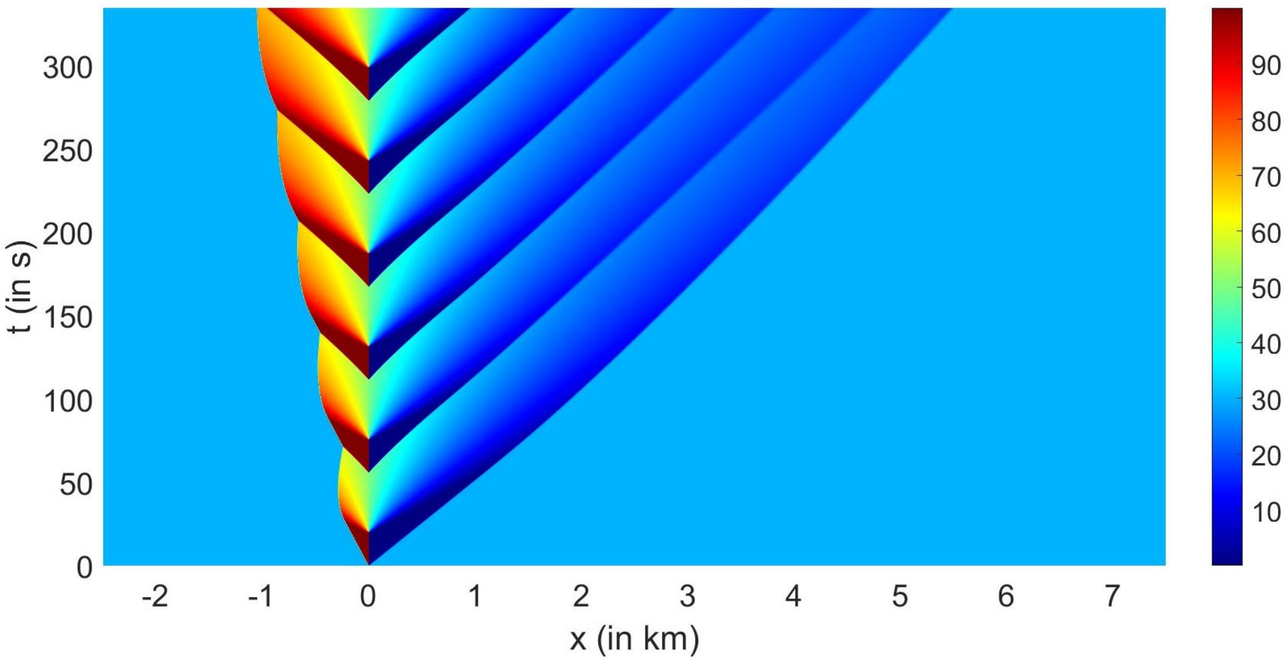

Figure 25: The characteristics for the traffic lights problem with $t_g = 35\,\mathrm{s}$

