On Shock Propagation in Financial Networks

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Abstract

This thesis develops a simplified financial network model for an interbank lending system which is then analyzed in terms of contagion when exposed to external liquidity shocks. The aim is to understand how individual institutions and the network structure affect the shock propagation and finding factors that increase respectively decrease the systemic risk of the network. The network structures analyzed are mainly the ring graph, the complete graph, and the directed tree graph, given an ex-post and an ex-ante perspective.

The first result indicates that traditional centrality measures are not capable of identifying systemically important institutions. The second result concerns the interconnections in the network structure, where it is concluded that if one institution or all institutions are subject to a certain shock, a complete structure always performs better than or equally as well as the denser structure of a ring graph, in terms of number of defaulting institutions, whereas if multiple institutions, but less than all of them, are exposed, the complete graph may perform worse. The last result shows that in acyclic tree graphs, a higher number of offspring in the k-regular tree graph and an offspring distribution with less variance in the random tree graph, can restrict the contagion respectively reduce the probability of shock propagation further down the tree.

Keywords: Financial Network, Financial Contagion, Systemic Risk, Shock Propagation, Network Structure

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Notations

The financial network model is constructed by a set of notations and definitions are stated below.

n	Number of nodes, representing financial institutions, in the network
W	Weight matrix of size $n \times n$, representing the weighted, directed links in the network
W_{ij}	Weighted link from i to j representing the claim of j towards i, i.e. the amount of money that j has lent to i $\Leftrightarrow$ The liability of i towards j, i.e. the amount of money that i has borrowed from j
w_i	Total liability of node i, i.e. out-degree
v_i	Total claim of node i, i.e. in-degree
P_{ij}	Fraction of total liability of i owed to j
x_i	Total repayment of node i to its creditors
x_i^g	Total repayment of node i in generation g to its creditors, applies only to the tree graph networks
a_i	External asset of node i
b_i	External liability of node i
ε_i	External shock on node i
s_i	Induced shock from node i to all its creditors
s_i^g	Induced shock from node i in generation g to all its creditors in the tree graph networks
s_{ij}	Induced shock from node i to j
s_{ij}^g	Induced shock from node i in generation g to j in the tree graph networks
ξ_i	Stochastic part of the induced shock s_i
β_i	Deterministic part of the induced shock s_i
c_i	Net worth of node i before shocks
d_i	Net worth of node i after shocks
k_i	Number of children of node i in the tree graph networks
ψ	Number of defaulting institutions in the network
Ψ	Number of defaulting generations of institutions in the tree network

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1

Introduction

The global financial economy today is highly interconnected and interdependent which have raised the question of how this affects the overall financial stability. The recent financial crises have shown that a shock in one market may cause instability globally, especially the financial crisis in 2008, where turmoil on the mortgage market in the US turned into global recession. During the last years there has been an increasing interest in and research of applying network theory to the financial system in order to gain understanding of how the structure of the network affects the economic stability and systemic risk and it has been heavily debated whether more densely or more sparsely connected network structures are the key amplifiers of shock contagion [Glasserman and Young, 2016].

The financial system is also characterized by an opaque and complex structure. The liabilities and claims between financial institutions are to a great extent confidential and the many ways of trading financial instruments create a delicate network of connections. The limited information available about the connections in the network makes it difficult to understand how the network structure affects the exposure to contagion in the system. This in turn makes it harder for policy makers and regulators to limit systemic risk and prevent large financial crises. One approach is to instead create simpler models of the financial network, according to graph theory, where the nodes represent financial institutions and the links in between represent the monetary connections. Then, the complex structure may be broken down and several established graph concepts can be exploited to form comprehensible conclusions that may be applied in reality. Centrality measures are one of these graph concepts and the question that has been discussed in the literature is whether these may capture the systemically importance of single nodes, in terms of spreading financial contagion [Acemoglu et al., 2015].

1.1 Purpose

The purpose of this thesis is to gain further understanding of both the financial network structure's role and the individual institutions' role within the financial network in influencing the systemic risk. The goal is to derive measures of contagion and conclusions, given ex-post and ex-ante shocks, which may be applied to the real-world financial system in order to comprehend factors that may amplify versus absorb negative shocks. Specifically, further analysis on the role of interconnections and cycles, respectively, in the network is performed on specific topologies with the aim to develop an understanding of how they affect the stability in presence of shocks.

1.2 Limitations

The thesis considers a purely theoretical financial network which aims to model a simplified interbank lending system. As it is simplified, several real-world factors are not taken into account. Also, since the time frame of the master thesis is restricted to 20 weeks, the scope and depth of the analysis has been limited by this.

1.3 Previous Work

The concept of understanding the role of the financial network's structure in spreading financial contagion by shocks emerged during the early millennium. There are primarily two significant papers that a great part of the later literature build upon. It is partly a work which examines simple interbank markets and conclude that a more complete structure is to prefer as it is more robust [Allen and Gale, 2000]. The other work examines a similar network but instead of shock contagion, they analyze stability given a partially or completely defaulting bank and come to similar conclusions of that a higher connectivity results in a more resilient network [Freixas et al., 2000].

The financial network model in this thesis is mainly build upon the model described in [Eisenberg and Noe, 2001]. In this paper, Eisenberg and Noe define and prove the existence and uniqueness for a payment clearing vector in a financial, cyclic network. With these findings, it is possible to develop a better framework and understanding of how contagion spreads and default cascades through the network. The model is basic and thus several suggestions of improvement have been developed afterwards.

Other important works for this thesis are the models and conclusions developed in [Acemoglu et al., 2015; Acemoglu et al., 2016]. The financial network models in these papers are closely related and inspired by the model by Eisenberg and

Noe. An analysis of the role of connections and network structure is performed to find an answer to whether more interconnectedness amplifies or absorbs contagion. In [Acemoglu et al., 2015], it is concluded that given a small enough magnitude of the shocks, a complete network performs better than more sparsely connected networks. In this case, the more connected a network is, the better it performs. This is in line with the conclusions of Allen and Gale and Freixas et al. However, in contrast to these works, Acemoglu et al. conclude that the complete network performs worse than more sparsely connected networks given a large enough magnitude of shocks and hence it is not always better with denser connections to provide a stable network.

There has also been a field of literature building upon numerical simulations on empirical data instead of theoretical frameworks. Even though this is not within the scope of the thesis, there are some interesting result considering the interconnections and centrality in a network worth to mention. In [Gai et al., 2011] a simple interbank system is modeled by a variety of complex network structures and it is analyzed how the connectivity and concentration in the network affect the stability and funding contagion. By experiments it is found that increasing the connectivity may increase the fragility of the system. An extensive review of developed simulation models of default contagion is found in [Upper, 2011] where it is concluded that the current models are merely tools in assessing risk and not a complete framework to prevent crises.

An extensive review of the current literature within financial contagion in networks is produced and compared in [Glasserman and Young, 2016]. Glasserman and Young also show examples of how to extend the model by Eisenberg and Noe and develop a measure to understand the importance of different nodes called Node Depth. The systemic importance as measured by standard centrality measures has been questioned in literature. Acemoglu et al. argues that these measures are insufficient and possibly misleading for nonlinear financial network models while there are, on the other hand, empirical findings on interbank systems where these centrality measures are found to capture the systemic importance well [Farooq Akram and Christophersen, 2010; Craig et al., 2015].

This literature review within the subject of financial contagion shows some contradictory results of the relationship between the interconnectedness and systemic risk in a network, and also the role of traditional centrality measures in a financial network as to capture the systemically important institutions.

1.4 Outline

The outline of this thesis is the following. In Chapter 2, the established graph theory is presented and the analyzed specific graph topologies are described. Also, classical centrality measures are defined and graph concepts important for the thesis are presented. The financial network model is presented in Chapter 3 with a detailed description and application. Performance measures for the financial network are defined and the model is then compared to similar models developed in previous work. Also, a payment equilibrium, given the presence of shocks, is proved to exist and being (generically) unique. Lastly, a notion called Node Depth is introduced and three heterogeneous network examples are analyzed to compare their performance in the presence of shocks to some traditional centrality measures as to find the systemically important nodes.

The analysis of contagion when ex-post shocks are present, given a specific homogeneous network structure, begins in Chapter 4. First, an interpretation and argumentation are given for the specific topologies. Both one deterministic and multiple deterministic shocks are analyzed on the three specific topologies and the contagion is measured by the number of defaulting institutions. In Chapter 5, the shock propagation analysis continues on ex-ante shocks. The acyclic specific topologies are analyzed given one stochastic and multiple independent stochastic shocks. The contagion is measured by a probability measure of the size of the induced shock from one node to another. In Chapter 6, the findings of the thesis are concluded and further work is presented.

2

Background on Graph Theory

This chapter describes basic graph theory with common notations and definitions of concepts that are important for this thesis. The financial network model is based on this graph theory. The specific graph topologies ring graph, complete graph, and tree graph are explicitly explained as they are later considered in the contagion analysis. More traditional, off-the-shelf centrality measures for networks are further described as well as the concept of cohesiveness. Lastly, since the Galton-Watson branching process is applied to generate the random tree graph, it is further described.

2.1 Networks as Graphs

The structure of a network can be described by a graph and in this section, graph theory is more thoroughly described. The theory is based on the Lecture Notes from Network Dynamics [Como, 2018]. A graph simply formalizes which nodes that are connected and to what extent they are connected to each other. In other words, it describes the pattern of links between the nodes.

Formally, a graph $\mathcal{G}$ consists of three sets, $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. The set $\mathcal{V}$ contains the nodes, or vertices, where every node is assigned a number such that $\mathcal{V} = \{1, 2, ..., n\}$. The set of links, or edges, is denoted by $\mathcal{E}$ and consists of ordered pairs of nodes (i, j), where the link points from node i to node j. W is the link weight matrix, or adjacency matrix. It is a square matrix with non-negative elements. If $(i, j) \in \mathcal{E}$, then $W_{ij} > 0$. Similarly, if $(i, j) \notin \mathcal{E}$, then $W_{ij} = 0$.

The nodes that are connected by links are called *neighbors*. For a node $i \in \mathcal{V}$, node $j \in \mathcal{V}$ is an *out-neighbor* if the link $(i, j) \in \mathcal{E}$ and an *in-neighbor* if the link $(j, i) \in \mathcal{E}$. The neighbors define the degree of each node. The *out-degree* of a node i

is defined as

$$w_i = \sum_{j \in \mathcal{V}} W_{ij} \tag{2.1}$$

The *in-degree* of a node i is defined as

$$v_i = \sum_{j \in \mathcal{V}} W_{ji} \tag{2.2}$$

The normalized weight matrix is denoted by P and measures the relative fraction of the out-degree of each node. In matrix notation it is defined as

$$P = D^{-1}W, \qquad D = \text{diag}(w) \tag{2.3}$$

Graphs can be either directed or undirected. If a graph is *directed*, at least one of the links has a direction, which means that there is a link that points from a node to another node but not vice versa. In an *undirected* graph on the other hand, all links point in both directions. In this report, both directed and undirected graphs are considered.

Graphs can also be either weighted or unweighted. In a *weighted* graph, every link is given some positive weight. An *unweighted* graph can be seen as a specific case of a weighted graph where all links have weights equal to one. In this report, both unweighted and weighted graphs are considered.

A graph is said to be *connected* if there exists some path between every pair of nodes in the graph, which means that every node is reachable from every other node. If this is not true, the graph is *disconnected*. A graph is *out-connected* if for every node in the graph, there exists a path that leads out from the graph. This may only occur if there is some path to a node with a link that does not point to another node in the graph but to some external entity. The term *cyclic* is related to connectivity and refers to a graph that contains at least one cycle, which means that it is possible to traverse the graph when starting and ending in the same node without passing through a previously visited node. If the graph does not contain any cycle, it is called *acyclic*.

In the next chapters, a financial network is modeled according to this graph theory to develop a framework for how liquidity shocks spread across networks of financial institutions. Different types of graphs can be combined to build more intricate topologies by this framework, which in a better way might reflect how actual interbank lending network really are structured. Some specific graph topologies are further investigated and these are the ring graph, the complete graph, and the tree graph. They are by definition *simple* graphs which are unweighted, undirected and without any self-loops, i.e. $W_{ii} = 0 \; \forall i$. However, the graphs will be altered in this report as described below.

Ring graph The considered ring graph is a specific case which can be described as a directed ring graph with $n > 3$ nodes. As it is directed, each node has one out-neighbor and one in-neighbor. The directed ring graph is the most sparsely connected graph. An example of the directed ring graph is shown in Figure 2.1 with $n = 10$ nodes.

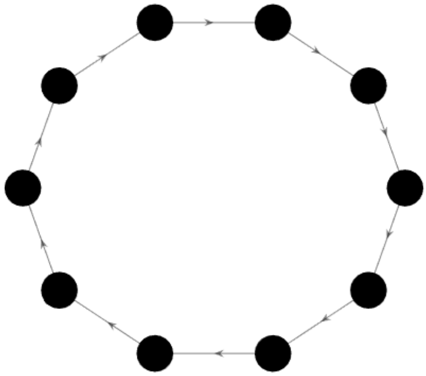

Figure 2.1 A directed ring graph with $n = 10$ nodes

Complete graph The complete graph considered is an undirected graph. In the complete graph, all nodes n are connected to each other, and is thus the maximally connected graph. That results in an equal degree for each node $i \in \mathcal{V}$ of size $w_i = v_i = n - 1$ and hence, the graph is regular. In this report, only complete graphs of $n > 3$ nodes are considered. See Figure 2.2 for an example of a complete graph with $n = 10$ nodes.

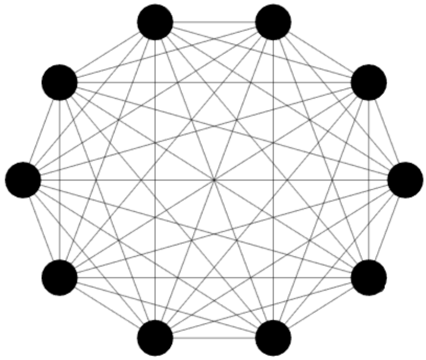

Figure 2.2 A complete graph with $n = 10$ nodes

Tree graph The tree graph considered is a directed and acyclic graph with $n-1$ links. It is acyclic as the graph does not contain any cycle. This tree is also disconnected as it is not possible to reach every node by any other node. The neighbors of degree-1 are called leaves and a tree may have a minimum of 2 (line graph) and a maximum of $n-1$ leaves (star graph). A line graph is shown in Figure 2.3 and may be seen as a special case of an acyclic ring graph. A single root node initialize the tree by letting directed links point to a certain number of neighbors k called children, or offspring. Then, each node i has a number of new links to k_i children. The link W_{ij} is thus directed from the parent node i to the child node j.

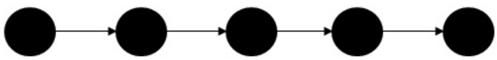

Figure 2.3 A directed line graph with $n = 5$ nodes

The tree may be k-regular and constructed by a fixed number of children of each node or random and generated by a probability distribution. The tree graph is structured according to generations where the root node belongs to generation $g = 0$, the children of the root node belongs to generation $g = 1$, and so on. In this context the tree is assumed to be infinite which means that $n \to \infty$ and all nodes are assumed to have children, meaning that leaves are non-existent. An example of a k-regular directed tree graph with $k = 3$ children and $g = 3$ generations is shown in Figure 2.4.

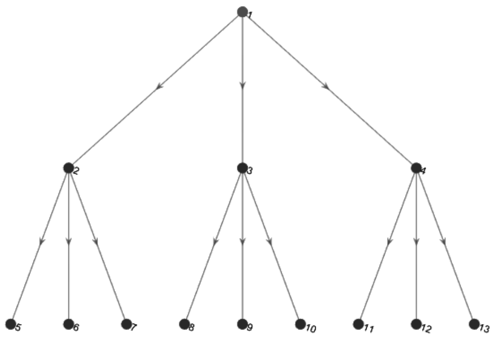

Figure 2.4 A directed k-regular tree graph with $k = 3$ children and $g = 3$ generations

2.2 Network Centrality

In network analysis, the concept of centrality measures aims to identify how central the different nodes of the network are. There are many possible ways of measuring node centrality. The different measures may or may not coincide and a measure may be more or less suitable for the given network. In this thesis, two strands of centrality measures will be considered, based on the in- or out-degree of nodes and variants of eigenvector centrality.

The conceptually simplest measure is the *degree centrality*, where the centrality of a node is given by its in- or out-degree. The in-degree of a node i is defined as the sum of weights of the links pointing to the node, that is $v_i = \sum_j W_{ji}$. In the same way the out-degree is the sum of the weights of the links pointing from node i, that is $w_i = \sum_j W_{ij}$. One may extend the degree centrality to the *Eigenvector centrality* vector π defined as

$$\lambda \pi = W' \pi \tag{2.4}$$

where λ is the leading eigenvalue to the leading eigenvector π and this centrality measure is unique if the graph $\mathcal{G}$ is assumed to be connected.

A variant of the eigenvector centrality is the *invariant distribution centrality*. It is defined as the leading left eigenvector of the normalized weight matrix P. Thus, it is the solution of

$$\pi = P' \pi \tag{2.5}$$

where P is defined as $P = D^{-1}W$ and $D = \text{diag}(w)$. The centrality vector is unique for connected graphs.

