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Anomaly-free Froggatt-Nielsen extensions of the Standard Model with two Higgs doublets

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Abstract

In this Master thesis we extend the Standard Model with an additional $U(1)$ gauge group and an additional Higgs doublet with a lepton specific $\mathbb{Z}_2$ symmetry. This is used in a gauged version of the Froggatt-Nielsen mechanism to describe the observed masses and mixings among fermions. To reproduce these observables and satisfy the anomaly constraints posed by the new gauge symmetry methods from algebraic geometry are used. By introducing three right-handed neutrinos, an anomaly-free model reproducing the observed masses, PMNS matrix and Cabibbo mixing in the quark sector was found.

Acknowledgements

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Populärvetenskaplig sammanfattning

År 2012 hittade man Higgs boson vid LHC i CERN, den här partikeln är nyckeln för att beskriva hur partiklar får sina massor i standardmodellen. Vad vi fortfarande inte vet, är vilka massor partiklarna får. Detta måste mätas experimentellt och är inte något vi kan teoretiskt förutsäga. Det visar sig att de observerade massorna är vitt skilda, vi har allt från toppkvarkens massa: 172 GeV $\approx 3 \cdot 10^{-25}$ kg till neutrinerna med en massa mindre än 1 eV $\approx 2 \cdot 10^{-36}$ kg. För att försöka förklara varför de fundamentala partiklarnas massor är så olika använder vi den så kallade Froggatt-Nielsen mekanismen. Den här mekanismen bygger på att vi introducerar en ny typ av laddning som vi kallar flavonladdning. Precis som elektrisk laddning gör att partiklar växelverkar med fotoner, vilket vi till vardags upplever som elektriska och magnetiska krafter, så gör flavonladdning att partiklarna växelverkar med vad vi kallar ett flavonfält. Om olika partiklar har olika flavonladdning kommer de växelverka olika mycket med flavonfältet. Partiklar med stor flavonladdning växelverkar mycket med flavonfältet och blir därför lättare än partiklar med mindre laddning som växelverkar mindre med fältet. På det här sättet kan vi förklara partiklarnas olika massor med att de har olika flavonladdning.

Vi vill inte bara hitta uppsättningar med flavonladdningar som ger de observerade massorna, utan det finns också ett teoretisk villkor som är att teorin måste vara fri från anomalier. Vad det här betyder är att det finns en uppsättning polynomekvationer som laddningarna behöver satisfiera för att modellen ska vara konsistent. I praktiken ser vi till att dessa villkor, tillsammans med de observerade massorna, blir uppfyllda genom att använda metoder från algebraisk geometri. Att hitta laddningsuppsättningar som både är anomalifria och ger de observerade massorna är svårt, vi utökar därför partikelinnehållet i standardmodellen till att innehålla en extra Higgsdublett och högerhänta neutriner. Med dessa tillägg är det möjligt att ha en anomalifri modell som reproducerar alla massor och nästintill all mixning i standardmodellen.

Contents

1 Introduction

Today's most complete model for describing Nature's fundamental interactions is the Standard Model (SM). Despite its tremendous success, there is a sense of incompleteness about this theory. Most striking is that it does not account for gravity nor the now well-established neutrino masses and oscillations. Including the neutrinos, the Standard Model has 26 free parameters that have to be decided by measurements. Among these 26 parameters are the 12 fermion masses and eight mixing angles. Nine of the masses are generated by the Higgs mechanism, the three remaining neutrino masses have unknown origin. The only fermion

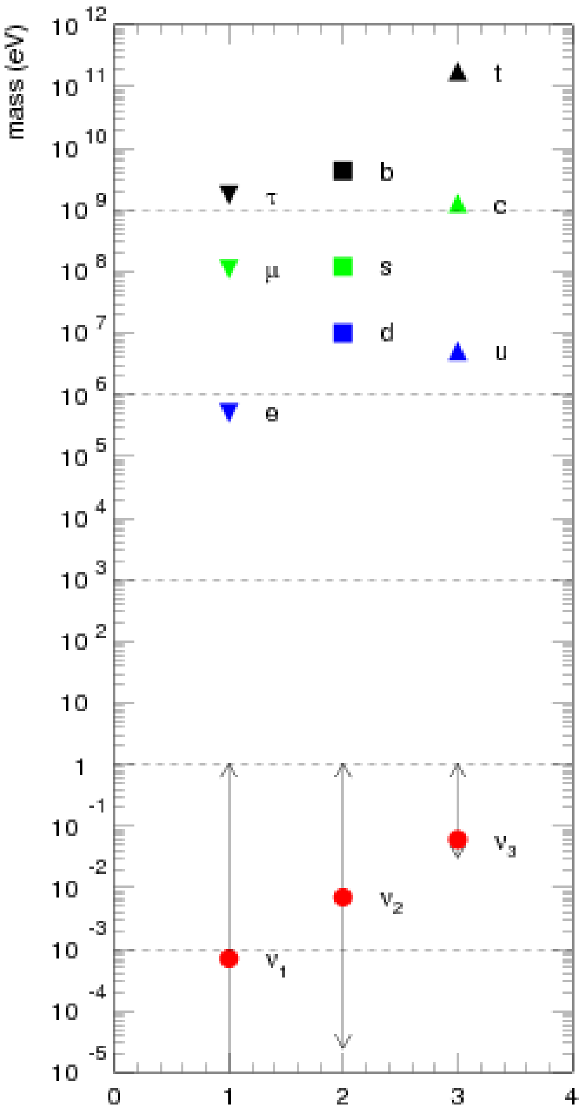


Figure 1: The different fermion masses in the Standard Model shown by generation. Taken from [1].

with mass of the same size as the vacuum expectation value of the Higgs boson is the top quark (Fig. 1). The other quark and charged fermion masses are clearly smaller than the top quark's mass. Since all the fermion masses are free parameters in the SM, there is no known fundamental reason for this mass hierarchy. In addition to this, the neutrino masses are many order of magnitudes smaller than the other fermion masses. This, together with the fact that we have not observed any right-handed neutrinos, are the reasons why it seems likely that neutrinos have their own mass generating mechanism.

One way of describing the mass hierarchies among the quarks was proposed by Froggatt and Nielsen in [2]. They used a global $U(1)$ symmetry and a set of very heavy fermions to this end. In this thesis, we will take on a similar approach but with a $U(1)$ gauge symmetry. In addition to this new symmetry, we also add a Higgs doublet and impose a lepton specific $\mathbb{Z}_2$ symmetry. The $\mathbb{Z}_2$ symmetry guarantees that the two Higgs doublet will not generate any flavor changing neutral currents and the only choice of $\mathbb{Z}_2$ symmetry consistent with all the other conditions of our models is the lepton specific (where one Higgs double couples to the quarks and the other to the leptons). We call the charge associated with this new gauge symmetry for flavon charge, this is assigned to all fermions and to the two Higgs doublets.

When a symmetry of a theory is broken due to quantum effects, we say that the symmetry is anomalous. If one of a theory's gauge symmetries is anomalous, the theory will be inconsistent; unitarity will break and non-physical degrees of freedom might become physical. For the gauge anomalies to vanish, the charges have to satisfy certian homogeneous polynomial equations; the anomaly conditions. In addition to the anomaly conditions for the flavon charges, we have the Froggatt-Nielsen conditions, which dictate the necessary relations between the charges to reproduce the observed masses and mixings. The Froggatt-Nielsen constraints are linear non-homogeneous polynomials. These polynomial equations make it natural to treat this problem in the context of algebraic geometry. By using Gröbner bases the problem is heavily reduced, and in our specific cases it often reduces to finding rational points on a curve.

This thesis is organized as follows. In the next section we discuss gauge anomalies in field theory and derive the constraints in different cases. In Section 3 we describe both the computational algebraic methods and theoretical results about existence of rational and integer charges. The physics of mass generation, two-Higgs doublet models and the Froggatt-Nielsen mechanism are described in Section 4. By adding a gauge group, the phenomenology of the gauge sector will change, this is discussed in Section 5. The results are described in Section 6 and some concluding remarks are given in Section 7.

2 Anomalies in Field Theory

Anomalies are the key to a deeper understanding of quantum field theory.
-Reinhold A. Bertlmann [3]

The Lagrangian in quantum field theory does not know if it describes a classical or quantum field theory. Therefore, a symmetry that is manifest in the Lagrangian may be broken by quantum effects, when this happens, we say that the symmetry is anomalous. When the involved symmetries are gauge symmetries, anomalies make the theory inconsistent since unitarity will break down and unphysical degrees of freedom will become physical. In this section we will study what constraints must be satisfied for a gauge theory to be anomaly-free. We will begin with discussing some fundamental aspects of Lie groups, then in Section 2.2 we derive the chiral anomaly from transformation of the path integral measure. In Section 2.3 we discuss the geometric and analytic meaning of the anomaly in terms of the Atiyah-Singer index theorem. We then relate the anomaly to triangle diagrams in Section 2.4 and study the anomaly cancelation in the Standard Model in Section 2.5. Finally we derive the constraints for the Standard Model extended with one $U(1)$ gauge symmetry and right-handed neutrinos to be anomaly-free.

2.1 Lie groups

Lie groups are a special class of groups that define smooth manifolds with a smooth group operation, for a general reference see e.g. [4]. Let G be a Lie group with neutral element $\mathbf{1}$, then any element $U \in G$ may be written as

$$U = \exp(i\theta^a T^a) \cdot \mathbf{1} \tag{2.1}$$

(summation over a implied) where θ^a are numbers and T^a are the group generators. The generators of a Lie group generate an algebra called the Lie algebra, $\mathfrak{g}$, defined by the bracket relation

$$\begin{aligned} [,]: \ \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ [T^a, T^b] &\mapsto i f^{abc} T^c \end{aligned} \tag{2.2}$$

where f^{abc} are known as the structure constants. A Lie group is Abelian if and only if $f^{abc} = 0$. We will only be interested in the classical matrix Lie groups ($U(N), SU(N)$, etc.), in these cases the Lie algebra can be identified with the tangent space at the neutral element, the elements of the Lie algebra may then be thought of as matrices and it therefore makes sense to interpret the bracket as a commutator relation

$$[T^a, T^b] = T^a T^b - T^b T^a. \tag{2.3}$$

A Lie group has infinitely many representations, but the two most important are; the fundamental and the adjoint representation. The simplest non-trivial representation is the fundamental representation, for $SU(N)$ it is the set of unitary $N \times N$ matrices with determinant 1, this corresponds to a Lie algebra generated by traceless Hermitian $N \times N$ matrices. Let ϕ_i be an element on which the fundamental representation acts, the infinitesimal group action is then given by

$$\phi_i \to \phi_i + i\alpha^a (T^a_{\text{fund}})_{ij}\phi_j \tag{2.4}$$

with α^a real numbers.

For $SU(2)$ the fundamental representation is generated by the Pauli matrices, we introduce the convention

$$T^a = \tau^a = \frac{\sigma^a}{2} \tag{2.5}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$

These matrices satisfies the Lie algebra relation $[T^a, T^b] = i\epsilon^{abc}T^c$ where ϵ^{abc} is the totally anti-symmetric symbol ($\epsilon^{123} = 1$). Another well-known example is $SU(3)$ where we have $T^a = \frac{1}{2}\lambda^a$ with λ^a being the Gell-Mann matrices.

Above we divided the Pauli and Gell-Mann matrices with two, this is a specific choice of normalization of the generators which in general is arbitrary. We will normalize the structure constants by

$$\sum_{c,d} f^{acd} f^{bcd} = N\delta^{ab}. \tag{2.7}$$

Once this is choosen the normalization of the generators is also fixed since $[T^a_R, T^b_R] = if^{abc}T^c_R$ must hold for any representation R. For the fundamental representation of $SU(N)$ this means that the generators are normalized to

$$\text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}. \tag{2.8}$$

When the generators are written as T^a, with no representation R specified, we will always mean the fundamental representation. In the fundamental representation of $SU(N)$, a product of the generators satisfies

$$T^a T^b = \frac{1}{2N}\delta^{ab} + \frac{1}{2}d^{abc}T^c + \frac{1}{2}if^{abc}T^c \tag{2.9}$$

where $d^{abc} = 2\text{tr}[T^a\{T^b, T^c\}]$ is a totally symmetric group invariant and $\{,\}$ denotes the anti commutator. Using this relation one can also show

$$\text{tr}[T^a T^b T^c] = \frac{1}{4}(d^{abc} + if^{abc}). \tag{2.10}$$

The other useful representation is the adjoint representation, this representation acts on the vector space spanned by the generators themselves. Since $SU(N)$ has $N^2 - 1$

generators, this is an N^2-1 dimensional representation. The matrices describing this representation are defined by $(T^a_{\text{adj}})^{bc} = -if^{abc}$, e.g. for $SU(2)$ we explicitly have

$$T^1_{\text{adj}} = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \quad T^2_{\text{adj}} = \begin{pmatrix} 0 & & i \\ & 0 & \\ -i & & 0 \end{pmatrix}, \quad T^3_{\text{adj}} = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}. \tag{2.11}$$

In the spirit of mathematics, we would like to have basis-independent ways of characterizing the representations. One such invariant is the quadratic Casimir $C_2(R)$ defined by

$$T^a_R T^a_R = C_2(R) \cdot \mathbf{1}. \tag{2.12}$$

To evaluate the quadratic Casimir we define the following inner product on the generators

$$\text{tr}(T^a_R T^b_R) = T(R)\delta^{ab} \tag{2.13}$$

where $T(R)$ is called the index of the representation. For the fundamental and adjoint representations we have

$$T(\text{fund}) = T_F = \frac{1}{2}, \quad T(\text{adj}) = T_A = N. \tag{2.14}$$

By putting $a = b$ in the inner product and summing over a we obtain

$$d(R)C_2(R) = T(R)d(G) \tag{2.15}$$

where $d(R)$ is the dimension of the representation ($d(\text{fund}) = N$ and $d(\text{adj}) = N^2 - 1$) and $d(G)$ is the number of group generators, for $SU(N)$, $d(SU(N)) = N^2 - 1$. The quadratic Casimirs for $SU(N)$ can now be written as

$$C_F = C_2(\text{fund}) = \frac{N^2-1}{2N}, \quad C_A = C_2(\text{adj}) = N. \tag{2.16}$$

Another invariant that characterizes $SU(N)$ representations which will be usefull later is the anomaly coefficient $A(R)$ defined as

$$\text{tr}[T^a_R\{T^b_R, T^c_R\}] = \frac{1}{2}A(R)d^{abc} = A(R)\text{tr}[T^a\{T^b, T^c\}]. \tag{2.17}$$

The anomaly coefficients satisfy some usfull properties:

- Conjugate representations satisfy $A(R) = -A(R^*)$ and the anomaly coefficient therefore vanishes for real representations.
- $A(R_1 \oplus R_2) = A(R_1) + A(R_2)$
- $A(R_1 \otimes R_2) = A(R_1)d(R_2) + d(R_1)A(R_2)$

2.2 Transformation of the path integral measure

One symmetry of the Lagrangian that is broken by quantization is the chiral symmetry. In this section we will use Fujikawa's approach [5] to anomalies and derive the Abelian chiral anomaly from a transformation of the path integral measure. A rigorous derivation should be performed in Euclidean space, here we will follow Weinberg [6] and proceed in a less rigorous way and stay in Minkowski space. Let $\psi(x)$ be a massless spin 1/2 fermion field that interacts non-chirally with a set of non-Abelian gauge fields $A_\mu^a(x)$. Moreover, let $U(x)$ be the local chiral transformation:

$$\psi(x) \to U(x)\psi(x) = \exp(i\alpha(x)\gamma^5 T)\psi(x) \tag{2.18}$$

where $\alpha(x)$ is an arbitrary real function of x, T a general Hermitian matrix and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4$. A chiral transformation acts in opposite way on left- and right-handed spinors, where handedness is defined as the eigenstates of the γ^5-matrix. Chiral transforamtions are an example of a classically conserved Noether currents that may obtain non-zero divergences due to quantum effects.