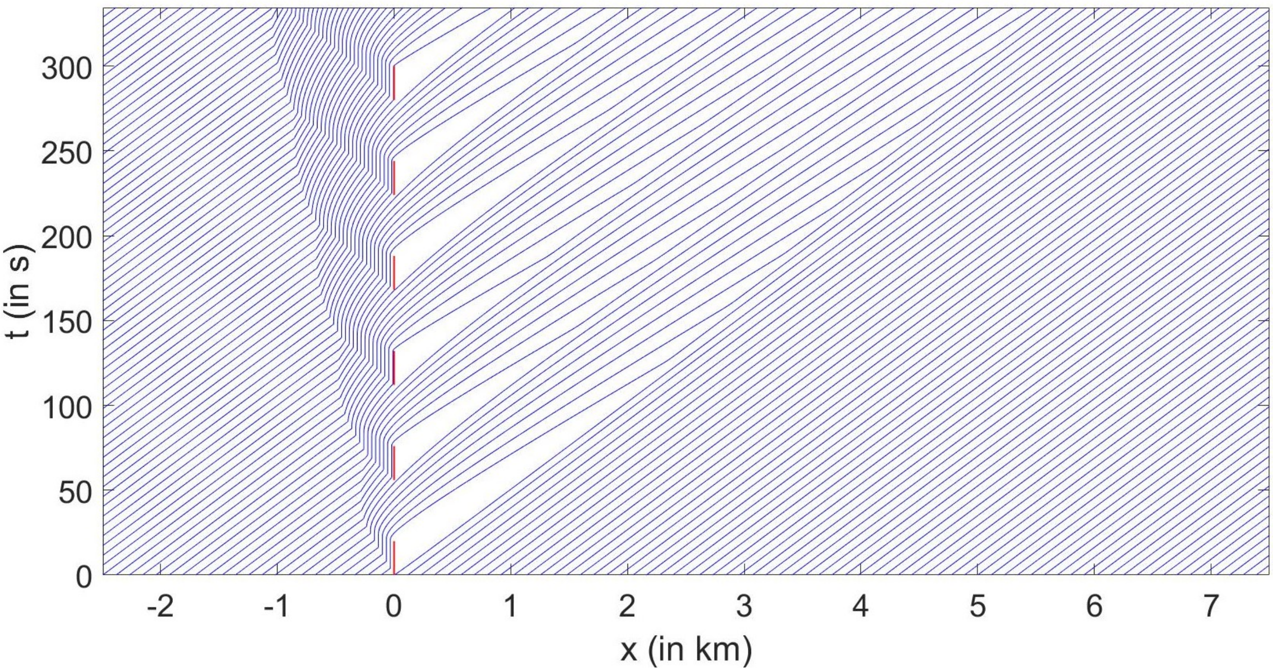

Figure 26: The trajectory of every third car on the road for the traffic lights problem with $t_g = 35\,\mathrm{s}$

4.8 Criterion 2: choice of t_g/t_r given $u_0 \in [0, u_{\max}]$

Consider the traffic lights problem with constant initial density $u_0 \in [0, u_{\max}]$. The length of a traffic lights cycle is $t_r + t_g$. Had there been no traffic lights at $x = 0$, the density of traffic would have been equal to u_0 at any place x and time t. Therefore the number of cars passing the traffic lights during a traffic lights cycle would have been:

$$\int_0^{t_r+t_g} f(u(0,t))dt = f(u_0)(t_r + t_g).$$

A reasonable requirement is that the number of cars that pass the traffic lights during any cycle be greater than or equal to the number of cars that would have passed had there not been any traffic lights. This means that the traffic lights, at worst will have no net effect on traffic flow between the left part of the road and the right part over a period of $t_r + t_g$; at best, the traffic lights will lead to a greater traffic flow between the two parts of the road.

The number of cars that pass the traffic lights during a traffic lights cycle is:

$$\int_0^{t_r+t_g} f(u(0,t))dt = \int_0^{t_r} f(u(0,t))dt + \int_{t_r}^{t_g} f(u(0,t))dt = 0t_r + f(\hat{u})t_g = f(\hat{u})t_g.$$

Then the requirement mentioned above can be restated as:

$$\begin{aligned} &(t_r + t_g)f(u_0) \leqslant t_g f(\hat{u}) \\ \Longrightarrow\ &\frac{t_g}{t_r} \geqslant \frac{f(u_0)}{f(\hat{u}) - f(u_0)}. \end{aligned} \tag{25}$$

Note that criterion 1 for the traffic lights problem considered earlier with the following parameters:

$$\begin{aligned} v_{\max} &= 100\text{km/h} \\ u_{\max} &= 100\text{cars/km} \\ u_0 &= 30\text{cars/km} \\ t_r &= 20\text{s} \end{aligned}$$