A modified version of the Eigenvector centrality is the *Katz centrality* [Katz, 1953], defined as the solution z of

$$z = \frac{1-\beta}{\lambda_w} W' z + \beta \mu \tag{2.6}$$

where the parameter β weights the inherent centrality to the graph topology and μ is defined as a nonnegative vector which measures the inherent centrality, typically $\mu = \mathbb{1}$. The dominant eigenvalue of W' is denoted by λ_w. For every $0 < \beta < 1$, the matrix $(I - \lambda_W^{-1}(1-\beta)W')$ is invertible and the Katz centrality vector z can be written as

$$z = (I - \lambda_W^{-1}(1-\beta)W')^{-1} \beta \mu \tag{2.7}$$

By using the normalized weight matrix P instead of the adjacency matrix W, it becomes the *Bonacich centrality* π, as introduced in [Bonacich, 1987]

$$\pi = (1-\beta) P' \pi + \beta \mu \tag{2.8}$$

A variant of the Bonacich centrality is the *PageRank centrality* which is defined in a similar way but typically with a β set to 0.15 [Brin and Page, 1998]. Then one may find the PageRank centrality as

$$\pi^{(\beta)} = (I - (1-\beta)P')^{-1} \beta \mu \tag{2.9}$$

2.3 Cohesiveness

The concept of *cohesiveness* is a measure of connectivity [Morris, 2000]. It finds the minimum normalized out-degree in a subset of the graph. In other words, given a subset in the graph, the cohesiveness measures the minimum fraction of connections to the other nodes in the subset relative to the connections going outside the subset. Thus, it finds how well a certain subset in the network is connected. The mathematical definition of cohesiveness on a subset $\mathcal{S} \subseteq \mathcal{V}$ is

$$\kappa(\mathcal{S}) = \min_{i \in \mathcal{S}} \frac{1}{w_i} \sum_{j \in \mathcal{S}} W_{ij} = \min_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} P_{ij} \tag{2.10}$$

A subset $\mathcal{S}$ is called *α-cohesive* for an $\alpha \geq 0$ if

$$\kappa(\mathcal{S}) \geq \alpha \Leftrightarrow \sum_{j \in \mathcal{S}} P_{ij} \geq \alpha, \qquad \forall i \in \mathcal{S} \tag{2.11}$$

The more cohesive a subset is, the more connections there are between the nodes in the subset compared to the connections to the external nodes outside the subset. The maximal cohesiveness $\alpha = 1$ is given by a subset disconnected to the rest of the network as there are no links between the subset and the nodes outside the subset. The minimum cohesiveness $\alpha = 0$ is given by a subset of disconnected nodes as there are no links between the nodes within the subset.

To illustrate with an example, the cohesiveness in the complete graph consisting of n nodes is shown. For any node in the complete graph the degree is $n-1$, as it is maximally connected. Given a subset $\mathcal{S}$ of $k < n$ nodes in the complete graph, there will be $k-1$ connections within the subset. Thus, the cohesiveness for this subset in the complete graph is

$$\kappa(\mathcal{S}) = \frac{k-1}{n-1} \tag{2.12}$$

2.4 Random Tree by Galton-Watson Branching Process

In a random tree graph, the number of children of each node depends on a certain offspring distribution. In this thesis, the random tree graph will be generated by the *Galton-Watson branching process*, as defined in [Como, 2018]. The process starts with one root node of generation $g = 0$ to which a random number of neighbors are added, called offspring or children. These neighbors belong to generation $g = 1$. To each of these neighbors, a random number of new neighbors are added and so it continues. The number of offspring to each node is an independent and identically distributed random variable which follows a certain probability distribution $\{p_k\}_{k\geq 0}$ with the mean

$$\mu = \sum_{k\geq 0} k p_k \tag{2.13}$$

The number of offspring may be denoted by k_i^g, where i is the member of the generation g. Thus, the offspring probability is defined as

$$Pr(k_i^g = k) = p_k, \qquad i \geq 1,\ g \geq 0,\ k \geq 0 \tag{2.14}$$

The number of nodes in generation g is X_g and may be recursively found as

$$X_0 = 1, \qquad X_{g+1} = \sum_{i=1}^{X_g} k_i^g, \qquad g \geq 0 \tag{2.15}$$

For the random tree graph generated by the Galton-Watson branching process, the number of children k_i of each node i follows a certain positive offspring distribution $\{p_k\}_{k>0}$. The offspring distribution considered in this thesis is either the Discrete Uniform or the Binomial distribution, each defined by a probability mass function $f_K(k)$:

$$\text{Discrete uniform:} \qquad f_K(k) = \begin{cases} \frac{1}{k_{max}}, & \text{if } k \in [1, k_{max}] \\ 0, & \text{if } k \notin [1, k_{max}] \end{cases} \tag{2.16}$$

$$\text{Binomial:} \qquad f_K(k;n;p) = \binom{n}{k} p^k (1-p)^{n-k} \tag{2.17}$$

Each node will have $k_i \in [1, k_{max}]$ children, where k_{max} is the maximum possible number of children of a node. There is no possibility of leaves in this graph, as the probability distribution is only defined for positive integers. A random tree graph created by the Galton-Watson branching process with the Discrete Uniform offspring distribution is shown in Figure 2.5 where the number of children is defined in the interval $k_i \in [1, 3]$ with equal probability of $1/3$.

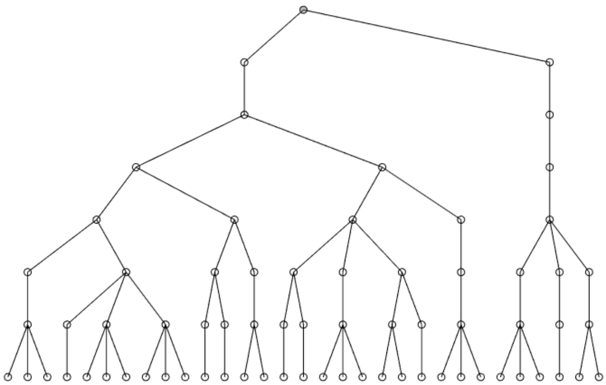

Figure 2.5 A random tree graph created by a Discrete Uniform probability distribution with $k_{max} = 3$ and $G = 7$ generations

3

A Network Model for Financial Contagion

In this chapter, the financial network model is defined in detail with important notions and concepts clearly explained. The model is then compared to previous established models. The main part of the financial network model is based on the basic model defined in [Eisenberg and Noe, 2001]. The model is also inspired by the "Financial Contagion" example in the general framework introduced in [Acemoglu et al., 2016]. The result is a simple and more general interbank lending network. The foundation of the financial network model is the previously defined graph theory, which this simplified interbank lending system is applied upon. Then, a couple of performance measures are defined to enable comparison between different networks.

Also, a payment equilibrium is defined. This is proved to always exist and to be (generically) unique in the presence of shocks. The result is important to carry on the analysis of shock propagation in the financial network model. Lastly, a centrality measure called Node depth is defined and used together with more traditional centrality measures in three different examples of heterogeneous networks. The centrality measures are then evaluated by their ability to find the systemically most important nodes, given by the comparison to the performance of the network.

3.1 Financial Network Model

Consider a graph consisting of n nodes where each node represents a financial institution, e.g. a bank or an insurance company, and a weight matrix W of size $n \times n$. Each element (i, j) in W corresponds to a weighted link in the graph, directed from i to j, and represents the extent to which institution i owes money to institution j. Thus, a link connecting two institutions represents an interbank lending, directed from borrower to creditor. This means that the size of node i's debt to node j is W_{ij}, which is equal to the size of the claim of node j from i. If element $W_{ij} = 0$, there is no link in the graph from node i to node j, i.e. node i does not owe any money to node j.

The financial network interacts with external non-financial entities. Each financial institution i is assumed to have an asset of net value a_i, which can be seen as a liability from non-financial entities to institution i. The net value a_i may consist of both claims on non-financial entities and senior liabilities to non-financial entities, but for simplicity it is called the external assets of node i. There may also exist non-senior external liabilities from institution i to non-financial entities, denoted by b_i. The asset is assumed to be positive, $a_i > 0 \; \forall i$. The external liability vector is nonnegative, $b \geq 0$, which means that the case of nonexistent external liabilities of all nodes is possible, $b_i = 0 \; \forall i$.

The financial network model of a network with two financial institutions is graphically displayed in Figure 3.1 to show the directions of claims and liabilities, both externally and internally.

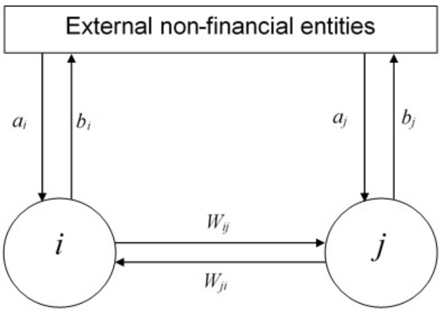

Figure 3.1 Financial network model

The total debt of a node i is denoted by w_i and the total claim is denoted by v_i. The total debt w_i is given as the out-degree of node i by the sum of the total liabilities

$$w_i = \sum_j W_{ij} + b_i \tag{3.1}$$

The total claim v_i is given as the in-degree of node i by the sum of the total claims

$$v_i = \sum_j W_{ji} + a_i \tag{3.2}$$

The fraction of liabilities of i to j of the total liabilities of i is given as

$$P_{ij} = \frac{W_{ij}}{w_i} \tag{3.3}$$

P is the relative liabilities matrix and if the external liabilities vector exists with at least one positive entry, the P matrix will be sub-stochastic. The loans are assumed to be of equal seniority, resulting in an equal, pro rata repayment relative the fraction of liabilities. The initial net worth of a node is defined as the inflow by claims and assets minus the outflow of liabilities by

$$c_i = v_i - w_i \tag{3.4}$$

and it is assumed that the net worth is nonnegative, i.e. $c_i \geq 0$. This assumption implies that all institutions can meet their liabilities before introducing any liquidity shocks.

3.2 Shocks and Payment Equilibrium

The network will be exposed to shocks $(\varepsilon_1, ..., \varepsilon_n)$, affecting one or several of the institutions. These shocks are modeled as a sudden reduction of liquidity, i.e. a reduction of the net value of external assets $(a_1, ..., a_n)$, for the affected institutions and can thereby affect the institutions' ability to meet their liabilities. The shocks are nonnegative, i.e. $\varepsilon_i \geq 0$. The simulation of shocks is carried trough to analyze how the contagion spreads and evaluate the systemic risk in the network. The net worth with shocks present is denoted by d and for a node i it is defined as

$$d_i = c_i - \varepsilon_i = v_i - w_i - \varepsilon_i \tag{3.5}$$

After the simulation of shocks, the loans in the network expire and the borrowers need to repay their lenders. The repayment of i to its creditors is defined as minimum zero and maximum w_i and is denoted by x_i. The repayment will depend on the size of the external asset a_i, the shock ε_i, and the sum of the received claim repayments to i by

$$x_i = sat_{w_i}(a_i - \varepsilon_i + \sum_j x_j P_{ji}) \tag{3.6}$$

where $sat_{w_i}(x)$ is a saturation function as shown in Figure 3.2 and defined as

$$sat_{w_i}(x) = \max\{\min\{x, w_i\}, 0\} \tag{3.7}$$

DEFINITION 1
A payment equilibrium x is a vector that satisfies Equation 3.6 for every entry i. □

The existence of such a vector is non-trivial for a cyclic network and is proved in Section 3.4. In the same section, such a vector is also proved to be generically unique or, under some mild restrictions, unique. For acyclic networks, the solution of the payment equilibrium is trivial.

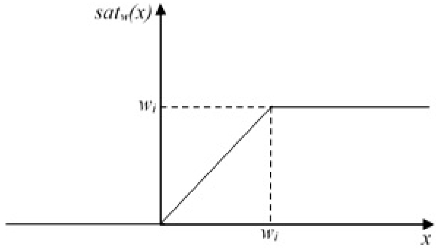

Figure 3.2 Saturation function

As the repayment x_i by definition cannot be negative, the shock ε_i will have a maximum impact of

$$\varepsilon_i^{max} = v_i \tag{3.8}$$

Even though the shock may be of any size, it will never cascade to the next nodes by more than ε_i^{max} because of the saturation function.

The total induced shock of i is denoted by s_i. It is the difference of the total liability w_i and the repayment x_i. The induced shock s_i describes to what extent node i is affected by the shocks in terms of the magnitude of the contagion it spreads to its creditors and is defined in the range $s_i \in [0, w_i]$. If $s_i = 0$, node i can repay its liabilities in full. If $s_i > 0 \Leftrightarrow x_i < w_i$, node i cannot meet its liabilities in full. The total induced shock may be rewritten as

$$\begin{aligned}
s_i &= w_i - x_i \\
&= sat_{w_i}(w_i + \varepsilon_i - a_i - \sum_j x_j P_{ji}) \\
&= sat_{w_i}(w_i + \varepsilon_i \underbrace{-a_i - \sum_j w_j P_{ji}}_{=-v_i} + \sum_j \underbrace{(w_j - x_j)}_{=s_j} P_{ji}) \\
&= sat_{w_i}(w_i - v_i + \varepsilon_i + \sum_j \underbrace{s_j P_{ji}}_{=s_{ji}})
\end{aligned} \tag{3.9}$$

The induced shock depends on both the fixed, known values of the total liabilities w_i and the total claims v_i and variable, possibly unknown values of a shock ε_i and the sum of induced shocks spreading from the borrowers of i, $\sum_j s_{ji}$.

The shock induced from i to solely one creditor j, s_{ij}, is the fraction of the total induced shock, computed as the difference between the single liability W_{ij} and the fraction of repayment to j. The shock induced from i to j may be written as

$$\begin{aligned} s_{ij} &= W_{ij} - x_i P_{ij} \\ &= P_{ij} \underbrace{(w_i - x_i)}_{=s_i} \\ &= P_{ij} sat_{w_i}(w_i - v_i + \varepsilon_i + \sum_h (w_h - x_h) P_{hi}) \\ &= P_{ij} sat_{w_i}(w_i - v_i + \varepsilon_i + \sum_h s_{hi}) \end{aligned} \tag{3.10}$$

The induced shock may expressed by one stochastic part ξ and one deterministic part β when analyzing how stochastic shocks affect the network. The induced shock may hence be defined as

$$s_i = sat_{w_i}(\underbrace{w_i - v_i - a_i}_{\beta_i} + \underbrace{\varepsilon_i + \sum_h s_{hi}}_{\xi_i}) = sat_{w_i}(\beta_i + \xi_i) \tag{3.11}$$

DEFINITION 2
An institution i suffers a

i) direct default if the net worth after shocks is negative

$$d_i < 0 \tag{3.12}$$

ii) complete default if the payment is zero and thus the induced shock maximal

$$x_i = 0 \Leftrightarrow s_i = w_i \tag{3.13}$$

iii) partial default if the payment is less than the actual liability and thus the induced shock positive

$$0 < x_i < w_i \Leftrightarrow 0 < s_i < w_i \tag{3.14}$$

□

If an institution partially defaults, it repays its liabilities on a pro rata basis by the remaining means available. Given a realized shock, the institutions in the network may be separated into two subsets: Institutions that do not default and institutions that partially or completely default. The set of defaulting institutions is denoted by $\mathcal{D}$ and the non-defaulting institutions is thus a part of the complementary set. This implies that

$$\begin{aligned} s_i &> 0 \qquad \forall i \in \mathcal{D} \\ s_i &= 0 \qquad \forall i \notin \mathcal{D} \end{aligned} \tag{3.15}$$

The general model presented in this section may be applied to any network constructed by a certain graph topology.

In order to understand how shocks affect the financial network model it is important to develop some kind of measure to be able to compare the performance between different network structures. An intuitive performance measure is to calculate the number of completely and partially defaults in the network as in [Acemoglu et al., 2015]. This measure is denoted by ψ and simply called number of defaults, given a certain shock. In Chapter 4, expressions for this measure are derived, given certain topologies.

However, in [Glasserman and Young, 2016] it is argued that this measure is not comprehensive enough and that the total systemic loss in a better way measures the total impact. The systemic loss is denoted by L and takes both the financial entities' ability to meet their liabilities as well as the non-financial entities' ability to meet their liabilities into account. The systemic loss is the sum of the external shocks and the induced shocks

$$L = \sum_{i=1}^{n} (\varepsilon_i + s_i) \tag{3.16}$$

These performance measures can then by applied to find the systemically most important nodes in a financial network, as defined below.

DEFINITION 3
A systemically important node is a node to which an external shock causes greater propagation of contagion, as measured by a performance measure, than when shocking other nodes in the network. □

The idea is that when exposing a systemically more important node to a certain shock, it will lead to a worse performance compared to when other nodes are hit.

3.3 Comparison to Previous Models

The model developed by Eisenberg and Noe very much inspires both the financial network model described in this thesis as well as other models that have been used in previous work in this field [Eisenberg and Noe, 2001]. An example of such previous work is the work done by Glasserman and Young who use a similar model to the one in this thesis [Glasserman and Young, 2016]. Their model takes both external assets and external liabilities into account and makes an assumption of out-connectivity. This means that from every node in the network, there exists a path out from the network to some node j which has a positive external liability $b_j > 0$. By making this assumption, there is a risk that an induced shock spreads out of the financial network which also implies that the shock can be absorbed externally.

Acemoglu et al. develops a similar but more complex model with features as investments and time periods considered [Acemoglu et al., 2015]. In this model, Acemoglu et al. also take external liabilities into account but assume that these liabilities are senior over the interbank liabilities. Although this assumption is questionable [Upper, 2011], it leads to another possible interpretation of the model in section 3.1. If the external liabilities are assumed to be senior and always met in full they can be excluded from the model, i.e. the vector of external liabilities can be set to zero, $b = 0$.

The general framework in [Acemoglu et al., 2016] is developed to understand how network interactions affect the performance of the economy. This framework is then specified to create a model of financial contagion which is closely related to the model in [Acemoglu et al., 2015]. Also in this model the external liabilities are set to zero.