The fermionic part of the path integral measure transforms as

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} \to |\mathcal{J}|^{-2}\mathcal{D}\psi\mathcal{D}\overline{\psi} \tag{2.19}$$

where $\mathcal{J}$ is the Jacobian $\det U$. To write $\mathcal{J}$ on a useful form, we use the formal equality

$$\mathcal{J} = \det U = \exp \operatorname{Tr} \log U \tag{2.20}$$

which gives

$$\mathcal{J} = \exp\left(i \int d^4x \; \alpha(x)\operatorname{Tr}[\gamma^5 T]\right). \tag{2.21}$$

The trace in this equation vanishes, which implies that the transformation of the measure becomes singular. To deal with this we have to introduce a regulator function, for the regularization to be gauge invariant we choose the function

$$e^{(i\not{D})^2/M^2} \tag{2.22}$$

where $D_\mu = \partial_\mu - iA_\mu^a T^a$ is the covariant derivative for the fermions interacting with A_μ^a and M a large mass we will take to infinity at the end. For computational convenience, we also introduce the one-particle Hilbert space $\{|x\rangle\}$ so we may write

$$U(x) = \langle x| U(\hat{x}) |x\rangle . \tag{2.23}$$

The Jacobian is now

$$\mathcal{J} = \lim_{M\to\infty} \exp\left(i \int d^4x \; \alpha(x)\operatorname{Tr}\left[\langle x| \gamma^5 T e^{(i\not{D})^2/M^2} |x\rangle\right]\right) \tag{2.24}$$

We are now going to examine the trace in detail, beginning with using the equality

$$(i\not{D})^2 = -D^2 + \frac{i}{4}\left[\gamma_\mu, \gamma_\nu\right] F_a^{\mu\nu} T^a = -D^2 + \frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}. \tag{2.25}$$

The trace may now be split into

$$\lim_{M\to\infty} \mathrm{Tr}\left[\langle x|\, \gamma^5 T e^{-D^2/M^2} \exp\left(\frac{(\sigma_{\mu\nu}F^{\mu\nu})^2}{4M^2}\right) |x\rangle\right] \tag{2.26}$$

For a trace of γ-matrices multiplied with γ^5 to be non-zero we need at least four γ-matrices. Since $\not{D}^2$ only contains products of two γ-matrices, the leading term in the expansion of the second exponential will be of order $1/M^4$. Expanding the second exponential to leading order, and ignoring the background gauge field in the other exponential, yields

$$\lim_{M\to\infty} \mathrm{Tr}\left[\langle x|\, \gamma^5 T e^{-\partial^2/M^2} \frac{1}{2!}\left(\frac{\sigma_{\mu\nu}F^{\mu\nu}}{2M^2}\right)^2 |x\rangle\right]. \tag{2.27}$$

The bra-ket now only affects $e^{-\partial^2/M^2}$ which is calculated by Wick-rotation as follows:

$$\langle x|\, e^{-\partial^2/M^2}\, |x\rangle = \int \frac{d^4k}{(2\pi)^4} e^{k^2/M^2} = i\int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2} = i\frac{M^4}{16\pi^2} \tag{2.28}$$

We now have

$$\begin{aligned}\lim_{M\to\infty} i\frac{M^4}{16\pi^2}\mathrm{Tr}\left[\gamma^5 T\frac{1}{2!}\left(\frac{\sigma_{\mu\nu}F^{\mu\nu}}{2M^2}\right)^2\right] &= i\frac{1}{128\pi^2}\mathrm{Tr}\left[\gamma^5 T(\sigma_{\mu\nu}F^{\mu\nu})^2\right] \\ = -\frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^b_{\alpha\beta}\mathrm{Tr}[TT^aT^b]\end{aligned} \tag{2.29}$$

where we have used $\mathrm{Tr}[\gamma^5[\gamma^\mu,\gamma^\nu][\gamma^\alpha,\gamma^\beta]] = 16i\epsilon^{\mu\nu\alpha\beta}$. Using this for the trace in the Jacobian yields

$$\mathcal{J} = \exp\left(-i\int d^4x\ \frac{1}{32\pi^2}\alpha(x)\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^b_{\alpha\beta}\mathrm{Tr}[TT^aT^b]\right) \tag{2.30}$$

This result may be associated with an anomalous axial current. Assume we have the path integral

$$Z = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left[i\int d^4\overline{\psi}(i\not{D}\psi)\right]. \tag{2.31}$$

Under the chiral transformation defined in the beginning of this section, the action transforms as

$$\int d^4x\overline{\psi}(i\not{D}\psi) \to \int d^4x\left[\overline{\psi}(i\not{D}\psi) + \alpha(x)\partial_\mu J^{5\mu}\right] \tag{2.32}$$

where $J^{5\mu} = \overline{\psi}\gamma^5 T\gamma^\mu\psi$. Joining this with the transformation of the measure, the transformed integral becomes

$$Z = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left[i\int d^4x\ +\overline{\psi}(i\not{D}\psi) + \alpha(x)\left\{\partial_\mu J^{5\mu} + \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^b_{\alpha\beta}\mathrm{Tr}[TT^aT^b]\right\}\right] \tag{2.33}$$

By varying the exponent with respect to $\alpha(x)$ we obtain the operator equation

$$\partial_\mu J^{5\mu} = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^b_{\alpha\beta}\mathrm{Tr}[TT^aT^b]. \tag{2.34}$$

2.3 Geometry and anomalies

Anomalies is an extremly profound concept in quantum field theory. Although originally calculated from perturbation theory [7, 8], the anomaly is in fact a non-perturbative topological effect. If this was not the case, anomalies would not be of the same importance since they would then change at every order in perturbation theory. Going to Euclidean spacetime, we will be able to connect anomalies to the famous Atiyah[1]-Singer index theorem [9, 10, 11].

We introduce the Euclidean fourth coordinate $x_4 = ix^0$ and perform a Wick rotation. In Euclidean space the Dirac operator $i\not{D}$ is Hermitian and therefore has an orthonormal set of spinor eigenfunctions ϕ_m:

$$i\not{D}\phi_m = \lambda_m\phi_m \tag{2.35}$$

with the normalization completeness relations:

$$\begin{aligned}\int d^4x_E \phi_m(x)^\dagger\phi_n(x) &= \delta_{mn},\\ \sum_m \phi_m(x)\phi_m^\dagger(y) &= \delta^4(x-y)\mathbf{1}.\end{aligned} \tag{2.36}$$

We assume here that T and $i\not{D}$ commute so we also have $T\phi_n = t_n\phi_n$. If we now return to the Jacobian from the previous section, we can write it as

$$\begin{aligned}\mathcal{J} &= \lim_{M\to\infty} \exp\left(i\int d^4x_E\ \alpha(x)\mathrm{Tr}\left[\gamma^5 T e^{(i\not{D})^2/M^2}\sum_m \phi_m(x)\phi_m^\dagger(x)\right]\right)\\ &= \lim_{M\to\infty} \exp\left(i\int d^4x_E\ \alpha(x)\sum_m\left[t_m e^{\lambda_m^2/M^2}\left\{\phi_m^\dagger(x)\gamma^5\phi_m(x)\right\}\right]\right)\end{aligned} \tag{2.37}$$

By the exact same type of calculations as we did in the previous section, the Jacobian becomes

$$\mathcal{J} = \exp\left(-i\int d^4x_E\ \alpha(x)\frac{1}{32\pi^2}\epsilon^E_{ijkl}F^a_{ij}F^b_{kl}\mathrm{Tr}[TT^aT^b]\right) \tag{2.38}$$

where i, j, k, l are Euclidean indices going from 1 to 4 and ϵ^E_{ijkl} is the totally anti-symmetric tensor with $\epsilon^E_{1234} = 1$.

Without the regulator function, the Jacobian becomes

$$\mathcal{J} = \exp\left(i\int d^4x_E\ \alpha(x)\sum_m\left[t_m\phi_m^\dagger(x)\gamma^5\phi(x)\right]\right) \tag{2.39}$$

and we are now going to study the properties of the sum in more detail. Given the eigenvalue problem $i\not{D}\phi_m(x) = \lambda_m\phi(x)$, there is the associated problem:

$$i\not{D}\gamma^5\phi_m(x) = -\lambda_m\gamma^5\phi_m(x). \tag{2.40}$$

[1]Michael Atiyah was awarded the Fields Medal in 1966 for, among other things, the proof of the index theorem for elliptic operators.

For $\lambda_m \neq 0$ this means that $\phi_m(x)$ and $\gamma^5\phi_m(x)$ are eigenfunctions to the same Hermitian operator but with different eigenvalues, they are therefore orthogonal:

$$\int d^4x_E\ \phi_m^\dagger(x)\gamma^5\phi_m(x) = 0. \tag{2.41}$$

In the case of $\lambda_m = 0$, $\phi_m(x)$ and $\gamma^5\phi_m(x)$ are eigenfunctions with the same eigenvalue. These eigenfunctions, called zero-modes, are not generally paired in a specific way. But since γ^5 anti-commutes with $i\not{D}$, they may be chosen to be simultaneous orthogonal eigenfunctions ϕ_{n+} and ϕ_{n-} of $i\not{D}$ with eigenvalue zero and of γ^5 with eigenvalues +1 and -1 respectively:

$$\begin{aligned} i\not{D}\phi_{n+} &= 0, \quad \gamma^5\phi_{n+} = \phi_{n+} \\ i\not{D}\phi_{n-} &= 0, \quad \gamma^5\phi_{n-} = -\phi_{n-} \end{aligned} \tag{2.42}$$

The sum in the Jacobian may now, thanks to the orthogonality relation for $\lambda_m \neq 0$, be written as

$$\sum_m t_m\phi_m^\dagger(x)\gamma^5\phi_m(x) = \sum_{n+} t_{n+}\left(\phi_{n+}^\dagger(x)\phi_{n+}(x)\right) - \sum_{n-} t_{n-}\left(\phi_{n-}^\dagger(x)\phi_{n-}(x)\right) \tag{2.43}$$

Using the normalization of $\phi_m(x)$, integrating the above equation gives

$$\int d^4x_E\ \sum_m t_m\phi_m^\dagger(x)\gamma^5\phi_m(x) = \sum_{n+} t_{n+} - \sum_{n-} t_{n-}. \tag{2.44}$$

In the special case that T is the identity matrix, these sums becomes the number of zero-modes n_+ and n_- with eigenvalues ± 1 for γ^5. The difference $n_+ - n_-$ is the index of the the Dirac operator:

$$\text{index } i\not{D} = n_+ - n_- \tag{2.45}$$

Combining this with the result in Eq. (2.38) we get

$$\text{index } iD_+ = -\frac{1}{32\pi^2}\int d^4x_E\ \epsilon^E_{ijkl}F^a_{ij}F^b_{kl}\text{Tr}[T^aT^b] \tag{2.46}$$

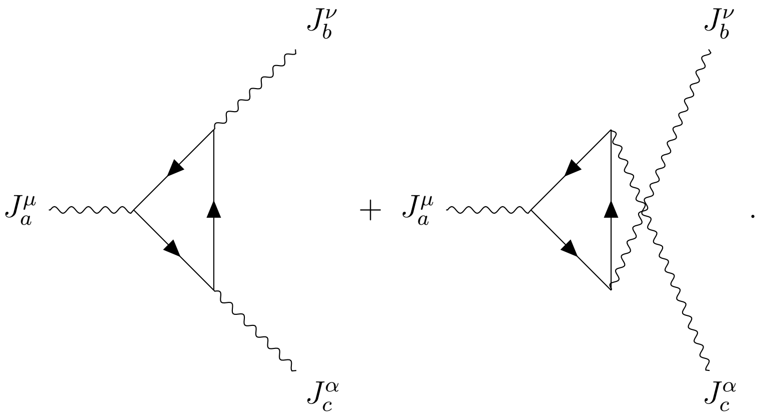
which is the Atiyah-Singer index theorem. In this way we may regard the anomalous current $\partial_\mu J^{5\mu}$ as a local index theorem. This shows how the anomaly, initially determined from local fields and perturbation theory, is connected to the topology of the gauge field configuration.

2.4 Triangle diagrams

The above calculation using Fujkawa's approach gave us the Abelian chiral anomaly for gauge theories with non-chiral gauge interactions. To deal with the non-Abelian chiral case, it might be easier to take another approach and derive the anomaly from triangle diagrams. We define the one-loop three-point function

$$\Gamma^{\mu\nu\alpha}_{abc}(x,y,z) = \langle\Omega|\, T\{J^\mu_a(x), J^\nu_b(y), J^\alpha_c(z)\}\, |\Omega\rangle \tag{2.47}$$

for massless left-handed fermions ψ interacting with gauge bosons A^μ_a, A^ν_b, A^α_c. The leading order contributions to this function are the diagrams



(2.48)

The anomalous contribution comes from calculating the divergence of $\Gamma^{\mu\nu\alpha}_{abc}$. As discussed above, the anomaly is a topological non-perturbative result and the contributions from one-loop diagrams is the only contribution [12]. Even though $\Gamma^{\mu\nu\alpha}_{abc}$ is convergent and thus independent of the labelling of the momentum in the fermion loop, the divergence of $\Gamma^{\mu\nu\alpha}_{abc}$ (which contains the anomalous contribution) involves divergent integrals that do depend on this labelling. This gives us a freedom to choose which current should be anomalous, but not enough freedom to remove the anomaly completely.

Calculating the divergence of $\Gamma^{\mu\nu\alpha}_{abc}$ with the fermion momentum chosen so J^μ_a is anomalous gives the anomalous contribution (see [6] chapter 22.3 for details):

$$\partial_\mu J^\mu_a = -\frac{1}{128\pi^2} d^{abc} \epsilon^{\mu\nu\alpha\beta} F^b_{\mu\nu} F^c_{\alpha\beta} \tag{2.49}$$

where $d^{abc} = 2\mathrm{Tr}[T^a\{T^b, T^c\}]$. The reason $\mathrm{Tr}[T^a\{T^b, T^c\}]$ appears is that we calculate the sum of the two triangle diagrams, each with contribution $\mathrm{Tr}[T^aT^bT^c]$ and $\mathrm{Tr}[T^aT^cT^b]$ respectively. Using Eq. (2.10), the non-symmetric parts cancel and only $\mathrm{Tr}[T^a\{T^b, T^c\}]$ remains.

We have already derived an anomaly in Eq. (2.34), and would now like to see that our new and more general result contains this. In a theory containing both left- and right-handed particles the anomalous contribution from each of the two chiral states is different because they couple to a chiral axial current. This current contains a γ^5-matrix which yields different sign for the two chiral states. In such a theory, we have

$$d^{abc} = 2\mathrm{Tr}[T^a_L\{T^b_L, T^c_L\}] - 2\mathrm{Tr}[T^a_R\{T^b_R, T^c_R\}] \tag{2.50}$$

here the subscripts L and R denote the representation under which the left-handed and right-handed fermions transform respectively. To obtain Eq. (2.34) we coupled an axial current to two non-chiral gauge currents. For the axial current we have $T_L = -T_R \equiv T$ and for the non-chiral gauge currents we have $T^b_L = T^b_R \equiv T^b$ and $T^c_L = T^c_R \equiv T^c$ respectively. Equation (2.49) now becomes

$$\partial_\mu J^\mu_5 = -\frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F^b_{\mu\nu}F^c_{\alpha\beta}\mathrm{Tr}[T\{T^b, T^c\}] \tag{2.51}$$

which is the same as Eq. (2.34), if we use in Eq. (2.34) that the only part of $\mathrm{Tr}[TT^bT^c]$ that contributes to the anomaly is the symmetric $\frac{1}{2}\mathrm{Tr}[T\{T^b, T^c\}]$.

2.5 Gauge anomalies in the Standard Model

Due to the electroweak interaction, the Standard Model is chiral and may therefore contain anomalies. For the theory to be consistant the anomalies have to vanish, and this is the case in the Standard Model thanks to an almost magical interplay between the lepton and quark sector. The gauge group in the Standard Model: $SU(3)_{QCD} \times SU(2)_L \times U(1)_Y$, have the associated currents J^μ_{QCD}, J^μ_L and J^μ_Y. Quantum chromodynamics (QCD) is a non-chiral theory and therefore couples equally to left- and right-handed fields so there is no anomaly associated with $SU(3)^3_{QCD}$. The generators of $SU(2)_L$ are the Pauli matrices $\tau^a = \sigma^a/2$, these satisfy $\{\tau^a, \tau^b\} = \frac{1}{2}\delta^{ab}\mathbf{1}$ so

$$d^{abc} = \delta^{bc}\mathrm{Tr}[\tau^a] = 0 \tag{2.52}$$

so there is no $SU(2)_L \times SU(2)_L \times SU(2)_L (= SU(2)^3_L)$ anomaly. For $SU(2)_L \times U(1)^2_Y$ we have

$$d^{abc} \propto 2\mathrm{Tr}[\tau^a\{\mathbf{1}, \mathbf{1}\}] = 4\mathrm{Tr}[\tau^a] = 0 \tag{2.53}$$

and in a similar way, any anomaly with one factor of $SU(2)_L$ or $SU(3)_{QCD}$ vanishes.

We must remember that even though quantum electrodynamics is a vector-like theory and thus anomaly-free, the current J^μ_Y is not a vector current since it couples differently to left- and right-handed fermions (this is an artefact from the electroweak unification). As in the previous section, the left- and right-handed fermions will contribute to the anomaly with different sign due to the γ^5-matrices in the hypercharge current. We denote the hypercharges in the Standard Model by Y_Q, Y_u, Y_d, Y_L and Y_e for the quark doublet, right-handed up singlet, right-handed down singlet, lepton doublet and charged right-handed lepton singlet respectively.

