*R&D and Economic Growth: A Model to Explain Why Increased Research Does Not Lead to Increased Growth*

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Abstract

In recent decades, industrialized economies have significantly increased their R&D efforts while economic growth rates have been relatively stable. This goes against the conclusions of most previous economic models of endogenous growth. In this paper, an endogenous growth model has been created in which technology is treated as an income-compensated factor of production. The purpose is to create a theoretical model in which economies must increase their research intensity in order to maintain economic growth rates. The model is able to predict the growth rate and level of GDP per capita to the correct order of magnitude, although not with a remarkable degree of accuracy. However, the model seems to necessitate continuous increases in research intensity in order to maintain the economic growth rate.

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## 1. Introduction

Over the past decades, research and development has grown significantly in importance, especially in developed economies. Between 1996 and 2014, the share of the labor force allocated to the R&D sector has almost doubled. During the same time period, industrialized economies have seen steady, or even slightly decreasing rates of economic growth. Economic growth models in which the level of technology is a central variable dictate that such a drastic increase in research intensity should lead to increased economic growth rates, at least in the short term. Observations on economic growth and research intensity tend to indicate that this is not necessarily the case.

The research question of this paper is the following: *Can treating technology as an income-compensated factor of production explain the increased number of researchers and the simultaneous absence of increased economic growth in the OECD?* It is hypothesized that industrialized economies today actually have to increase the allocation of resources to research and development in order to maintain economic growth. Furthermore, this method can only be effective up to a certain point, at which further increasing the research intensity will no longer have this positive effect. A simple model of economic growth, with technology as the central variable, has been developed in this paper. The defining characteristic of the model is that it considers technology, as well as capital and labor, to be a factor of production to the owners of which accrue income, henceforth referred to as an income-compensated factor of production. Technology is thus included in the production function itself and encompasses new technology that is still under patent protection. The aim of the model presented in this paper is to internalize this hypothesis into a theoretical and mathematical framework.

The model presented in this paper is an attempt at reconciling the problem of increased research intensity without any significant change in average per capita GDP growth. In building this model, this paper hopes to create a theoretical framework in which industrialized economies such as the OECD and its member countries must actually continuously increase its research efforts to maintain a stable economic growth rate, as stipulated in the previous paragraph.

Certain characteristics of the model presented in this paper will be described here in order to enable a basic understanding of the model. A complete description and solution of the model is presented in section 4. The purpose of developing the model that this paper describes is to – to a more satisfactory level than Jones' model, described below – reconcile the increased research intensity in developed economies with the absence of increased economic growth over the past decades. The model that this paper develops differs from previous models primarily by considering the technology variable as an income-compensated factor of production, resulting in a different production function compared with other models. Once this important difference is established, the model is solved in a similar manner to other economic growth models. However, the focus of this paper's model lies more heavily on the variables relating to the division of the labor force into production of goods and services on the one hand, and research and development on the other hand.

Tests of the model will be conducted by comparing data on the level and growth rate of GDP per capita with the values that the model predicts. This will be done for the years 1997-2014. The predictions will be based on equations derived from the model describing these levels and growth rates. The variables that are necessary for the model to make its predictions will be taken from data from the World Bank Group. The accompanying parameters will be appropriately estimated, using information from previous studies where possible.

When testing the model, the OECD will be considered as a unit in this paper and will be the subject of the model estimations and the theoretical reasoning. There are two reasons for this. Firstly, the OECD is the largest group of industrialized economies in the world that is generally grouped together. Since the development of new technology occurs mainly in developed economies such as these, it makes sense to place the focus on these economies. Secondly, the significance of the spread of technology, as opposed to domestic innovation, is minimized if the OECD is treated as one economy. Seeing as this paper's model centers on the development of new technology rather than the adoption of foreign technologies, the prevalence of external innovation should be minimized. Because the OECD member countries stand for a large portion of new technology worldwide, aggregating the OECD into one unit reduces the risk of disturbances to the model due to technology spreading.

Following this introductory section, section 2 (Background) will introduce some empirical data related to the importance of the research question. Section 2 also briefly outlines some key characteristics of this paper's model and how they relate to previous growth models, as well as introducing some notes on the data collection. Section 3 (Previous Research) outlines the development of economic growth models over time and focuses especially on the models on which this paper is most closely based. Section 4 (The Model) introduces the model in its mathematical form and derives the equations necessary to test the model empirically. In section 5 (Testing the Model), the equations describing the growth rate and the level of GDP per capita are tested in order to evaluate the model's fit with earlier observations. Finally, section 6 (Conclusion) summarizes the main conclusions of the reasoning made and the tests performed in this paper.

## 2. Background

### 2.1: Empirical Background Information

The following points are illustrated in figure 1. Between the years 1996 and 2014, the aggregated GDP per capita of all current OECD member countries has almost doubled, going from about $23,000 to about $39,000, measured in current US dollars (World Bank [1]). (The in-text citations to the World Bank are described in more detail in Appendix II.) This corresponds to an average growth rate in real per capita GDP of 3.0%. With the notable exception of the years 2008 and 2009 – the worst years of the global financial crisis known as the Great Recession – the OECD growth rates in real GDP per capita have remained relatively constant.

During this same time period, the number of researchers in R&D per million people has increased from about 2,500 to about 4,000 (World Bank [2]); a dramatic increase to say the least. The aggregated population of the current OECD member countries saw an increase of around 13% (World Bank [3]). Taking this population increase into account, we can see that the total number of researchers in the population increased by just over 81% during this time period.