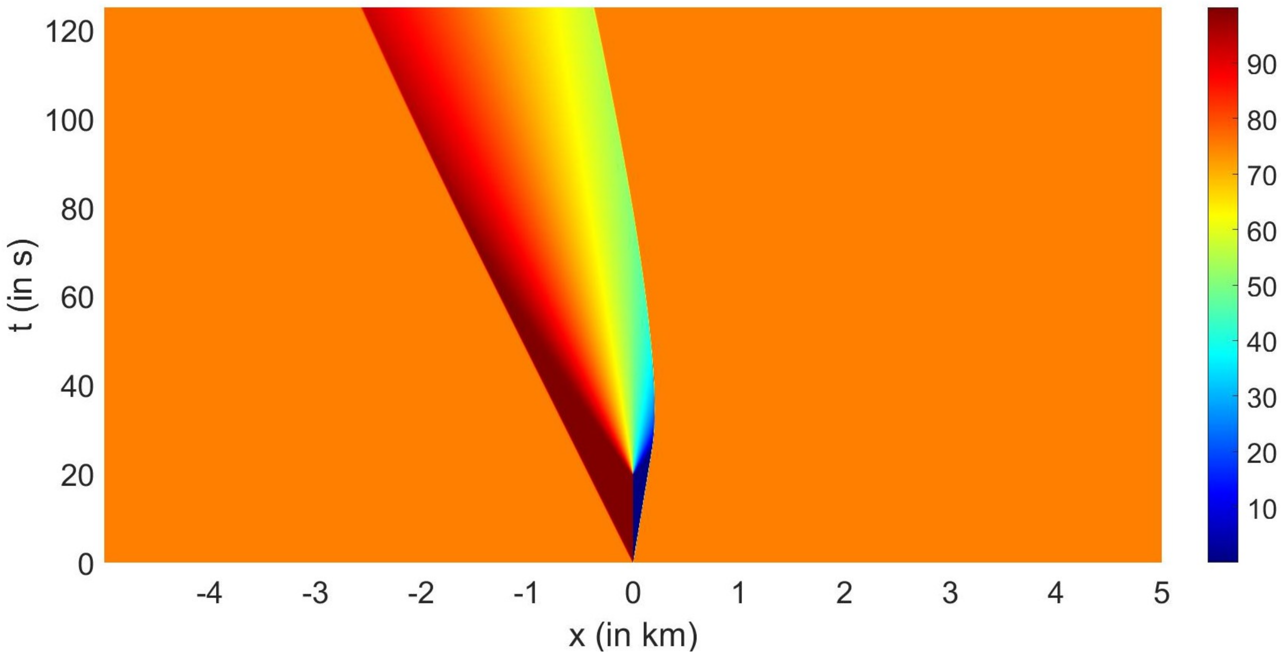

Figure 27: The characteristics of a traffic lights problem with high initial constant density during one traffic lights cycle

recommends $t_g \geqslant 105$s. On the other hand, criterion 2 recommends that (25) is satisfied. This means that $t_g/t_r \geqslant 5.25$, and therefore $t_g \geqslant 5.25t_r = 105$s. The trajectories of a third of the cars on the road can be seen in Figure 24. Note that there was no traffic accumulation from cycle to cycle, as the maximum length of stationary cars in each cycle is constant.

Also note that criterion 2 can be computed directly using the formula, while criterion 1 needs to find t_{shock} first. Moreover, criterion 2 is more versatile as it does not assume anything about the value of u_0.

It is worth mentioning, however, that as u_0 tends to $\hat{u}$, t_g/t_r tends to ∞, therefore it is not very practical to use this criterion when u_0 is close to $\hat{u}$.

4.9 The solution over one traffic lights cycle with high initial constant density

Consider the traffic lights problem with the following parameter values:

$$\begin{aligned} v_{\max} &= 100\text{km/h} \\ u_{\max} &= 100\text{cars/km} \\ u_0 &= 75\text{cars/km} \\ t_r &= 20\text{s} \\ t_g &= 105\text{s}. \end{aligned}$$

The numerical solution of this problem is shown graphically in Figure 27.

At $t = 0$ s the traffic lights turn red. Two shock waves emerge from $x = 0$ km. The one on the left is due to a jump from $u^- = u_0 = 75$ cars/km (in orange) to $u^+ = u_{\max} = 100$ cars/km (in dark red). The one on the right is due to a jump from $u^- = 0$ cars/km (in dark blue) to $u^+ = u_0 = 25$ cars/km (in orange).

At $t_r = 20$ s the traffic lights turn green. An expansion wave starts at $x = 0$ km to continuously decrease the density from $u^- = u_{\max} = 100$ cars/km to $u^+ = 0$ cars/km. This expansion wave can be