The financial network model defined and applied in this thesis can be seen as a more general case of these two models as it is a combination of both. There is no restriction on the external liabilities as they may be both positive and set to zero.

3.4 Existence and Uniqueness of Payment Equilibrium

To study the impact of shocks and to determine how well the financial institutions can meet their liabilities, it is essential to find the payment equilibrium. In this section the existence and uniqueness of a payment equilibrium vector x is proved. By Equation 3.6, the payment equilibrium satisfies

$$x = sat_w(a - \varepsilon + P'x) \tag{3.17}$$

Two propositions are presented and proved below that given any realized vector of shocks ε and structure of the financial network, the equilibrium exists and is (generically) unique.

PROPOSITION 1
There exists a payment equilibrium in any financial network, exposed to any liquidity shocks. □

Proof It should be proven that there exists a payment vector $x^* \subseteq \mathbb{R}^n$ which satisfies $x^* = sat_w(a - \varepsilon + P'x^*)$. A mapping function $\Phi : S \to S$ is defined as

$$(\Phi_{(w)}(x))_i = \max\{\min\{a_i - \varepsilon_i + \sum_j P_{ji}x_j, w_i\}, 0\} \tag{3.18}$$

where $S = \prod_i [0, w_i]$, thus making S convex and compact. The problem of proving the existence of a payment equilibrium is rewritten as proving the existence of a payment vector $x^* \subseteq \mathbb{R}^n$ that fulfills the fixed point

$$x^* = \Phi_{(w)}(x^*) \tag{3.19}$$

Since Φ is also continuous, by using Brouwer's fixed point theorem, such a payment vector must exist. □

PROPOSITION 2

i) If the financial network is out-connected, the payment equilibrium is unique.

ii) If all external liabilities are zero ($b = 0$) and the financial network is connected, the payment equilibrium is generically unique. □

Proof i)
If the network of financial institutions is out-connected, then uniqueness can be proved by the contraction mapping theorem. For the sake of simplicity, the first part of the proposition will only be proved for the special case when all elements of the external liability vector is larger than zero, $b_i > 0\ \forall i$.[1] The contraction mapping theorem says that given a mapping function $\Phi : S \to S$, a metric space (S,d) and a constant $c \in [0,1)$, the mapping function is a contraction mapping iff

$$d(\Phi(x),\Phi(\hat{x})) \leq cd(x,\hat{x}), \qquad \forall x,\hat{x} \in S \tag{3.20}$$

If this is true, there exists a unique fixed point as defined in Equation 3.19 on the complete metric space.

Assume that d defines the distance by the Matrix infinity-norm, defined as $d(x) = ||x||_\infty$, and given any $x,\hat{x} \in \mathbb{R}^n$. The distance of the mapping function can be written as

$$||\Phi(x) - \Phi(\hat{x})||_\infty = ||sat_w(a - \varepsilon + P'x) - sat_w(a - \varepsilon + P'\hat{x})||_\infty \leq ||P'x - P'\hat{x}||_\infty \tag{3.21}$$

where the last inequality comes from the restriction on the minimum and maximum value, described by the saturation function. Without the saturation function, the asset and the shock would cancel each other out and the only expression left would be the P-matrix and the payment vector. With the saturation function, it will always be less than or equal to this. The rows of the relative liabilities matrix P sum up to less than one when the external payment vector b is positive for all entries. This can be written as

$$||P'||_\infty = ||P||_1 = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |P_{ij}| < 1 \tag{3.22}$$

[1] The proof could be extended to the general case, where the network is out-connected. In this case $||P||_1$ in Equation 3.22 can be equal to one. Though, the out-connectivity makes the spectral radius of P less than one and to extend the proof to the general case, the idea is to replace $||P||_1$ with $||P^k||_1$ which, for large enough k, is less than one.

Thus, the mapping function can be proved to be a contraction mapping with the constant $c = ||P'||_\infty$ by writing Equation 3.21 as

$$||\Phi(x) - \Phi(\hat{x})||_\infty \leq ||P'||_\infty ||x - \hat{x}||_\infty \tag{3.23}$$

and the fixed point is proven to be unique. □

Proof ii)
This proof follows the ideas of the proof of Proposition 1 in [Acemoglu et al., 2015]. For a zero external liabilities vector, $b = 0$, the payment equilibrium can only be shown to be generically unique and only for networks that are connected. As the b-vector is zero, the P-matrix is stochastic with row and column sums equal to 1. Assume now that there exist two payment equilibria vectors x and $\hat{x}$. These equilibria are distinct and not equal, $x \neq \hat{x}$. Since they are payment equilibria, they fulfill Equation 3.17. Looking at one entry, representing any node i, the absolute value of the difference between the payment equilibria is

$$|x_i - \hat{x}_i| = |sat_{w_i}(a_i - \varepsilon_i + (P'x)_i) - sat_{w_i}(a_i - \varepsilon_i + (P'\hat{x})_i)) \leq |(P'x)_i - (P'\hat{x})_i| \tag{3.24}$$

where the last inequality comes from the saturation function as described earlier. The only two cases of a tight inequality are when the payment equilibria are equal, $x_i = \hat{x}_i$, or when the payment equilibria are within the saturation interval, $x_i, \hat{x}_i \in [0, w_i]$. Rewriting for the whole payment vectors, the expression becomes

$$||x - \hat{x}||_1 \leq ||P'(x - \hat{x})||_1 \leq ||P'||_1 \cdot ||x - \hat{x}||_1 = ||x - \hat{x}||_1 \tag{3.25}$$

where the last equality comes from that $||P||_1 = 1$ by definition, since the external payment vector is $b = 0$. Thus, it is proved that the inequality must be tight for all entries and that the last inequality in Equation 3.25 is actually an equality. This leads to that, for any node i, either

$$\begin{aligned} (P'x)_i = (P'\hat{x})_i \quad (1) \\ \text{or} \\ 0 \leq a_i - \varepsilon_i + (P'x)_i, a_i - \varepsilon_i + (P'\hat{x})_i \leq w_i \quad (2) \end{aligned} \tag{3.26}$$

For case (1) to occur, the only solution is $x_i = \hat{x}_i$. For case (2) to occur, the payment entry must be within the saturation interval, $x_i, \hat{x}_i \in [0, w_i]$. By denoting the set of nodes that satisfies case (2) by $\mathcal{B}$, the payment vector x_i for all nodes $i \in \mathcal{B}$ can be rewritten without the saturation function as

$$x_i = a_i - \varepsilon_i + (P'x)_i \tag{3.27}$$

Since the asset and shock cancel each other out, this results in a difference equal to

$$(P'x)_i - (P'\hat{x})_i = x_i - \hat{x}_i, \quad \forall i \in \mathcal{B} \tag{3.28}$$

For all nodes $i \notin \mathcal{B}$, the difference must be equal to zero as it goes under case (1). Thus, the total difference must be equal to

$$||P'(x-\hat{x})||_1 = ||x_B - \hat{x}_B||_1 = ||x-\hat{x}||_1 \tag{3.29}$$

where the last equality comes from Equation 3.25 and the subscript B indicates the payment for nodes in the set $\mathcal{B}$. This means it is necessary that $x_i = \hat{x}_i \ \forall i \notin \mathcal{B}$ for the previous equations to hold. By defining a submatrix P_{BB} of P, representing only the nodes belonging to the set $\mathcal{B}$, the following relationship can be established

$$P'_{BB}(x_B - \hat{x}_B) = x_B - \hat{x}_B \tag{3.30}$$

By making the assumption of a connected network, the matrices P and P_{BB} are irreducible and Equation 3.30 cannot hold for $x \neq \hat{x}$ unless the complementary set of $\mathcal{B}$ is empty, $\mathcal{B}^C = \emptyset$. If all nodes belong to $\mathcal{B}$, Equation 3.27 is true for all nodes, and the sum of payments becomes

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (a_i - \varepsilon_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij} x_j \tag{3.31}$$

This implies that $a_i = \varepsilon_i \ \forall i$ and this problem is non-generic with no unique solution. Thus, the other case is $x_i = \hat{x}_i$ and that results in a payment equilibrium that is generically unique. □

3.5 Node Depth

Glasserman and Young develops a type of loss amplification measure called Node depth [Glasserman and Young, 2016]. This measure is based on a Markov chain applied on the defaulting set of nodes $\mathcal{D}$. The Node depth of a node i is denoted by u_i and is given as the expected number of steps to exit the default set $\mathcal{D}$ when starting at node i. The mathematical definition, given a realized shock ε, is

$$u_i = \begin{cases} ((I_D - P_D)^{-1} \mathbb{1}_D)_i = ((I_D + P_D + P_D^2 + \ldots)\mathbb{1}_D)_i, & \text{for } i \in \mathcal{D} \\ 0, & \text{for } i \notin \mathcal{D} \end{cases} \tag{3.32}$$

where I_D is the identity matrix of size $D \times D$, P_D is the sub-matrix of P of the default set $\mathcal{D}$, and $\mathbb{1}_D$ is the vector of ones of size D. The notation of the first line in Equation 3.32 should be read as the element of the vector $(I_D + P_D + P_D^2 + \ldots)\mathbb{1}_D$ that corresponds to element i and not necessarily as the i'th element of the vector. The notion of Node depth is in one sense related to the eigenvector centrality. As will be shown below, this measure can be used to understand how the loss is amplified through the network.

The performance of a network, given a realized shock ε, can be evaluated by the total loss in value of both the direct loss in liquidity by ε and the indirect loss of induced shocks by s. This performance is called systemic loss L and defined in Equation 3.16. This performance measure can be combined with the Node depth by the following derivation. Assume that the shocks are restricted according to $0 \leq \varepsilon_i \leq a_i$. Also assume that the subset of defaulting nodes $\mathcal{D}$ is out-connected, either by a path to the set of non-defaulting nodes or by a path to the outside of the network. Rewrite the shock as

$$\begin{aligned}
s_i &= w_i - x_i \\
&= w_i - sat_{w_i}(a_i - \varepsilon_i + \sum_{i \neq j} P_{ji} x_j) \\
&\overset{\varepsilon_i \leq a_i}{=} w_i - a_i + \varepsilon_i - \sum_{j \in \mathcal{D}} P_{ji} \underbrace{x_j}_{= w_j - s_j} - \sum_{j \notin \mathcal{D}} P_{ji} w_j \\
&= w_i - a_i + \varepsilon_i + \sum_{j \in \mathcal{D}} P_{ji} s_j - \sum_j \underbrace{P_{ji} w_j}_{W_{ji}} \\
&= \sum_{j \in \mathcal{D}} P_{ji} s_j - \underbrace{(v_i - \varepsilon_i - w_i)}_{d_i} \qquad (3.33)
\end{aligned}$$

As the shock is zero for all nodes outside the default set, i.e. $s_i = 0 \; \forall i \notin \mathcal{D}$, the induced shock can be expressed in matrix form for all nodes in the default set. The induced shock for the defaulting nodes is denoted by s_D and can thus be written as

$$s_D = P_D' s_D - d_D = P_D' s_D - (c_D - \varepsilon_D) \qquad (3.34)$$

where c_D defines the net worth before a realized shock and ε_D the realized shock of defaulting nodes in set $\mathcal{D}$. The sub-matrix P_D is sub-stochastic which makes the spectral radius less than one and the induced shock can be rewritten as

$$\begin{aligned}
s_D &= (I_D - P_D')^{-1} (\varepsilon_D - c_D) \\
\Rightarrow \mathbb{1}_D' s_D &= \underbrace{\mathbb{1}_D' (I_D - P_D')^{-1}}_{u_D'} (\varepsilon_D - c_D) \qquad (3.35)
\end{aligned}$$

This result implies that the performance given by L can be rewritten as

$$\begin{aligned} L &= \sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} s_i \\ &= \sum_{i=1}^{n} \varepsilon_i + \sum_{i\in\mathcal{D}} s_i \\ &= \sum_{i=1}^{n} \varepsilon_i + \mathbb{1}_D' s_D \\ &= \sum_{i=1}^{n} \varepsilon_i + u_D'(\varepsilon_D - c_D) \\ &= \sum_{i=1}^{n} \varepsilon_i + \sum_{i\in\mathcal{D}} u_i(\varepsilon_i - c_i) \end{aligned} \tag{3.36}$$

The Node depth u_i for a node i can be described as a measure of shock amplification for large ε_i but on the other hand, if the net worth c_i is larger than the shock, a larger node depth implies a larger shock absorption capability.

Node depth can be bounded by a lower and an upper bound. These bounds can be found by the cohesiveness measure $\kappa(\mathcal{S})$ for a subset of the nodes $\mathcal{S} \subseteq \mathcal{V}$. By applying cohesiveness on the subset of defaulting nodes $\mathcal{D}$, it can be called α_D-cohesive if

$$\sum_{j\in\mathcal{D}} P_{ij} \geq \alpha_D, \qquad \forall i \in \mathcal{D} \tag{3.37}$$

The cohesiveness measure α_D represents the minimum fraction of liabilities a node in $\mathcal{D}$ owes to other nodes in $\mathcal{D}$ and thus shows the minimum connectivity between nodes in the default set. As the node depth is a measure for how long it takes to exit the default set, then $1-\alpha_D$ can be seen as the maximum probability for a node in the default set to exit. Thus, the lower bound for the node depth is given as

$$u_i \geq (1-\alpha_D)^{-1}, \qquad \forall i \in \mathcal{D} \tag{3.38}$$

The more cohesive the set is, the deeper the node depth becomes as the probability for leaving the set reduces. For the upper bound, the connectivity measure β_i is used. For a node i, the connectivity β_i shows the fraction of liabilities to all nodes within the financial network. Define the maximum connectivity of the default set as

$$\beta_D = \max_{i\in\mathcal{D}} \beta_i = \max_{i\in\mathcal{D}} \sum_{j\in\mathcal{V}} P_{ij} \tag{3.39}$$

Then the minimum probability for a node in the default set to exit to the external entities is $(1-\beta_D)$ and bounds the node depth by

$$u_i \leq (1-\beta_D)^{-1}, \qquad \forall i \in \mathcal{D} \tag{3.40}$$

It is possible to bound the node depth for the whole set of nodes $\mathcal{V}$ by an upper bound without knowledge of the present shocks which result in a default set. Then the connectivity for the whole financial network is calculated as

$$\beta^+ = \max_{i\in\mathcal{V}} \beta_i = \max_{i\in\mathcal{V}} \sum_{j\in\mathcal{V}} P_{ij} \tag{3.41}$$

The minimum probability of leaving the financial network is thus $(1-\beta^+)$ which creates an overall upper bound for the node depth

$$u_i \leq (1-\beta^+)^{-1}, \qquad \forall i \in \mathcal{V} \tag{3.42}$$

3.6 Centrality in Example Networks

In this section, it is investigated if it is possible to find and measure which nodes that are systemically more important in a network, in terms of systemic contagion. The analysis is applied on more general, heterogeneous example networks to understand the role of individual nodes of spreading financial contagion in any network structure. More traditional centrality measures, as defined in Section 2.2, are compared to the performance measures. Also the new notion of Node Depth, introduced in previous section, is used and serves both as an indication of centrality and as a part of calculating the systemic loss L.

For a centrality measure to be considered well-performing, it should hold that a shock of a certain size to a more central node should lead to a worse performance of the network compared to shocking a less central node. In this section, three arbitrarily chosen examples of networks, all of them with $n = 4$ nodes, are considered. This setup is inspired by Glasserman and Young to make a simple analysis. For every network, the following centrality measures are calculated for each node: The internal in-degree $\sum_j W_{ji}$, the total in-degree v_i, the internal out-degree $\sum_j W_{ij}$, the total out-degree w_i, the PageRank centrality π_i, and the Katz centrality z_i. For both the PageRank centrality and the Katz centrality, values of $\beta = 0.15$ and $\mu = \mathbb{1}$ are used. The P-matrix used in calculating PageRank centrality does not take out-connections in the network into account. This is to mainly study the internal network structure but also because including the out-connections did not improve the performance of PageRank. Also the topological measure Node depth u_i is calculated and shown in separate tables.

For every network, four simulations are run. In the first simulation, node 1 is hit by a shock of a certain size. In the second simulation, node 2 is hit by a shock of the same size, and so on. After each simulation, the payment equilibrium is found and thus also the set of defaulting nodes. Two performance measures are then found by the default set, the number of defaulting nodes ψ and the systemic loss L, through Node depth. Then, the centrality measures are compared to the performance measures to get an idea of how well the centrality measures actually captures the systemic importance of the nodes.

The comparison between a centrality measure and the two performance measures is made by comparing the centrality of the four nodes to the performance of the network in the four different simulations. As an example, let us say that a centrality measure indicates that node 1 is the most central, node 2 is the second most central, followed by node 3 and then node 4. For the centrality measure to be considered good, the performance of the network should be the worst (i.e. the number of defaults or the systemic loss should be the largest) for the case when node 1 is shocked, second worst when node 2 is shocked, and so on. If this order does not hold, the centrality measure is concluded to be insufficient. Of course, if the opposite is true, i.e. that the centrality measure seem to satisfy the above mentioned criteria for the example networks, the only conclusion that could be drawn is that the centrality measure is satisfying for these three example networks but in order for it to be considered a good centrality measure for financial networks in general, further studies would be needed.

The three networks are shown in Figure 3.3, Figure 3.4, and Figure 3.5, respectively. The considered networks are all both connected and out-connected. The shocks affecting each node in the networks are chosen within the interval $\varepsilon_i \in (0, a_i]$.