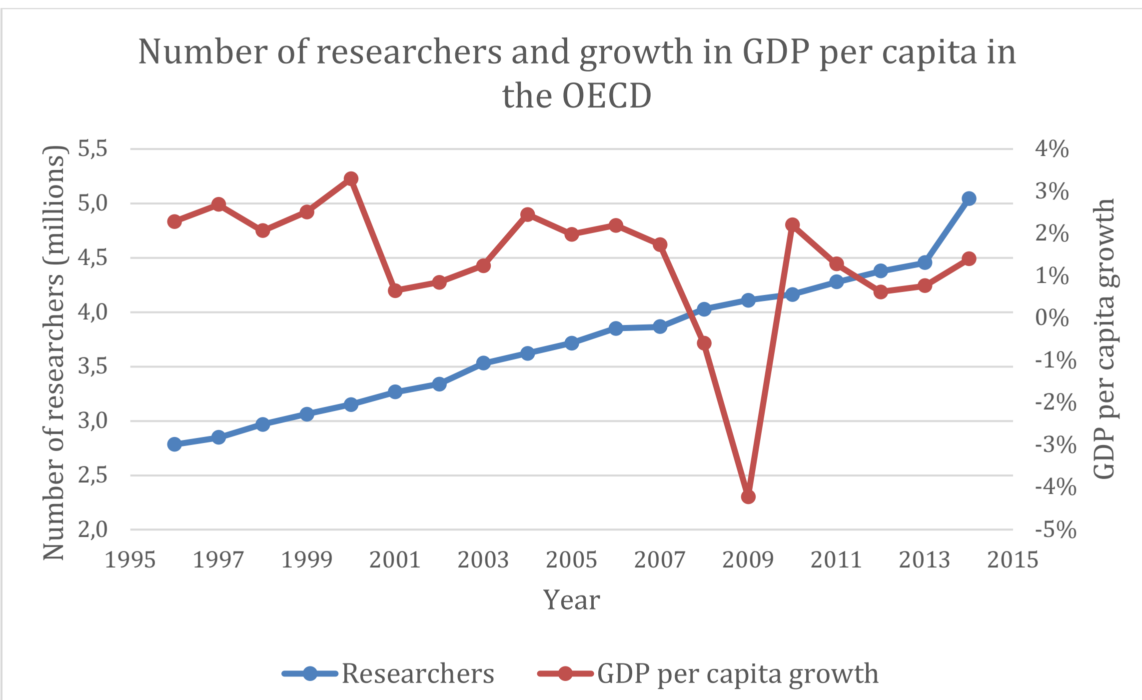


*Figure 1.*

The model presented in this paper uses the total number of researchers in the population as well as the portion of researchers that make up the labor force as important variables. It neglects, however, the magnitude of monetary spending on research and development. The model would become many times more complicated if both research spending *and* the number of researchers were to be accounted for. On the other hand, R&D spending is not a factor that can reasonably be discarded as an insignificant detail. Therefore, even though the model itself and the solution of said model do not take this variable into account, R&D spending must be examined in order to gain a more holistic view of the development of the research sector in the OECD over the past couple of decades. In all brevity, research and development spending in the OECD doubled between 1996 and 2014 (World Bank [4]). This development approximately matches the rise in the number of researchers within the OECD, and even supersedes it somewhat. It is impossible to posit that while the number of researchers has increased significantly, the absence in the rise of research spending can be seen as an explanation to the absence of increased economic growth. Rather, the remarkable increase in research spending has meant that slightly *more* money is spent on every researcher in the OECD member countries. This fact strengthens the problematic nature of the increase in research intensity without a resulting increase in economic growth.

### 2.2: Notes on Data Collection

Due to the nature of the Romer Model, both in its original form and in the altered version by Jones – namely, the assumption that all new technology is developed domestically – the OECD will be analyzed as a unit in this paper. The main arguments will rest on conclusions drawn from aggregated OECD data. The reason behind this is that individual member countries cannot reasonably be assumed to produce all, or even a majority, of their new technological advancements domestically. In reality, even all of the OECD member countries put together do not produce all new innovations that come about. However, the vast majority of patents filed worldwide do come from the OECD (World Bank [5]). Granted, the portion of the world's patent applications coming from the OECD has declined in recent decades, but still constitute a significant portion of worldwide patent applications. An economic model is, by definition, a simplified representation of reality. Thus, for the purposes of the model presented in this paper, it is assumed that the OECD develops all new technology within its member countries' borders.

Due to the limited availability of data on the number of researchers before the year 1996, this section and the assertions within will be based on the time period 1996-2014. Calculations that rely on growth rates in the number of researchers in the labor force have been made for the years 1997-2014, as these growth rates cannot, with the currently available data, be calculated for the year 1996. Data has been gathered from the World Bank. Another viable source of data was the OECD. However, certain factors complicate the calculation of the values of the relevant variables if the OECD database were to be used. Firstly, the OECD data on the number of researchers is given per 1000 employed. Thus, in order to calculate the total number of researchers in the population, one would have to take into account population, the labor force participation rate, and the unemployment rate; each of these steps in the calculation exacerbates the risk of rounding errors and human error. The World Bank database, on the other hand, provides data for the number of researchers per million inhabitants, which minimizes the significance and the risk of rounding and human errors, respectively. Seeing as the total number of researchers in the population is a highly critical variable in the model presented in this paper, it is preferable to use the World Bank database for the purposes of this paper. Secondly, the World Bank provides data on real GDP per capita more readily than does the OECD. For the same reasons as the ones stated above, it is therefore preferable to use the World Bank database.

### 2.3: Additional Notes

A simplifying assumption is made in this paper's model, similarly to the previously existing models described in section 3. This assumption states that the labor force participation rate is constant, or at least sufficiently invariable to be able to equate population growth with labor force growth. This assumption will carry into this paper, as the differences therein do not constitute the focus of the problem.

Therefore, population growth and labor force growth will henceforth be used interchangeably. Also, the *number* of researchers in the population is equivalent to the number of researchers in the labor force. Note, however, that when calculating the *portion* of the labor force allocated to either production or research, population size will not be considered. Instead, the size of the labor force will be the denominator in these calculations.

## 3. Previous Research

This section includes an overview of previous models of economic growth that have been important for the understanding, development, and accompanying theoretical reasoning of this paper's model. The classical growth theories have been included here in order to introduce the concept of diminishing marginal returns. The subsection on the neoclassical Solow-Swan Model has been included as it is the basis of all later models mentioned. The focus of this section is on endogenous growth theories, since these models are most closely related to the model in this paper. Finally, a note is made on technology adoption models to justify the focus of this paper on the OECD.

### 3.1: Classical Growth Theories

The focus of economic growth theorists has, over the past few centuries, seen a number of shifts. The classical growth theories presented by such figures as Thomas Malthus and David Ricardo focused on increasing the factors of production to attain a higher aggregate growth (Bjork, 1999, p. 297-8). The law of diminishing returns was present here in each of the factors of production if the others were held constant. This concept of diminishing marginal returns was, and remains, an important mechanism in production theory and economic growth theory. What the law does is allow for an explanation as to why we cannot exponentially, or even proportionally, increase our production through a one-sided increase in a given production factor. In other words, more is better, but at a decreasing rate.

### 3.2: The Solow-Swan Model

The neo-classical Solow-Swan Model, in its simplest form, laid its focus on capital accumulation as a productivity enhancer for the labor force. The same mechanism for capital accumulation has been used in later models and will also be used in this paper. Here, the law of diminishing returns applies to the accumulation of capital, but not its depreciation. Thus, the simplest version of the Solow-Swan Model allows for no per capita economic growth in steady state. For this reason, technological change was introduced, although in an exogenous form. The implication of this condition of a globally uniform rate of innovation is that all economies should grow at the same rate in the steady state. (Solow, 1956)

The results of the Solow-Swan Model with technological change provide an answer to one of the fundamental questions posed in economic growth theory; why do economies grow? One could argue that the model also allows for an explanation as to why some economies grow faster than others. Economies with growth rates higher than the rate of technological change lie under their steady-state levels, and vice versa. It can take several years, or even decades, for economies to reach a new steady state. However, the model does imply that absolute convergence of per capita income should be observed between economies, if they have the same savings rate. The discrepancy between these implied results of the model and empirical evidence regarding convergence suggests that the Solow-Swan Model does not answer to a satisfactory degree the questions of why some countries are richer than others and why economies grow at different rates. Furthermore, if technology is the explaining factor for long run growth, economies should aim to explain it, and not take it as exogenously given.

### 3.3: Endogenous Growth Theory I: The Romer Model

The absence of reasoning behind technological advancement prompted Paul Romer to develop endogenous growth theory. The model endogenized innovation as well as human capital accumulation. Seeing as this paper focuses on technology as a driver of economic growth, this variable will bear the focus of the description of this model. Although human capital is undeniably an important factor in long-term economic growth - one which to a great extent goes hand in hand with technology - this paper will consider the Romer Model without human capital in order to narrow its focus to the technology variable.