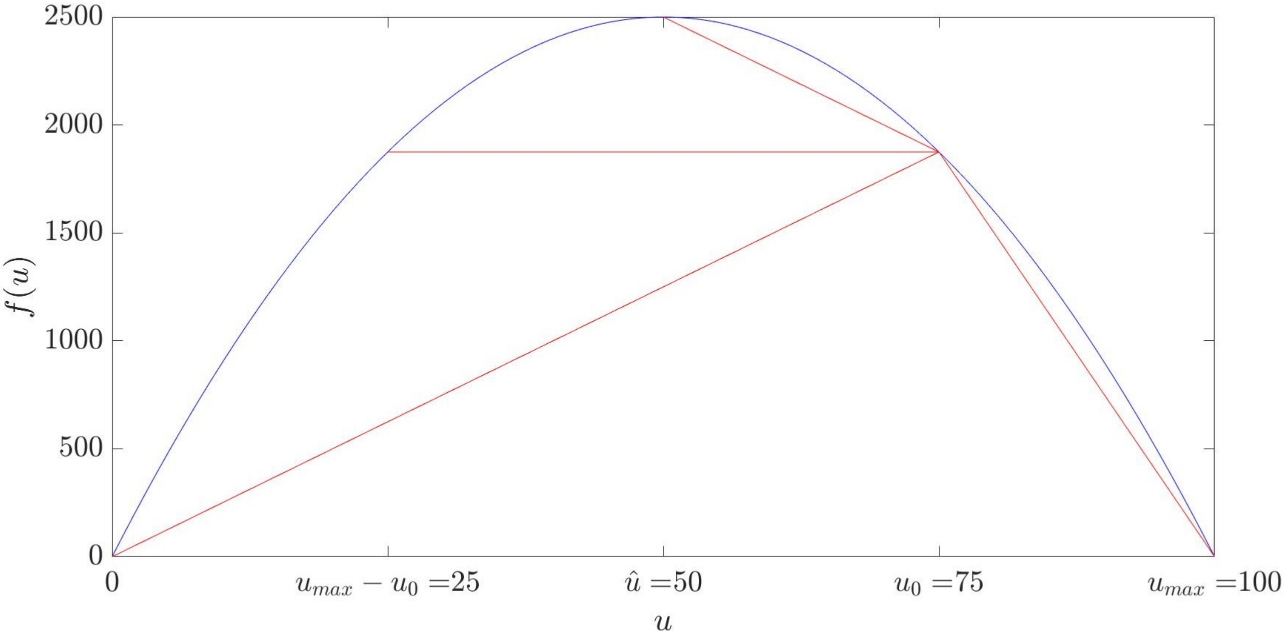

Figure 28: The secants of f between u_0 (high density) and $u \in [0, u_{\max}]$

divided into a left part which includes the signals with densities in $[u_0, u_{\max}]$ and a right part which includes the signals with densities in $[0, u_0)$.

The shock wave on the right first intersects paths with the expansion wave when it meets the first signal of the expansion wave from the right, which has density 0. This is the top right point of the dark blue triangle in Figure 27. The discontinuity is between $u^- = 0$ and $u^+ = u_0$. After this point the shock wave meets signals with densities continuously increasing from 0 to u_0.

The speed of the shock wave between u^- and u^+ is s which is the slope of the secant between u^- and u^+ (given that $u^- < u^+$). Therefore the change of the speed of the left shock wave can be seen in Figure 28.

As the shock wave meets signals with densities continuously increasing from 0 cars/km to $u_{\max} - u_0 =$ 25 cars/km, the speed of the shock wave continuously decreases from the positive value of $s = 25$ km/h (the shock wave is traveling to the right) to $s = 0$ km/h (the shock wave is stationary). After that the shock wave meets signals with densities continuously increasing from 25 cars/km to $\hat{u} = 50$ cars/km. The speed of the shock wave decreases continuously from $s = 0$ km/h to $s = -25$ km/h (the shock wave is moving to the left). Finally, as the shock wave meets signals continuously decreasing from $\hat{u}$ to u_0, its speed continues to decrease until it reaches a minimum value of $f'(u_0) = -50$ km/h. This happens asymptotically as t_g tends to ∞.

The shock wave on the left first intersects paths with the expansion wave when it meets the first signal of the expansion wave from the left, which has density $u_{\max}$. This is the top right point of the dark red triangle in Figure 27. The discontinuity is between $u^- = u_0 = 75$ cars/km and $u^+ = u_{\max} = 100$ cars/km. After this point the shock wave meets signals with densities continuously decreasing from 100 cars/km to 75 cars/km. At this point the speed of the shock wave is $s = -75$ km/h. As the densities of the signals intersecting with the left shock wave decrease, the speed of the shock wave increases until it reaches a maximum of $f'(u_0) = -50$ km/h. This also happens asymptotically as t_g tends to ∞.