Network 1

The centrality measures in Table 3.1 favors node 1 to be the most central node, except for the Katz centrality giving slightly more centrality to node 2 and the internal out-degree which indicates that node 4 is the most central node. All nodes are then subject to a shock $\varepsilon_i = 0.3$. When comparing the centrality measures to the performance measures given this shock in Table 3.3, it turns out that shocking node 1 causes the least number of defaults and smallest systemic loss in the network. Instead, an equal shock to node 4 results in the highest number of defaults and the largest systemic loss. None of the centrality measures seem to capture the systemic risk in this network in a good way.

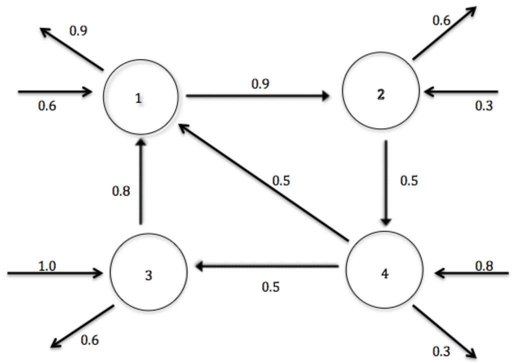

Figure 3.3 Network 1

Node	Internal In-degree	Total In-degree	Internal Out-degree	Total Out-degree	PageRank	Katz
1	**1.3**	**1.9**	0.9	**1.8**	**1.148**	1.201
2	0.9	1.2	0.5	1.1	1.126	**1.350**
3	0.5	1.5	0.8	1.4	0.620	0.643
4	0.5	1.3	**1.0**	1.3	1.101	0.894

Table 3.1 Centrality measures for each node in Network 1

Ext. Shock	Node Depth			
	u_1	u_2	u_3	u_4
$\varepsilon_1 = 0.3$	1.000	0.000	0.000	0.000
$\varepsilon_2 = 0.3$	0.000	1.455	0.000	1.000
$\varepsilon_3 = 0.3$	1.000	0.000	1.571	0.000
$\varepsilon_4 = 0.3$	1.000	0.000	1.571	1.989

Table 3.2 Node depth for each node given the external shock in Network 1

Ext. Shock	Number of Defaulting Nodes ψ	Systemic Loss L
$\varepsilon_1 = 0.3$	1 (node 1)	0.500
$\varepsilon_2 = 0.3$	2 (node 2 and 4)	0.591
$\varepsilon_3 = 0.3$	2 (node 1 and 3)	0.514
$\varepsilon_4 = 0.3$	**3** (node 1, 3, and 4)	**0.640**

Table 3.3 Performance measures given the external shock for Network 1

Network 2

In Network 2, both node 1 and 4 are favored to be more central than the others, as seen in Table 3.4. In this simulation, the shock affecting all nodes is $\varepsilon_i = 0.5$. When affecting node 1 with this shock, it causes three nodes to default which is equal to the number of defaulting nodes when exposing node 3 to the same shock. However, as stated in Table 3.6, the worst systemic loss is reached when shocking node 3. Shocking node 2 by this shock causes the least damage with only one default and the least systemic loss. Node 2 is also given the least centrality by PageRank, Katz, and both out-degrees. None of the centrality measures seem to match the performance measures in a satisfying way.

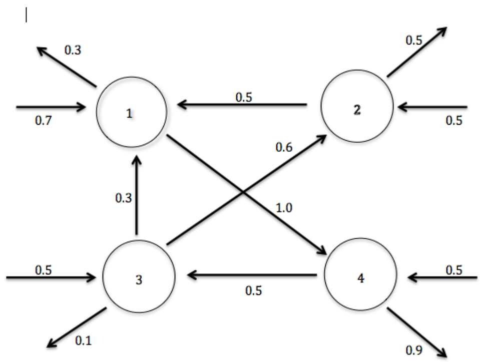

Figure 3.4 Network 2

Node	Internal In-degree	Total In-degree	Internal Out-degree	Total Out-degree	PageRank	Katz
1	0.8	**1.5**	**1.0**	1.3	**1.095**	0.973
2	0.6	1.1	0.5	1.0	0.756	0.819
3	0.5	1.0	0.9	1.0	1.069	0.933
4	**1.0**	**1.5**	0.5	**1.4**	1.081	**1.312**

Table 3.4 Centrality measures for each node in Network 2

Ext. Shock	Node Depth			
	u_1	u_2	u_3	u_4
$\varepsilon_1 = 0.5$	3.093	0.000	4.820	2.722
$\varepsilon_2 = 0.5$	0.000	1.000	0.000	0.000
$\varepsilon_3 = 0.5$	1.000	1.500	2.200	0.000
$\varepsilon_4 = 0.5$	0.000	0.000	1.000	1.357

Table 3.5 Node depth for each node given the external shock in Network 2

Ext. Shock	Number of Defaulting Nodes ψ	Systemic Loss L
$\varepsilon_1 = 0.5$	**3** (node 1, 3, and 4)	1.156
$\varepsilon_2 = 0.5$	1 (node 2)	0.900
$\varepsilon_3 = 0.5$	**3** (node 1, 2, and 3)	**1.250**
$\varepsilon_4 = 0.5$	2 (node 3 and 4)	1.043

Table 3.6 Performance measures given the external shock for Network 2

Network 3

The last example network shows the largest spread of highest centrality and node 1, 3, and 4 are favored by some centrality measure, given by Table 3.7. Node 2 is deemed to be the least central of all nodes by all centrality measures. Given an equal shock $\varepsilon_i = 0.3$, the performance measures in Table 3.9 shows that shocking both node 1 and 3, respectively, results in 2 defaults but the largest systemic loss is implied when shocking node 3. This coincides with the largest centrality as given by PageRank, Katz, and internal in-degree centrality. Both when shocking node 2 and 4, respectively, an equal systemic loss is incurred and only the one shocked node defaults. The only ranking consistent with this performance is the ranking by PageRank centrality.

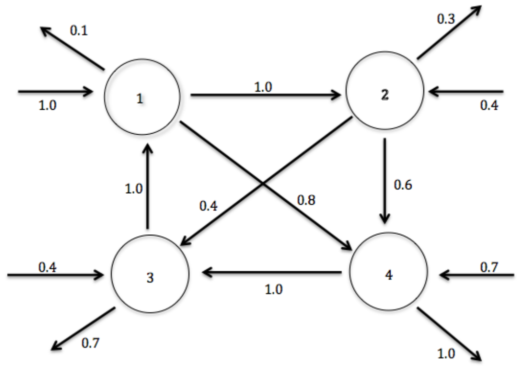

Figure 3.5 Network 3

Node	Internal In-degree	Total In-degree	Internal Out-degree	Total Out-degree	PageRank	Katz
1	1.0	2.0	**1.8**	1.9	1.163	0.958
2	1.0	1.4	1.0	1.3	0.699	0.832
3	**1.4**	1.8	1.0	1.7	**1.192**	**1.135**
4	**1.4**	**2.1**	1.0	**2.0**	0.946	1.051

Table 3.7 Centrality measures for each node in Network 3

Ext. Shock	Node Depth			
	u_1	u_2	u_3	u_4
$\varepsilon_1 = 0.3$	1.526	1.000	0.000	0.000
$\varepsilon_2 = 0.3$	0.000	1.000	0.000	0.000
$\varepsilon_3 = 0.3$	1.000	0.000	1.588	0.000
$\varepsilon_4 = 0.3$	0.000	0.000	0.000	1.000

Table 3.8 Node depth for each node given the external shock in Network 3

Ext. Shock	Number of Defaulting Nodes ψ	Systemic Loss L
$\varepsilon_1 = 0.3$	**2** (node 1 and 2)	0.505
$\varepsilon_2 = 0.3$	1 (node 2)	0.500
$\varepsilon_3 = 0.3$	**2** (node 1 and 3)	**0.518**
$\varepsilon_4 = 0.3$	1 (node 4)	0.500

Table 3.9 Performance measures given the external shock for Network 3

Result

Given these three simple heterogeneous networks, there is no proof that any of the considered centrality measures are able to find the systemically most important node, in terms of performance of the network when shocks are present. Neither in the first example nor in the second example the centrality measures are able to predict which node that causes the worst performance, not even the PageRank and Katz centrality which are designed to take more information into account. However, in the second example they are able to predict the systemically least important node by giving it the least centrality. The centrality measures coincided somewhat better with the performance of the third network example, where especially PageRank managed to rank the nodes similarly to the performance measures.

By these examples, it is shown that the traditional, off-the-shelf centrality measures fail to capture the systemically important nodes in a financial network subject to shocks. This result is in line with [Acemoglu et al., 2015] which shows that due to the nonlinearities in the financial network model for interbank lending, traditional centrality measures are not sufficient to find the systemically most important nodes. The measure of Node depth is comprehensible and given an ex-post perspective, it captures the systemically important nodes better. However, this measure is not possible to apply to a financial network to capture the inherent risk of individual nodes, before any shocks has been realized, neither to a financial network that is not out-connected. Though, as shown in the previous section by Equation 3.42, it is possible to find an overall upper bound of the Node depth without realizations of shocks, which illustrates the worst case of loss amplification for any node in a financial network.

4

Ex-post Shock Propagation Analysis

In this chapter, an analysis in an ex-post perspective will be carried through of how deterministic shocks propagate through specific, homogeneous networks where no node has a larger influence than anyone else to understand the network structure's role in financial contagion. Mainly three different, specific networks are considered, constructed by the directed ring graph, the complete graph, and the directed k-regular tree graph. These specific model topologies are modeled according to the network model defined in Chapter 3 and a further motivation and implementation is given in Section 4.1. The random tree graph will not be considered in this chapter as it is not possible to find an exact measure due to its random nature.

To study how the contagion spreads in the different networks, two different cases of shocks will be considered. As it is an ex-post analysis, the shocks are modeled as deterministic and known. First, one deterministic shock on one node is analyzed and then, both shocks on all nodes and multiple shocks on some nodes. To measure the contagion and the performance of the network, the number of defaults are calculated, denoted by ψ. The number of defaults consists of both completely and partially defaulting nodes in the network and are counted to later compare the different network structures in terms of performance.

4.1 Specific Model Topologies and Assumptions

The main reason for choosing the directed ring graph as one of the investigated topologies is the fact that it is the most sparsely connected graph. It has been discussed by many papers whether a higher or lower connectivity is to prefer in terms of stability and robustness in the network when shocks are present [Glasserman and Young, 2016]. It is also cyclic which implies that shocks may cascade around the ring in more than one cycle. The implication of applying a directed ring graph on the financial network model is the fact that each node has only one borrower and one creditor, as there are only one in-neighbor and one out-neighbor. That results in a generalization of the fraction of the repayment to the total repayment as $x_{ij} = x_i$ and the fraction of the induced shock to the total induced shock as $s_{ij} = s_i$.

The complete graph is chosen by a similar reason as the ring graph. It is the most connected of all graphs and is disputed to be the most stable or the least stable financial network graph. It is also cyclic and the maximum connectivity implies that a shock may cascade through many cycles simultaneously.

The tree graph is chosen for its acyclic nature where shock contagion cannot spread through more than one cycle. In the directed tree graph, the liabilities are directed downwards, which means that a parent node i owes a total of w_i to its k_i children. This means that the claims are directed in the opposite direction, upwards. All nodes, except the root node, has one claim from its parent. A fixed number of children means that all nodes in the graph have the same number of children, i.e. $k_i = k$, while the number of children is a random variable in the random tree graph. This randomness implies that it is not possible to compute an exact outcome but only a probability, which is the reason for it to not be considered in this ex-post analysis.

The main assumption for all topologies is a homogeneous network where all nodes have equal net worth $c_i = c$. The reason for this restriction is to limit the shock propagation analysis to the case where the contagion is only dependent on the specific network structure and not on any heterogeneity in the network, like different sizes or leverages of nodes. It is also assumed that the external liability vector is nonexistent, i.e. $b_i = 0 \ \forall i$. This may be interpreted as that all external liabilities have already been paid and is of no interest. Lastly, the external asset is assumed to be less than the total liabilities, i.e. $a_i < w_i$, as to make propagation of contagion possible even when only one shock is present. For each specific topology, a couple of more assumptions are made which are clearly defined in the next section.

The topologies and assumptions described above are also used when investigating shock propagation in an ex-ante perspective in Chapter 5.

4.2 Contagion by One Deterministic Shock

To understand what happens when a financial network is exposed to an external liquidity shock, it is first assumed that only one shock hits the network at node 1, resulting in $\varepsilon_1 > 0$ and $\varepsilon_i = 0$ for $i = 2, \ldots, n$. To measure the direct impact of the shock, it is initially given that the borrower(s) of the shocked node 1 can meet its liabilities in full. The analysis starts with the directed ring graph, followed by the complete graph, and lastly the k-regular directed tree graph.

Ring network

To make the ring network homogeneous, the out-degree is assumed to be equal to the in-degree. This makes the total internal liability, and thereby the total internal claim, the same size when modeling the ring graph as the financial network model, i.e. $\sum_j W_{ij} = \sum_j W_{ji}$. The size of total claims and total liabilities of each institution is normalized to $w_i = v_i - a_i = 1 \ \forall i$ to simplify the analysis. Thus, the weight matrix can be illustrated as:

$$W = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ \vdots & 0 & 1 & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & 1 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & \dots & \dots & 0 \end{bmatrix} \tag{4.1}$$

For homogeneity, it is assumed that the external asset is of the same size for all nodes, i.e. $a_i = a \ \forall i$. As node 1 is initially assumed to receive its claim in full, institution n meets its liability to institution 1 in full, i.e. $x_n = 1$.

Given these assumptions as initial starting values, the induced shock s_i from Equation 3.9 may be redefined for the ring network as

$$s_i = \begin{cases} sat_1(\underbrace{w_1}_{=1} + \varepsilon_1 - a_1 - \underbrace{x_n}_{=1}) = sat_1(\varepsilon_1 - a) & \text{for } i = 1 \\ sat_1(\underbrace{w_i - v_i}_{=-a_i} + \underbrace{\varepsilon_i}_{=0} + s_{i-1}) = sat_1(s_{i-1} - a) & \text{for } i = 2, \dots, n \end{cases} \tag{4.2}$$

The number of defaults in the network depends on how the contagion spreads, measured by s_i. A positive induced shock, $s_i > 0$, implies a default of node i and the contagion will continue to spread until the induced shock is zero, $s_i = 0$. As can be seen in Equation 4.2 above, the induced shock mainly depends on the size of the asset and the external shock, relative to each other. Given the initial assumptions, there are two possible scenarios for the affected node:

- Affected node absorbs shock if $\varepsilon_1 \leq a$
  Node 1 can meet its liability to node 2 in full if $\varepsilon_1 \leq a$ since then, according to Equation 3.6, $x_1 = sat_1(a - \varepsilon_1 + 1) = 1$. In this case, the induced shock will be $s_1 = 0$ and no contagion will spread. All the other nodes will also be able to meet their liabilities in full and no nodes in the network defaults, i.e. $x_i = 1$ $\forall i \Leftrightarrow s_i = 0 \ \forall i$.

- Affected node spreads shock if $\varepsilon_1 > a$
  On the other hand, if $\varepsilon_1 > a$, then the net worth after the shock is negative, $d_1 < 0$, and the repayment is thus $x_1 < 1$, which causes node 1 to a directly default as it is not able to pay its full liabilities. The shock will then propagate to node 2 as the induced shock will be $s_{12} = s_1 = sat_1(\varepsilon_1 - a) > 0$, from equation 4.2. For the shock to spread even further, the induced shock must be larger than the asset. Hence, if $s_1 > a$, the contagion spreads to the next node and $s_2 = sat_1(s_1 - a) > 0$, which causes node 2 to default as well.

However, there is a maximum impact by the external shock due to the saturation function of the repayment, defined by Equation 3.8. Even though the shock may be of any size, the maximum impact for this ring network is $\varepsilon_1^{max} = 1 + a$. This is of high importance when finding an expression for the number of defaulting institutions. The number of defaulting institutions may be found as a function of the asset a and the shock ε_1 and is denoted by ψ. In the ring network it is equal to

$$\psi = \min(\left\lceil \frac{\varepsilon_1}{a} - 1 \right\rceil, \left\lceil \frac{1}{a} \right\rceil, n) \tag{4.3}$$

This expression is derived from the definition of a partial default by the induced shock, $s_i > 0$. The number of defaults can be found as the minimum of i so that $s_{i+1} = 0$. For each node, there is an external asset a which serves as an absorption. The condition for the first node to default is $\varepsilon_1 > a \Leftrightarrow \frac{\varepsilon_1}{a} > 1$. For the next node to default, the condition becomes $\varepsilon_1 > 2a \Leftrightarrow \frac{\varepsilon_1}{a} > 2$, and so on. The strict inequality results in a ceiling function of the fraction minus one to separate the cases when the shock is equal to the asset. The ceiling function $f(x) = \lceil x \rceil$ rounds up non-integer numbers to the next integer. There are three cases in Equation 4.3. The first two cases depend on the size of ε_1. In the second case, the shock is restricted to its maximum impact $\varepsilon_1^{max} = 1 + a$. The third case comes from the fact that the number of defaults cannot exceed the number of institutions in the network n.