The original presentation of the Romer Model makes two assumptions regarding the growth of the level of technology from which Jones' presentation of the model, described below, departs. The equation describing innovation is as follows:

$$\dot{A} = \theta {L_A}^{\lambda} A^{\phi} \qquad (3a)$$

Firstly, the parameter to which the technology level is raised in equation (3a) is assumed to be equal to one. That is, the current level of technology has no impact on the rate of change in the level of technology. Synonymously, the absolute change in technology per time period is exactly proportional to the level of technology, holding constant the number of researchers. Secondly, the parameter to which the number of researchers is raised is also assumed to be equal to one. This means that an increase in the number of researchers will lead to a proportional increase in the rate of technological progress. These two assumptions result in the absence of all diminishing returns in production as a function of technology. In this model, it is concluded that per capita output can increase without bound, and that the production of goods and services as a function of the level of technology exhibit increasing returns to scale. (Romer, 1986)

A resulting conclusion of the Romer Model is that adding more researchers implies a faster rate of technological change. Within the equation describing innovation, the number of researchers is the only steady state growth variable that can be affected with economic policy. A positive level effect on per capita income can be achieved by reducing the proportion of researchers in the population. A positive growth effect, on the other hand, can be achieved by increasing the number of researchers in the population (Romer, 1986). Note here that both of these changes can occur simultaneously if the labor force grows. Therefore, in theory, countries that continuously increase their research intensity should see a continuously increasing steady state rate of economic growth.

### 3.4: Endogenous Growth Theory II: The Jones Model

The discrepancy between the growth-effect conclusions of Romer's original model and a reality of increasing R&D but relatively steady growth rates needed to be resolved. Charles I. Jones relaxed the assumptions made by the former researchers regarding the values of the parameters in the technology growth equation. Instead of the parameters described above being equal to one, Jones assumed that the values of these parameters lay between zero and one. In this presentation of the model, diminishing returns are exhibited both in the number of researchers and the level of technology (Jones, 1995). The former variable, which in reality can be thought of as a proxy for all resources spent on research within an economy, exhibits diminishing returns due to the fact that the research of each individual researcher may overlap with the research of others. This is consistent with the classical economists' presentation of diminishing marginal returns. The risk of this overlap occurring becomes greater as the number of researchers increases. The technological level is in itself thought to be a limiting factor to innovation because the simple ideas are discovered first. This idea is sometimes referred to as the "fishing out" process. As technology advances, further advancement is contingent on increasingly complicated innovations. The solution of this model leads to the conclusion that adding more labor to the pool of researchers will lead to a positive level effect, but no growth effect, on per capita national output.

### 3.5: Technology Adoption

All countries do not independently develop their own technology. There is a certain degree of technology adoption present. This phenomenon was modeled by authors such as Easterly et al (1994). In this type of model, all economies lie some distance from the technological frontier, and are able to adopt differing levels of technology based on variables such as the respective number of researchers in each country. Unlike the endogenous growth theory, which is best applied to developed economies, the models of technology transfer best explain growth in developing countries. This is because developing countries rely to a far greater extent on adoption of technology than do industrialized countries. Conversely, domestic innovation plays a large role in industrialized countries. This type of model is noteworthy for the purpose of this paper because it illustrates why the focus of this paper lies with industrialized countries, as well as why the OECD is treated on the aggregate level.

As previously stated, the subsections on endogenous growth theory are the most important in this section. The theories described therein are the ones that most closely relate to the model presented in this paper. The technology adoption subsection is included in order to emphasize the point that this paper does not deal with this source of technological advancement. It is also included to justify the selection of the OECD as the subject of the model tests and the treatment of the OECD as a unit.

# 4. <u>The Model</u>

## 4.1: Introductory Notes

The major difference between the model presented here and Jones' presentation of the Romer Model, upon which the model in this paper is most closely based, is that the level of technology is treated as a factor of production for which the owners receive a portion of the national income. Instead of considering two such factors of production – labor and capital – we now consider three. The reasoning behind this is that although the large majority of national incomes accrue to the owners of either capital or labor, there is a certain portion that goes to what one could describe as the "owners" of technology. In this paper, the owners of technology are considered to be the current owners of monopoly rights on some innovation. Any technology for which this monopoly right has expired is considered to be in the public domain. Therefore, any income that arises from the application of non-patented technology is not included in the accompanying parameter to technology. Specifically, the incomes from technology are made up of royalty payments to patent holders. Just as it might be insufficient to treat only capital – and neglecting labor – as an income-compensated factor of production in an economic growth model, it is possible that neglecting to factor in technology income erodes the model's ability to predict reality. The resulting discrepancy between model and reality might be exacerbated with the ever-increasing importance and prevalence of R&D and new, patented products. Treating technology as an income-compensated factor of production has little precedence, but has been considered by, for instance, Parkin et al (2007).

The production function of the Romer Model ensures that the model exhibits constant returns to scale in capital and labor combined. In other words, doubling both the amount of capital and labor while holding the level of technology constant results in a doubled national output. In this paper's model, however, the production function is set up in such a way that there are constant returns to scale in capital, labor, and technology combined. The hope is that this altered condition will limit economic growth due to technological advancement, and thus give the model a better fit with reality.

The definition of the steady state has been developed in this paper to include a "weak" steady state and a "final" steady state. In the weak steady state, the growth rate in the labor force can be assumed to be constant. The savings rate, $s$, described in subsection 4.2, is also constant. The weak steady state does not mean that the growth rate in the number of researchers is constant, which is where this idea departs from the classical steady state of previously mentioned models. This, in turn, means that the growth rate in technology can fluctuate. Therefore, in the weak steady state, the growth rates of GDP per capita and capital per capita can also change over time. In the final steady state, however, the growth rate of the number of researchers is constant and equal to the labor force growth rate. Consequently, the other aforementioned growth rates are also constant. It is thought that, given the currently increasing proportion of researchers in OECD member countries, these countries today have not yet reached this final steady state.

All derivations of equations have been placed in Appendix I.

### 4.2: Assumptions Taken from Previous Models

Output will, in this paper, be presented in the aggregate. In previous models, there is quite a complex plethora of equations and conditions relating to the intermediate goods sector, the final goods sector and, in models of endogenous technological change, the research sector. Since these sectors work no differently than in the predecessors of this paper's model, the calculations therein will not be described or analyzed in this paper. Please note, however, that the assumptions on these sectors are, for the purposes of this paper, identical to those within previous models. The final goods sector is characterized by a large number of perfectly competitive firms, which produce homogenous goods. Research and development leads to new types of capital goods that can be used in the production of goods and services. The intermediate goods sector is made up of monopolists each selling one type of capital good to the market. The development of ideas – the product of the research sector –

is free for anyone to pursue, and results in new capital goods becoming available to the market. (Jones & Vollrath, 2013)

An additional assumption relating to the research sector in previous models of endogenous growth is that patents last forever. The purpose of this assumption is to simplify the analysis of the research sector and its effects on the final and intermediate goods sector. In this paper's model, patents eventually expire. However, for the analytical purposes of this paper, the assumption does not need to be changed. The research sector is assumed to work the same way as in other endogenous growth models.