Let us begin with the anomaly associated with $U(1)^3_Y$, the symmetric trace is now given by

$$d^{abc} \propto \mathrm{Tr}[Y^L\{Y^L, Y^L\}] - \mathrm{Tr}[Y^R\{Y^R, Y^R\}] \propto \sum_F (Y^L_F)^3 - \sum_F (Y^R_F)^3 \tag{2.54}$$

where the sums are over the left- and right-handed fermions respectively. If the difference between these two sums is zero, the anomaly vanish. This gives the following constraint on the hypercharges:

$$\mathcal{A}_{111}: \quad 3(2Y_Q^3 - Y_u^3 - Y_d^3) + 2Y_L^3 - Y_e^3 = 0 \tag{2.55}$$

where the factor 3 in front of the quarks is a color factor and the factors of 2 represents the doublets.

Let us now study the contribution from $SU(3)_{QCD}^2 U(1)_Y$. The generators of any $SU(N)$ satisfies (with our normalization in Eq. (2.14)) $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$, using this we get for the symmetric trace

$$d^{abc} \propto 2\delta^{ab} \left(\sum_{F,\text{ color}} Y_F^L - \sum_{F,\text{ color}} Y_F^R \right) \tag{2.56}$$

where the sums are over the left- and right-handed colored states respectively. This anomaly vanishes if

$$\mathcal{A}_{331}: \quad 2Y_Q - Y_u - Y_d = 0. \tag{2.57}$$

The $SU(2)_L^2 U(1)_Y$ anomaly is similar to the anomaly just calculated, but instead of summing over all colored states, we sum over all left-handed states:

$$d^{abc} = 2\delta^{ab} \sum_F Y^L = 2\delta^{ab}(6Y_Q + 2Y_L). \tag{2.58}$$

This anomaly vanishes if

$$\mathcal{A}_{221}: \quad 3Y_Q + Y_L = 0. \tag{2.59}$$

There is one more potential family of anomalies, those from triangle diagrams with gravitons. In four-dimensional spacetime, the only possible gravitational gauge anomalies are with two gravitons (g), the anomalous current will then have the divergence

$$\partial_\alpha J_a^\alpha \propto \mathrm{Tr}[T^a] \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\rho\sigma} R_{\alpha\beta\rho\sigma} \tag{2.60}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. Since the trace of an $SU(N)$ generator is zero, the only possible gravitational anomaly is $g^2 U(1)_Y$. Gravity couples to all fermions, so this anomaly is simply the sum over all fermions:

$$\mathcal{A}_{gg1}: \quad 3(2Y_Q - Y_u - Y_d) + 2Y_L - Y_e = 0. \tag{2.61}$$

For the Standard Model to be anomaly-free, the hypercharges of the fermions thus must satisfy:

$$\begin{aligned}
\mathcal{A}_{111}: &\quad 3(2Y_Q^3 - Y_u^3 - Y_d^3) + 2Y_L^3 - Y_e^3 = 0 \\
\mathcal{A}_{331}: &\quad 2Y_Q - Y_u - Y_d = 0 \\
\mathcal{A}_{221}: &\quad 3Y_Q + Y_L = 0 \\
\mathcal{A}_{gg1}: &\quad 3(2Y_Q - Y_u - Y_d) + 2Y_L - Y_e = 0
\end{aligned} \tag{2.62}$$

Fermion	Q_f	T_3	Y_L	Y_R
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	-1	0
e^-, μ^-, τ^-	-1	$-\frac{1}{2}$	-1	-2
u, c, t	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{4}{3}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{2}{3}$

Table 1: Electric charge Q_f, isospin T_3 and hypercharge Y for the fermions in the Standard Model.

These equations may easily be solved using the algebraic methods which will be presented in Section 3. Using these methods here, we find that there are three distinct solutions for the hypercharges:

$$\begin{array}{llll} Y_Q = -Y_e/6, & Y_u = -2Y_e/3, & Y_d = Y_e/3, & Y_L = Y_e/2 \\ Y_Q = -Y_e/6, & Y_u = Y_e/3, & Y_d = -2Y_e/3, & Y_L = Y_e/2 \\ Y_Q = 0, & Y_u = -Y_d, & Y_L = 0, & Y_e = 0 \end{array} \tag{2.63}$$

All these solutions depends on one free parameter: Y_e in the two first and one of the quark singlet charges in the second. This freedom comes from the fact that hypercharges always appear together with the coupling g'. The object that needs to have a specific value is the product Yg', if the charge is made twice as large, this can be compensated by making the coupling half as large. These solutions tell us that hypercharge, and thus electric charge, has to be quantized. However, we can not determine the charges from first principles, we have to use experiments to determine which of these three solutions corresponds to what is observed in nature.

Let Q be the electric charge and T_3 the third component of isospin. We adopt the following normalization for hypercharge:

$$Y = 2(Q - T_3). \tag{2.64}$$

In Table 1 are the observed values of the different charges with this normalization. This means that the correct solution for the anomalies is the first one:

$$Y_Q = -Y_e/6,\ Y_u = -2Y_e/3,\ Y_d = Y_e/3,\ Y_L = Y_e/2 \tag{2.65}$$

with $Y_e = -2$.

2.6 Extending the Standard Model

In this thesis we will work with an extension of the Standard Model where the gauge group is $SU(3)_{QCD} \times SU(2)_L \times U(1)_Y \times U(1)'$. It will also be necessary for us to introduce right-handed neutrinos, these may only interact with the new $U(1)'$ field and with gravity. The hypercharges in the Standard Model are generation independent, we will need more degrees of freedom than that, so we will have generation dependent $U(1)'$ charges. We denote these charges $Q_i,\ u_i,\ d_i,\ L_i,\ e_i$ and ν_i which are the charges for the quark doublet, the two quark singlets, the lepton doublet, the charged lepton singlet and the right-handed neutrino fields where i is the generation index. The anomaly constraints for this model may be derived in a similar way as for the SM and are given by:

$$\begin{aligned}
\mathcal{A}_{1'1'1}: &\quad \sum_{j=1}^{3}\left(Q_j^2 - 2u_j^2 + d_j^2 - L_j^2 + e_j^2\right) = 0 \\
\mathcal{A}_{1'11}: &\quad \sum_{j=1}^{3}\left(Q_j - 8u_j - 2d_j + 3L_j - 6e_j\right) = 0 \\
\mathcal{A}_{331'}: &\quad \sum_{j=1}^{3}\left(2Q_j - u_j - d_j\right) = 0 \\
\mathcal{A}_{221'}: &\quad \sum_{j=1}^{3}\left(3Q_j + L_j\right) = 0 \\
\mathcal{A}_{1'1'1'}: &\quad \sum_{j=1}^{3}\left(6Q_j^3 - 3u_j^3 - 3d_j^3 + 2L_j^3 - e_j^3 - \nu_j^3\right) = 0 \\
\mathcal{A}_{gg1'}: &\quad \sum_{j=1}^{3}\left(2L_j - e_j - \nu_j\right) = 0
\end{aligned} \tag{2.66}$$

where we have used that the quark part of $\mathcal{A}_{gg1'}$ is the same as $\mathcal{A}_{331'}$.

3 Algebraic Geometry

Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

-Hilbert's 10th problem, **David Hilbert** [13]

Caution: In this section a field is a commutative ring with identity such that every element has a multiplicative inverse.

For the mathematically oriented reader the mathematics introduced in the beginning of this section should be familiar, see [14] for a general reference. But for this thesis to be self-contained and accessible to a wider audience, the basic definitions will be given here. In Section 3.2 and 3.3 some more advanced results from Diophantine geometry will be discussed (see [15] for a general reference) which might be of interest even for mathematicians.

In the following, let K denote a infinite field and $K[x_1, \dots, x_n]$ the polynomial ring in n variables over this field.

Definition 1. *If $f_1, \dots, f_s \in K[x_1, \dots, x_n]$ we call the set*

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in K^n \mid f_i(a_1, \dots, a_n) = 0 \ \forall \ 1 \leq i \leq s\}$$

the ***affine variety*** *defined by $f_1, \dots, f_s$.*

It is clear from the definition that the affine variety is the set of solutions to the system of polynomial equations:

$$\begin{cases} f_1(x_1, \dots, x_n) & = 0 \\ \vdots & \vdots \\ f_s(x_1, \dots, x_n) & = 0 \end{cases}.$$

The algebraic object defining varieties are **ideals**:

Definition 2. *A set $I \subseteq K[x_1, \dots, x_n]$ is an* ***ideal*** *if it satisfies:*

(i) $0 \in I$

(ii) If $f, g \in I$, then $f + g \in I$

(iii) If $f \in I$ and $h \in K[x_1, \dots, x_n]$ then $hf \in I$

By the Hilbert basis theorem (Chapter 2§5, Theorem 4 in [14]), every ideal has a finite generating set, that is, we may write every ideal in $K[x_1, \dots, x_n]$ in the form

$$I = \langle f_1, \dots, f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i \ \middle| \ h_i \in K[x_1, \dots, x_n] \ \forall \ 1 \leq i \leq s \right\}. \tag{3.1}$$

The connection between ideals and varieties is given by

$$\begin{aligned}\mathbf{I}(V) &= \{f \in K[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \ \forall \ (a_1, \ldots, a_n) \in V\} \\ \mathbf{V}(I) &= \{(a_1, \ldots, a_n) \in K^n \mid f(a_1, \ldots, a_n) = 0 \ \forall \ f \in I\}. \end{aligned} \tag{3.2}$$

For polynomials in one variable, $K[x]$, it is natural to order the monomials they consist of according to their degree, either increasing or decreasing. In several variables however, there is no such natural ordering, let us choose the lexicographic ordering (LEX): $x_1 > x_2 > \ldots > x_n$ and $x_j^k > x_j^l$ if $k > l$. In this ordering $x_1 x_3 > x_2^{55}$ since $x_1 > x_2$. With an ordering, each polynomial has a well-defined leading term, i.e., the biggest monomial in the polynomial. We define the **ideal of leading terms** as follows:

Definition 3. *Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal other than $\{0\}$.*

(i) Let $LT(I)$ denote the set of leading terms of elements in I, i.e.

$$LT(I) = \{c x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mid \exists \ f \in T \text{ such that } LT(f) = c x_1^{\alpha_1} \ldots x_n^{\alpha_n}\}$$

(ii) Let $\langle LT(I) \rangle$ denote the ideal generated by elements in $LT(I)$.

All the above definitions are necessary when we now construct the key notion of computational algebraic geometry, that of **Gröbner bases**.

Definition 4. *For a fix monomial order, a finite subset $G = \{g_1, \ldots, g_t\} \subset I$, I an ideal, is said to be a Gröbner basis if*

$$\langle LT(g_1), \ldots, LT(g_t) \rangle = \langle LT(I) \rangle \, .$$

As the name suggests, a Gröbner basis is a basis for I and every ideal except $\{0\}$ admits such a basis [14]. One important application of Gröbner bases is that they eliminate variables and give the most reduced version of the system you are studying. If a variety is described by polynomials containing different variables, the Gröbner basis of the ideal generated by these polynomials will consist of polynomials where one or many of the variables have been eliminated. The following example clarifies this.

Example.
Consider the variety $\mathbf{V}(f_1, f_2, f_3) \in \mathbb{C}^3$ where

$$\begin{aligned} f_1 &= x^2 + y^2 + z^2 - 1 \\ f_2 &= x^2 + z^2 - y \\ f_3 &= x - z. \end{aligned}$$

The ideal of this variety is $I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{C}[x, y, z]$ and we want to find all points in the variety. To do this we calculate the Gröbner basis of I, which is

$$G = \{4z^4 + 2z^2 - 1, y - 2z^2, x - z\}.$$

Note that the first term only consists of z's, the zeros of this polynomial can be found by standard techniques. These solutions can then be substituted one by one into the other polynomials to find their zeros. The variety $\mathbf{V}(f_1, f_2, f_3) \cap \mathbb{R}^3$ is visualized in Fig. 2.

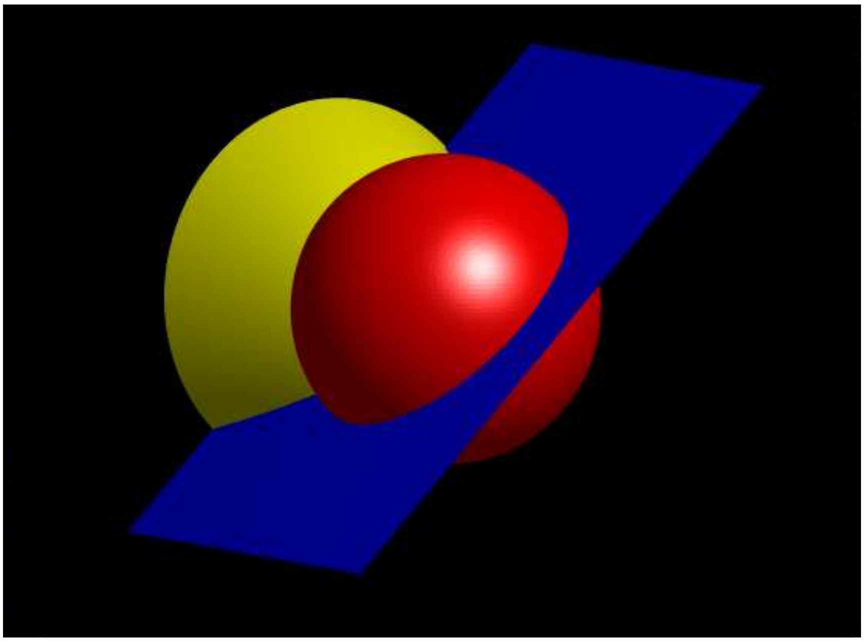


Figure 2: The intersection of these three surfaces is the variety given in the Gröbner basis example restricted to $\mathbb{R}^3$.

3.1 Some abstract notions

It is often the case in geometry that life becomes easier if one adds "points at infitiy", like how we in complex analysis often prefer to work over the Riemann sphere rather than the complex plane. Towards this end we introduce the notion of **projective spaces**.

Definition 5. *The projective n-space $\mathbb{P}^n$ over K is the set of lines through the origin in K^{n+1}. In symbols:*

$$\mathbb{P}^n = \frac{K^{n+1} \setminus \{0\}}{\sim} \tag{3.3}$$

where $\sim$ is the equivalence relation defined by

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \iff (x_0, \dots, x_n) = \lambda(y_0, \dots, y_n) \tag{3.4}$$

for some non-zero $\lambda \in K$.

The most familiar example is the previously mentioned Riemann sphere, which is the projective line $\mathbb{P}^1$ over $\mathbb{C}$. As we defined affined varieties above, we may now define **projective varieties**.

Definition 6. *Let K be a field and let $f_1, \dots, f_s \in K[x_0, \dots, x_n]$ be homogeneous polynomials. We set*

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_0, \dots, a_n) \in \mathbb{P}^n \,| f_i(a_0, \dots, a_n) = 0 \ \forall \ 1 \leq i \leq s\}. \tag{3.5}$$

$\mathbf{V}(f_1, \dots, f_s)$ *is called the projective variety defined by $f_1, \dots, f_s$.*

Now that we have varieties we want to extend our toolbox to maps between varieties. This is important for our study of algebraic curves later, especially the curves of genus 0. Let us first introduce the notation $\mathbb{A}$ for the algebraic closure $\overline{K}$.

Definition 7. *Let X be a variety and x' a point on X. A function $f : X \to \mathbb{A}$ is* ***regular*** *at x' if there exists an open affine neighborhood $U \subset X$ of x', say $U \subset \mathbb{A}^n$, and two polynomials $P, Q \in \overline{K}[x_1, \dots x_n]$ such that $Q(x') \neq 0$ and $f(x) = P(x)/Q(x)$ for all $x \in U$. The function f is regular on X if it is regular at every point of X. The* ***ring of regular functions*** *on X is denoted $\mathcal{O}(X)$.*

Note that the definition of regularity is local, this means that even though f may be regular on all of X, there are in general no fixed polynomials P and Q such that $f = P/Q$ at every point on X. In general, one can write X as a finite union of affine open subsets U_i, and one can find polynomials P_i, Q_i such that $f(x) = P_i(x)/Q_i(x)$ for all $x \in U_i$.

Definition 8. *Let x be a point on a variety X. The* ***local ring*** *of X at x is the ring of functions that are regular at x, where we identify two such functions if they coincide on some open neighborhood of x. This ring is denoted by $\mathcal{O}_{x,X}$ or simply $\mathcal{O}_x$*

Even more generally, we can define a ring of functions regular along a subvariety of X.