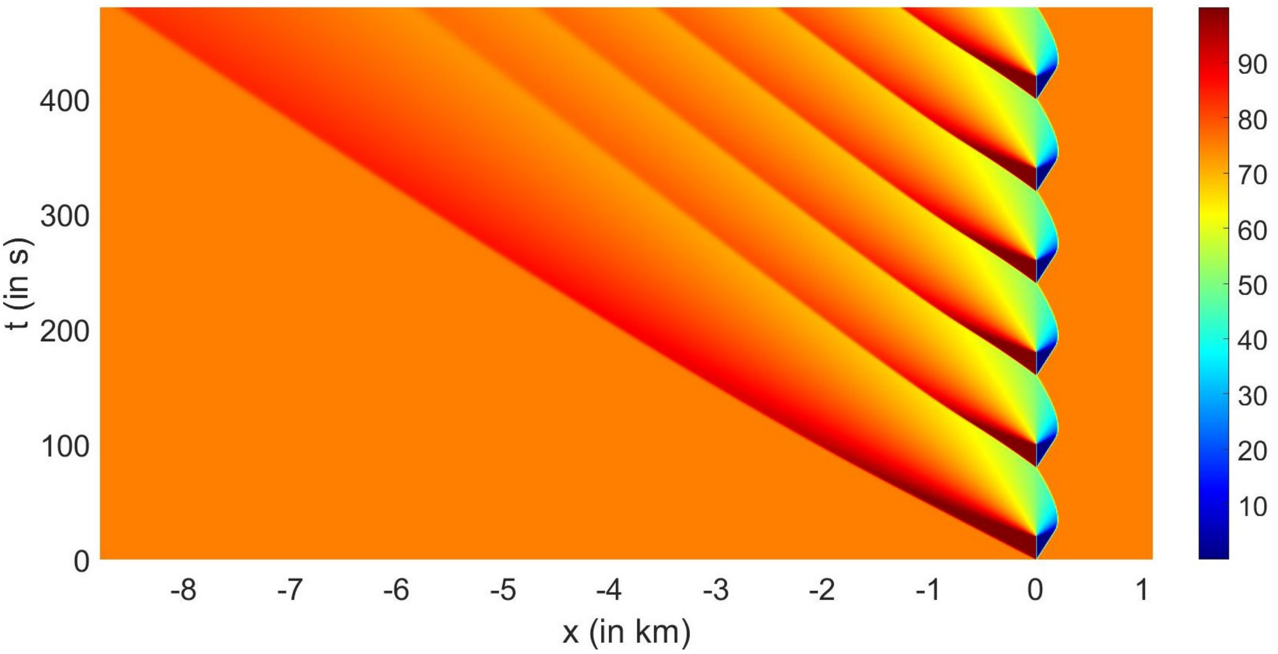

Figure 29: The characteristics for the traffic lights problem with $t_g = 60\,\mathrm{s}$

4.10 The solution over several traffic lights cycles with high initial constant density

Consider the traffic lights problem with the following parameter values:

$$\begin{aligned} v_{\max} &= 100\,\mathrm{km/h} \\ u_{\max} &= 100\,\mathrm{cars/km} \\ u_0 &= 75\,\mathrm{cars/km} \\ t_r &= 20\,\mathrm{s}. \end{aligned}$$

The choice of t_g and t_r satisfies criterion 2 if (25) is satisfied, that is if $t_g/t_r \geqslant 3$. Therefore, $t_g \geqslant 3t_r = 60\,\mathrm{s}$. To see what happens when a choice of t_g and t_r satisfies criterion 2, simulation 1 is ran with $t_g = 60\,\mathrm{s}$. The solution can be seen in Figure 29. Moreover, the trajectory of every tenth car on the road is plotted to get Figure 30.

To see what happens when a choice of t_g and t_r doesn't satisfy criterion 2, simulation 2 is ran with $t_g = 30\,\mathrm{s}$. The solution can be seen in Figure 31. Moreover, the trajectory of every tenth car on the road is plotted to get Figure 32.

For high densities, a the choice t_g/t_r that satisfies criterion 2 leads to a solution which mimics itself every cycle on the right part of the road. This can be seen in Figure 29 and not in Figure 31.

As a result, the length of the part of the road where there are no cars, remains constant for all cycles with such a t_g/t_r as can be seen in Figure 30. On the other hand when t_g/t_r does not satisfy criterion 2, the the length of the part of the road where there are no cars, grows with each cycle as can be seen in Figure 32.