Complete network

Modeling a financial, homogeneous network using the complete graph implies equal internal liabilities and claims for all nodes. As in the previous analysis of ring graph, the size of internal claims and liabilities for each node are equal and normalized to $w_i = v_i - a_i = 1\ \forall i$. This means that the elements of the weight matrix W, and of the normalized adjacency matrix P, in the complete network are

$$W_{ij} = P_{ij} = \begin{cases} \frac{1}{n-1}, & i \neq j \\ 0, & i = j \end{cases} \tag{4.4}$$

For homogeneity reasons, the additional asset is also assumed to be equal for all nodes, $a_i = a\ \forall i$. As in the ring network, it is initially given that node 1 receives its claims in full from the other nodes. The total repayment from a node i is $x_i \in [0,1]$ and as long as $x_i = 1$, it does not default. For node 1 to default, the same applies as in the ring network, which is that the external shock must be larger than the asset value, i.e. $\varepsilon_1 > a$. Then, the induced shock $s_1 = sat_1(\varepsilon_1 - a) > 0$, i.e. node 1 will not be able to meet its liabilities in full and the shock will propagate to all the other nodes in equal proportion.

All the other nodes will receive an equal amount of money from node 1, which implies that the induced shock from node 1 to an individual node will be the same, $s_{12} = s_{13} = ... = s_{1n}$. The induced shock from node 1 to any other node $j \neq 1$ is $s_{1j} = P_{1j}s_1 = s_1/(n-1)$. Thus, the expression for the induced shock can be redefined for the complete network as

$$s_i = \begin{cases} sat_1(\varepsilon_1 - a) & \text{for } i = 1 \\ sat_1(\frac{s_1}{n-1} - a) & \text{for } i = 2, ..., n \end{cases} \tag{4.5}$$

This expression can be found in a similar way as for the ring graph by rewriting Equation 3.9. In order for the other nodes to default, the induced shock from these nodes must be larger than 0, i.e. $s_j > 0$ for $j = 2, .., n$. This happens when the shock that propagates to any node j from node 1 is larger than the asset a, i.e. $s_{1j} > a$. Since the conditions are equal for all nodes, the condition for default applies to all nodes and is defined by

$$s_j > 0 \iff s_{1j} > a \iff \min\left(\frac{\varepsilon_1 - a}{n-1}, \frac{1}{n-1}\right) > a \qquad \text{for } j = 2, ..., n \tag{4.6}$$

The shock must hence be $\varepsilon_1 > a(n-1) + a \Leftrightarrow \varepsilon_1 > na$ to cause a complete default of the network. Since the induced shock s_{1j} that propagates cannot be larger than $1/(n-1)$, it implies that the maximum impact of the shock is $\varepsilon_1^{max} = 1 + a$ on the other nodes. This means that if $\varepsilon_1 \geq 1 + a$, any increase of ε_1 will not affect the other nodes and this is the reason for the two different cases in Equation 4.6.

The final result is that there are only three possible outcomes of the number of defaults in the complete network:

- Zero defaults when $\varepsilon_1 \leq a$
  If the shock is smaller than or equal to the asset, the affected node can absorb the shock and prevent further contagion.
- One default of the affected node 1 when
  - $a < \varepsilon_1 \leq na$
    If the shock is larger than the asset but smaller than the complete default point na, only the affected node 1 will suffer a direct default and

the shock that spreads will not be large enough to affect the rest of the network.

- $\varepsilon_1 > na$ and $na \geq 1+a$
  If the shock is larger than its maximum possible impact ε_1^{max}, it does not matter that the shock is larger than the complete default point na as the impact cannot be larger than $1+a$. This implies that this case occurs when the asset is large enough, i.e. $a \geq \frac{1}{n-1}$ since $na \geq 1+a$.

- n defaults of all nodes in the network when $\varepsilon_1 > na$ and $na < 1+a$
  If the shock is within its impact range and large enough, the whole network defaults. This case can only occur when the asset is small enough, i.e. $a < \frac{1}{n-1}$ since $na < 1+a$.

In summary, the number of defaults ψ in the complete network is equal to

$$\psi = \begin{cases} 0, & \text{if } \varepsilon_1 \leq a \\ 1, & \text{if } \varepsilon_1 > a \text{ and } a \geq \frac{1}{n-1} \\ n, & \text{if } \varepsilon_1 > na \text{ and } a < \frac{1}{n-1} \end{cases} \tag{4.7}$$

K-regular tree network

For the directed k-regular tree network, the total liability of a node i to its children is normalized to $w_i = 1$ and is assumed to be equally distributed between its k children, to keep the homogeneity restriction. The total claim v_j of a child j consists of only one internal claim, the claim on its parent, and the external asset a_i. As the liability of the parent is equally distributed between its children, a node $i > 1$ has a total claim of

$$v_i = \frac{1}{k} + a_i \tag{4.8}$$

To make the initial conditions for all nodes equal and keep the homogeneity constraint, the root node has an extra asset of value 1 in addition to the a' and the other nodes have an extra asset of value $\frac{k-1}{k}$ in addition to the a'. The extra assets hence make the net worth c of all nodes equal. The external assets of the nodes are

$$a_i = \begin{cases} a' + 1, & \text{for } i = 1 \\ a' + \frac{k-1}{k}, & \text{for } i = 2, \ldots, n \end{cases} \tag{4.9}$$

When the k-regular tree network is subject to a shock $\varepsilon_1 > 0$ on the root node, the shock possibly propagates from the root node to the next generations. Each node in one generation is hit equally hard since the claims are identical. This makes the repayment for all nodes in one generation equal and the repayment can be generalized for all nodes in the same generation. A consequence of this generalization is an extension of the notation for the repayment and induced shock by a superscript defining the generation, i.e. x_i^g and s_i^g. The generation index g simply explains to which generation node i belongs to. The root node generation is denoted generation zero and so on. Equation 3.6 is hence redefined as

$$\begin{aligned} x_i^g &= \begin{cases} sat_{w_1}(a_i - \varepsilon_i), & g = 0, i = 1 \\ sat_{w_i}(a_i + x_h^{g-1} P_{ji}), & g \geq 1, h \geq 1, i \geq k+1 \end{cases} \\ \Rightarrow x_i^g &= \begin{cases} sat_1(a' + 1 - \varepsilon_1), & g = 0, i = 1 \\ sat_1(a' + \frac{k-1}{k} + \frac{x_h^{g-1}}{k}), & g \geq 1, h \geq 1, i \geq k+1 \end{cases} \end{aligned} \tag{4.10}$$

where the node h is the parent of node i.

Equation 3.9, i.e. the total induced shock from a node in a given generation to all its creditors, is redefined as

$$s_i^g = \begin{cases} sat_1(\varepsilon_1 - a'), & g = 0, i = 1 \\ sat_1(\frac{s_h^{g-1}}{k} - a'), & g \geq 1, h \geq 1, i \geq k+1 \end{cases} \tag{4.11}$$

where the node h is the parent of node i, just as in Equation 4.10.

Since the condition is equal for all nodes in the same generation, it is intuitive that if a node does not default, then the other nodes of that generation do not default either. Neither does any generation further down the tree default, as there is no shock left to spread. Thus, for each generation, all nodes will default or none will default. A generation g of nodes will default if, for any node i in the generation, the induced shock $s_i^g > 0$.

The extreme case of no defaults at all occurs if and only if the size of the induced shock from the root node is zero, i.e. if $s_1^0 = 0 \Leftrightarrow a' \geq \varepsilon_1$. Since it is assumed that the net worth $c_i \geq 0$ and thereby that $a' > 0$, it is clear that for every generation g, the additional asset a_i absorbs a part of the shock and the induced shock s^g decreases until it reaches zero. This implies that the most interesting cases where ε_1 has potential to affect the network is when $a' < \varepsilon_1$. But as in previous cases, due to the non-possibility of negative repayments, there is still a maximum impact of the shock, $\varepsilon_1^{max} = 1 + a'$.

The number of defaults in the tree network depends on how fast s_i^g reaches zero, which in turn depends on the initial asset size a', the number of children k, and the size of the shock ε_1. The number of defaults in the network can be generalized to the number of generational defaults, as the nodes in the same generation have equal condition. For the whole network, the number of generational defaults Ψ is given by

$$\Psi = \begin{cases} 0, & \text{if } s_1^0 = 0 \\ max(g+1 | s_i^g > 0), & \text{if } s_1^0 > 0 \end{cases} \tag{4.12}$$

If Ψ generations default, the number of defaulting nodes is

$$\psi = \sum_{i=0}^{\Psi-1} k^i \tag{4.13}$$

The maximum impact the external shock may have is $\varepsilon_1^{max} = a_1 = a' + 1$ which means that even if the shock is larger than this, it will still have the same, maximum impact on the network. A better expression than Equation 4.12 for the number of generational defaults Ψ, if the root node defaults (i.e. $\varepsilon_1 > a'$), can be found by maximizing Ψ subject to the following restriction

$$\Psi = \max(\Psi | \varepsilon_1^* > a' \sum_{i=0}^{\Psi-1} k^i) \tag{4.14}$$

where $\varepsilon_1^* = \min(\varepsilon_1, \varepsilon_1^{max})$. This expression is derived from Equation 4.10 by iteratively finding each generation's repayment, given the previous generation's repayment. If $x_1^0 = sat_1(a' + 1 - \varepsilon_1^*) < 1$, the next generation will repay $x_i^1 = sat_1(\frac{a'+1-\varepsilon_1^*}{k} + a' + \frac{k-1}{k}) = sat_1(\frac{a'(k+1)+k-\varepsilon_1^*}{k})$. For this generation to default, the repayment must be $x_i^1 < 1$ and hence $\varepsilon_1^* > a'(k+1)$. If this is true, the repayment for the next generation will be $x_j^2 = sat_1(\frac{a'(k+1)-\varepsilon_1^*+k}{k^2} + a' + \frac{k-1}{k})$ and the condition for this generation to default is $\varepsilon_1^* > a'(k^2+k+1)$. Thus, the number of generational defaults can be seen as a relationship between the shock and a geometric sum.

The restriction in Equation 4.14 can be rewritten to find a formula for the number of generational defaults in the network by making use of the geometric sum.

$$\begin{aligned}
& \varepsilon_1^* > a' \sum_{i=0}^{\Psi-1} k^i \\
\overset{k\neq 1}{\Longrightarrow} \quad & \varepsilon_1^* > a' \frac{k^\Psi - 1}{k-1} \\
\Longleftrightarrow \quad & \varepsilon_1^*(k-1) > a'(k^\Psi - 1) \\
\Longleftrightarrow \quad & \frac{\varepsilon_1^*(k-1)}{a'} > k^\Psi - 1 \\
\Longleftrightarrow \quad & \ln(\frac{\varepsilon_1^*(k-1)+a'}{a'}) > \Psi \ln(k) \\
\Longleftrightarrow \quad & \Psi < \frac{ln(\varepsilon_1^*(k-1)/a'+1)}{ln(k)}
\end{aligned} \tag{4.15}$$

If $k = 1$, the tree is reduced to a directed line graph, similar to the ring graph defined earlier, and the geometric sum is equal to

$$\begin{aligned} & \varepsilon_1^* > a' \sum_{i=0}^{\Psi-1} k^i \\ \overset{k=1}{\Longrightarrow} \quad & \varepsilon_1^* > a'\Psi \\ \Longleftrightarrow \quad & \Psi < \frac{\varepsilon_1^*}{a'} \end{aligned} \tag{4.16}$$

So, the number of generational defaults is found by taking the largest integer value of Ψ that fulfills Equation 4.15 (if $k > 1$) or Equation 4.16 (if $k = 1$). The number of generational defaults Ψ is hence given by the equation

$$\Psi = \begin{cases} \lceil \frac{\varepsilon_1^*}{a'} - 1 \rceil, & \text{if } k = 1 \\ \lceil \frac{\ln(\varepsilon_1^*(k-1)/a'+1)}{\ln(k)} - 1 \rceil, & \text{if } k > 1 \end{cases} \tag{4.17}$$

For this equation to hold, only the "interesting" cases of $a' < \varepsilon_1$ are considered (if $a' \geq \varepsilon_1$, no nodes default) and the shock is still restricted to its maximum impact $\varepsilon_1^* = \min(\varepsilon_1, \varepsilon_1^{max})$.

Comparison

Given the three topologies considered for one deterministic shock, a comparison between the networks is presented below. The condition for the shocked node 1 to default is equal for all of the network topologies. Since it is assumed that node 1 receives its claims in full or, as for the k-regular tree, has an additional asset representing this, the shock must be larger than a or a' for one node to default. This outcome is important since it makes the initial condition equal for further contagion in the networks. How the contagion then spreads is only a result of the structure of the network and is not dependent on any other factor.

In Figure 4.1, the ring and complete network are compared given four different parameters for the number of nodes n and the size of the asset a. The contagion is measured as a function of the number of defaults given the size of the shock. By also looking at the graphs, the result becomes clear. The complete graph performs always at least as good as the ring graph. As earlier concluded, the shock required for one default is equal for both networks. Also, the shock required for all nodes to default is equal for both networks. However, for any shocks within the interval of these extreme cases, the ring network always perform worse by linearly increasing the number of defaults while the complete network absorbs the shock for small enough shocks. It is especially clear in the plot on the bottom left where the number of nodes and the asset are large enough to completely absorb any shock in the complete network and thus never causes more than one default, whereas the ring network keeps increasing the number of defaults.

This result is in line with [Acemoglu et al., 2015], who states that given a large enough shock relative to the asset and the size of the network, the complete and the ring network perform equally bad. As long as the shock is lower than this threshold, the complete network is strictly more stable and resilient than the ring network, omitting the shocked node and only looking at the rest of the non-shocked nodes.

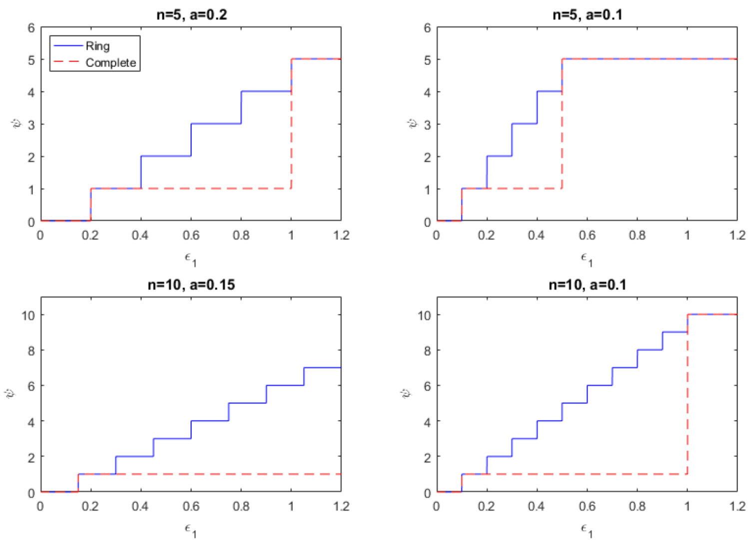

Figure 4.1 Comparison of the performance in terms of number of defaults as a function of size of the shock by ring and complete network

Figure 4.2 shows the result of number of generational defaults Ψ and Figure 4.3 shows the number of defaulting nodes ψ when shocking the root node in the k-regular tree network by a shock ranging as $\varepsilon_1 \in [0, 1+a']$, when $a' = 0.15$. Four different scenarios are tested with number of children equal to $k = 1,2,3,4$. These scenarios result in a different distribution of generational defaults. For $k = 1$ children, the tree graph is simply a line of nodes and behaves like the ring graph earlier described. The maximum number of generational defaults is $\Psi = 7$, which coincides with the number of node defaults $\psi = 7$, as each generation only has one node. By increasing the number of children to $k = 2$, the claim reduces to $1/2$ but the asset also increases to $a' + 1/2$ which makes the network more stable as the nodes can rely on their asset a_i to a larger extent. The maximum number of generational defaults is $\Psi = 3$ and the maximum number of nodes defaulting is $\psi = 7$. With $k = 3$ children, the default curve is even lower and reaches a maximum of $\Psi = 2$ generational defaults, which implies a maximum of $\psi = 4$ node defaults. When the number of children is $k = 4$, the number of generational defaults is the same as with $k = 3$ children, but it requires a larger shock to reach the maximum of $\Psi = 2$ generational defaults, which translates to $\psi = 5$ node defaults.

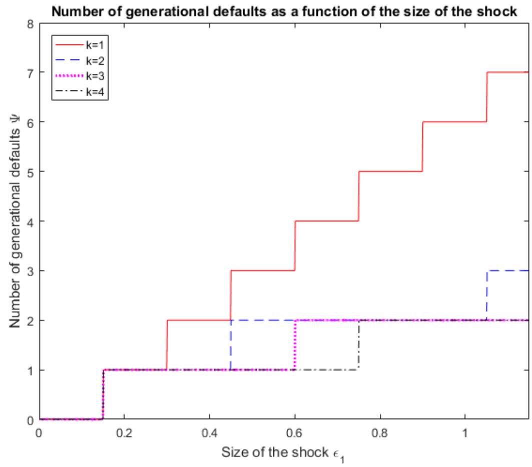

Figure 4.2 Comparison of the performance in terms of number of generational defaults by four different tree graphs with $k = 1, 2, 3, 4$ and $a' = 0.15$.

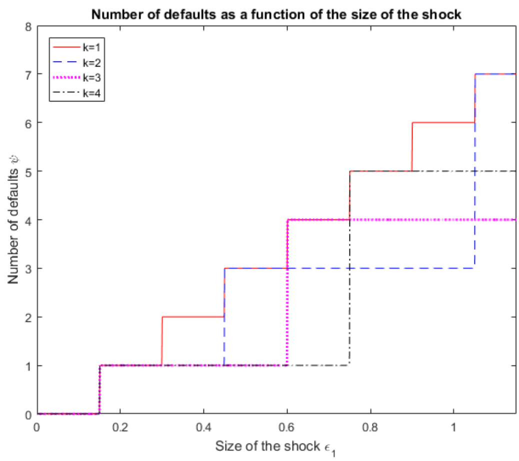

Figure 4.3 Comparison of the performance in terms of number of defaults by four different tree graphs with $k = 1, 2, 3, 4$ and $a' = 0.15$.