Definition 9. *Let X be a variety and $Y \subset X$ a subvariety. The local ring of X along Y, denoted by $\mathcal{O}_{X,Y}$, is the set of pairs (U, f), where U is an open subset of X with $U \cap Y \neq 0$ and $f \in \mathcal{O}(U)$ is a regular function on U, and where we identify two pairs $(U_1, f_1) = (U_2, f_2)$ if $f_1 = f_2$ on $U_1 \cap U_2$.*

A special case of local rings are the **function fields**, denoted $\overline{K}(X)$, which are defined to be $\mathcal{O}_{X,X}$, i.e. the local ring of X along X. We can now define maps between varieties.

Definition 10. *A map $\phi : X \to Y$ between varieties is a* ***morphism*** *if it is continuous, and if for every open set $U \subset Y$ and every regular function f on U, the function $f \circ \phi$ is regular on $\phi^{-1}(U)$. A map is regular at a point x if it is a morphism on some neighborhood of x.*

We call such a map **rational** if it is a morphism on some non-empty subset of X. And a **birational map** is a rational map with a rational inverse. Two varieties are said to be **birationally equivalent** if there exists a birational map between them.

3.2 Algebraic curves

Thanks to the Gröbner basis, the problem of finding rational or integer charges that satisfy the anomaly and Froggatt-Nielsen constraints reduces to finding rational or integer points on algebraic curves. To make a systematic study of curves we would like to find some classification of them, the most obvious way to do this is by their degree. However, from an arithmetic point of view, this classification is insufficient. Take for example the two affine curves

$$C_1 : \; y^2 = x^5 + x^4, \quad \text{and} \quad C_2 : \; y^2 = x^5 + x. \tag{3.6}$$

These are both curves of degree five, a classification based on degree would then suggest that these curves have the same arithmetic properties, e.g. existence of rational solutions. As it happens, C_1 has infinitely many solutions in rational x and y, while C_2 only has finitely many. It turns out that the interesting invariant of a curve is its genus g. For a non-singular projective curve C, the genus is number the of handles of the Riemann surface $C(\mathbb{C})$.

Many curves are not smooth, e.g. $y^2 = x^3$ has a cusp at the origin. In general, a curve C is a variety of dimension one, this means that its function field $K(C)$ is algebraic over any subfield $K(x)$ generated by a non-constant function $x \in K(C)$. We may therefore write $K(C) = K(x, y)$ where x and y are non-constant functions on C satisfying an algebraic relation $P(x, y) = 0$. Now, let $C_0 \subset \mathbb{A}^2$ denote the affine plane curve defined by P, and let $C_1 \subset \mathbb{P}^2$ denote the projective curve defined by $Z^{\deg P} P(X/Z, Y/Z)$. By definition both C_0 and C_1 are birational to C, we call such curves **models** of C. Such models may very well have singularities, like the affine curve $y^2 = x^3$. However, it turns out that all algebraic curves have a smooth model:

Theorem 1. *Any algebraic curve is birational to a unique (up to isomorphisms) smooth projective curve.*

Proof. See e.g. Fulton [16], Section 7.5 Theorem 3. □

An algebraic curve and its smooth projective model can, at most, differ by a computable finite set of points (points associated with singularities and at infinity). We may therefore, without loss of generality, always assume that C is a smooth projective curve. For smooth projective curves over any number field K we have the following trichotomy classified by the genus g:

- $g = 0$:
  Here we have two choices: either $C(K) = \emptyset$ or $C(K)$ is non-empty which means that C is isomorphic over $\mathbb{Q}$ to the projective line $\mathbb{P}^1$. Any such isomorphism defines a parameterization of $C(\mathbb{Q})$ in terms of rational functions in one variable, which is easily computable. For example, all rational points on the unit circle $x^2 + y^2 = 1$ are given by
  $$(x(t), y(t)) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right) \tag{3.7}$$
  for $t \in \mathbb{P}^1(\mathbb{Q})$.

- $g = 1$:

  Theorem 2. ***Mordell-Weil:*** *For any Abelian variety the set of K-rational points form a finitely generated group.*

  Proof. For the original proof for elliptic curves by Mordell, see [17], and for the generalization to Abelian varieties by Weil, see [18]. □

For $K = \mathbb{Q}$ this means that the only genus 1 curves with rational points are the elliptic curves.

- $g \geq 2$:
  For these higher genus curves, Mordell [17] conjectured and Falting [19] later proved that the set K-rational points is finite.

3.2.1 Genus zero

From the trichotomy we have that a smooth projective curve C over $\mathbb{Q}$ of genus zero has infinitely many rational points (which we may describe by an explicit parameterization) if it contains one rational point. Practically, given a curve we can ask a computer algebra system such as Singular [20] or Macaulay 2 [21] to find the parameterization, if it does not succeed, then there are no rational points. If we do not want to check if there exists an explicit parameterization, we can also use the Hasse principle to check theoretically if there are any rational points without having to actually find them.

For curves of genus zero there are also well-developed methods to find integer solutions. This will be useful for us when we study neutrinos since integer charges will allow them to have effective Majorana masses.

Let $f(x, y) = 0$ be an absolutely irreducible polynomial with integer coefficient defining a plane curve of genus zero. By C we denote the projective model defined by $F(X, Y, Z) = 0$, where $F(X, Y, Z)$ is the homogenization of $f(x, y)$. We denote the algebraic closure of $\mathbb{Q}$ by $\overline{\mathbb{Q}}$ and function field of C by $\overline{\mathbb{Q}}(C)$. For a point P on C the local ring is denoted by $\mathcal{O}_P(C)$ and Σ_∞ is the set of discrete valuation rings at infinity. In [22], Poulakis and Voskos gave an algorithm to explicitly find all integer solutions of $f(x, y) = 0$ for $|\Sigma_\infty| \geq 3$. They later extended this to $|\Sigma_\infty| \leq 2$ in [23]. These cases differ since the latter may have infinitely many integer solutions, while for $|\Sigma_\infty| \geq 3$ there are only finitely many.

Let N be the degree of $F(X, Y, Z)$ and remember that the existence of one rational point on C is equivalent to the existence of a birational map over $\mathbb{Q}$ between C and $\mathbb{P}^1$.

Lemma 1. *Let $u(S, T)$, $v(S, T), w(S, T) \in \mathbb{Z}[S, T]$ be homogeneous polynomials of the same degree with no common non-constant factor such that the correspondence*

$$(S, T) \mapsto (u(S, T), v(S, T), w(S, T)) \tag{3.8}$$

defines a birational map ϕ over $\mathbb{Q}$ of $\mathbb{P}^1 \to C$. Then ϕ is a birational morphism of $\mathbb{P}^1$ onto C and $\deg u(S, T) = \deg v(S, T) = \deg w(S, T) = N$. If $(x : y : 1)$ is a non-singular point of $C(\mathbb{Q})$, then there exists $s, t \in \mathbb{Z}$ with $s \geq 0$ and $\gcd(s, t) = 1$ such that $x = u(s, t)/w(s, t)$ and $y = v(s, t)/w(s, t)$.

Proof. See Poulakis and Voskos [22] Lemma 2.1. □

Let ϕ be as above, then the correspondence $f \to f \circ \phi$ induces an isomorphism $\tilde{\phi}$ over $\mathbb{Q}$ between the function fields $\overline{\mathbb{Q}}(C)$ and $\overline{\mathbb{Q}}(\mathbb{P}^1)$.

Lemma 2. *The correspondence* $P \to \tilde{\phi}^{-1}(\mathcal{O}_P(\mathbb{P}^1))$ *defines a bijection between the set of zeros of* $w(s,t)$ *and* Σ_∞.

Proof. See Poulakis and Voskos [22] Lemma 2.2. □

What this lemma means is that the number of distinct zeros of $w(s,t)$ is equal to $|\Sigma_\infty|$. This is of practical importance if one wants to use the following theorem.

Theorem 3. *The set* $C(\mathbb{Z})$ *is infinite if and only if one of the following two conditions is satified:*

(i) Σ_∞ *consists of one element and* $C(\mathbb{Z})$ *has at least one simple integer point.*

(ii) Σ_∞ *consists of two elements which are conjugate over a real quadratic field and* $C(\mathbb{Z})$ *has at least one simple integer point.*

Proof. See Poulakis and Voskos [23] Theorem 5.2. □

For a curve with genus 0 we have thus learned that if it contains one rational point, we may find a parameterization for all the infinitely many other rational points. For a curve with $|\Sigma_\infty| \leq 2$ we have found a way to determine if it has infinitely many integer points or not.

3.2.2 Genus one

The only Abelian varieties over $\mathbb{Q}$ are the elliptic curves. Hence, to determine if a curve of genus 1 has rational points one only has to check if it is elliptic. Any elliptic curve can be written on Weierstrass normal form:

$$y^2 = x^3 + ax + b. \tag{3.9}$$

To test if a curve is elliptic in practice one could ask a high-level language, such as Maple, to try to write the given curve on this form.

For integer solutions, we have Siegel's theorem, that states for curves with genus ≥ 1 there are only finitely many integer solutions [24].

4 Masses and Higgs physics

The advantage of this procedure over the elementary interpretation of the Dirac equations is that there is no reason to presume the existence of antineutrons or antineutrinos.
-Ettore Majorana [25]

In the Standard Model (SM), all fermions and massive vector bosons obtain their masses via the Higgs mechanism [26, 27, 28, 29]. Within the SM framework neutrino masses are usually not treated, even though we know since many years that neutrinos are in fact massive [30, 31, 32]. In 2012 the ATLAS [33] and CMS [34] experiment discovered a new resonance around 125 GeV with properties resembling the expected properties of a Higgs boson with that mass.

The electroweak sector of the SM is invariant under the $SU(2)_L \times U(1)_Y$ gauge group and the Higgs mechanism provides a way of keeping the structure of these gauge interactions invariant at high energies while also generating the masses for the W and Z bosons. The Higgs mechanism is generated by a self-interacting complex scalar doublet whose CP-even neutral component is the observed Higgs field which acquires a vacuum expectation value (VEV) of $v \approx 246$ GeV.

In the SM, the scalar $SU(2)_L$ doublet Φ generating the Higgs mechanism has the potential

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \tag{4.1}$$

For this potential to have a well-defined global minimum λ has to be greater than zero, and moreover, if $\mu^2 > 0$ there is only one minimum obtained at $\Phi = 0$ but if $\mu^2 < 0$ there is a parametrized family of minima. All the minima for $\mu^2 < 0$ are equivalent, but a convenient choice of minimum is

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{4.2}$$

Since Φ is a complex doublet, it has four real degrees of freedom, three of which may be gauged away (these three appears as longitudinal degrees of freedom for the $W^\pm$ and Z bosons) while the fourth is the physical Higgs field. In this gauge (the unitary gauge) the doublet is simply written as

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{4.3}$$

where $h(x)$ is the physical Higgs boson.

The Higgs mechanism is not only responsible for generating the masses of the electroweak gauge bosons but it also generates the masses of the fermions via Yukawa interactions. We denote the weak eigenstates of the SM fermions as

$$Q_L^i = \begin{pmatrix} U_L \\ D_L \end{pmatrix}^i, \quad L_L^i = \begin{pmatrix} \nu_L \\ E_L \end{pmatrix}^i, \quad U_R^i, \quad D_R^i, \quad E_R^i. \tag{4.4}$$

Using this notation the Yukawa interactions may be written as

$$-\mathcal{L}_Y = \overline{Q}_L^i \tilde{\Phi} Y_{ij}^U U_R^j + \overline{Q}_L^i \Phi Y_{ij}^D D_R^j + \overline{L}_L^i \Phi Y_{ij}^L E_R^j + \text{h.c.} \tag{4.5}$$

where $\tilde{\Phi} = i\sigma_2\Phi^\dagger$ is the Lorentz invariant conjugate, and $Y_{ij}^U,\ Y_{ij}^D,\ Y_{ij}^L$ are 3×3 matrices containing the Yukawa couplings. These interactions generate mass matrices looking as

$$M_{ij}^F = \frac{v}{\sqrt{2}} Y_{ij}^F, \quad F = U, D, L. \tag{4.6}$$

These mass matrices are in general complex and non-Hermitian, therefore, they have to be diagonalized via biunitary transformations. For the quarks we have

$$m^U = V_L^U M^U V_R^{U\dagger} \tag{4.7}$$
$$m^D = V_L^D M^D V_R^{D\dagger} \tag{4.8}$$

where m^U and m^D are diagonal matrices containing the physical masses, e.g. $m_u = m_{11}^U$. Since M^U and M^D are diagonalized by different biunitary transformations and the up- and down-sector are mixed in weak interactions, the electroweak currents do not have to be invariant under biunitary transformations. One can easily check that the electromagnetic and neutral currents are invariant under these transformations. However, the charged current gets changed by $V_{\text{CKM}} = V_L^U V_L^{D\dagger}$ known as the Cabibbo-Kobayashi-Maskawa (CKM) matrix. This matrix encodes couplings between the different quark generations via $W^\pm$ bosons.

A similar matrix, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, exists in the lepton sector and encodes neutrino oscillations (we discuss neutrino masses in detail in Section 4.3). These matrices are roughly [35]:

$$V_{\text{CKM}} \approx \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}, \quad V_{\text{PMNS}} \approx \begin{pmatrix} 0.85 & 0.50 & 0.17 \\ 0.35 & 0.60 & 0.70 \\ 0.35 & 0.60 & 0.70 \end{pmatrix} \tag{4.9}$$

where the elements are the absolute values of each element.

As seen above, the CKM matrix is nearly diagonal and it can be parametrized by the four parameters $\lambda,\ A,\ \rho$ and η in the following way

$$V_{\text{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{4.10}$$

which is known as the Wolfenstein parameterization [36]. The parameters are

$$\lambda = \sin\theta_c = 0.225,\ A = 0.81,\ \rho = 0.13 \text{ and } \eta = 0.35. \tag{4.11}$$

4.1 Froggatt-Nielsen mechanism

The observed mass hierarchy among quarks and leptons are not necessarily a problem, the Yukawa couplings could be fundamental constants with values we just have to accept, or their hierarchy could be an indication of new physics. In this thesis we use the latter alternative and thus search for some new physics capable of explaining the observed hierarchy. This is done by an extended version of the Froggatt-Nielsen (FN) mechanism [2]. In their original work Froggatt and Nielsen tried to explain the mass hierarchies using a spontaneously broken global $U(1)$ symmetry. Together with this symmetry one also have to assume the existence of many super heavy vector-like[2] fermions with different values of the new "flavon" charge associated with the new $U(1)$ symmetry. At the high energy scale where these heavy FN-fermions live (we denote this scale Λ_{FN}) the observed particles are effectively massless and what we observe are just low energy tails of the physics defined at this high scale. The heavy FN-fermions attain their masses through a Higgs mechanism with a neutral Higgs scalar Φ' whose dynamics we do not specify further. Now, assume that the observed mass hierarchies are generated by a symmetry breaking mechanism caused by a scalar field S ("flavon") with flavon charge 1 attaining a VEV.

We denote the $U(1)$-charges Q_i, u_i, d_i, L_i, e_i and H where Q and L denotes the left-handed quark and lepton doublets while u, d and e denotes the right-handed quark and lepton singlets. Typical examples of the FN-mechanism are illustrated in Fig. 3. In Fig. 3(a) a right-handed down-type quark interacts with the SM Higgs doublet, but instead of transforming into a right-handed quark as in a SM Yukawa interaction, a heavy FN-fermion with mass $\Lambda_{FN} \sim \langle \Phi' \rangle$ is generated. This fermion interacts with the flavon field and then a right-handed quark is created. Evolving this process down to observable energy levels, the low energy tail of the Froggatt-Nielsen interactions may be described by the symmetry breaking parameter:

$$\epsilon = \frac{\langle S \rangle}{\Lambda_{FN}} \approx 0.2 \tag{4.12}$$

where $\epsilon \approx 0.2$ is choosen so we can identify the Wolfenstein parameter λ with ϵ. For the $U(1)$-charge to be conserved in the process shown in Fig. 3(a), we realize that the $U(1)$-charges must satisfy $Q_i - d_j - H = 1$ since we have the insertion of one flavon field. In Fig. 3(b) we have the insertion of one conjugated flavon field and the charges must satisfy $Q_i - u_j + H = -1$. We denote the number flavon insertions in a process n_{ij}, so in the above cases we have $Q_i - u_j + H = n_{ij}$ and $Q_i - d_j - H = n_{ij}$. There is some ambiguity in the choice of S or S^*. We get around this by using S as our default field for insertions, and if this leads to $n_{ij} < 0$ we interpret this as we should insert S^* instead.