Full employment is assumed in this model in order to simplify the analysis. With the help of this assumption, unemployment does not have to be taken into consideration. Furthermore, on a more mathematical basis, the assumption enables the condition $L = L_A + L_Y$ to be set. In words, this means that every unit of the labor force (every worker) is allocated either to the production of goods and services or to research and development.

The monetary value of the capital stock in the economy exhibits the same characteristics as in the Solow-Swan Model and the models of endogenous growth. The absolute growth of the capital stock is expressed in the following way:

$$\dot{K} = sY - \delta K \tag{0}$$

The capital stock grows according to some percentage of the national output which is saved by the population, less some fixed portion of the existing capital stock due to the depreciation of capital.

### 4.3: The Core Equations

Considering technology as a factor of production while keeping earlier researchers' condition of constant returns to scale intact, the production function is as follows:

$$Y = K^{\alpha} A^{\beta} {L_Y}^{1-\alpha-\beta} \tag{1}$$

$Y$ represents total output; the homogenous product of the three factors of production. $K$ represents capital, $A$ represents the level of technology, and $L_Y$ represents the number of units of labor dedicated to production.

The difference between this setup and that of Jones - being the classification of technology as an income-compensated factor of production - necessitates consideration of income gained by the "owners" of technology. The magnitude of this income is necessarily the total income subtracted by labor and capital incomes. The measurement of technology income that makes most sense in this context would be patent incomes. These would be made up of income gained by the sale of patented goods and services. This would reasonably be a very small portion of total national income. Thus, even adding this new factor of production does not violate the approximate portions of national income made up by capital and labor income, respectively. Generally, capital income makes up around a third of national income, with labor income constituting roughly the remaining two thirds.

A per capita version of the production function can be derived by dividing both sides of the equation by the labor force variable, $L$. Lower-case letters represent per capita values of the respective variables:

$$y = k^{\alpha}(\frac{A}{L})^{\beta}(\frac{L_Y}{L})^{1-\alpha-\beta} \qquad (2)$$

As in the Romer Model á la Jones, income per capita depends partly on the amount of capital per capita as well as the portion of the labor force dedicated to production. The difference between equation (2) in this paper and the corresponding equation of the Jones Model is that the technological level divided by the population, and not the technological level itself, is a determining factor in income per capita. Therefore, as the population grows, innovation is necessitated to secure a steady per capita income level.

This model endogenizes innovation in the same way as Jones' Model:

$$\dot{A} = \theta {L_A}^{\lambda} A^{\phi} \qquad (3a)$$

$$\frac{\dot{A}}{A} = g_A = \theta {L_A}^{\lambda} A^{\phi - 1} \tag{3b}$$

$\dot{A}$ represents the change in the level of technology, so $g_A$ represents the rate of change in that variable. The letter $g$ with a subscript will henceforth represent the rate of change in the variable in the subscript.

### 4.4: Solving the Model

An expression describing the rate of change in technology independent of the current level of technology is required. In order to achieve this, equation (3b) is logarithmized and then differentiated with respect to time, and the *rate of change in the rate of change* in technology is set to 0 - as the final steady state condition requires. Since the rate of change in the parameter $\theta$ is 0, the rate of change in technology can be isolated to give:

$$g_A = \left(\frac{\lambda}{1-\phi}\right) g_{L_A} \tag{4}$$

Both the Solow-Swan and the Romer Models dictate that in steady state, the growth rate of capital per capita equals that of output per capita, due to the condition:

$$g_K = \frac{sY - \delta K}{K} = s\frac{Y}{K} - \delta$$

Since, by definition, $g_K$ is constant in the final steady state, and $s$ and $\delta$ are constant parameters, $Y$ and $K$ must grow at the same rate. Because $y$ and $k$ are equal to these variables divided by $L$, respectively, $y$ and $k$ must also grow at the same rate in the steady state.

Isolating the growth rate of per capita income from logarithmizing equation (2) and differentiating with respect to time, we get:

$$g_y = \frac{\beta}{1-\alpha}\left(\left(\frac{\lambda}{1-\phi}\right) g_{L_A} - g_{L_Y}\right) + \left(g_{L_Y} - g_L\right) \tag{5}$$

Since full employment is assumed, and research & development and production are the only two sectors in the economy, the following condition can be set:

$$g_{L_Y} = \frac{L}{L_Y} g_L - \frac{L_A}{L_Y} g_{L_A} \tag{6}$$

$$\frac{L}{L_Y} = 1 + \frac{L_A}{L_Y} \tag{7}$$

Substituting (7) into (6), we can find $g_{L_Y}$ in terms of the labor force growth rate, the growth rate of the number of researchers, and the ratio of labor in research to labor in production. This gives:

$$g_{L_Y} = g_L + (\frac{L_A}{L_Y})(g_L - g_{L_A}) \tag{8}$$

It is relevant to find a per capita income growth equation as a function of the aforementioned ratio, the necessary parameters, and the growth rates of the labor force and the number of researchers. This can be done by substituting (8) into (5). The resulting equation is as follows:

$$g_y = \frac{\beta}{1-\alpha}(g_{L_A}(\frac{\lambda}{1-\phi} + \frac{L_A}{L_Y}) - g_L(1 + \frac{L_A}{L_Y})) + (\frac{L_A}{L_Y})(g_L - g_{L_A}) \tag{9}$$

At last, we are presented with an equation that in some way illustrates what might happen in a final steady state situation, in which population grows at the same pace as the number of researchers. In this scenario, the last term in equation (9) is equal to zero. The first term, and thus the equation as a whole, can be positive if the term containing the research-related parameters is greater than one. That is, if:

$$\frac{\lambda}{1-\phi} > 1$$

Even when this condition is met, equation (9) illustrates one of the main points of this paper; as the proportion of researchers increases and the growth rates of the labor force and the number of researchers approach equivalence, the benefits of having a researcher growth rate that is higher than the population growth rate diminish. This is true because the last expression in the equation – $(\frac{L_A}{L_Y})(g_L - g_{L_A})$ – increases in magnitude, while the first expression decreases in magnitude. Once the final steady state is reached, and $g_L = g_{L_A}$, the economic growth rate will be lower than before, albeit still positive.

As in any economic growth model, one wishes not only to find the growth rate of per capita income in steady state, but also its level. In this case, the optimal ratio of researchers to labor force, which maximizes per capita income, is also sought after. Expressions for these two values are presented below.