If the concern is how far down in the tree, in terms of generations, the contagion spreads, the ability to absorb shocks increases with the number of children in the k-regular tree network, given the assumption on homogeneity and the structure of the asset. Compared to the complete network, the k-regular tree network does not have a certain threshold where the shock causes a complete default which may be concluded to result in a higher stability. Also the fact that the tree is acyclic, which means that defaults can only cascade in one direction, downwards, and does not cycle back to any nodes, can be seen as factor of creating a more stable network. Though, only looking at the number of generational defaults might be misleading as for each additional child in the tree, the generation becomes larger and a generational default implies a larger number of nodes defaulting.

4.3 Contagion by Multiple Deterministic Shocks

In this section, it is assumed that a number of shocks m are present. The shocks are assumed to be homogeneous and deterministic. First a contagion analysis is performed on the three networks when $m = n$ shocks are present and thus all nodes are exposed. The shock can then be rewritten as $\varepsilon_i = \varepsilon \ \forall i$, since it is equal for all nodes. Then an example of exposing a ring and complete network with $m = 2 < n$ homogeneous shocks is shown.

Ring network

The analysis starts from the same initial condition as earlier, where it is given that node 1 receives its claim in full from node n. As found in the previous section, node 1 does not directly default if the shock is less than its asset, i.e. $\varepsilon < a$. All nodes are now subject to the same shock and if the shock is $\varepsilon < a$, all nodes' net worth after the shock is positive by $d = 1 - 1 + a - \varepsilon > 0$ from Equation 3.5. Thus, no nodes default in this case. The case of $\varepsilon = a$ is omitted since the solution is not unique, as given from Proposition 2 in Section 3.4. Then the final case is when the shock is larger than the asset $\varepsilon > a$ and this implies a complete default of the whole network since all nodes' net worth is negative and no nodes are able to pay their liabilities. Hence, the total number of defaults only depends on the relation between the shock and the asset. The result is either no node defaults or all nodes n default.

$$\psi = \begin{cases} 0, & \text{if } \varepsilon < a \\ n, & \text{if } \varepsilon > a \end{cases} \tag{4.18}$$

Complete network

If the initial condition of node 1 receiving its total claim and if all nodes in the complete network are exposed to the same shock ε, the consequence is the same as in the ring network. As long as the shock is smaller than the asset, all nodes will have a positive net worth and there will not exist any induced shock propagating between any nodes since the asset absorbs all shocks. For the other case when the shock is larger than the asset, $\varepsilon > a$, all nodes in the network will directly default as the net worth is negative. The size of the network has no effect on the number of defaults. Thus, the number of defaults in the complete network equals the number of defaults in the ring network and ψ is equal to Equation 4.18.

K-regular tree network

From the analysis from the one deterministic shock case, it is still true that the root node does not default if the shock is smaller than the extra asset, i.e. $\varepsilon \leq a'$. Then the root node can repay its liabilities in full as $x^0 = sat_1(a' + 1 - \varepsilon) = 1$ by Equation 4.10. Even though the nodes in the next generation also are subject to this shock, they will also be able to repay their liabilities in full as

$x^1 = sat_1(a' + \frac{k-1}{k} + \frac{1}{k} - \varepsilon) = 1$. And so it continues through the tree in each generation and no nodes will default. Since the tree is disconnected and acyclic, the payment equilibrium is also unique when the shock is equal to the asset and all nodes may still meet their liabilities in this case.

The other case of when the shock is larger than the extra asset, $\varepsilon > a'$, causes the root node to default as it cannot repay its liabilities when its net worth is negative $d = a' + 1 - 1 - \varepsilon < 0$. This is also true for the other nodes in all the next generations as their net worth also is negative $d = a' + \frac{k-1}{k} + \frac{1}{k} - 1 - \varepsilon < 0$. The number of default in the k-regular network depends thus on the relation between the extra asset and the shock.

$$\begin{aligned} \psi &= 0, \quad \text{if } \varepsilon \leq a' \\ \psi &\to \infty, \quad \text{if } \varepsilon > a' \end{aligned} \tag{4.19}$$

Comparison

When homogeneous shocks hits each node in the financial network, the ring and complete network performs equally in terms of number of default. The only factor determining if all nodes or no nodes default is how large the external asset a is relative to the shock ε. For the k-regular tree network the conclusion is similar but with the size of the extra asset a' relative to the shock ε determining a complete or a zero default of the network instead.

Comparing the k-regular tree network with the complete and ring network in this case is difficult as the external assets are modeled differently. The total external asset for the root node in the tree network is always $1/k$ larger than the external assets for the rest of the nodes. The greater number of children k, the less difference. The extra asset a' is always smaller than the total asset a and given a similar total external asset a in the three networks, one may argue that smaller, multiple shocks are required to cause a complete default in the k-regular tree network compared to the complete and ring network.

Example of 2 shocks in the ring and complete network

This example is to illustrate a scenario where the complete network performs worse in terms of number of defaults compared to the ring network. Assume a complete and ring network with $n = 5$ nodes ,modeled by the same homogeneous assets and normalized liabilities and claims as earlier. Figure 4.4 illustrates these two networks. Assume that the external asset is $a = \frac{2}{5}$ for all nodes in both networks. Then let node 1 and node 2 in both networks be exposed to a shock $\varepsilon_1 = \varepsilon_2 = a + \delta$, where $\delta \in (0,1)$ is a positive quantity keeping the shock within its maximum impact. As the shock now is larger than the asset, both node 1 and node 2 will directly default in both networks.

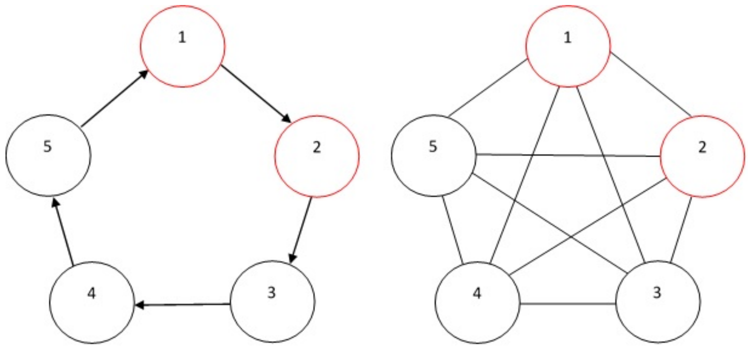

Figure 4.4 The ring network and the complete network in the Example of 2 shocks

Starting in the ring network on node 1, which, as earlier, initially receives its full claim, the induced shock from node 1 becomes $s_1 = sat_1(1-1-a+a+\delta) = \delta$, from Equation 3.9. For the next node 2 the induced shock becomes $s_2 = sat_1(2\delta)$ and node 2 thus repays $x_2 = sat_1(1-2\delta)$ to node 3. The worst case scenario occurs if $\delta \geq 1/2$ which implies that $x_2 = 0$ and node 2 suffers a complete default, repaying zero to node 3. Then, node 3 can only repay its asset $x_3 = a < 1$ and thus also defaults. For the next node 4, the repayment is $x_4 = sat_1(a+a) = 2a < 1$ and node 4 also defaults. However, node 5 does not default as its repayment is equal to $x_5 = sat_1(3a) = 1$. The maximal number of defaults is $\psi = 4$.

For the complete network, the shocks will affect the network simultaneously. To initialize the process, it is given that node 3, 4, and 5 repays their liabilities in full, e.g. $x_3 = x_4 = x_5 = 1$. Then node 1 and node 2 will produce the same induced shock and also repay the other nodes the same amount. The induced shock for node 1 and node 2 is found by Equation 3.9 as

$$\begin{aligned} s_1 &= sat_1(1-1-a+a+\delta+\frac{s_2}{n-1}) = sat_1(\delta+\frac{s_2}{4}) \\ s_2 &= sat_1(1+1-a+a+\delta+\frac{s_1}{n-1}) = sat_1(\delta+\frac{s_1}{4}) \end{aligned} \tag{4.20}$$

By a simple equation system, the total induced shock is found as $s_1 = s_2 = sat_1(\frac{4\delta}{3})$. The individual induced shock from node 1 then becomes $s_{1i} = \frac{1}{4}sat_1(\frac{4\delta}{3})$ $\forall i = 2,..,5$ and the individual induced shock is the same for node 2. Given these induced shocks, the repayment for the other nodes is now equal to $x_3 = x_4 = x_5 = sat_1(1+a-\frac{1}{2}sat_1(\frac{4\delta}{3}))$ and they can only meet their liabilities in full if $a \geq \frac{1}{2}sat_1(\frac{4\delta}{3})$. However, when $\delta > \frac{3}{5}$, the induced shock becomes larger than the asset $a = \frac{2}{5}$ and the whole complete network defaults. In this setup, the result is a maximum of $\psi = 5$ defaults in the complete network.

By this example, it is proven that the complete network may perform worse than the ring network. This is true under some circumstances for when the network is hit by $1 < m < n$ shocks. The result can be seen as an effect of the complete structure which spreads the induced shocks to all nodes in the network and thus amplifies the initial external shocks. The ring network, on the other hand, only spreads the shock to one node at a time which in this case helps to absorb the initial external shock.

5

Ex-ante Shock Propagation Analysis

In this chapter, the contagion is analyzed on specific acyclic graph topologies subject to ex-ante shocks, given the same initial assumptions as described earlier. The networks are modeled by the financial network model and are homogeneous with equal initial net worths. The external liabilities are still assumed to be zero. However, since it is now an ex-ante perspective, the shocks are modeled as independent random variables. The analysis is firstly performed with one stochastic shock, only affecting one node, and then continues to multiple independent stochastic shocks, affecting all nodes.

When analyzing the contagion of stochastic shock(s), the shock propagation will be examined by looking at the shape and size of the complementary cumulative distribution function (CCDF). It is defined as the probability of the induced shock from a node i to one of its creditors j being larger than some α, $Pr(s_{ij} \geq \alpha)$. From this probability measure, the probability of default is found as $Pr(s_{ij} > 0)$ and the performance by each topology can be compared.

The examined graph topologies are the line graph, which represents an acyclic ring graph, the k-regular directed tree graph, and the random directed tree graph. For each type of shock, the considered graph topologies are evaluated and compared to each other given the resulted contagion measure. As the graph topologies are acyclic, the shock propagation is exactly the direct impact which is the reason for this analysis to be limited to only acyclic graphs.

5.1 Contagion by One Stochastic Shock

The stochastic shock is only affecting node 1 and is now considered to be a random variable distributed uniformly according to $\varepsilon_1 \sim \mathcal{U}(0, \varepsilon_1^{max})$ and $\varepsilon_i = 0 \ \forall i \geq 2$. The range of ε_1 is constructed so that the shock is restricted to its maximum impact. The distribution of ε_1 may be altered to fit the distribution of a real-world external shock but for now, the uniform distribution is used as it shows the possible consequences of an external shock in a simple and intuitive way. The shocks are assumed to be independent. The cumulative distribution function (CDF) of ε_1 is thus defined as

$$F_{\varepsilon}(x) = Pr(\varepsilon_1 < x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{1+a}, & \text{if } 0 \leq x < 1+a \\ 1, & \text{if } x \geq 1+a \end{cases} \tag{5.1}$$

The propagated contagion can be described by the individual induced shock s_{ij} from node i to node j. A positive shock $s_{ij} > 0$ implies a default of node i and a measure of how much of the contagion spreads to the next node j. Since the shock is stochastic, it is logical to evaluate the propagation by a probability measure. Thus, to continue the analysis, the contagion will be evaluated by $Pr(s_{ij} \geq \alpha)$ which defines the probability of the individual propagated shock being larger than some α. This can be seen as the probability of default, given that $\alpha > 0$. By Equation 3.10 and 3.11, the definition of this probability is equal to

$$\begin{aligned} Pr(s_{ij} \geq \alpha) &= Pr(P_{ij} sat_{w_i}(w_i - v_i + \varepsilon_i + \sum_h s_{hi}) \geq \alpha) \\ &= Pr(P_{ij} sat_{w_i}(\beta_i + \xi_i) \geq \alpha) \end{aligned} \tag{5.2}$$

where β_i represents the deterministic part and ξ_i represents the stochastic part. The stochastic parts $\xi_i = \varepsilon_i + \sum_h s_{hi}$ are also assumed to be independent of each other. Finding the probability measure for each induced shock from the previous probabilities of induced shocks will be performed by a recursive process implemented in MATLAB.

The analysis will be carried out on the directed line graph, the k-regular directed tree graph, and lastly the random directed tree graph created by the Galton-Watson branching process.

Line network

In this example, the ring graph is acyclic and can thus be represented by a line graph, where the first node has received its total claim. For the line network, each node has only one creditor which makes the individual induced shock equal to the total, i.e. $s_{ij} = s_i$, which is the same implication as for the ring network. By finding a probability measure for the size of s_i compared to some α, $Pr(s_i \geq \alpha)$, it is possible to further understand how the contagion is distributed. For this line network, the probability as a complementary cumulative distribution function (CCDF) is defined as

$$Pr(s_i \geq \alpha) = \begin{cases} Pr(sat_1(\varepsilon_i - a) \geq \alpha), & \text{for } i = 1 \\ Pr(sat_1(s_{i-1} - a) \geq \alpha), & \text{for } i = 2,..,n \end{cases} \tag{5.3}$$

by applying the definition of the induced shock s_i as given by Equation 4.2. As the probability is found by an iterative process, it is intuitive to first find the probability $Pr(s_1 \geq \alpha)$ of node 1 and then continue to node 2 and so on. As the external shock is $\varepsilon_1 \sim \mathcal{U}(0, 1+a)$, then $(\varepsilon_1 - a) \sim \mathcal{U}(-a, 1)$. The probability for the induced shock of node 1 being larger than some α is given as

$$\begin{aligned} Pr(s_1 \geq \alpha) &= Pr(sat_1(\varepsilon_1 - a) \geq \alpha) \\ &= \begin{cases} 1, & \text{if } \alpha \leq 0 \\ 1 - Pr(\varepsilon_1 < \alpha + a), & \text{if } 0 < \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases} \end{aligned} \tag{5.4}$$

where the probability of $Pr(\varepsilon_1 < \alpha + a)$ is given by the CDF $F_\varepsilon(\alpha + a)$ of ε_1, redefined from Equation 5.1 as

$$Pr(\varepsilon_1 < \alpha + a) = F_\varepsilon(\alpha + a) = \begin{cases} 0, & \text{for } \alpha < -a \\ \frac{\alpha + a}{1+a}, & \text{for } -a \leq \alpha < 1 \\ 1, & \text{for } \alpha \geq 1 \end{cases} \tag{5.5}$$

Thus, the final expression for the probability as a CCDF of the first shocked node is

$$Pr(s_1 \geq \alpha) = \begin{cases} 1, & \text{for } \alpha \leq 0 \\ 1 - \frac{\alpha + a}{1+a}, & \text{for } 0 < \alpha \leq 1 \\ 0, & \text{for } \alpha > 1. \end{cases} \tag{5.6}$$

The probability of the induced shock of the next node 2 as a CCDF becomes

$$\begin{aligned} Pr(s_2 \geq \alpha) &= Pr(sat_1(s_1 - a) \geq \alpha) \\ &= \begin{cases} 1, & \text{if } \alpha \leq 0 \\ Pr(s_1 \geq \alpha + a), & \text{if } 0 < \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases} \end{aligned} \tag{5.7}$$

where the distribution of s_1 from Equation 5.6 is used to find the probability. Continuing this iteration, for each next node, there is one more asset a added which absorbs the shock and reduces the probability. The general expression for the probability of the induced shock is thus

$$Pr(s_i \geq \alpha) = \begin{cases} 1, & \text{if } \alpha \leq 0 \quad \forall i \\ Pr(s_{i-1} \geq \alpha + a), & \text{if } 0 < \alpha \leq 1 \text{ and } i \geq 2 \\ Pr(\varepsilon_i \geq \alpha + a), & \text{if } 0 < \alpha \leq 1 \text{ and } i = 1 \\ 0, & \text{if } \alpha > 1 \quad \forall i \end{cases} \tag{5.8}$$

K-regular tree network

In the directed tree graph with a fixed number of children $k > 1$, the induced shock from node i to its child j can be found from Equation 3.10 as

$$s_{ij}^{g} = \frac{1}{k} sat_{w_i}(w_i - v_i + \varepsilon_i + s_{hi}^{g-1}) = \frac{1}{k} sat_{w_i}(-a' + \varepsilon_i + s_{hi}^{g-1}) \tag{5.9}$$

where node h is the parent of node i. The one external stochastic shock hits the root node. The external shock ε_1 is modeled as a random independent variable that is uniformly distributed from 0 to ε_1^{max}, $\varepsilon_1 \sim \mathcal{U}(0, 1+a')$.