This means that the conservation of $U(1)$-charge determines the number of inserted flavons. Since each inserted flavon generates a factor ϵ at low energies, this process may be used to generate the observed hierarchy among the Yukawa couplings in the SM. The Yukawa interactions in Eq. (4.5) may now be expressed as

$$Y_{ij}^F = g_{ij}^F \epsilon^{|n_{ij}|} \tag{4.13}$$

[2]This means that the FN fermions do not contribute to the anomalies.

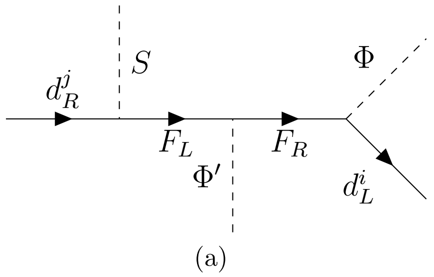


Figure 3: The Froggatt-Nielsen mechanism with one flavon insertion in the quark sector. For the flavon charges in (a) to be conserved they must satisfy $d_j + H + n_{ij} = Q_i$ where $n_{ij} = 1$ in this case. In (b) we insert the conjugate field S^* instead (just as an example). For this process, the charges must satisfy $u_j - H + n_{ij} = Q_i$ with $n_{ij} = -1$.

where g^F_{ij} are complex random constants of order one and $|n_{ij}|$ is the number of flavon insertions needed for charge conservation. Considering a straight forward generalization of the process in Fig. 3 gives the following expressions for the necessary number of insertions for charge conservation:

$$\begin{aligned} \mathrm{Up-quarks:} \quad & n_{ij} = Q_i - u_j + H, & (4.14)\\ \mathrm{Down-quarks:} \quad & n_{ij} = Q_i - d_j - H, & (4.15)\\ \mathrm{Leptons:} \quad & n_{ij} = L_i - l_j - H. & (4.16) \end{aligned}$$

These expressions are derived with insertions of S in mind. If n_{ij} becomes negative, it means that we insert S^* instead.

From the Wolfenstein parameterization of the CKM matrix in Eq. (4.10) it is clear that, if we identify λ with ϵ, it has the leading order structure

$$\begin{pmatrix} 1 & \epsilon & \epsilon^3 \\ \epsilon & 1 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}. \tag{4.17}$$

Froggatt and Nielsen showed that the entries of the CKM matrix goes as

$$V^{ij}_{\mathrm{CKM}} \sim \epsilon^{|Q_i - Q_j|}. \tag{4.18}$$

So to reproduce the CKM matrix we need the three constraints:

$$|Q_1 - Q_2| = 1, \quad |Q_2 - Q_3| = 2, \quad |Q_1 - Q_3| = 3. \tag{4.19}$$

Or, if we impose some ordering on the Q_i charges, say $Q_1 < Q_2 < Q_3$, it is enough with the two constraints:

$$Q_2 - Q_1 = 1 \text{ and } Q_3 - Q_2 = 2. \tag{4.20}$$

Now going to the PMNS matrix (Eq. (4.9)), which will appear when the SM is extended with massive neutrinos, we note that there is no clear ϵ-structure and all elements are almost of order one. For the elements to not have any ϵ-suppression we impose the constraints

$$L_2 - L_1 = 0 \text{ and } L_3 - L_2 = 0. \tag{4.21}$$

The parameters in the PMNS matrix are therefore only determined by the random coefficients g_{ij}.

4.2 2HDM

The Higgs mechanism in the SM is mediated by the simplest possible scalar sector, one $SU(2)$ doublet. Since one scalar doublet is sufficient to explain all observed data there is no real reason to extend the scalar sector to contain more fields. However, we do not know of any fundamental reason for there to only be one scalar doublet, hence there could very well be more scalar fields that we have yet to observe. One of the simplest possible extension of the SM is the two-Higgs-doublet-model (2HDM) [37], which is the SM with one extra scalar doublet, see ref. [38] for a review.

Denote the two Higgs doublets Φ_1 and Φ_2, then a general renormalizable potential can be written as

$$\begin{aligned} V =& m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - (m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c}) \\ &+ \frac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \frac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\ &+ \left\{ \frac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \left[\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)\right] (\Phi_1^\dagger \Phi_2) + \text{h.c} \right\} \end{aligned} \tag{4.22}$$

where m_{11}^2, m_{22}^2 and $\lambda_{1,2,3,4}$ are real while m_{12}^2 and $\lambda_{5,6,7}$ may be complex which may cause CP-violation. The VEVs of the two Higgs fields are given by

$$\langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} e^{i\theta_1} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} e^{i\theta_2} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \tag{4.23}$$

where it is also useful to define $\tan\beta = v_2/v_1$. A particularly useful basis is the Higgs basis where only one of the Higgs fields acquires a VEV. This basis is obtained from the generic basis by a rotation of angle β:

$$\begin{aligned} H_1 &= \cos\beta \Phi_1 + \sin\beta e^{-i\theta} \Phi_2, \\ H_2 &= -\sin\beta \Phi_1 + \cos\beta e^{-i\theta} \Phi_2, \end{aligned} \tag{4.24}$$

where $\theta = \theta_2 - \theta_1$. By defining $v^2 = v_1^2 + v_2^2$ the VEVs in the Higgs basis may be written:

$$\langle H_1 \rangle = \frac{1}{\sqrt{2}} e^{i\theta_1} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle H_2 \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.25}$$

The two Higgs fields are complex doublets so in total there are eight real degrees of freedom; three of which are the longitudinal degrees of freedom of the weak bosons and the remaining five correspond to physical Higgs bosons: three neutral and one charged pair.

4.2.1 Yukawa sector

The Yukawa interactions in 2HDM are given by

$$\begin{aligned}-\mathcal{L}_Y =&\overline{Q}_L\tilde{\Phi}_1\eta_1^U U_R + \overline{Q}_L\Phi_1\eta_1^D D_R + \overline{L}_L\Phi_1\eta_1^L E_R \\ &+ \overline{Q}_L\tilde{\Phi}_2\eta_2^U U_R + \overline{Q}_L\Phi_2\eta_2^D D_R + \overline{L}_L\Phi_2\eta_2^L E_R + \text{h.c.}\end{aligned} \tag{4.26}$$

where η_i^F are the Yukawa couplings for $F = U, D, L$. This expression for the Yukawa couplings is given in a generic basis, if we rotate it to the Higgs basis using Eq. (4.24) we obtain

$$\begin{aligned}-\mathcal{L}_Y =&\overline{Q}_L\tilde{H}_1\kappa_0^U U_R + \overline{Q}_L H_1\kappa_0^D D_R + \overline{L}_L H_1\kappa_0^L E_R \\ &\overline{Q}_L\tilde{H}_2\rho_0^U U_R + \overline{Q}_L H_2\rho_0^D D_R + \overline{L}_L H_2\rho_0^L E_R + \text{h.c.}\end{aligned} \tag{4.27}$$

where the new Yukawa matrices are given by

$$\begin{aligned}\kappa_0^U &= \cos\beta\ \eta_1^U + \sin\beta\left(e^{-i\theta}\eta_2^U\right), \\ \kappa_0^D &= \cos\beta\ \eta_1^D + \sin\beta\left(e^{i\theta}\eta_2^D\right), \\ \kappa_0^L &= \cos\beta\ \eta_1^L + \sin\beta\left(e^{i\theta}\eta_2^L\right),\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\rho_0^U &= -\sin\beta\ \eta_1^U + \cos\beta\left(e^{-i\theta}\eta_2^U\right), \\ \rho_0^D &= -\sin\beta\ \eta_1^D + \cos\beta\left(e^{i\theta}\eta_2^D\right), \\ \rho_0^L &= -\sin\beta\ \eta_1^L + \cos\beta\left(e^{i\theta}\eta_2^L\right).\end{aligned} \tag{4.29}$$

In this basis the H_2 field has no VEV, meaning that the fermion masses are generated by the couplings to the H_1 field. As in the Yukawa sector in the SM the Yukawa couplings are in general not diagonal and thus have to be diagonalized to yield the masses. We denote the diagonalized Yukawa matrices κ^F and these are obtained via a bi-unitary transformation:

$$\kappa^F = V_L^F\kappa_0^F V_R^{F\dagger}. \tag{4.30}$$

The bare mass of, for example, the up quark is thus given by

$$m_u = \frac{v}{\sqrt{2}}\kappa_{11}^U \tag{4.31}$$

and similar expressions yield the masses for the remaining fermions. Now, applying the same bi-unitary transformation to the ρ-matrices does not in general diagonalize them, i.e.

$$\rho^F = V_L^F\rho_0^F V_R^{F\dagger} \tag{4.32}$$

are in general not diagonal. This leads to flavor changing neutral currents (FCNC) at tree level which are heavily suppressed in the SM and by experiments. On the other hand,

Type	U_R	D_R	L_R	ρ^U	ρ^D	ρ^L
I	$+$	$+$	$+$	$\kappa^U \cot\beta$	$\kappa^D \cot\beta$	$\kappa^L \cot\beta$
II	$+$	$-$	$-$	$\kappa^U \cot\beta$	$-\kappa^D \tan\beta$	$-\kappa^L \tan\beta$
III/Y	$+$	$-$	$+$	$\kappa^U \cot\beta$	$-\kappa^D \tan\beta$	$\kappa^L \cot\beta$
IV/X	$+$	$+$	$-$	$\kappa^U \cot\beta$	$\kappa^D \cot\beta$	$-\kappa^L \tan\beta$

Table 2: Different types of $\mathbb{Z}_2$-symmetric 2HDM models where Φ_1 is assumed to be $\mathbb{Z}_2$-even (i.e. have $\mathbb{Z}_2$-charge $+1$) and Φ_2 is assumed to be $\mathbb{Z}_2$-odd. The right-handed fermions' $\mathbb{Z}_2$-charges are given in the F_R columns and the ρ^F matrices in the three last columns. Note that with a $\mathbb{Z}_2$-symmetry $\rho^F \propto \kappa^F$ which of course means that they are simultaneously diagonalizable and thus there are no FCNC.

flavor changing charged currents, are well known and, as in the SM, described by the CKM matrix

$$V_{CKM} = V_L^U V_L^{D\dagger}. \tag{4.33}$$

One way of suppressing the FCNC is to construct the theory such that only one of the Higgs doublet couples to each type of fermion [39]. This can be achieved by imposing a $\mathbb{Z}_2$ symmetry on the theory, meaning that one of the Higgs doublets and some of the fermions are $\mathbb{Z}_2$-odd while the other are $\mathbb{Z}_2$-even. The different models are summarized in Table 2 where we also note that the Type-II model is the same Higgs sector as in the Minimal Supersymmetric Standard Model (MSSM).

4.3 Neutrino masses

In the Standard Model neutrinos are taken to be massless, however, a series of experiments have now established the occurrence of neutrino oscillations implying that the neutrinos are massless. The SM must therefore be extended to include neutrino masses. One way of doing this is to let them get masses through the Higgs mechanism. Even though the neutrino masses are not known, one may, from cosmological data and baryon acoustic oscillations obtain an upper limit on the sum of neutrino masses[3] [40, 41]:

$$\sum_{i=1}^{3} m_i < 0.170 \text{ eV}, \quad 95\% \text{ CL}. \tag{4.34}$$

This total mass is many orders of magnitude smaller than any of the other masses generated by the Higgs. It may therefore be the case that the neutrino masses are generated by a different mechanism. An appealing way of obtaining the small neutrino masses is the seesaw mechanism. For this mechanism to work it is necessary that neutrinos are Majorana fermions, i.e. they are their own anti-particles.

For the neutrinos to have masses generated by the Higgs mechanism there has to exist right-handed chiral states. If these right-handed neutrinos exist they have to be sterile in

[3] This limit might be too restrictive depending on the exact mass generating process.

all SM gauge interactions. However, as first discovered by Majorana [25], it is possible to construct mass terms using only left-handed (or only right-handed) chiral states. This comes from the observation that the charge-conjugate field $\psi_L^c = C\overline{\psi}_L^T$ transforms like a right-handed particle where $\psi_L = P_L\psi = 1/2(1-\gamma^5)\psi$. We can now define the Majorana field as

$$\psi = \psi_L + \psi_R = \psi_L + C\overline{\psi}_L^T = \psi_L + \psi_L^c \tag{4.35}$$

which has the important implication that $\psi^c = \psi$, i.e., a Majorana particle is its own anti-particle. A Majorana mass term constructed from the left-handed chiral neutrinos is written as

$$\mathcal{L}_{ML} = -\frac{1}{2}M_L\overline{\nu_L^c}\nu_L + \text{h.c.} \tag{4.36}$$

and similarly for a right-handed state:

$$\mathcal{L}_{MR} = -\frac{1}{2}M_R\overline{\nu_R^c}\nu_R + \text{h.c.} \tag{4.37}$$

With both left- and right-handed Majorana neutrinos we may also have Dirac masses:

$$\mathcal{L}_D = -m_D\overline{\nu_R}\nu_L + \text{h.c.} = -\frac{1}{2}m_D(\overline{\nu_R}\nu_L + \overline{\nu_L^c}\nu_R^c) + \text{h.c.} \tag{4.38}$$

where we have introduced the mass terms for the charge-conjugated fields. This must be the same mass as $\overline{\nu_R}\nu_L$ since the total fields are $\nu_L + \nu_L^c$ and $\nu_R^c + \nu_R$. The total mass term for the neutrinos is now given by

$$\begin{aligned}\mathcal{L}_\nu &= \mathcal{L}_{ML} + \mathcal{L}_{MR} + \mathcal{L}_D \\ &= -\frac{1}{2}M_L\overline{\nu_L^c}\nu_L - \frac{1}{2}M_R\overline{\nu_R^c}\nu_R - \frac{1}{2}m_D(\overline{\nu_R}\nu_L + \overline{\nu_L^c}\nu_R^c) + \text{h.c.} \\ &= -\frac{1}{2}\begin{pmatrix}\overline{\nu_L^c} & \overline{\nu_R}\end{pmatrix}\begin{pmatrix}M_L & m_D \\ m_D & M_R\end{pmatrix}\begin{pmatrix}\nu_L \\ \nu_R^c\end{pmatrix} + \text{h.c.}\end{aligned} \tag{4.39}$$

Since ν_L carries hypercharge, $M_L = 0$ for the SM neutrinos. In this case, the mass matrix (for one generation) is equal to

$$M = \begin{pmatrix}0 & m_D \\ m_D & M_R\end{pmatrix} \tag{4.40}$$

which, if $M_R >> m_D$, has the two approximate eigenvalues

$$m_1 \approx \frac{m_D^2}{M_R}, \quad m_2 \approx M_R \tag{4.41}$$

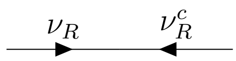
where $m_1 << m_2$. This is the seesaw mechanism which in a natural way explains the smallness of the neutrino mass. So if there exist some heavy right-handed neutrino states, it becomes natural for the other neutrino states to be extremely light. The physical neutrinos may now be written

$$\nu \approx (\nu_L + \nu_L^c) - \frac{m_D}{M_R}(\nu_R + \nu_R^c), \quad N \approx (\nu_R + \nu_R^c) + \frac{m_D}{M_R}(\nu_L + \nu_L^c) \tag{4.42}$$

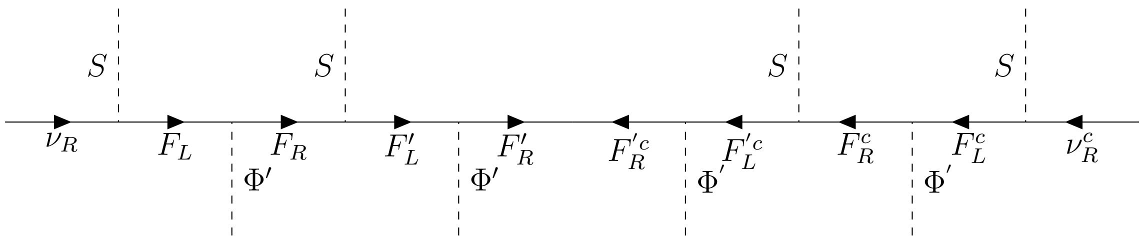
where the light physical neutrino, ν, almost entirely consists of the Majorana field constructed from the left-handed states, $\nu_L + \nu_L^c$, whereas the heavy physical neutrino, N, almost entirely consists of the Majorana field constructed by the right-handed states. This means that the **physical states are Majorana particles**. The above discussion generalizes directly to multiple neutrino flavors with m_D and M_R becoming matrices.