Firstly, a general equation for per capita income, based on the production function, is needed. This is found in equation (2). Next, technology needs to be endogenized in this equation and substituted into (2). The technology variable, $A$, can be isolated after equating equations (3b) and (4):

$$A = (\frac{\theta {L_A}^{\lambda}(1-\phi)}{\lambda g_{L_A}})^{\frac{1}{1-\phi}} \tag{10}$$

Substituting (10) into (2), we get:

$$y = k^{\alpha}(\frac{(\frac{\theta {L_A}^{\lambda}(1-\phi)}{\lambda g_{L_A}})^{\frac{1}{1-\phi}}}{L})^{\beta}(\frac{L_Y}{L})^{1-\alpha-\beta} = k^{\alpha}(\frac{\theta {L_A}^{\lambda}(1-\phi)}{\lambda g_{L_A}})^{\frac{\beta}{1-\phi}}(\frac{{L_Y}^{1-\alpha-\beta}}{L^{1-\alpha}}) \tag{11}$$

Since $L_Y = L - L_A$, equation (11) can be expressed in terms of $L$ and $L_A$ in the following way:

$$y = k^{\alpha}(\frac{\theta(1-\phi)}{\lambda g_{L_A}})^{\frac{\beta}{1-\phi}}(L_A)^{\frac{\lambda\beta}{1-\phi}}(\frac{(L-L_A)^{1-\alpha-\beta}}{L^{1-\alpha}}) \tag{12}$$

Note that the variable representing the number of researchers in the economy has been isolated from the term with the parameters and the growth of said variable. This has been done due to the nature of the next step in the solution of the model. The model presented in this paper focuses strongly on the number of researchers in the population as well as the ratio of this variable to the total labor force. Therefore, it is of interest to optimize this ratio in order to maximize income per capita. The isolating maneuver in equation (12) facilitates the differentiation that is required for this optimization. Income per capita is differentiated with respect to the number of researchers.

$$L_A = L\left(\frac{\frac{\lambda\beta}{1-\phi}}{\frac{\lambda\beta}{1-\phi}+1-\alpha-\beta}\right) \tag{13}$$

$$\frac{L_A}{L} = \frac{\frac{\lambda\beta}{1-\phi}}{\frac{\lambda\beta}{1-\phi}+1-\alpha-\beta} \tag{14}$$

The left-hand sides of equations (13) and (14) represent the optimal values of the number of researchers in the population and the ratio of researchers to the labor force, respectively. As stated earlier in this section, the parameter $\beta$ is thought to be a very small fraction of 1. Remember that this was assumed partly in order to roughly preserve the conditions that the portion of national income going to capital is around one third, and that to labor is roughly two thirds. Thus, we know that $1 - \alpha - \beta$ takes a value around two thirds. $\frac{\lambda}{1-\phi}$ is thought to take a value close to, but above, 1. If one is inclined to somewhat simplify equation (14), $\frac{\lambda\beta}{1-\phi}$ can be approximated to $\beta$. While this maneuver is not strictly true to the model, it can provide an idea of the optimal portion of the labor force that should be dedicated to research. The resulting expression is as follows:

$$\frac{L_A}{L} = \frac{\beta}{1-\alpha} \tag{15}$$

From this simplified expression, two conclusions can - tentatively - be drawn. Firstly, the greater the importance of new technology in the economy's production of goods and services, the greater the proportion of researchers in the labor force should be. Furthermore, in cases wherein capital income constitutes a large portion of national income, the optimal ratio of researchers to labor force will be high.

## 5. Testing the Model

### 5.1: Description of Tests Performed

The testing of the model presented in this paper will be done by estimating the predicted per capita GDP growth and the level of per capita GDP as described in equations (9) and (12), respectively. These tests will be done over the course of the period 1997-2014. The values of the relevant variables will be drawn from the World Bank database where possible and appropriately estimated in other cases. The necessary parameters will in a similar fashion be estimated. The values resulting from the tests will then be compared to the actual levels and growth rates of per capita GDP over this time period. Tables of results for each of the tests can be found in the respective subsections.

In order to test the model, estimations of the parameters in the equation technological change are required. While attempts have been made at estimating these parameters, there exists little consensus within academia regarding their values (Kruse-Andersen, 2017). Furthermore, many estimations have been made using modified versions of whichever model is being analyzed. For this reason, the estimations made differ significantly in their respective results. This paper also performs testing for a modified model, which makes any estimations previously performed of questionable relevance and reliability for the purposes of this paper. Because the model developed in this paper differs from Jones' model, the implications of the parameters in Jones' model will possibly be quite different from those in the model in this paper.

The values established for the relevant parameters are as follows:

$\alpha = {}^{1}/_{3}$ (Valentinyi & Herrendorf, 2008)

$\beta = 0{,}2616\%$ (EconStats)

$\beta$-coefficient = 18 – This coefficient is explained in subsection 5.2.

$\lambda = 0{,}9$

$\phi = 0{,}9$

$\theta = 10$

$\delta = 5{,}9\%$ (Nadiri & Prucha, 1993)

$\alpha$ has been measured by Valentinyi and Herrendorf (2008) by taking the weighted average of the capital shares of various sectors in the economy. The capital shares of the respective sectors have been measured using producer prices. $\beta$ has been measured using the EconStats data on royalty and license fee payments. EconStats has collected its data from the World Bank. $\beta$ is proven in the following subsections to be too small for the model to provide reasonable predictions. For this reason, the $\beta$-coefficient has been added. The magnitude of this coefficient was chosen based on which value resulted in predictions of the correct order of magnitude. $\lambda$, $\phi$, and $\theta$ were given values that were deemed reasonable based on which values gave the most reasonable predictions. $\lambda$ and $\phi$ are parameters for which little consensus has been reached (Kruse-Andersen, 2017). $\theta$ was shown to have little effect on the magnitude of the predictions of GDP per capita. The estimation of $\delta$ by Nadiri and Prucha (1993) is based on the US manufacturing sector between the years 1960 and 1988.

### 5.2: Testing for the Economic Growth Rate

The test performed for this subsection resulted in values for GDP per capita growth that were of the correct order of magnitude. There were significant discrepancies during periods of economic crises. This resulted in a high standard deviation in the ratio between the predicted and the observed growth values. While the predictions were approximately correct on average, they varied in value to a higher degree than the observed growth rates. The results are illustrated in figure 2. In the following paragraphs, a line of reasoning is described in performing this first test and in the interpretations of the resulting values.

| Year | GDP p.c. growth (predicted) | GDP p.c. growth | Ratio, predicted/observed |
|---|---|---|---|
| 1996 | - | 2,28% | - |
| 1997 | 1,42% | 2,68% | 0,53 |
| 1998 | 2,58% | 2,07% | 1,25 |
| 1999 | 2,01% | 2,51% | 0,80 |
| 2000 | 1,73% | 3,29% | 0,52 |
| 2001 | 2,26% | 0,65% | 3,49 |
| 2002 | 1,40% | 0,85% | 1,64 |
| 2003 | 3,52% | 1,24% | 2,83 |
| 2004 | 1,62% | 2,45% | 0,66 |
| 2005 | 1,60% | 1,98% | 0,81 |
| 2006 | 2,24% | 2,19% | 1,02 |
| 2007 | 0,19% | 1,74% | 0,11 |
| 2008 | 2,57% | -0,60% | - 4,30 |
| 2009 | 1,27% | -4,23% | - 0,30 |
| 2010 | 0,74% | 2,20% | 0,34 |
| 2011 | 1,73% | 1,28% | 1,35 |
| 2012 | 1,39% | 0,62% | 2,25 |
| 2013 | 1,07% | 0,77% | 1,39 |
| 2014 | 8,29% | 1,40% | 5,93 |
| | | Ratio mean | 1,13 |
| | | Ratio standard deviation | 1,99 |

*Figure 2.*

In testing the model, the OECD has been aggregated to constitute one economy, with a collective research and development effort. Equation (9) has been solved for per capita GDP growth and then compared to the actual growth numbers for the OECD over the period 1996-2014. Over this time period, the model developed in this paper consistently underestimates the economic growth for the OECD unless the assumption that $0 \leq \lambda \leq 1$ is completely relaxed. With a value for $\phi$ of between 0.6 and 0.9, the value for $\lambda$ has to be increased to between 15 and 30 to reasonably fit the actual annual growth numbers. Note that a higher value for $\phi$ lowers the necessary corresponding value of $\lambda$ because of the nature of the fraction including these two parameters in equation (9).