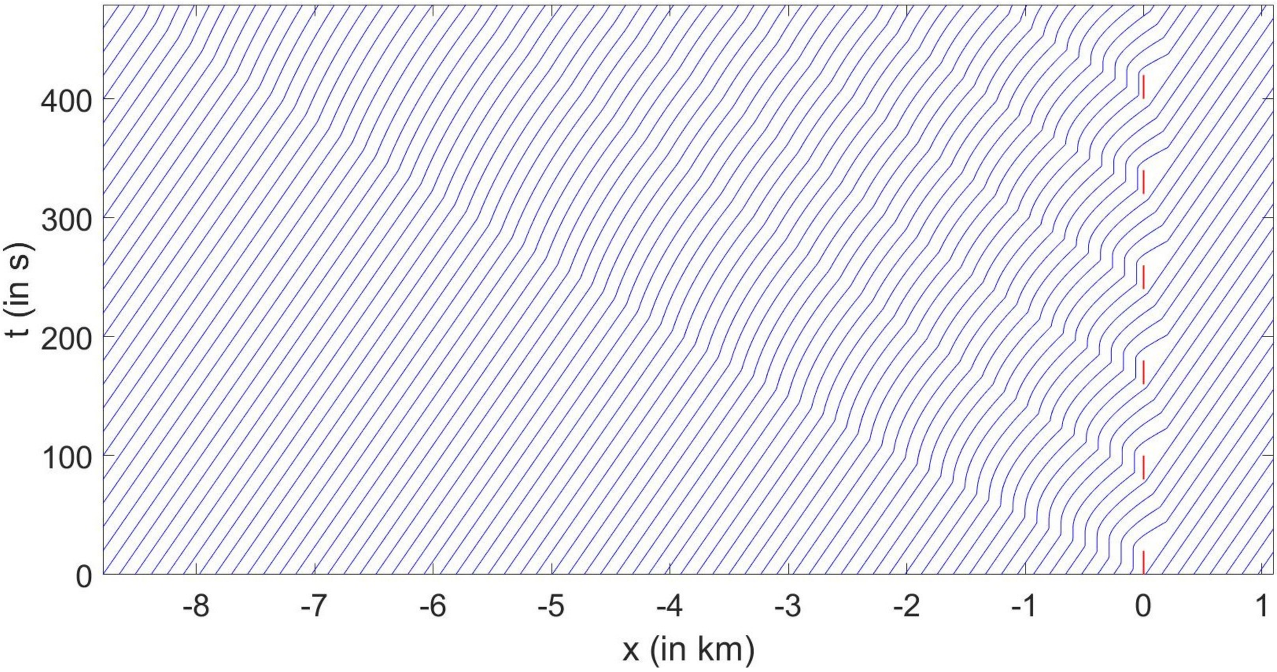

Figure 30: The trajectory of every tenth car on the road for the traffic lights problem with $t_g = 60\,\mathrm{s}$