Important for the possibility of Majorana masses is that the fields carry no quantum number that has to be conserved. The only quantum numbers that are truly conserved in the SM are the gauge charges, so for

$$\overline{\nu_R^c}\nu_R = \tag{4.43}$$



to be a fundamental vertex, ν_R must have all gauge charges equal to zero. However, we will later use these right-handed states to cancel gauge anomalies by giving them a charge under a new gauge group $U(1)'$. This means that this fundamental vertex will not conserve this new charge; the flavon charge. If ν_R has integer flavon charge, we may couple it to the Froggatt-Nielsen mechanism. This means that $\overline{\nu_R^c}\nu_R$ would be a low-energy effective operator comming from something looking like:



where ν_R has flavon charge -2 and turns into a ν_R^c with flavon charge 2. In the middle there is a true Majorana fermion; the Froggat-Nielsen fermion F_R', whose mass we denote M_{FN}.

Later we will use three right-handed neutrinos ν_R^i, $i = 1, 2, 3$, with integer flavon charges ν_i. The effective Majorana mass matrix is then given by

$$M_R^{ij} = M_{FN}\epsilon^{|\nu_i|+|\nu_j|}, \quad i, j = 1, 2, 3 \tag{4.44}$$

where $\epsilon = \langle S\rangle / \Lambda_{FN}$.

5 Extending the SM gauge group

Gauge symmetries, both "global" (i.e. space-time independent) and "local" (space-time dependent) became key issues in elementary particles. [...] The guiding principles in this work were symmetry and elegance of the magnificent edifice that we call our universe. And the most important symmetry was gauge symmetry.

-Gerard 't Hooft [42]

Extending the SM gauge group $SU(3)_{QCD} \times SU(2)_L \times U(1)_Y$ with an additional spontaneously broken $U(1)'$ symmetry generates a new electrically neutral color-singlet gauge boson Z'. In general, when several $U(1)$ groups are present kinetic mixing between the gauge bosons becomes possible, however, at tree-level this mixing can be rotated away at any given scale. Thus, kinetic mixing only has to be dealt with at loop-level.

The way the extra gauge group was added above is not unique; there are two options for the group structure and the symmetry breaking. One way is to start with the gauge group $SU(3)_{QCD} \times SU(2)_W \times U(1)_Y \times U(1)'$ and break $U(1)'$ at some high scale while breaking $SU(3)_{QCD} \times SU(2)_W \times U(1)_Y \to SU(3)_{QCD} \times U(1)_{EM}$ at the electroweak symmetry breaking (EWSB) scale as in the SM. Another way is to consider the group $SU(3)_{QCD} \times SU(2)_W \times U(1)_1 \times U(1)_2$ and first break $U(1)_1 \times U(1)_2 \to U(1)_Y$ at some high scale and then proceed by electroweak symmetry breaking (EWSB). However, these two possibilities are always equivalent by redefining the gauge fields and couplings [43].

Here we follow ref. [43], generalized to two Higgs doublets. The new gauge group $U(1)'$ is spontaneously broken by a complex scalar singlet S acquiring a VEV v_S. We will later take this field to be the flavon field mediating the Froggatt-Nielsen (FN) mechanism [2]. By redefining the $U(1)'$ coupling g_Z we may set the charge of S under $U(1)'$ to be 1. We also extend the Higgs sector to contain two Higgs doublets $\Phi_{1,2}$ with charges $H_{1,2}$ under $U(1)'$, so we have a 2HDM as discussed in Section 4.2. After symmetry breaking we thus have mixing between Z and Z' bosons. The kinetic terms for S and $\Phi_{1,2}$ are obtained from the covariant derivatives;

$$\left|\left(\partial^\mu - i\frac{g}{2}W^\mu - i\frac{g'}{2}B_Y^\mu - iH_{1,2}\frac{g_Z}{2}B_Z^\mu\right)\Phi_{1,2}\right|^2 + \left|\left(\partial^\mu - i\frac{g_Z}{2}B_Z^\mu\right)S\right|^2 \tag{5.1}$$

where W^μ, B_Y^μ and B_Z^μ are the gauge fields with couplings g, g' and g_Z associated with $SU(2)_W$, $U(1)_Y$ and $U(1)'$ respectively. Denoting the VEV for the Higgs doublets v_1 and v_2 respectively, the mass terms for the different fields, except $W^\pm$, are

$$\frac{v_1^2}{8}\left(gW^{3\mu} - g'B_Y^\mu - H_1 g_Z B_Z^\mu\right)^2 + \frac{v_2^2}{8}\left(gW^{3\mu} - g'B_Y^\mu - H_2 g_Z B_Z^\mu\right)^2 + \frac{v_S^2}{8}g_Z^2 B_Z^\mu B_{Z\mu} \tag{5.2}$$

after EWSB where $v_1^2 + v_2^2 = (246\text{GeV})^2$. Introducing the quotient $\tan\beta = v_2/v_1$ makes it possible to express the Higgs' VEVs as $v_1 = v\cos\beta$ and $v_2 = v\sin\beta$. The mass-square matrix, which is twice the above expression, may now be written as:

$$M^2 = \frac{g^2 v^2}{4\cos\theta_W} \times U^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\cos^2\beta(H_1 + H_2\tan^2\beta)t_z\cos\theta_W \\ 0 & -\cos^2\beta(H_1 + H_2\tan^2\beta)t_z\cos\theta_W & (r + \cos^2\beta[H_1^2 + H_2^2\tan^2\beta])t_z^2\cos^2\theta_W \end{pmatrix} U \tag{5.3}$$

where

$$U = \begin{pmatrix} \cos\theta_W & \sin\theta_W & 0 \\ -\sin\theta_W & \cos\theta_W & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.4}$$

and $t_z = g_Z/g$, $\tan\theta_W = g'/g$, $r = v_S^2/v^2$. If either of the Higgs have non-zero $z_{1,2}$ the diagonalization of the mass-square matrix will cause $Z - Z'$ mixing. This mixing can be characterized by an angle θ'. Together with the Weinberg angle θ_W defined above, the gauge fields can be written in terms of the physical fields as

$$\begin{pmatrix} B_Y^\mu \\ W^{3\mu} \\ B_Z^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W\cos\theta' & \sin\theta_W\sin\theta' \\ \sin\theta_W & \cos\theta_W\cos\theta' & -\cos\theta_W\sin\theta' \\ 0 & \sin\theta' & \cos\theta' \end{pmatrix} \begin{pmatrix} A^\mu \\ Z^\mu \\ Z'^\mu \end{pmatrix} \tag{5.5}$$

where A is the massless photon, Z the observed massive boson and Z' the unobserved heavy boson. From the diagonalization we get that the mixing angle must satisfy

$$\tan 2\theta' = \frac{2\cos^2\beta(H_1 + H_2\tan^2\beta)t_z\cos\theta_W}{(r + \cos^2\beta[H_1^2 + H_2^2\tan^2\beta])t_z^2\cos^2\theta_W - 1} \tag{5.6}$$

and that the massive bosons' masses are given by:

$$M_{Z,Z'} = \frac{gv}{2\cos\theta_W} \times \left[\frac{1}{2}[(r + \cos^2\beta[H_1^2 + H_2^2\tan^2\beta])t_z^2\cos^2\theta_W + 1] \pm \frac{\cos^2\beta(H_1 + H_2\tan^2\beta)t_z\cos\theta_W}{\sin 2\theta'}\right]^{1/2}. \tag{5.7}$$

The M_Z mass is given by taking the minus sign in the above expression. If $(r+\cos^2\beta[H_1^2 + H_2^2\tan^2\beta])t_z^2\cos^2\theta_W > 1$ then $M_{Z'} > M_Z$ and in the opposite case $M_{Z'} < M_Z$.

We see here that the mixing disapears in the limit $r >> 1$, i.e. $v_S^2 >> v^2$, or in the limit $t_z << 1$, i.e. $g_Z << g$, and in this case $M_{Z'} = g_Z v_S/2$.

5.1 Running of the coupling constant

An important theoretical constraint on Z' physics is the possible existence of low-lying Landau poles. The renormalization group equation (RGE) at one-loop for the $U(1)'$ coupling

g_Z is

$$\frac{d\alpha_Z}{d\log\mu^2} = b\alpha_Z^2 \tag{5.8}$$

where $\alpha_Z(\mu^2) = g_Z^2(\mu^2)/4\pi$ and b is the beta function containing the charges:

$$b = \frac{1}{4\pi}\left[\frac{2}{3}\sum_{i=1}^{3}\left(Q_i^2 + u_i^2 + d_i^2 + L_i^2 + e_i^2 + \nu_i^2\right) + \frac{1}{3}\left(2\sum_{i=1}^{2} H_i^2 + 1^2\right)\right] \tag{5.9}$$

where we sum all the fermion and Higgs charges and the term 1^2 is from the flavon [44]. Solving the RGE yields a Landau pole at

$$\Lambda_{LP} = M_{Z'}\exp\left[\frac{1}{2b\alpha_Z(M_{Z'}^2)}\right]. \tag{5.10}$$

This contrains the allowed flavon charges for given Z' parameters.

6 Examples of models

The first point is that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it. Second, it is just this uncanny usefulness of mathematical concepts that raises the question of the uniqueness of our physical theories.

-Eugene Wigner [45]

In this section we give examples of different models, both anomalous and anomaly-free, which to varying degree reproduce the observed masses and mixings. Using Dirac neutrinos in Section 6.3.2 we manage to find an anomaly-free model describing all the observed masses and mixing except mixing with the third generation in the quark sector.

To derive the Froggatt-Nielsen constraints we use the fermion masses at the Z-mass from ref. [46]:

$$\begin{aligned} &m_u = 1.27 \text{ MeV} \sim \epsilon^7, \; m_d = 2.9 \text{ MeV} \sim \epsilon^7, \;\; m_s = 55 \text{ MeV} \sim \epsilon^5, \\ &m_c = 0.62 \text{ GeV} \sim \epsilon^3, \; m_b = 2.9 \text{ GeV} \sim \epsilon^2, \;\; m_t = 172 \text{ GeV} \sim \epsilon^0, \\ &m_e = 0.5 \text{ MeV} \sim \epsilon^8, \;\; m_\mu = 103 \text{ MeV} \sim \epsilon^5, \; m_\tau = 1746 \text{ MeV} \sim \epsilon^3. \end{aligned} \tag{6.1}$$

Since we are using a 2HDM model the VEVs that go together with the ϵ factors are not 174 GeV as in the SM but rather $\langle \Phi_1 \rangle = \cos\beta \, \langle \Phi \rangle$ and $\langle \Phi_2 \rangle = \sin\beta \, \langle \Phi \rangle$ where Φ is the SM Higgs doublet and $\Phi_{1,2}$ the 2HDM doublets. If we take $\tan\beta = 1$, then $\cos\beta = \sin\beta = 1/\sqrt{2} \approx 0.7$ which does not change the powers of ϵ from the SM. We will assume this choice of $\tan\beta$ throughout this section. This is in no way a limitation of the method, one may use any $\tan\beta$ as one likes, as long as the powers of ϵ are adjusted accordingly.

6.1 Anomaly-free model with irrational charges

In this model we only use the SM particle content together with two Higgs doublets. For the theory to be anomaly-free the $U(1)'$ charges of the fermions must satisfy the anomaly conditions in Eq. (2.66), for the model at hand they are given by:

$$\begin{cases} \mathcal{A}_{1'1'1} : \; \sum_{j=1}^{3} (Q_j^2 - 2u_j^2 + d_j^2 - L_j^2 + e_j^2) = 0 \\ \mathcal{A}_{1'11} : \; \sum_{j=1}^{3} (Q_j - 8u_j - 2d_j + 3L_j - 6e_j) = 0 \\ \mathcal{A}_{331'} : \; \sum_{j=1}^{3} (2Q_j - u_j - d_j) = 0 \\ \mathcal{A}_{221'} : \; \sum_{j=1}^{3} (3Q_j + L_j) = 0 \\ \mathcal{A}_{1'1'1'} : \; \sum_{j=1}^{3} (6Q_j^3 - 3u_j^3 - 3d_j^3 + 2L_j^3 - e_j^3) = 0 \\ \mathcal{A}_{gg1'} : \; \sum_{j=1}^{3} (2L_j - e_j) = 0 \end{cases} \tag{6.2}$$

The Froggatt-Nielsen conditions are then given by:

$$\begin{aligned}
m_u: &\quad (Q_1 - u_1 + H_2) + 7 = 0 \\
m_c: &\quad (Q_2 - u_2 + H_2) - 3 = 0 \\
m_t: &\quad (Q_3 - u_3 + H_2) \qquad = 0 \\
\\
m_d: &\quad (Q_1 - d_1 - H_2) - 7 = 0 \\
m_s: &\quad (Q_2 - d_2 - H_2) + 5 = 0 \\
m_b: &\quad (Q_3 - d_3 - H_2) - 2 = 0 \\
\\
m_e: &\quad (L_1 - e_1 - H_1) - 8 \ = 0 \\
m_\mu: &\quad (L_2 - e_2 - H_1) - 5 \ = 0 \\
m_\tau: &\quad (L_3 - e_3 - H_1) - 3 \ = 0
\end{aligned} \tag{6.3}$$

Here we have imposed a lepton specific $\mathbb{Z}_2$ symmetry on the two Higgs doublets. This symmetry serves two purposes; first it removes FCNCs (Section 4.2) and secondly, it ensures that the sum of all the variables in the quark mass conditions equals the $SU(3) \times SU(3) \times U(1)'$ anomaly $\mathcal{A}_{331'}$. The sign of the charges for the quarks are then chosen so that the sum of all the coefficients equals zero:

$$+7 - 3 - 7 + 5 - 2 = 0. \tag{6.4}$$

With these choices, $\mathcal{A}_{331'}$ is naturally satisfied and thus becomes redundant. This might seem convenient at first, but it is actually necessary to impose the lepton specific $\mathbb{Z}_2$ symmetry and choose the signs of the charges in this way for the ideal not to define an empty variety. Lastly, we have to ensure that the charges reproduce the CKM matrix (Eq. (4.9)), this is done by (Eq. (4.20)):

$$\begin{aligned}
\mathrm{CKM}_1: &\quad Q_2 - Q_1 - 1 = 0 \\
\mathrm{CKM}_2: &\quad Q_3 - Q_2 - 2 = 0
\end{aligned}$$

To find a possible set of charges that satisfies these 17 constraints we calculate the Gröbner basis of the corresponding ideal using Sage [47]:

$$\begin{aligned}
I =& \langle \mathcal{A}_{1'1'1}, \mathcal{A}_{1'11}, \mathcal{A}_{331'}, \mathcal{A}_{221'}, \mathcal{A}_{1'1'1'}, \mathcal{A}_{gg1'}, m_u, m_c, m_t, m_d, m_s, m_b, \\
& m_e, m_\mu, m_\tau, \mathrm{CKM}_1, \mathrm{CKM}_2 \rangle \\
=& \langle Q_1 - 1/3 \cdot H_2 + 8/9, Q_2 - 1/3 \cdot H_2 - 1/9, Q_3 - 1/3 \cdot H_2 - 19/9, \\
& u_1 - 4/3 \cdot H_2 - 55/9, u_2 - 4/3 \cdot H_2 + 26/9, u_3 - 4/3 \cdot H_2 - 19/9, \\
& d_1 + 2/3 \cdot H_2 + 71/9, d_2 + 2/3 \cdot H_2 - 46/9, d_3 + 2/3 \cdot H_2 - 1/9, \\
& L_1 - 2/3 \cdot e_3 - 1/3 \cdot H_2 - 23/9, L_2 + 5/3 \cdot e_3 + 13/3 \cdot H_2 + 50/9, L_3 - e_3 - H_2 + 1, \\
& e_1 - 2/3 \cdot e_3 + 2/3 \cdot H_2 + 13/9, e_2 + 5/3 \cdot e_3 + 16/3 \cdot H_2 + 59/9, \\
& e_3^3 + 6 \cdot e_3^2 \cdot H_2 + 317/30 \cdot e_3^2 + 12 \cdot e_3 \cdot H_2^2 + 634/15 \cdot e_3 \cdot H_2 + 1777/90 \cdot e_3 + \\
& 8 \cdot H_2^3 + 634/15 \cdot H_2^2 + 1777/45 \cdot H_2 - 6172/45, H_1 - H_2 + 4 \rangle
\end{aligned} \tag{6.5}$$

Q_1	-8/9	d_1	-71/9	e_1	$\frac{2}{3}e_3 - 17$
Q_2	1/9	d_2	46/9	e_2	$-\frac{5}{3}e_3 + 25$
Q_3	19/9	d_3	1/9	e_3	Eq. (6.8)
u_1	55/9	L_1	$\frac{2}{3}e_3 + \frac{23}{9}$	H_1	-4
u_2	-26/9	L_2	$-\frac{5}{3}e_3 - \frac{50}{9}$	H_2	0
u_3	19/9	L_3	$e_3 - 1$	–	–

Table 3: Irrational charges obtained for the SM with 2HDM and an additional $U(1)$ gauge symmetry. This set of charges is free from anomalies and satisfies the Froggatt-Nielsen conditions for mass hierarchies and CKM matrix..