A value for $\lambda$ that is drastically larger than one has strong implications for the theoretical nature of the research and development sector. Such a high value implies that there is no significant problem of overlapping research when the number of researchers increases. Rather, this effect is strongly outweighed by some mechanism of increasing returns to scale in the research sector with respect to the number of researchers. Either this divergence from previous models with exogenously advancing technology is highly problematic, or the nature of this parameter must be rethought.

There turns out to be another way of looking at this issue. In testing the model, the ratio $\frac{\lambda}{1-\phi}$ must be very high in order to result in positive and reasonable growth rates. This, however, is to a large extent due to the fact that the ratio $\frac{\beta}{1-\alpha}$ is extremely small with the current method of calculating $\beta$. It is a possibility that the problem with the model lies not in the interpretation of $\lambda$, but in the calculation of $\beta$.

For this reason, a positive coefficient has been added to equation (9), which has been multiplied with $\beta$. Considering the novelty of and the corresponding lack of knowledge about the model presented in this paper, it is difficult to adequately justify the size – or even the existence – of this coefficient. Nevertheless, after implementing this coefficient, a line of reasoning must be made in order to justify its presence and its magnitude.

Once this coefficient is introduced and set to an appropriate magnitude, the model is able to somewhat accurately predict the economic growth rates within the OECD. More specifically, the coefficient leads to predictions of the correct order of magnitude. As described in the following paragraphs, there are significant fluctuations and discrepancies in the predictions, which are not reflected in the actual data. This said, when the coefficient is given a magnitude of 18, equation (9) provides predictions that fit reality to a satisfactory, although not remarkable, degree. The fact that $\beta$ turned out to be too small when measured by total annual royalty and license fee payments indicates that these payments do not fully encompass the importance of technology as a compensated factor of production in developed economies such as the OECD.

Too little research has been done within this specific area, and any conclusions about the actual meaning of the $\beta$-parameter must therefore be accompanied by caution. That said, it is possible that royalty payments do not encompass the full importance of new technology simply because not all innovators allow other economic agents to use their innovations, even for a fee. For instance, a pharmaceutical company that develops a new medication generally monopolizes the production and sale of said medication during the validity period of the patent. They typically do not allow rival companies to sell their medication for a royalty fee. Thus, there seems to be a large number of innovations that are never included in the $\beta$-parameter in its current method of measurement. Perhaps if this condition were to be remedied – that is, if these innovations, too, were to be included in the $\beta$-parameter – the model would predict reality with some accuracy even without this multiplicative coefficient.

Even with the coefficient, there are some noteworthy discrepancies between the predicted annual per capita GDP growth rates and the actual growth rates. The most apparent of these discrepancies lies in the years during and after the global financial crisis of 2008. This result, and the following points, are illustrated in figure 2. After the crash, this paper's model quite consistently overestimates the growth rates of per capita GDP. During the years following the Great Recession, economic growth rates in the OECD member countries have been sluggish. It seems as though the model presented here fails to take into account various factors that have increased in relevance since the recession, which negatively affect economic growth.

Another significant discrepancy is apparent in the year 2007, for seemingly different reasons. In 2007, the model vastly underestimates the rate of economic growth in the OECD. This, in itself, is not very peculiar; the model may require some refining measures in order to improve its year-by-year fit with reality. However, the core reason for the low predicted growth rate for this year is that the growth in the number of researchers in the OECD was very low. According to the model in its current form, this temporary dip in the growth of the number of researchers should have led to a corresponding decrease in per capita GDP growth. A similar but opposite anomaly is present in the years 2001-2003, when there was a high rate of growth in the number of researchers but a low level of per capita GDP growth. This period is, however, also characterized by a financial crisis – namely, the burst of the dot-com bubble. The high growth rate in the number of researchers in these years should, according to the current form of the model, have corresponded to a high economic growth rate.

The reasoning in the paragraph above is, in reality, quite unreasonable. If this line of reasoning is followed literally, it implies that any given quarter, month or even week with low rates of growth in the number of researchers should see very little, or negative, growth in per capita GDP. The absurdity in this claim leads the reasoning to the possibility of timeframe issues. There are two apparent possibilities, which are not mutually exclusive. Firstly, this model may be better applicable over longer periods of time. That is, this model may be better applicable in the comparison of growth rates between decades than between years. The second possibility is that this paper's model might fit better with reality if one considered the average growth rate in the number of researchers over the last, say, five years rather than a year-by-year measurement of this growth rate. Since data on the number of researchers is not available for very many decades in the past, this second possibility provides a higher probability of the applicability of this model.

The model presented in this paper undoubtedly exhibits some shortcomings empirically, especially on a year-by-year basis. However, apart from the aforementioned periods of weak predictions, the model predicts the economic growth rates in the OECD with an adequately satisfactory degree of accuracy. It seems as though the model performs adequately in the absence of large financial shocks. The model seems to perform better before the 2008 financial crisis and the current era of aggressive expansionary monetary policy. It is likely not this monetary policy that is the cause of the weaker fit of the model in later years. It is more probable that the weak fit of the model and the contemporary monetary policy share some cause.

### 5.3: Testing for the Level of GDP Per Capita

The model tests on the level of GDP per capita exhibit a similar problem to those conducted on GDP per capita growth; the central importance of the labor force and number of researchers and the related variables lead to a very high volatility in the predictions made by the model, as illustrated in figure 3. For years during which the number of researchers in the OECD member countries barely grew, such as 2007, the model predicts levels of GDP per capita that far exceed any reasonable value. It is reasonable to assert that low growth in the personnel of the research sector might have a short-term benefit to the size of the economic pie. However, the model presented in this paper seems to be far too dependent on this variable.

| Year | GDP per capita (predicted) | GDP per capita | Ratio, predicted/observed |
|---|---|---|---|
| 1996 | - | 22 821 | - |
| 1997 | 34 226 | 22 311 | 1,53 |
| 1998 | 22 736 | 22 384 | 1,02 |
| 1999 | 28 086 | 23 353 | 1,20 |
| 2000 | 32 030 | 23 687 | 1,35 |
| 2001 | 26 280 | 23 305 | 1,13 |
| 2002 | 37 541 | 24 146 | 1,55 |
| 2003 | 20 216 | 26 847 | 0,75 |
| 2004 | 37 582 | 29 669 | 1,27 |
| 2005 | 39 205 | 31 142 | 1,26 |
| 2006 | 31 827 | 32 593 | 0,98 |
| 2007 | 147 926 | 35 361 | 4,18 |
| 2008 | 30 320 | 37 140 | 0,82 |
| 2009 | 48 210 | 34 500 | 1,40 |
| 2010 | 69 732 | 35 907 | 1,94 |
| 2011 | 40 953 | 38 386 | 1,07 |
| 2012 | 47 974 | 38 079 | 1,26 |
| 2013 | 58 110 | 38 289 | 1,52 |
| 2014 | 13 435 | 38 828 | 0,35 |
| | | Ratio mean | 1,37 |
| | | Ratio standard deviation | 0,79 |

*Figure 3.*

Although the focus of this paper is on the economic growth rate rather than the level of GDP per capita, it is necessary to be able to test the model for the level as well, if the model is to be considered complete. No claim is made here that the model developed here is flawless or complete. However, in order to holistically evaluate the model, it must be estimated with respect to the level, as well as the growth, of GDP per capita. Values for GDP per capita are presented in current US$.