As assumed earlier, the total liability is normalized to $w_i = 1 \ \forall i$. This means that the CCDF for the induced shock, s_{ij}, is

$$\begin{aligned} Pr(s_{ij}^{g} \geq \alpha) &= Pr(\frac{1}{k} sat_{w_i}(\underbrace{w_i - v_i}_{\beta_i} + \underbrace{\varepsilon_i + s_{hi}^{g-1}}_{\xi_i}) \geq \alpha) \\ &= Pr(sat_{w_i}(\beta_i + \xi_i) \geq \alpha k) = \\ &= Pr(sat_1(-a' + \xi_i) \geq \alpha k) \\ &= \begin{cases} 0, & \text{if } \alpha > \frac{1}{k} \\ 1, & \text{if } \alpha \leq 0 \\ Pr(sat_1(\xi_i - a') \geq \alpha k), & 0 < \alpha \leq \frac{1}{k} \end{cases} \end{aligned} \tag{5.10}$$

As for the line graph in the previous analysis, it is intuitive to start the iteration in the shocked root node, i.e. by finding $Pr(s_{1j}^{0} \geq \alpha)$. Rewriting Equation 5.10 by $\xi_1 = \varepsilon_1$, it is equal to

$$\begin{aligned} Pr(s_{1j}^{0} \geq \alpha) &= Pr(sat_1(\varepsilon_1 - a') \\ &= \begin{cases} 0, & \text{if } \alpha > \frac{1}{k} \\ 1, & \text{if } \alpha \leq 0 \\ Pr(sat_1(\varepsilon_1 - a') \geq \alpha k), & 0 < \alpha \leq \frac{1}{k} \end{cases} \\ &= \begin{cases} 0, & \text{if } \alpha > \frac{1}{k} \\ 1, & \text{if } \alpha \leq 0 \\ 1 - F_{\varepsilon}(\alpha k + a'), & 0 < \alpha \leq \frac{1}{k} \end{cases} \end{aligned} \tag{5.11}$$

where F_{ε} is the CDF of ε_1 defined in Equation 5.1 and given a fixed number $k > 1$ of children.

For a node $j > 1$ in generation 1, the probability of the induced shock to a child

$l > k+1$ in generation 2 is

$$Pr(s^1_{jl} \geq \alpha) = Pr(\frac{1}{k} sat_1(-a' + s^0_{1j}) \geq \alpha)$$
$$= \begin{cases} 0, & \text{if } \alpha > \frac{1}{k} \\ 1, & \text{if } \alpha \leq 0 \\ Pr(sat_1(s^0_{1j} - a') \geq \alpha k), & \text{if } 0 < \alpha \leq \frac{1}{k} \end{cases} \quad (5.12)$$

and for each new generation the probability is found by the probability CCDF for the previous generation.

Random tree network by branching process

The shock propagation analysis is carried out both on a random tree network with a Discrete Uniform offspring distribution,with the PMF defined in Equation 2.16, and with a Binomial offspring distribution, with the PMF defined in Equation 2.17. The different distributions are chosen in order to analyze how different distributions, with the same mean but different variances, affect the contagion conclusions.

In this random tree network, as in the k-regular tree network, all nodes have children and hence liabilities. The total liability of each node to its children is still normalized to $w_i = 1$. The internal claim of a node j from its parent i is $v_i - a_i = 1/k_i$, thus depending on the number of children for node j's parent. To make the network homogeneous by equal net worth, the initial condition is made equal for the root node, which has zero in claims, and the rest of the nodes in the tree, which have both liabilities and claims. The asset a_i for node i is hence equal to

$$a_i = \begin{cases} a' + 1, & \text{for } i = 1 \\ a' + \frac{k_j - 1}{k_j}, & \text{for } i = 2, ..., n \end{cases} \quad (5.13)$$

where k_j is the number of children of node i's parent j. It is still assumed that $a_i > 0$ for all possible values of k_j. Since the smallest possible value of k_j is 1, only positive values of a' are considered $a' > 0$.

To initialize the analysis of how contagion spreads in this random tree network, it is important to understand how the random elements will affect the result. As both the number of children k_i and the shock ε_i are random variables and belong to a certain distribution, the concluded induced shock for a node i will also be distributed according to these random values. An important assumption is that the number of children k_i is independent from the shock ε_i. In a directed random tree, if j is a child of i and i is a child of h, the expression for the individual induced shock is

$$s^g_{ij} = \frac{1}{k_i} sat_{w_i}(\underbrace{w_i - v_i}_{\beta_i} + \underbrace{\varepsilon_i + s^{g-1}_{hi}}_{\xi_i}) \quad (5.14)$$

It means that the size of the induced shock from node i to j depends on the fixed variable β_i and the random variables k_i and ξ_i. It is assumed that k_i is independent from all elements in ξ_i. Given this assumption, the probability of s^g_{ij} being larger than or equal to some α, the CCDF becomes

$$Pr(s^g_{ij} \geq \alpha) = \sum_{k=1}^{k_{max}} f_K(k) Pr(sat_{w_i}(\beta_i + \xi_i) \geq \alpha k) \tag{5.15}$$

where the second probability is given as

$$Pr(sat_1(\beta_i + \xi_i) \geq \alpha k) = \begin{cases} 0, & \text{if } \alpha > \frac{1}{k} \\ 1, & \text{if } \alpha \leq 0 \\ Pr(\xi_i \geq \alpha k - \beta_i), & 0 < \alpha \leq \frac{1}{k} \end{cases} \tag{5.16}$$

The saturation function causes a zero probability in the first case and full probability in the second case. Then it may be disregarded in the third case since the interval of α for this outcome is defined within the saturation interval. Choosing one of the two given offspring distributions above, $f_K(k)$ is defined and known. So, if $Pr(\xi_i \geq \alpha)$ is also known for all values of α, then $Pr(sat_1(\beta_i + \xi_i) \geq \alpha k)$ and thereby $Pr(s^g_{ij} \geq \alpha)$ can be computed from Equation 5.16 and 5.15 respectively. Now, when $Pr(s^g_{ij} \geq \alpha)$ is computed, it is possible to find an expression for $Pr(\xi_j \geq \alpha)$ since $\xi_j = \varepsilon_j + s_{ij}$. Thus, when $Pr(\xi_j \geq \alpha)$ is known it is possible to move on to the next generation in the tree and calculate the CCDF for the induced shock s^g_{jl} from node j to a child l. Iterating through the generations makes it possible to see how the CCDF of the induced shock changes.

The external shock to the root node is modeled as a uniform distribution, $\varepsilon_1 \sim \mathcal{U}(0, 1 + a')$. In this case, where only the root node is hit by an external shock, it follows that $\xi_1 = \varepsilon_1$, since the root node has no claims and thereby no incoming induced shock. The rest of the nodes are not hit by any external shock which means that the total shock for these nodes only consists of the incoming induced shock. That is, $\xi_i = s^{g-1}_{hi}$ for $i > 1$ and $g > 0$, where node h is the parent of node i.

For the root node, the induced shock is given by the offspring distribution $f_K(k)$ and the probability $Pr(sat_1(\beta_1 + \xi_1) \geq \alpha k) = Pr(sat_1(\varepsilon_1 - a') \geq \alpha k)$ which can be found using the same methodology as in previous analysis. Together with Equation 5.15 and 5.16, the CCDF of the induced shock from the root node to a child j can be explicitly calculated. The result is

$$P(s^0_{1j} \geq \alpha) = \begin{cases} 0, & \text{if } \alpha > 1 \\ \sum_{k=1}^{k_{max}} f_K(k)(1 - F_\varepsilon(\alpha k + a')), & \text{if } 0 < \alpha \leq 1 \\ 1, & \text{if } \alpha \leq 0 \end{cases} \tag{5.17}$$

where F_ε is the CDF of ε_1 and is defined in Equation 5.1.The CCDF of the next generation's induced shock, i.e. of the induced shock s^1_{jl} from node j to a child l, can be found from the previous induced shock by

$$Pr(s^1_{jl} \geq \alpha) = \begin{cases} 0, & \text{if } \alpha > 1 \\ \sum_{k=1}^{k_{max}} f_K(k) Pr(s^0_{1j} > \alpha k + a') & \text{if } 0 < \alpha \leq 1 \\ 1, & \text{if } \alpha \leq 0 \end{cases} \tag{5.18}$$

By iteratively calculating the probability for each generation, moving further down the tree, it is possible to get an understanding of and visualize how the shape of the CCDF of the induced shock changes when moving further down the tree.

Comparison

To see the result graphically and to be able to compare the different network topologies, the probability $Pr(s_{ij} \geq \alpha)$ is plotted for $\alpha \in [0,1]$ in the figures below with similar initial parameters.

For the line network, Figure 5.1 shows the probability when the asset is $a = 0.2$. As it can be seen in the plot, the probability of a higher induced shock decreases for each next node and when reaching node 4, the probability is zero for all $\alpha > 0$. This translates to a zero probability of default for all nodes further than two steps away from the shocked node. As there is an asset a added for each next node, the probability for a positive induced shock decreases. The result is consistent with the case of one deterministic shock on the ring network, as the number of defaults depend on the relation between the asset a and the shock ε_1. When the asset decreases (increases), the probability increases (decreases) for each next node and the number of defaulting nodes increases (decreases).

An example of the probability in the k-regular tree network when the additional asset is $a' = 0.05$ and the number of children is $k = 3$ is given. In Figure 5.2, the probability of the induced shock $Pr(s^g_{ij} \geq \alpha)$ is shown for the three first generations. There is a probability of default, i.e. $s^g_{ij} > 0$, only in generation 0 and 1 and for all the next generations, the probability is zero. Even though the asset is small, the shock is well absorbed and eliminated already by the third generation. As earlier concluded, it is mainly the number of children that determines the default probability. The more children the less probability of default as larger generations absorb the shock better.

For the random tree graph, an example is shown below of how two different offspring distributions with the same mean affects the probability of the induced shock. In both cases, the asset is $a' = 0.05$ and the maximal number of children is $k_{max} = 10$. In Figure 5.3, the offspring distribution follows the Discrete Uniform distribution defined in Equation 2.16. In Figure 5.4, the offspring distribution follows the Binomial distribution defined in Equation 2.17 with $p = 0.5$. The number

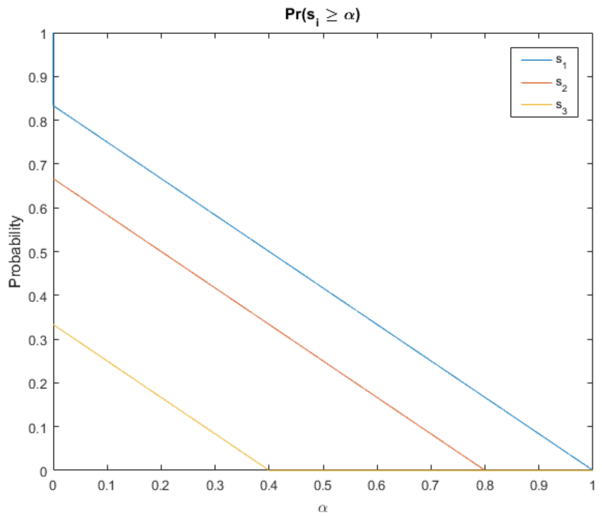

Figure 5.1 The probability of $s_i \geq \alpha$ in the line network when $a = 0.2$ and $\varepsilon_1 \sim \mathcal{U}(0, 1+a)$ shown in a CCDF

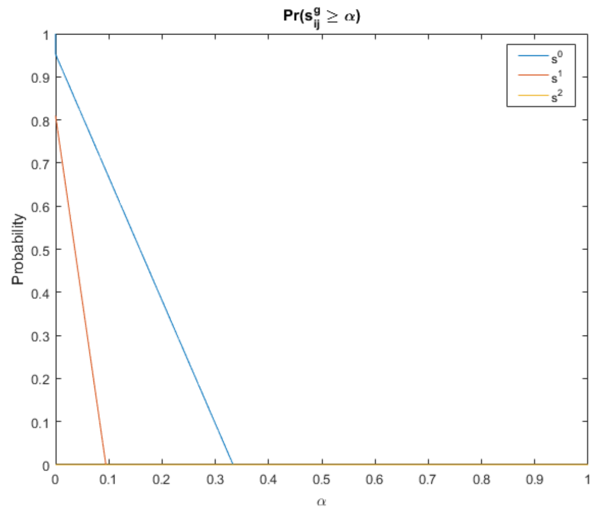

Figure 5.2 The probability of $Pr(s_{ij}^{g} \geq \alpha)$ for a k-regular tree network with $k = 3$ children, $a' = 0.05$ and one shock $\varepsilon_1 \sim \mathcal{U}(0, 1+a')$

of children k are first drawn from a binomial distribution ranging from 0 to 9 and then one children is added which makes the k vary from 1 to $k_{max} = 10$. This approach makes the mean equal for both the uniform and the binomial distribution. The mean is $\mu = \frac{k_{max}+1}{2} = 5.5$ in both cases.

As can be seen in the figures, the highest probability for default is in the first two generations and thereafter the probability is strongly reduced and only positive for small α. For the Binomial offspring distribution, the contagion may spread to a maximum of 8 generations and for the Uniform offspring distribution, the contagion may spread to a maximum of 10 generations. After the 8th and 10th generation respectively, there is a zero probability of default. The main difference between the two random tree networks is that the probability of a larger induced shock is higher for the network with a Uniform offspring distribution.

A possible conclusion of the differences of shock contagion can be drawn from the different variances of the distributions. For the Uniform distribution, the variance is

$$\sigma_k^2 = \frac{(k_{max} - 1)^2}{12} = 6.75 \tag{5.19}$$

For the Binomial distribution, the variance is

$$\sigma_k^2 = np(1-p) = 2.25 \tag{5.20}$$

A higher variance of the number of children k implies a higher systemic risk in the network as a larger induced shock is more probable. Given the distributions examined, the result is restricted for symmetric distributions without skewness. The reason for this difference may be that a higher variance means a higher probability for a lower number of children. This is the main risk for shock propagation in a tree, since a lower number of children absorbs less than a larger number of children. Although, a higher variance also means a higher probability for a large number of children, which would lower the risk of contagion further down the tree, the net effect seems to be a higher systemic risk.

The implication of variance can also be seen when comparing the k-regular and the uniformly generated random tree network. The probability for a higher α is significantly larger for this random tree network, with the interval of $k_i \in [1,10]$, compared to the k-regular tree network, with a fixed $k = 3$ children. Comparing the binomially generated random tree network, the probability for larger α is overall smaller than for the k-regular tree network. However, there is a larger probability for spreading the contagion further down the tree through more generations in both of the random tree networks compared to the k-regular tree network.

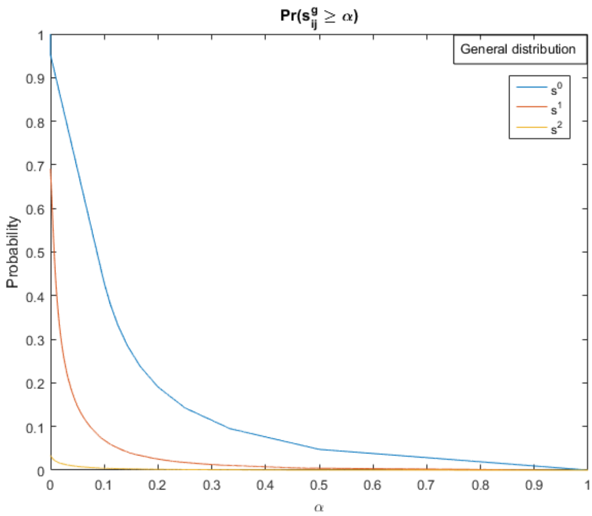

Figure 5.3 The probability of $Pr(s^g_{ij} \geq \alpha)$ for a random tree graph created by a uniform offspring distribution with $k_{max} = 10$, $a' = 0.05$ and one shock $\varepsilon_1 \sim \mathcal{U}(0, 1 + a')$

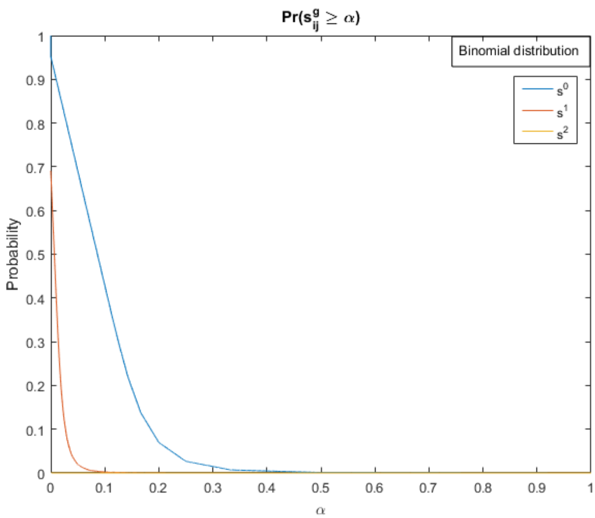

Figure 5.4 The probability of $Pr(s^g_{ij} \geq \alpha)$ for a random tree graph created by a binomial offspring distribution with $k_{max} = 10$, $a' = 0.05$ and one shock $\varepsilon_1 \sim \mathcal{U}(0, 1 + a')$

5.2 Contagion by Multiple Independent Shocks

In this case, all nodes are subject to a stochastic independent shock of size $\varepsilon_i > 0\ \forall i$. The shock is uniformly distributed as $\varepsilon_i \sim \mathcal{U}(0, \varepsilon_i^{max})$. The same probability measure of $Pr(s_{ij} \geq \alpha)$ is applied to understand how severe and probable the contagion is. However, since all nodes are subject to a shock, there will be two stochastic variables, both the external shock and the induced shock, to take into consideration. These are assumed to be independent of each other. To find the total probability, the concept of convolution between the two distributions of shocks will be applied.

The analysis will be performed on the directed line graph, the k-regular directed tree graph, and lastly the random directed tree graph, generated by the Galton-Watson branching process by two offspring distributions.