This ideal defines a one-dimensional variety which in principle could contain $\mathbb{Q}$-rational points. To check if this is the case, we study the second to last equation in the Gröbner basis:

$$e_3^3 + 6 \cdot e_3^2 \cdot H_2 + 317/30 \cdot e_3^2 + 12 \cdot e_3 \cdot H_2^2 + 634/15 \cdot e_3 \cdot H_2 + 1777/90 \cdot e_3 + \\ 8 \cdot H_2^3 + 634/15 \cdot H_2^2 + 1777/45 \cdot H_2 - 6172/45 = 0. \tag{6.6}$$

This is a non-singular curve of genus one. The Mordell-Weil theorem (Theorem 2) tells us that the set of K-rational points on Abelian varieties forms a finitely generated group. The only one-dimensional Abelian varieties are the elliptic curves, and a curve is elliptic if and only if it has genus one and one rational point. However, the above curve is non-elliptic and we can therefore not find any rational charges. Since we have no hope of finding rational points in this variety we might as well put $H_2 = 0$ which simplifies the cubic equation to

$$e_3^3 + 317/30 \cdot e_3^2 + 1777/90 \cdot e_3 - 6172/45 = 0 \tag{6.7}$$

This equation has three algebraic solutions but only one real, the real solution is:

$$\begin{aligned} e_3 =& \left(\frac{11}{16200}\sqrt{485967667}\sqrt{15} + \frac{10871773}{182250}\right)^{\frac{1}{3}} + \\ & \frac{47179}{8100\left(\frac{11}{16200}\sqrt{485967667}\sqrt{15} + \frac{10871773}{182250}\right)^{\frac{1}{3}}} - \frac{317}{90} \\ & \approx 2.56622573989773 \end{aligned} \tag{6.8}$$

All the other charges are either fractions or expressed in terms of e_3, see Table 3. Irrational charges are not frequently discussed in BSM models. This is because it becomes difficult to embed them in a larger gauge group in a GUT [48]. But if we do not consider what possible GUT this theory may come from, irrational charges are perfectly valid.

6.2 Anomalous model with rational charges

If the reader find the above result with irrational charges unsatisfactory, there are two ways to get rid of them; we either add more particles (e.g. right-handed neutrinos) or we lift one of the anomaly conditions, of which we will do the latter in this section. Lifting one of the anomaly conditions comes at a high price, there will now be an energy scale where unitarity breaks down. To maximize the chance of obtaining rational solutions we remove the cubic anomaly: $\mathcal{A}_{1'1'1'}$.[4] This yields an effective field theory below the scale [49, 50]

$$M_{Z'}\left(\frac{64\pi^3}{|g_Z^3\mathcal{A}_{1'1'1'}|}\right) \tag{6.9}$$

where $M_{Z'}$ is the mass of the new neutral boson and g_Z its coupling. Assuming that the Froggatt-Nielsen mechanism happens at a scale $\sim M_{Z'}$, we need the factor in the parenthesis to be bigger than one for the model to be valid.

Doing the calculations as above; with SM fermion content and two Higgs doublets with lepton specific $\mathbb{Z}_2$ symmetry, the allowed charges form a two-dimensional variety parametrized by e_3 and H_2:

$$\begin{aligned}&\langle Q_1 - 1/3\cdot H_2 + 8/9,\ Q_2 - 1/3\cdot H_2 - 1/9,\ Q_3 - 1/3\cdot H_2 - 19/9,\\ &u_1 - 4/3\cdot H_2 - 55/9,\ u_2 - 4/3\cdot H_2 + 26/9,\ u_3 - 4/3\cdot H_2 - 19/9,\\ &d_1 + 2/3\cdot H_2 + 71/9,\ d_2 + 2/3\cdot H_2 - 46/9,\ d_3 + 2/3\cdot H_2 - 1/9,\\ &L_1 - 2/3\cdot e_3 - 1/3\cdot H_2 - 23/9,\ L_2 + 5/3\cdot e_3 + 13/3\cdot H_2 + 50/9,\ L_3 - e_3 - H_2 + 1,\\ &e_1 - 2/3\cdot e_3 + 2/3\cdot H_2 + 13/9,\ e_2 + 5/3\cdot e_3 + 16/3\cdot H_2 + 59/9,\ H_1 - H_2 + 4\rangle\end{aligned} \tag{6.10}$$

Every point on this variety given by rational H_2 and e_3 is clearly rational, we thus have plenty of rational charges satisfying the FN conditions and all but the cubic anomaly. The only constraint left is from possible unitarity breaking, the region of allowed charges is seen in Fig. 4.

6.3 Anomaly-free model with Dirac neutrinos

6.3.1 No neutrino mixing

Instead of breaking one of the anomaly conditions we can add more fermions and try to find rational charges that way. A natural choice, that also allows us to incorporate massive neutrinos, is to add three right-handed neutrinos. In this way we may give the neutrinos Dirac masses just as all the other fermions in the SM. Since right-handed neutrinos have to be sterile with respect to the SM gauge interactions they only enter in the cubic and gravitational anomaly:

$$\begin{aligned}\mathcal{A}_{1'1'1'}:&\quad \sum_{j=1}^{3}(6Q_j^3 - 3u_j^3 - 3d_j^3 + 2L_j^3 - e_j^3 - \nu_j^3) = 0\\ \mathcal{A}_{gg1'}:&\quad \sum_{j=1}^{3}(2L_j - e_j - \nu_j) = 0\end{aligned} \tag{6.11}$$

[4] One could also argue that this anomaly should not be included in the first place. Since it only contains the unobserved Z' boson there could very well be particles only coupling to this boson and not to any SM bosons, we would then have no idea what the anomaly condition actually would look like [6].

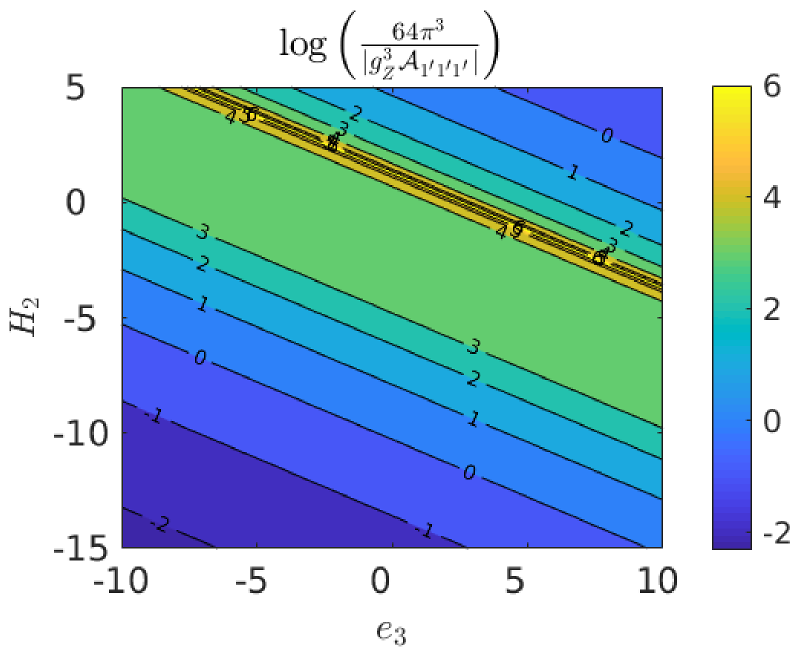


Figure 4: The logarithm of the anomaly breaking parameter with $g_Z = 0.5$. If this value is bigger than zero unitarity breaks above the FN scale and the corresponding charges are a good choice for the model.

The smallness and hierarchy of the neutrino masses are explained by a large number of flavon insertions. Assuming normal hierarchy, $m_{\nu 1} << m_{\nu 2} < m_{\nu 3}$ we get the masses $m_{\nu 3} \approx 0.0506$ eV and $m_{\nu 2} \approx 0.0086$ eV with $m_{\nu 1}$ arbitrary as long as it is much smaller than the other two. These masses corresponds to 18 and 19 flavon insertion for $m_{\nu 3}$ and $m_{\nu 2}$ respectively. We impose the FN mass constraints as:

$$\begin{aligned} m_{\nu 1}: &\quad (L_1 - \nu_1 + H_1) - 21 = 0 \\ m_{\nu 2}: &\quad (L_2 - \nu_2 + H_1) + 19 = 0 \\ m_{\nu 3}: &\quad (L_3 - \nu_3 + H_1) + 18 = 0 \end{aligned} \tag{6.12}$$

The choice of 21 insertions for m_1 is not arbitrary. Previously we chose the signs of the quarks' flavon charges such that the anomaly with $SU(3)$ became redundant. Similarly here, we chose the signs of the leptons' charges and m_1 such that the gravitational anomaly becomes redundant.

Introducing neutrino masses means that we should also introduce neutrino mixing. However, the constraints needed to reproduce the PMNS matrix conflicts with the constraint for the CKM matrix. In this section we will therefore not consider the PMNS matrix but continue to ensure the CKM matrix. In the next section however, we will only consider Cabibbo mixing in the quark sector and this allows for the PMNS matrix to be reproduced.

Doing the calculation with SM particle content plus three right-handed neutrinos with two Higgs doublets with lepton specific $\mathbb{Z}_2$ symmetry yields the following variety of charges:

$$\begin{aligned}
&\langle Q_1 - 1/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\nu_3 - 3,\ Q_2 - 1/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 - 4,\\
&Q_3 - 1/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 - 6,\\
&u_1 - 4/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 - 10,\ u_2 - 4/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 - 1,\\
&u_3 - 4/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 - 6,\\
&d_1 + 2/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 + 4,\ d_2 + 2/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 - 9,\\
&d_3 + 2/3\cdot H_2 + 1/8\cdot \nu_2 + 5/24\cdot \nu_3 - 4,\\
&L_1 + H_2 - 1/8\cdot \nu_2 - 7/8\cdot \nu_3 + 10,\ L_2 + H_2 - \nu_2 + 15,\ L_3 + H_2 - \nu_3 + 14,\\
&e_1 + 2\cdot H_2 - 1/8\cdot \nu_2 - 7/8\cdot \nu_3 + 14,\ e_2 + 2\cdot H_2 - \nu_2 + 16,\ e_3 + 2\cdot H_2 - \nu_3 + 13,\\
&H_1 - H_2 + 4,\ \nu_1 - 1/8\cdot \nu_2 - 7/8\cdot \nu_3 + 35,\\
&\nu_2^2 - 406/867\cdot \nu_2\nu_3 - 5720/867\cdot \nu_2 - 461/867\cdot \nu_3^2 + 16328/289\cdot \nu_3 - 239488/289\rangle
\end{aligned} \tag{6.13}$$

This ideal defines a two-dimensional variety of allowed charges. To check if there are any rational charges we look at the last equation in the Gröbner basis:

$$\nu_2^2 - 406/867\cdot \nu_2\nu_3 - 5720/867\cdot \nu_2 - 461/867\cdot \nu_3^2 + 16328/289\cdot \nu_3 - 239488/289 = 0 \tag{6.14}$$

This equation defines a non-singular curve of genus zero in the $\nu_2\nu_3$-plane. For curves of genus zero defined over $\mathbb{Q}$ there are only two options when it comes to the existence of rational points, either there are none, or the curve is isomorphic to the projective line $\mathbb{P}^1$. Such an isomorphism will always yield a rational parameterization of the curve. In this case, all rational points on the above curve (Fig. 5) is described by

$$(\nu_2(t), \nu_3(t)) = \left(\frac{-461t^2 + 48984t - 718464}{1328t - 43264}, \frac{867t^2 + 5720t - 718464}{1328t - 43264}\right). \tag{6.15}$$

The allowed rational charges are now described by two parameters: t in the above equation and one of the Higgs charges, say H_2 for definiteness. This model generates the mass matrices:

$$\begin{aligned}
m_u &= \begin{pmatrix} \epsilon^7 & \epsilon^2 & \epsilon^3 \\ \epsilon^6 & \epsilon^3 & \epsilon^2 \\ \epsilon^4 & \epsilon^5 & \epsilon^0 \end{pmatrix}, \quad m_d = \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^1 \\ \epsilon^8 & \epsilon^5 & \epsilon^0 \\ \epsilon^{10} & \epsilon^3 & \epsilon^2 \end{pmatrix},\\
m_e &= \begin{pmatrix} \epsilon^8 & \epsilon^{|10-\frac{7}{8}(\nu_2-\nu_3)|} & \epsilon^{|7+\frac{1}{8}(\nu_2-\nu_3)|} \\ \epsilon^{|3+\frac{7}{8}(\nu_2-\nu_3)|} & \epsilon^5 & \epsilon^{|2+(\nu_2-\nu_3)|} \\ \epsilon^{|4-\frac{1}{8}(\nu_2-\nu_3)|} & \epsilon^{|6-(\nu_2-\nu_3)|} & \epsilon^3 \end{pmatrix},\\
m_\nu &= \begin{pmatrix} \epsilon^{21} & \epsilon^{|-14-\frac{7}{8}(\nu_2-\nu_3)|} & \epsilon^{|-14+\frac{1}{8}(\nu_2-\nu_3)|} \\ \epsilon^{|16+\frac{7}{8}(\nu_2-\nu_3)|} & \epsilon^{19} & \epsilon^{|-19+(\nu_2-\nu_3)|} \\ \epsilon^{|17-\frac{1}{8}(\nu_2-\nu_3)|} & \epsilon^{|-18-(\nu_2-\nu_3)|} & \epsilon^{18} \end{pmatrix}.
\end{aligned} \tag{6.16}$$

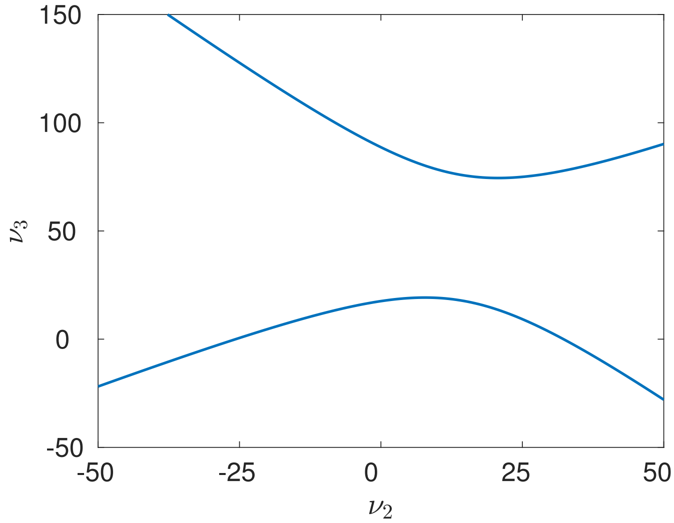


Figure 5: The charges for ν_2 and ν_3 given by Eq. (6.15).

The off-diagonal elements in the lepton matrices have in general non-integer exponents and are in that case zero. For some specific choice of t however, some of them may be integers, but there is not enough freedom to reproduce the PMNS matrix.

There are two potential problems we have yet to address; large off-diagonal elements in the mass matrices and Landau poles for g_Z. A large off-diagonal element, like the 23-element ϵ^0 in m_d above, will have too much weight when calculating the masses which causes the Froggatt-Nielsen mechanism to not work properly. That is, what is assigned to be the masses in the input might not be the mass eigenstates of the generated matrices. This can be dealt with by the g_{ij} factors in the matrices. Arguably, one now re-introduces some fine tuning in the theory, but this will still be much smaller of an adjustment than needed in the SM.

The energy scale of the Landau pole was derived in Eq. (5.10). For $t = 0$, the log-log of the Landau pole normalized to the Z' mass is shown in Fig. 6. We see in this figure that for couplings $g_Z < 0.1$ the Landau pole poses no problem.