While it would be interesting and beneficial to estimate the model in terms of equation (12) – that is, compare the actual GDP per capita values with the values extracted from the model – caution must be heeded, and constraints made clear, before any such estimation is conducted. The difference between equations (9) and (12) and the corresponding estimation tasks lies in the number and significance of hard-to-estimate parameters. Worth noting is that both equations also rely heavily on variables which, if misestimated or misinterpreted, may drastically change the outcome of the model's prediction. Equation (9), which determines per capita economic growth, depends on the parameters $\lambda$ and $\phi$, as well as the accompanying variables of the growth rates of the number of researchers and the labor force, as well as the sizes of the parts of the labor force dedicated to research and production, respectively. The parameters are, to be sure, important in determining both the sign and the magnitude of the economic growth rates predicted by the model.

In order to test the model with respect to the level of GDP per capita, equation (12) must be estimated. For this to be possible, the level of capital per person must be estimable. For this reason, the variable $\tilde{k}$ has been created, where $\tilde{k} = \frac{K}{AL}$. The variable thus represents the monetary value of the capital per person and unit of technology in the economy. The variable has been created because it is constant in the final steady state, in contrast to the variable $k$ (capital per person), which may grow as technology advances. The reason that this is important is because $k$ can be isolated in the equation describing the change in $\tilde{k}$ over time, $\dot{\tilde{k}}$. The process is described in Appendix 1. The resulting equation is:

$$k = \frac{sy}{\delta + \frac{\lambda g_{L_A}}{1-\phi} + g_L} \qquad (16)$$

With the help of equation (16), the capital stock per worker can be estimated using the parameter values chosen in the estimation of equation (9). Furthermore, equation (12) depends on not only the parameters $\lambda$ and $\phi$, but on $\theta$ as well. Granted, the former two parameters are difficult to estimate, as illustrated by the lack of academic consensus regarding their true values. $\lambda$ and $\phi$ are, however, parameters relevant at the margin, meaning that they can be semi-readily estimated with the help of changes in the technology variable. It is, after all, changes in the level of technology that the field of economic growth is primarily concerned with. The actual numerical level of technology is secondary in importance. This brings us to the major problem associated with estimating equation (12); $\theta$. This parameter is merely a catch-all productivity measure in the field of research and development. It has no unit, and no firm connection to real values. This is what makes it so difficult to estimate.

As it turns out, the $\theta$-parameter does not weigh very heavily in the equation, and any changes in its value has only very limited effects on the end result. The reason for this is that the parameter is multiplied with a very low number, namely $1 - \phi$. This is fortunate, considering the ambiguity in definition and value of $\theta$.

The coefficient applied multiplicatively to $\beta$ must also be included in the tests of equation (12) in order for the resulting GDP per capita predictions to be in the correct order of magnitude. Fortunately, the appropriate size of this coefficient seems to be – at least approximately – the same when testing for both income growth and income levels.

## 6. Conclusion

In this paper, an attempt has been made to internalize the growing importance of the research and development sector over the past decades into a theoretical framework. This has been done by considering technology as an income-compensated factor of production. The goal of the model was to better account for the remarkable increase in R&D efforts in developed economies, given that this increase has not resulted in raised economic growth rates. This paper has aimed to more satisfactorily than many previous endogenous growth theories illustrate the causal relationship between economic growth and research intensity. The OECD was used as the subject for the tests in this paper, since this group of countries provide a high portion of innovations that are then used internally. The hypothesis on which this paper rests is that economies, in our modern world, are actually required to increase the resources allocated to R&D in order to maintain current economic growth rates. Furthermore, these increases are thought to be effective only for a finite amount of time, after which further increasing the research intensity would be of neutral or negative value to economic growth.

In its original form, the model consistently predicted economic growth rates during the examined period that were notably lower than the actual growth rates. Similarly, it consistently underpredicted the level of GDP per capita. It was therefore reasoned that the $\beta$-parameter – representing the portion of the national income that accrues to the owners of patented technology – was smaller than it needed to be in order to create fitting predictions. Subsequently, a multiplicative coefficient was attached to $\beta$ in order to square the model with reality.

When the $\beta$-coefficient was included, the model generally predicted growth rates and levels of GDP per capita that were of the correct order of magnitude. However, the predictions were highly sensitive to fluctuations in the growth rate of the number of researchers, which led to great volatility in the predictions. The model was also unable to predict dips in economic growth rates due to economic crises.

Any attempt to refine the model presented in this paper should include efforts to make the model's predictions less sensitive to the growth rate in the number of researchers. In the long term, it is probable that this growth rate affects the growth and level of GDP per capita to a significant extent, but it should not affect the year-by-year predictions to the extent that it does in the model's current state. This problem could be partially solved either by using a longer-term average researcher growth rate or by focusing the model on decade-by-decade predictions. However, it is likely that the problem persists even after these alterations. For this reason, the reliance on the growth rate in the number of researchers should be reduced somewhat in order to make its predictions more reliable.

As for the model's ability to account for economic crises, two viewpoints should be considered. Either a mechanism for internalizing the effects of economic shocks should be introduced into the model, or the model should be used mainly for long-term predictions on economic growth. Given that data on research intensity is limited in availability far back in time, an economic-shock component should be implemented. However, this might render the model too complex to be manageable. If so, any use of the model should be applied to the long term.

One area that may be of interest in further studies on this subject is the assumptions relating to the research sector. This sector is described in subsection 4.2, and in its current form follows the same assumptions as the research sector in previous models of endogenous technological change. One simplifying assumptions in these previous models, which has also been adopted in this paper, is that patents last forever. This means that any one capital good is produced by a monopolist on an indefinite basis. Because the model presented in this paper places research in the center of attention, this assumption should be reexamined in follow-up studies. What happens to the research sector and the intermediate goods sector when the assumption of indefinite patents is relaxed?