Line network

For the line network, it is still initially given that the first node 1 receives its total claims and is not affected by any induced shocks. This results in a probability $Pr(s_1 \geq \alpha)$ equal to Equation 5.6. For the next node 2, the probability will be similar as in previous analysis but now with an additional stochastic shock ε_2 to consider.

$$
\begin{aligned}
Pr(s_2 \geq \alpha) &= Pr(sat_1(\underbrace{s_1 + \varepsilon_2}_{\xi_2} - \underbrace{a}_{\beta_2}) \geq \alpha) \\
&= \begin{cases} 1, & \text{if } \alpha \leq 0 \\ Pr(\xi_2 \geq \alpha + a), & \text{if } 0 < \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases}
\end{aligned}
\tag{5.21}
$$

As both the external shock and the induced shock are stochastic, the distribution of the total ξ_2 will be stochastic and is a sum of the two independent distributions of the shock ε_2 and the induced shock s_1. The distribution of ε_i is known as Uniform and the distribution of s_1 is calculated for the first node. The CDF of a sum of two independent random variables $Z = X + Y$ may be found by using convolution

$$
H(z) = Pr(Z \leq z) = Pr(X + Y \leq z) = \int_{-\infty}^{\infty} F_Y(z - x) f_x(x) dx = F_Y * f_X \tag{5.22}
$$

By rewriting the formula for $Z = \xi_2$, $X = \varepsilon_2$ and $Y = s_1$, it becomes

$$
\begin{aligned}
F_{\xi_2}(\alpha) &= Pr(\xi_2 \leq \alpha) \\
&= Pr(\varepsilon_2 + s_1 \leq \alpha) \\
&= \int_{-\infty}^{\infty} F_{s_1}(\alpha - x) f_\varepsilon(x) dx \\
&= F_{s_1} * f_\varepsilon
\end{aligned}
\tag{5.23}
$$

Thus, the CDF of ξ_2 is a convolution of the CDF F_{s_1} and the PDF f_ε defined as

$$F_{s_1}(\alpha) = 1 - Pr(s_1 \geq \alpha) \tag{5.24}$$

$$f_\varepsilon(\alpha) = \begin{cases} \frac{1}{1+a}, & \text{if } 0 \leq \alpha < 1+a \\ 0, & \text{otherwise} \end{cases} \tag{5.25}$$

By computing iteratively the convolution of each induced shock and external shock, it is possible to find each next node's probability by

$$\begin{aligned} Pr(s_i \geq \alpha) &= Pr(sat_1(\xi_i + \beta_i) \geq \alpha) \\ &= \begin{cases} 1, & \text{if } \alpha \leq 0 \\ 1 - F_{\xi_i}(\alpha + a), & \text{if } 0 < \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases} \qquad \forall i > 1 \end{aligned} \tag{5.26}$$

K-regular tree network

When there are several shocks present in the k-regular tree network, the analysis is carried through similarly as in the previous case of one stochastic shock. The shocks are modeled uniformly as $\varepsilon_i \sim \mathcal{U}(0, 1+a')$ $\forall i$ and the analysis starts with the root node. The probability of $Pr(s^0_{1j} \geq \alpha)$ for the root node is calculated as in Equation 5.11. Then, for the next generations, the probability will be calculated by convolutions since there are two stochastic independent variables, i.e. $\xi_i = s^g_{hi} + \varepsilon_i$ $\forall i > 1$. By using and rewriting Equation 5.26, the following equation can be calculated for finding the probability of the next generation of nodes

$$Pr(s^g_{jl} \geq \alpha) = \begin{cases} 1, & \text{if } \alpha \leq 0 \\ 1 - F_{\xi_j}(\alpha k + a), & \text{if } 0 < \alpha \leq \frac{1}{k} \\ 0, & \text{if } \alpha > \frac{1}{k} \end{cases} \qquad j, l > 1, g > 0 \tag{5.27}$$

where the Equation 5.23 is used to find F_{ξ_j} as the convolution of $F_{s_i^{g-1}}(\alpha) = 1 - Pr(s_i^{g-1} \geq \alpha)$ and $f_\varepsilon(\alpha)$.

Random tree network by branching process

The implication of several shocks in the network is similar to the k-regular tree network, except for the additional distribution $f_K(k)$ of number of children k_i. The shock is uniformly distributed as $\varepsilon_i \sim \mathcal{U}(0, 1+a')$ $\forall i$. The probability for the first root node is equal to Equation 5.17. The probability for the next nodes of generation $g > 0$ can be found by rewriting Equation 5.27 and is equal to

$$Pr(s^g_{ij} \geq \alpha) = \begin{cases} 0, & \text{if } \alpha \geq 1 \\ \sum_{k=1}^{k_{max}} f_K(k)(1 - F_{\xi_i}(\alpha k + a)) & \text{if } 0 < \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases}$$
$$g > 0, i, j > 1 \tag{5.28}$$

where the CDF $F_{\xi_i}(\alpha)$ is the convolution of $F_{s_{hi}^{g-1}}(\alpha) = 1 - Pr(s_{hi}^{g-1} \geq \alpha)$ and $f_\varepsilon(\alpha)$.

Comparison

The probability of an induced shock being larger than some α is shown below for the three first out-neighbors connected by liability for the directed line network, the k-regular and random tree network. All nodes are subject to an equally distributed independent stochastic shock ε.

By using the same example for the line network in the previous analysis of one stochastic shock, where all nodes have an asset of size $a = 0.2$, and then exposing all nodes to a random shock distributed as $\varepsilon_i \sim \mathcal{U}(0, 1+a)$, the result is shown as a CCDF for the three first nodes in line in Figure 5.5. The probability for a higher induced shock increases fast for each node. For the third node in line, the probability is close to 1 for the induced shock being larger than zero and a default is almost certain. The probability continues to stay high even for larger α.

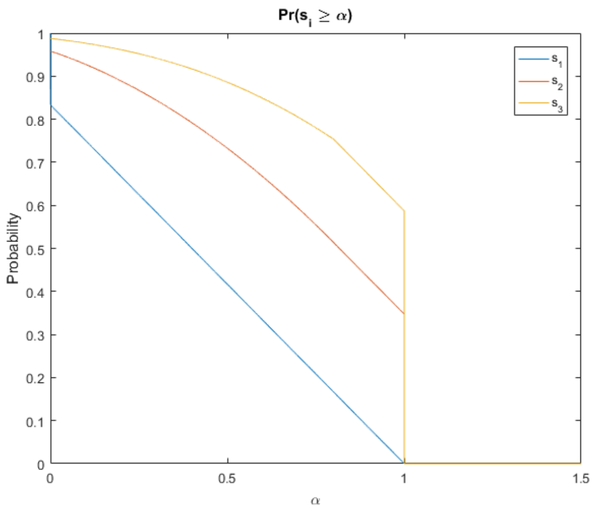

Figure 5.5 The probability of $Pr(s_i \geq \alpha)$ for a line network with $a = 0.2$ and several shocks $\varepsilon_i \sim \mathcal{U}(0, 1+a)$

For the k-regular tree network, the probability for the first three generations is shown as a CCDF in Figure 5.6 with the same parameters as last example, i.e. $k = 3$ children and an asset of the size $a' = 0.05$. The root node has an identical probability as when only one shock is present but then the probability for an induced shock increases for each next generation. The induced shock may not be larger than

$\alpha > 1/k = 1/3$ which makes the probability to be reduced to zero for larger α. For the third generation, the probability of the induced shock is larger than zero is close to one which implies an almost certain default of that generation. For each next generation, the probability will continue to increase for larger α, given that the nodes are shocked by the uniform ε_i.

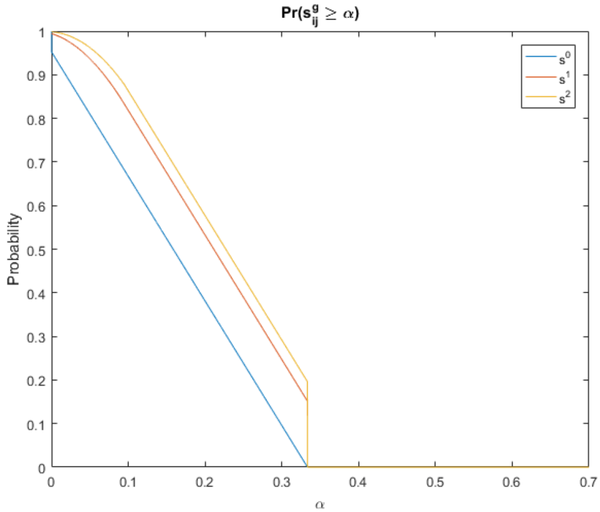

Figure 5.6 The probability of $Pr(s_{ij}^{g} \geq \alpha)$ for a k-regular tree network with $k = 3$, $a' = 0.05$ and several shocks $\varepsilon_i \sim \mathcal{U}(0, 1 + a')$

When exposing the two random tree networks, created by a Discrete Uniform and a Binomial offspring distribution respectively, to several uniformly distributed shocks, the result can be seen in Figure 5.7 and 5.8. The parameters is given as earlier, where each node has an asset $a' = 0.05$ and a maximum of $k_{max} = 10$ children. The plots show only the three first generations but it is clear that the probability increases for each next generation. The pattern is similar as for the k-regular tree graph but with a slightly lower increase in magnitude and also with a larger probability for the Uniform offspring distribution for larger α and a smaller probability for the Binomial offspring distribution. The Uniform offspring distribution still generates a higher probability than the Binomial distribution, which can be originating from the higher variance previously discussed.

The noncontinuous tendencies in the plots are due to the fact that for each possible number of children, the probability is restricted to $\alpha \leq 1/k_i$. For $\alpha > 1/k_i$, the probability is zero, which creates the gaps of reduced probability for each

$k_i \in [1, k_{max}]$.

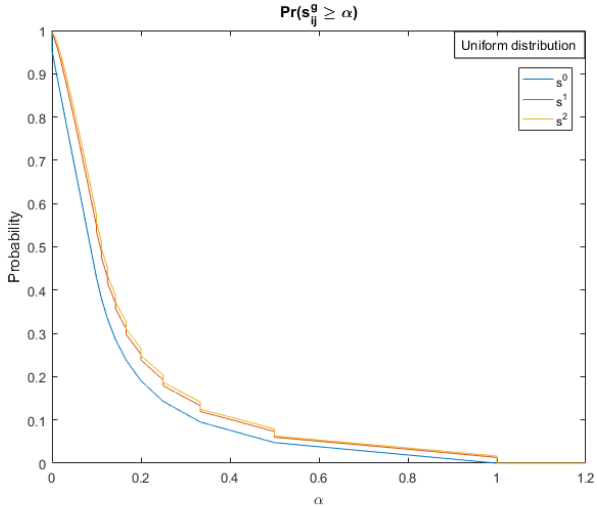

Figure 5.7 The probability of $Pr(s_{ij}^g \geq \alpha)$ for a random tree graph created by a uniform offspring distribution with $k_{max} = 10$, $a' = 0.05$ and several shocks $\varepsilon_i \sim \mathcal{U}(0, 1 + a')$

Given these examples, it is clear that each next node after the first shocked node will spread a larger induced shock by a higher probability. Especially for the line network the probability increases fast.

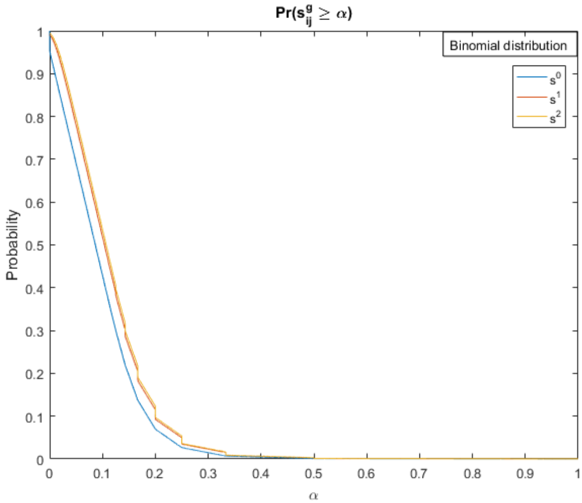

Figure 5.8 The probability of $Pr(s_{ij}^g \geq \alpha)$ for a random tree graph created by a binomial offspring distribution with $k_{max} = 10$, $a' = 0.05$ and several shocks $\varepsilon_i \sim \mathcal{U}(0, 1 + a')$

6

Conclusions

In this last chapter, the findings of the thesis are explicitly concluded. The fulfillment of the purpose and goal of the thesis, as defined in the first chapter, as well as the contribution to the subject of contagion in financial networks are further presented. The conclusions of the thesis may be divided into two parts: one of smaller magnitude regarding the role of individual nodes, as measured by traditional centrality measures, and one of greater magnitude regarding the role of the financial network's structure in spreading contagion. Also, future work continuing on the result of this thesis and within this strand of literature is proposed.

6.1 Summary of Work

In this thesis, a theoretical financial network model is developed to mimic the complex real-world interbank lending system in a simplified way. It is based on the well-established graph theory with several of its concepts applicable to the model. However, the notion of traditional centrality measures, concerning variants of degree and eigenvector centrality, is indicated to not be sufficient in finding systemically important institutions, to which a negative shock causes greater contagion than when shocking other nodes. The new notion of Node depth may find the worst case loss amplification but cannot specifically say anything about the centrality of the institutions in a network that is not out-connected or when knowledge of realized shocks is unavailable.

Furthermore, in homogeneous specific network structures, where no node can be deemed to be more central than any other, the role of interconnections in an ex-post perspective is clarified. By analyzing the sparsely connected ring graph and the densely connected complete graph, it is concluded that, in terms of number of defaults, the complete graph performs better or equally as well as the ring graph when there is only one shock present. When there are n shocks of the same size present, the complete and the ring graph perform equally well. In the presence of m shocks of the same size, where $1 < m < n$, it is shown that there are cases where the complete graph performs worse than the ring graph and thus the complete structure to a greater extent amplifies the shocks instead of absorbing them. Hence, more interconnections mainly work as shock absorbers but under some circumstances, sparser interconnections cause less contagion and number of defaults.

Also, acyclic features of the homogeneous network structures are examined by the tree graph, both the k-regular and random directed tree, in the presence of ex-ante shocks. The k-regular tree is also examined in the presence of ex-post shocks. Given the assumed model in this thesis, it is concluded that a higher number of offspring, equal to a higher out-degree, is preferred to ensure a more stable network in terms of number of generational defaults.

When generating the random tree graph, an offspring distribution with a lower variance seems to cause less contagion when looking at number of generational defaults. If the variance of the offspring distribution is small it is less likely that the contagion spreads as far down the tree as it would do if the offspring distribution with the same mean has a larger variance. The reason for this is assumed to be the following. If the variance is low, the probability for a high number of children is small. This would imply a less stable network in terms of generational defaults. However, a low variance also gives a lower probability for a smaller number of offspring. This implies a more stable network. The effect of the latter implication is concluded to be stronger than the effect of the first implication.

Comparing the acyclic tree graph with the cyclic ring and complete graphs in the presence of ex-post shocks, the main conclusion is that the acyclic feature might help to prevent further contagion. This is because in the cyclic graphs, the contagion may cascade through several cycles before a payment equilibrium is found whereas in the acyclic graphs, the equilibrium is directly given. This is especially clear in the presence of one deterministic shock where the shock in the k-regular tree graph quickly is absorbed as the number of children increases and there is no "threshold" that may turn the whole network into default, as in the complete graph. However, to compare the performance of these two features in this context is difficult since the structures are modeled differently. In the case of n deterministic shocks, the result is more ambiguous and it is not possible to say that a topology definitely performs better than any other.

The main contribution of this thesis concerns the contagion analysis of the acyclic tree graphs, which has not been found in previous literature. Also, the further clarification and exemplification of how interconnections affect the shock propagation helps to increase the understanding of their role in financial networks. Moreover, the probability measure of the induced individual shock, as developed for ex-ante shocks, gives an interesting perspective on the performance of the network and may be further applied for risk assessment before any shocks has been realized.

6.2 Future Work

Suggestions for future work firstly concerns further analysis similar to Chapter 4 and Chapter 5 on other homogeneous specific graph topologies, possibly with differing degree distribution between nodes like the barbell graph or the star graph. Also the assumption of homogeneity concerning assets and liability could be relaxed in future work. Yet another way of expanding the analysis could be to consider other distributions of external shocks and include the possibility of dependent shocks, as the correlation for shocks in financial networks is high in reality. The ex-ante shock propagation analysis may be extended to include cyclical graphs as well.

Further suggestions involve a development of the financial network model described, to include more parameters and extend the scope to cover more financial instruments traded between banks. This will obviously increase the complexity but also increase the resemblance to the real financial network which might make the conclusions more applicable. Also, more studies is needed to develop a centrality measure applicable to all kinds of financial networks before any realizations of shocks, which succeeds to capture the systemically important nodes.

Lastly, if the actual global financial network manages to increase its transparency, future work may include using real data to a larger extent, to compare the results in this report and further validate the theoretical framework.

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Title and subtitle

On Shock Propagation in Financial Networks

Abstract

This thesis develops a simplified financial network model for an interbank lending system which is then analyzed in terms of contagion when exposed to external liquidity shocks. The aim is to understand how individual institutions and the network structure affect the shock propagation and finding factors that increase respectively decrease the systemic risk of the network. The network structures analyzed are mainly the ring graph, the complete graph, and the directed tree graph, given an ex-post and an ex-ante perspective.

The first result indicates that traditional centrality measures are not capable of identifying systemically important institutions. The second result concerns the interconnections in the network structure, where it is concluded that if one institution or all institutions are subject to a certain shock, a complete structure always performs better than or equally as well as the denser structure of a ring graph, in terms of number of defaulting institutions, whereas if multiple institutions, but less than all of them, are exposed, the complete graph may perform worse. The last result shows that in acyclic tree graphs, a higher number of offspring in the k-regular tree graph and an offspring distribution with less variance in the random tree graph, can restrict the contagion respectively reduce the probability of shock propagation further down the tree.

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