6.3.2 Neutrino mixing with only Cabibbo mixing

As mentioned earlier it is difficult to impose constraint to ensure recreation of both the CKM and PMNS matrix. Since all elements in the PMNS matrix are relatively large while some elements in the CKM matrix are very small (Eq. (4.9)) we choose here to only impose Cabibbo mixing in the quark sector and rather ensuring the entire PMNS matrix. The lack of hierarchy in the PMNS matrix translates to the constraints

$$\begin{aligned} \mathrm{PMNS}_1: &\quad L_1 - L_2 = 0 \\ \mathrm{PMNS}_2: &\quad L_2 - L_3 = 0. \end{aligned} \tag{6.17}$$

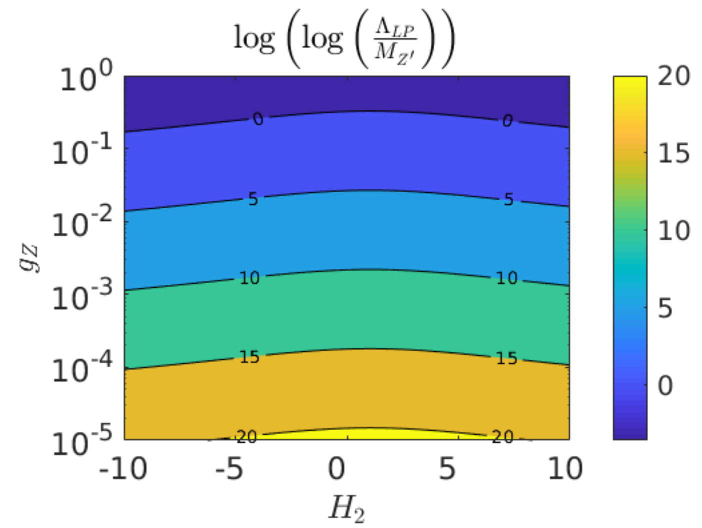


Figure 6: The energy scale of the Landau pole normalized to the Z' mass for Dirac neutrinos with no recreation of the PMNS matrix and $t = 0$ in Eq. (6.15). Note that the line at 0 corresponds to $\Lambda_{LP} = 2.7 M_{Z'}$ while the line at 5 corresponds to $\Lambda_{LP} = 2.85 \cdot 10^{64} M_{Z'}$. For couplings around 0.1 and smaller the Landau pole will never cause a problem.

Imposing this and only CKM$_1$ gives the following Gröbner basis:

$$\begin{aligned}
&\langle Q_1 - 1/3 \cdot H_2 + 2927/5484,\ Q_2 - 1/3 \cdot H2 - 2557/5484, Q_3 - 1/3 \cdot H_2 + 16637/5484,\\
&u_1 - 4/3 \cdot H_2 - 35461/5484,\ u_2 - 4/3 \cdot H_2 + 13895/5484,\ u_3 - 4/3 \cdot H_2 + 16637/5484,\\
&d_1 + 2/3 \cdot H_2 + 41315/5484,\ d_2 + 2/3 \cdot H_2 - 29977/5484,\ d_3 + 2/3 \cdot H_2 + 27605/5484,\\
&L_1 + H_2 - 5669/1828,\ L_2 + H_2 - 5669/1828,\ L_3 + H_2 - 5669/1828,\\
&e_1 + 2 \cdot H_2 + 1643/1828,\ e_2 + 2 \cdot H_2 - 3841/1828, e_3 + 2 \cdot H_2 - 7497/1828,\\
&H_1 - H_2 + 4,\ \nu_1 + 40031/1828,\ \nu_2 - 33089/1828,\ \nu_3 - 31261/1828\rangle
\end{aligned} \tag{6.18}$$

which is parametrized by one of the Higgs charges, say H_2 for definiteness. This model generates the mass matrices:

$$\begin{aligned}
m_u &= \begin{pmatrix} \epsilon^7 & \epsilon^2 & 0 \\ \epsilon^6 & \epsilon^3 & 0 \\ 0 & 0 & \epsilon^0 \end{pmatrix}, \quad m_d = \begin{pmatrix} \epsilon^7 & \epsilon^6 & 0 \\ \epsilon^8 & \epsilon^5 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix} \\
m_e &= \begin{pmatrix} \epsilon^8 & \epsilon^5 & \epsilon^3 \\ \epsilon^8 & \epsilon^5 & \epsilon^3 \\ \epsilon^8 & \epsilon^5 & \epsilon^3 \end{pmatrix}, \quad m_\nu = \begin{pmatrix} \epsilon^{21} & \epsilon^{19} & \epsilon^{18} \\ \epsilon^{21} & \epsilon^{19} & \epsilon^{18} \\ \epsilon^{21} & \epsilon^{19} & \epsilon^{18} \end{pmatrix}.
\end{aligned} \tag{6.19}$$

The Landau pole, Eq. (5.10), for this set of charges is shown in Fig. 7.

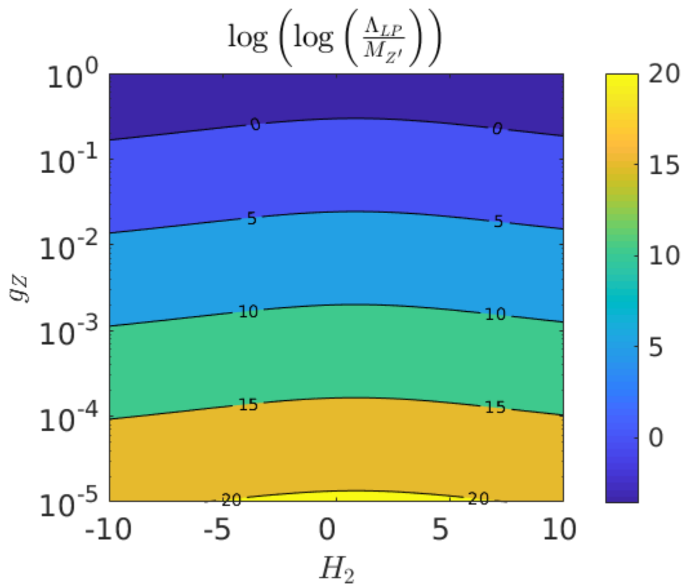


Figure 7: The energy scale of the Landau pole normalized to the Z' mass with only Cabibbo mixing.

6.4 Anomaly-free model with Majorana neutrinos

To be able to give the neutrinos Majorana masses using the Froggatt-Nielsen mechanism, it is necessary for the right-handed neutrino fields to have integer charges, see Section 4.3. We use a setup very similar to the above case; SM particle content with three right-handed neutrino fields and two Higgs doublets with a lepton specific $\mathbb{Z}_2$ symmetry. Here we assume that the Dirac masses for the neutrinos are of the same magnitude as the other particles' and with normal hierarchy:

$$\begin{aligned} m_{\nu 1}: &\quad (L_1 - \nu_1 + H_1) - 3 = 0 \\ m_{\nu 2}: &\quad (L_2 - \nu_2 + H_1) + 2 = 0 \\ m_{\nu 3}: &\quad (L_3 - \nu_2 + H_1) + 1 = 0 \end{aligned} \tag{6.20}$$

and we also change the charged lepton mass conditions to

$$\begin{aligned} m_e: &\quad (L_1 - e_1 - H_1) + 8 = 0 \\ m_\mu: &\quad (L_2 - e_2 - H_1) - 5 = 0 \\ m_\tau: &\quad (L_3 - e_3 - H_1) - 3 = 0 \end{aligned} \tag{6.21}$$

so that the coefficients in the neutrino and charged lepton conditions sum to zero individually. This, of course, also implies that they sum to zero together which means that the gravitational anomaly becomes redundant. The Gröbner basis for this ideal is:

$$\begin{aligned}
&\langle Q_1 + 1/3 \cdot H_2 + 13/9 \cdot \nu_1 - 13/18 \cdot \nu_2 - 7/18 \cdot \nu_3 + 89/9, \\
&Q_2 + 1/3 \cdot H_2 + 13/9 \cdot \nu_1 - 13/18 \cdot \nu_2 - 7/18 \cdot \nu_3 + 80/9, \\
&Q_3 + 1/3 \cdot H_2 - 23/9 \cdot \nu_1 + 16/9 \cdot \nu_2 + 10/9 \cdot \nu_3 - 157/9, \\
&u_1 + 4/3 \cdot H_2 + 13/9 \cdot \nu_1 - 13/18 \cdot \nu_2 - 7/18 \cdot \nu_3 + 152/9, \\
&u_2 + 4/3 \cdot H_2 + 13/9 \cdot \nu_1 - 13/18 \cdot \nu_2 - 7/18 \cdot \nu_3 + 53/9, \\
&u_3 + 4/3 \cdot H_2 - 23/9 \cdot \nu_1 + 16/9 \cdot \nu_2 + 10/9 \cdot \nu_3 - 157/9, \\
&d_1 - 2/3 \cdot H_2 + 13/9 \cdot \nu_1 - 13/18 \cdot \nu_2 - 7/18 \cdot \nu_3 + 26/9, \\
&d_2 - 2/3 \cdot H_2 + 13/9 \cdot \nu_1 - 13/18 \cdot \nu_2 - 7/18 \cdot \nu_3 + 125/9, \\
&d_3 - 2/3 \cdot H_2 - 23/9 \cdot \nu_1 + 16/9 \cdot \nu_2 + 10/9 \cdot \nu_3 - 175/9, \\
&L_1 - H_2 - \nu_1 - 13/3,\ L_2 - H_2 - \nu_2 + 2/3,\ L_3 - H_2 - \nu_3 - 1/3, \\
&e_1 - 2 \cdot H_2 - \nu_1 - 41/3,\ e_2 - 2 \cdot H_2 - \nu_2 + 13/3,\ e_3 - 2 \cdot H_2 - \nu 3 + 4/3, \\
&H_1 - H_2 - 4/3, \\
&\nu_1^2 - 126/95 \cdot \nu_1\nu_2 - 82/95 \cdot \nu_1\nu_3 - 101/285 \cdot \nu_1 + 77/190 \cdot \nu_2^2 + 61/95 \cdot \nu_2\nu_3 - \\
&113/285\nu_2 + 27/190 \cdot \nu_3^2 - 88/57 \cdot \nu 3 - 4722/95\rangle
\end{aligned} \tag{6.22}$$

which defines a three-dimensional variety. To find integer points on this variety we look at the last equation in the Gröbner basis (cleared to integer coefficients):

$$\begin{aligned}
&570\nu_1^2 - 756\nu_1\nu_2 - 492\nu_1\nu_3 - 202\nu_1 + 231\nu_2^2 + 366\nu_2\nu_3 - 226\nu_2 + \\
&81\nu_3^2 - 880\nu_3 - 28332 = 0.
\end{aligned} \tag{6.23}$$

This is a surface, to use our methods developed for curves we fix ν_3 and then systematically vary it.

For ν_3=0 the resulting curve is not parameterizable over the rationals, but over the finite field extension $\mathbb{Q}(\sqrt{1246})$, so there are not even any rational points on this curve.

For ν_3=1 there is still only a parameterization over $\mathbb{Q}(\sqrt{1246})$. However, for ν_3 an integer greater than one there seems to always exist integer solutions. These will be curves of genus 0 with two point evaluations at infinity, by Theorem 3 there not only exist integer solutions but actually infinitely many which may be described by a parameterization. Even though there exists infinitely many integer solutions the smallest are usually of magnitude 10^{10} (with an accidental exception for $\nu_1 = -17$, $\nu_2 = -43$, $\nu_3 = 3$) and therefore not so interesting from the viewpoint of a physicist.

Going now in the other direction, let us start with $\nu_3 = -1$, this curve is only parameterizable over $\mathbb{Q}(\sqrt{1246})$. For $\nu_3 < -1$ there always exists a rational parameterization, but integer points only exist for even integers. These integer solutions are usually of the order 10^{10} and above. One exceptional case is $\nu_1 = -16$, $\nu_2 = -10$, $\nu_3 = -10$ which defines the following Gröbner basis:

$$\begin{aligned}&\langle Q_1 + 1/3 \cdot H_2 - 19/9,\ Q_2 + 1/3 \cdot H_2 - 28/9,\ Q_3 + 1/3 \cdot H_2 - 49/9,\\ &u_1 + 4/3 \cdot H_2 + 44/9,\ u_2 + 4/3 \cdot H_2 - 55/9,\ u_3 + 4/3 \cdot H_2 - 49/9,\\ &d_1 - 2/3 \cdot H_2 - 82/9,\ d_2 - 2/3 \cdot H_2 + 17/9,\ d_3 - 2/3 \cdot H_2 - 67/9,\\ &L_1 - H_2 + 35/3,\ L_2 - H_2 + 32/3,\ L_3 - H_2 + 29/3,\\ &e_1 - 2 \cdot H_2 + 7/3,\ e_2 - 2 \cdot H_2 + 43/3,\ e_3 - 2 \cdot H_2 + 34/3,\\ &H_1 - H_2 - 4/3,\ \nu_1 + 16,\ \nu_2 + 10,\ \nu_3 + 10\rangle \end{aligned} \tag{6.24}$$

One of the Higgs charges is still a free parameter, we may for example choose this charge so that the entire lepton sector has integer charges. Take for example $H_1 = 0$, then the complete set of charges is:

$$\begin{array}{lll} Q_1 = 23/9, & Q_2 = 32/9, & Q_3 = 53/9, \\ u_1 = -28/9, & u_2 = 71/9, & u_3 = 65/9, \\ d_1 = 74/9, & d_2 = -25/9, & d_3 = 59/9, \\ L_1 = -13, & L_2 = -12, & L_3 = -11, \\ e_1 = -5, & e_2 = -17, & e_3 = -14, \\ \nu_1 = -16, & \nu_2 = -10, & \nu_3 = -10 \\ H_1 = 0, & H_2 = -4/3 & \end{array} \tag{6.25}$$

For this model the mass matrices are:

$$\begin{aligned} m_u &= \begin{pmatrix} \epsilon^7 & \epsilon^4 & 0 \\ \epsilon^8 & \epsilon^3 & 0 \\ 0 & 0 & \epsilon^0 \end{pmatrix}, \quad m_d = \begin{pmatrix} \epsilon^7 & \epsilon^4 & 0 \\ \epsilon^6 & \epsilon^5 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix} \quad m_e = \begin{pmatrix} \epsilon^8 & \epsilon^4 & \epsilon^1 \\ \epsilon^7 & \epsilon^5 & \epsilon^2 \\ \epsilon^6 & \epsilon^6 & \epsilon^3 \end{pmatrix}, \\ m_D &= \begin{pmatrix} \epsilon^3 & \epsilon^3 & \epsilon^3 \\ \epsilon^4 & \epsilon^2 & \epsilon^2 \\ \epsilon^5 & \epsilon^1 & \epsilon^1 \end{pmatrix}, \quad M = \begin{pmatrix} \epsilon^{32} & \epsilon^{26} & \epsilon^{26} \\ \epsilon^{26} & \epsilon^{20} & \epsilon^{20} \\ \epsilon^{26} & \epsilon^{20} & \epsilon^{20} \end{pmatrix}, \end{aligned} \tag{6.26}$$

where m_u and m_d represent the quarks, m_e the charged leptons, m_D and M the Dirac resp. Majorana mass matrices for the neutrinos.

Note that $\epsilon^{20} \sim 10^{-14}$, so even if the Majorana mass were naturaly at the GUT scale, the large number of flavon insertions forces the magnitude down to electroweak energies. This means that there will not be a seesaw mechanism in this model.

In this model we have no handles on the PMNS matrix. In this example we were also unable to use the seesaw mechanism to explain the small neutrino mass scale, the Majorana masses was even smaller than the Dirac masses. What this example teaches us is that there might exist integer solutions for the flavon charges, allowing for Majorana masses, but these integers are usually so large so that they are not of interest. There might be accidental smaller integer solutions, but even these yield small Majorana masses and thus no seesaw mechanism.

7 Conclusions and Outlooks

There's a fine line between wrong and visionary. Unfortunately you have to be a visionary to see it.
-Sheldon Cooper [51]

In this thesis we have explained the observed mass hierarchies and mixings using a gauged $U(1)$ Froggatt-Nielsen mechanism and two Higgs doublets. Using a gauge symmetry means that the flavon charges not only have to generate the observed masses but also cancel all gauge anomalies. This leads to a set of polynomial equations for the flavon charges to satisfy. We impose a $\mathbb{Z}_2$ symmetry on the Higgs doublets to remove FCNCs. The only choice of $\mathbb{Z}_2$ symmetry consistent with the anomaly and mass constraints is a lepton specific $\mathbb{Z}_2$ symmetry. We also studied the position of the Landau pole for the new $U(1)$ coupling to make sure the model not becomes strongly coupled below the Froggatt-Nielsen scale. To find charges that satisfies all the constraints from anomalies and reproduces the observed masses and mixings, methods from algebraic geometry was used.

This succeeded very well, especially with Dirac neutrinos. For this case, the neutrino mass differences and mixing angles, as well as the overall smallness of the neutrino masses, could be explained. Also the masses for the SM fermions were completely reproduced using only Cabibbo mixing in the quark sector. Since it was difficult to include a seesaw mechanism in this framework, only giving the neutrinos Dirac masses is the minimalistic most natural way of explaining the observed smallness and hierarchy.

There are many possible ways one could continue this work. One could, for example, use different seesaw mechanisms or use some other additional gauge symmetry. However, the most interesting direction to go is probably to use our parametrized solutions for the charges and try to embed them into a larger GUT group, like for example $SO(10)$.

What to really take away from this thesis is the extreme usefulness and power of algebraic methods when dealing with these kind of problems. This thesis will hopefully serve as a useful guide in how to use algebraic geometry when solving problems related to anomalies and charges in, for example, BSM physics.

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