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## Appendix I

**Deriving equation (4) from equation (3b):**

$g_A = \theta {L_A}^{\lambda} A^{\phi-1}$ (3b)

$\ln(g_A) = \ln\left(\theta {L_A}^{\lambda} A^{\phi-1}\right)$

$\ln(g_A) = \ln(\theta) + \lambda \ln(L_A) + (\phi - 1)\ln(A)$

$\frac{d\ln(g_A)}{dt} = \frac{d\,(\ln(\theta))}{dt} + \lambda \frac{d(\ln(L_A))}{dt} + (\phi - 1)\frac{d(\ln(A))}{dt}$

$g_{g_A} = g_\theta + \lambda g_{L_A} + (\phi - 1)g_A = 0$ - Note, $\theta$ is a constant parameter, so $g_\theta = 0$

$(1 - \phi)g_A = \lambda g_{L_A}$

$g_A = \frac{\lambda g_{L_A}}{(1-\phi)}$ (4)

**Deriving equation (5) from equation (2):**

$y = k^{\alpha}(\frac{A}{L})^{\beta}(\frac{L_Y}{L})^{1-\alpha-\beta}$ (2)

$\ln(y) = \ln(k^{\alpha}\left(\frac{A}{L}\right)^{\beta}(\frac{L_Y}{L})^{1-\alpha-\beta})$

$\ln(y) = \alpha\ln(k) + \beta\left(\ln(A) - \ln(L)\right) + (1 - \alpha - \beta)(\ln(L_Y) - \ln(L))$

$\frac{d\ln(y)}{dt} = \alpha\frac{d\ln(k)}{dt} + \beta\left(\frac{d\ln(A)}{dt} - \frac{d\ln(L)}{dt}\right) + (1 - \alpha - \beta)\left(\frac{d\ln(L_Y)}{dt} - \frac{d\ln(L)}{dt}\right)$

$g_y = \alpha g_k + \beta(g_A - g_L) + (1 - \alpha - \beta)(g_{L_Y} - g_L)$ - Note, $g_y = g_k$ in the weak steady state. Therefore:

$g_y = \alpha g_y + \beta(g_A - g_L) + (1 - \alpha - \beta)(g_{L_Y} - g_L)$

$(1 - \alpha)g_y = \beta(g_A - g_L) + (1 - \alpha - \beta)(g_{L_Y} - g_L)$

$(1 - \alpha)g_y = \beta(g_A - g_L) + \beta\left(g_L - g_{L_Y}\right) + (1 - \alpha)(g_{L_Y} - g_L)$

$(1 - \alpha)g_y = \beta\left(g_A - g_{L_Y}\right) + (1 - \alpha)(g_{L_Y} - g_L)$

$g_y = \frac{\beta}{(1-\alpha)}\left(g_A - g_{L_Y}\right) + (g_{L_Y} - g_L)$ - Note, $g_A = \frac{\lambda g_{L_A}}{(1-\phi)}$ (4)

$g_y = \frac{\beta}{1-\alpha}((\frac{\lambda}{1-\phi})g_{L_A} - g_{L_Y}) + (g_{L_Y} - g_L)$ (5)

**Deriving equation (6):**

$L = L_A + L_Y$

$g_L = \frac{L_A}{L_Y} g_{L_A} + \frac{L_Y}{L_A} g_{L_Y}$

$g_{L_Y} = \frac{L}{L_Y} g_L - \frac{L_A}{L_Y} g_{L_A}$ (6)

**Deriving equation (9) from equations (5) and (8):**

$g_y = \frac{\beta}{1-\alpha}((\frac{\lambda}{1-\phi})g_{L_A} - g_{L_Y}) + (g_{L_Y} - g_L)$ (5)

$g_{L_Y} = g_L + (\frac{L_A}{L_Y})(g_L - g_{L_A})$ (8)

$g_y = \frac{\beta}{1-\alpha}(g_{L_A}(\frac{\lambda}{1-\phi}) - g_L - (\frac{L_A}{L_Y})(g_L - g_{L_A})) + (g_L + (\frac{L_A}{L_Y})(g_L - g_{L_A}) - g_L$

$g_y = \frac{\beta}{1-\alpha}(g_{L_A}(\frac{\lambda}{1-\phi} + \frac{L_A}{L_Y}) - g_L(1 + \frac{L_A}{L_Y})) + (\frac{L_A}{L_Y})(g_L - g_{L_A})$ (9)

**Deriving equation (10):**

$g_A = \frac{\lambda g_{L_A}}{1-\phi} = \theta {L_A}^{\lambda} A^{\phi-1}$

$A^{\phi-1} = \frac{\lambda g_{L_A}}{\theta {L_A}^{\lambda}(1-\phi)}$

$A = (\frac{\lambda g_{L_A}}{\theta {L_A}^{\lambda}(1-\phi)})^{\frac{1}{\phi-1}}$

$A = (\frac{\theta {L_A}^{\lambda}(1-\phi)}{\lambda g_{L_A}})^{\frac{1}{1-\phi}}$ (10)

**Deriving equations (13) and (14):**

$$\frac{dy}{dL_A} = k^{\alpha}(\frac{\theta(1-\phi)}{\lambda g_{L_A}})^{\frac{\beta}{1-\phi}}[(\frac{\lambda\beta}{1-\phi})(L_A)^{\frac{\lambda\beta}{1-\phi}-1}(\frac{(L-L_A)^{1-\alpha-\beta}}{L^{1-\alpha}})$$

$$-(1-\alpha-\beta)(L_A)^{\frac{\lambda\beta}{1-\phi}}(\frac{(L-L_A)^{-\alpha-\beta}}{L^{1-\alpha}})] = 0$$

$(\frac{\lambda\beta}{1-\phi}))(L_A)^{\frac{\lambda\beta}{1-\phi}-1}\frac{(L-L_A)^{1-\alpha-\beta}}{L^{1-\alpha}} = (1-\alpha-\beta)(L_A)^{\frac{\lambda\beta}{1-\phi}}(\frac{(L-L_A)^{-\alpha-\beta}}{L^{1-\alpha}})$

$(\frac{\lambda\beta}{1-\phi})(L-L_A) = L_A(1-\alpha-\beta)$

$L_A(\frac{\lambda\beta}{1-\phi}+1-\alpha-\beta) = L(\frac{\lambda\beta}{1-\phi})$

$L_A = L(\frac{\frac{\lambda\beta}{1-\phi}}{\frac{\lambda\beta}{1-\phi}+1-\alpha-\beta})$ (13) <==> $\frac{L_A}{L} = \frac{\frac{\lambda\beta}{1-\phi}}{\frac{\lambda\beta}{1-\phi}+1-\alpha-\beta}$ (1

**Deriving equation (16):**

$\dot{\tilde{k}} = (\frac{\dot{K}}{AL}) = (\frac{K}{AL})(\frac{\dot{K}}{K}-\frac{\dot{A}}{A}-\frac{\dot{L}}{L})$

$= \tilde{k}\left(\frac{sY-\delta K}{K}-g_A-g_L\right) = \tilde{k}\left(s\frac{Y}{K}-\delta-g_A-g_L\right) = 0$ in steady state.

Since we know that $\tilde{k} \neq 0$, the terms within the parentheses must be equal to zero.

$s\frac{Y}{K}-\delta-g_A-g_L = 0$

$s\frac{Y}{K} = \delta+g_A+g_L$

$\frac{Y}{K} = \frac{\delta+g_A+g_L}{s}$

$\frac{K}{Y} = \frac{s}{\delta+g_A+g_L}$

$K = \frac{sY}{\delta+g_A+g_L}$

$k = \frac{K}{L} = \frac{sy}{\delta+g_A+g_L}$

$k = \frac{sy}{\delta+\frac{\lambda g_{L_A}}{1-\phi}+g_L}$ (16)

# Appendix II

(World Bank [1]): *GDP per capita (Current US$) | Data*

(World Bank [2]): *Researchers in R&D (per million people) | Data*

(World Bank [3]): *Population, total | Data*

(World Bank [4]): *Research & development expenditure (% of GDP) | Data*

(World Bank [5]): *Patent applications, residents | Data*