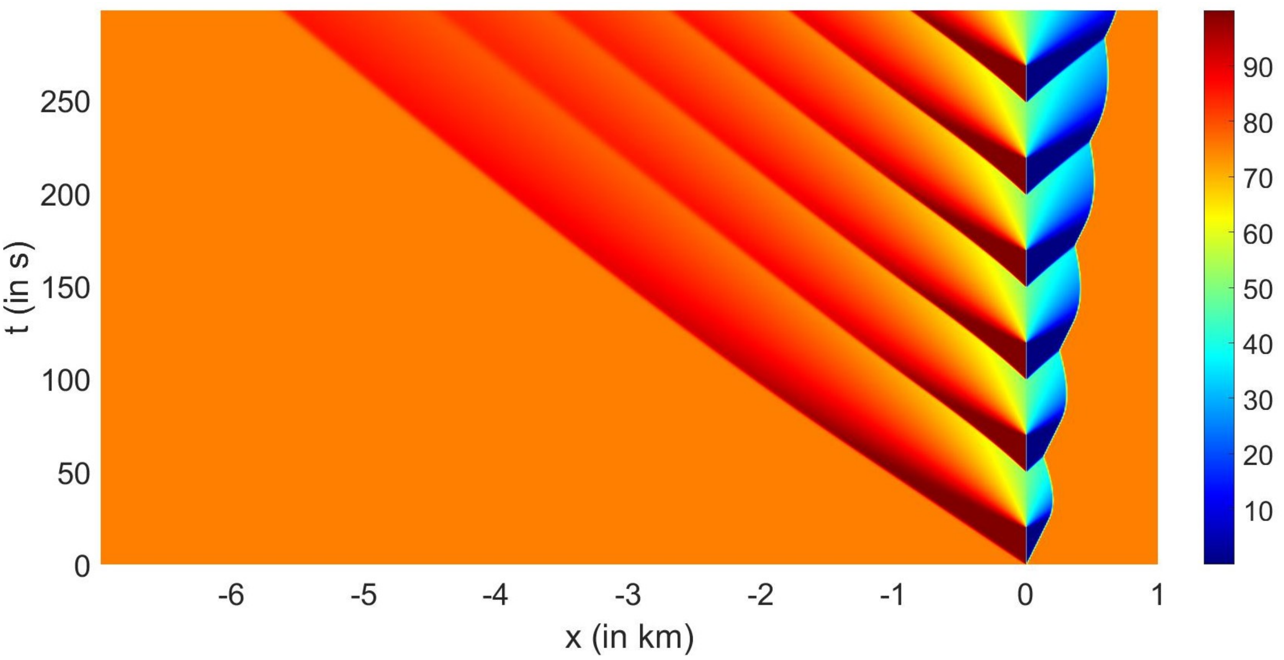

Figure 31: The characteristics for the traffic lights problem with $t_g = 30\,\mathrm{s}$

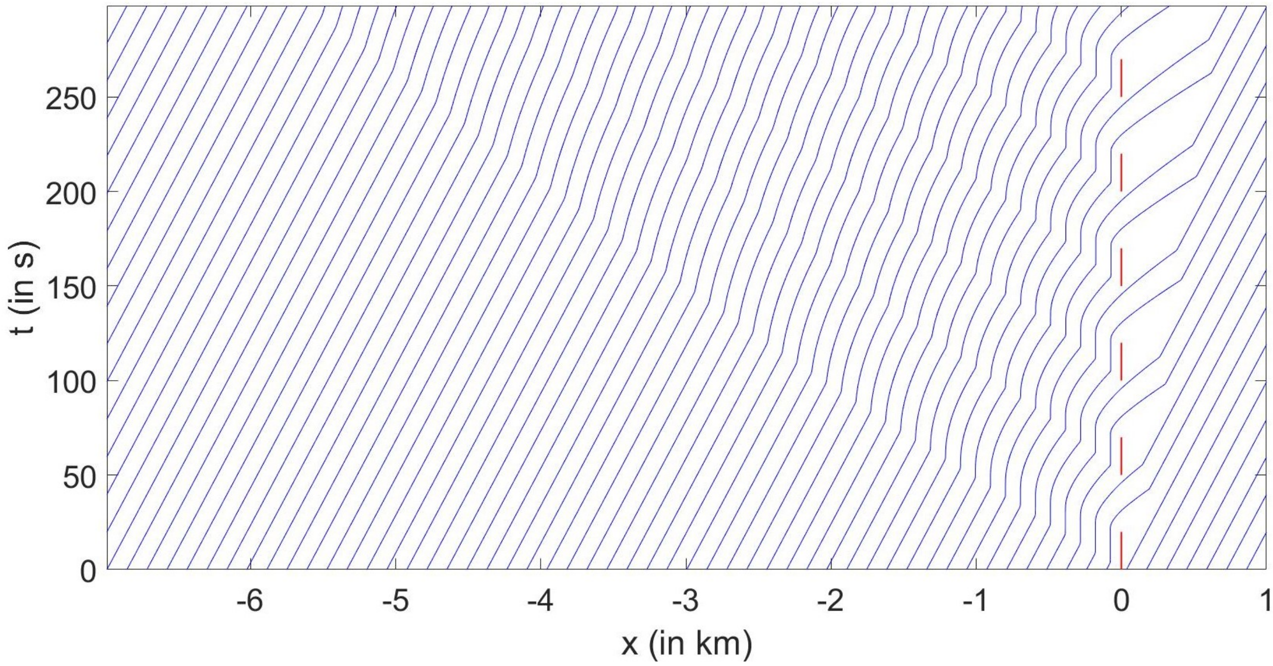

Figure 32: The trajectory of every tenth car on the road for the traffic lights problem with $t_g = 30\,\mathrm{s}$

5 Conclusions

Throughout this report, it was seen how the traffic flow problems can be mathematically modeled and numerically solved. The model and the numerical method are important because they allow for the simulation of traffic flow. The model used in this report required that the velocity function v be chosen so that the flux function $f(u) = uv(u)$ is strictly concave. This is a somewhat strong assumption, and the model can be made more general by relaxing the condition that f is a strictly concave function to f is a unimodal function.

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