

# On Robustness of Equilibria in Transportation Networks

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# Abstract

As infrastructural systems become ever more complex and interconnected, they may also become ever more vulnerable to system-wide faults due to local disturbances. As such it is of great importance to design these system to be resilient, i.e. to be able to withstand and recover from disturbances or new conditions. In the case of traffic networks, while much work has been done to analyze the stability of these systems, there is still little work to analyze their resilience.

This thesis analyzes a variant of Daganzo's Cell Transmission Model to explore the robustness of equilibria in dynamical flow networks in response to various perturbations. In particular it tries to characterize the set of perturbations which force a freeflow equilibrium out of freeflow. Since any such equilibrium is locally asymptotically stable, the retention of freeflow would thus ensure a retention of stability.

The report first finds the smallest necessary size (in  $\ell_1$ -norm and for arbitrary affine cost functions) of any deterministic perturbations to the exogenous inflows to violate freeflow. Second it finds bounds for the probability of the equilibrium flows to violate freeflow due to stochastic exogenous inflows; either normally or independent, exponentially distributed. Third it finds the new equilibrium matrix and a condition for the retention of freeflow following a single-cell routing perturbation. Finally it simulates a simple network's performance in response to periodic exogenous inflows and cell mass increments, where it is shown that exogenous inflows with feasible averages may still cause system-wide faults and that mass increments are more disruptive the further away the affected cell is from a drain cell.

# Acknowledgements

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## Notation

Listed below is a short-hand for most of the notation used in this thesis.

$\mathbb{1}$	An all-one vector of appropriate dimension.
$\mathbb{R}_+$	The set of non-negative real numbers.
$\mathbb{R}^{n \times m}$	An $n$ by $m$ matrix with only real-valued elements. If $m = 1$ that argument may be omitted, i.e. $\mathbb{R}^{n \times 1} = \mathbb{R}^n$ .
$\alpha_i$	The cost of additional exogenous inflow to cell $i$ .
$A$	The associated cost of a perturbation.
$\gamma_i(x)$	The flow control parameter for cell $i$ using FIFO routing rule.
$\Gamma(x)$	The diagonal $\mathbb{R}_+^{n \times n}$ flow control matrix using FIFO routing rule.
$\Lambda_F$	The set of exogenous inflows for which the (freeflow) equilibrium flows are all below capacity.
$\sigma_i(x_i)$	The supply function for cell $i$ .
$\phi_i(x_i)$	The demand function for cell $i$ .
$C_i$	The capacity of cell $i$ .
$\mathcal{E}$	The set of links/edges in the graph or network.
$\mathcal{F}$	The freeflow region, where there are no upstream congestion effects for any cell.
$\mathcal{G}$	A graph consisting of (at least) a set of nodes and a set of links.
$h_{ij}$	The equilibrium weights which gives the effect of inflow $u_j$ on the outflow of cell $i$ at equilibrium.
$H$	The $\mathbb{R}_+^{n \times n}$ equilibrium matrix describing the effects of the exogenous inflows on the flow equilibrium.
$n$	The number of nodes/cells in the network, $n =  \mathcal{V} $ .
$R_{ij}$	The fraction of the total outflow from cell $i$ that goes to cell $j$ .
$R$	The $\mathbb{R}_+^{n \times n}$ routing matrix describing how flow is diverted within the network.
$\mathcal{R}$	The set of cells who accept exogenous flow.
$\mathcal{S}$	The set of cells who diverts part of their outflow out of the network.
$u_i$	The exogenous inflow to cell $i$ .
$\mathcal{V}$	The set of nodes/cells in the network.
$x_i$	The state/mass of cell $i$ .
$z_i$	The total outflow from cell $i$ .
$z_i^*$	The total outflow from cell $i$ at equilibrium.
$a \leq b$	In the case of vectors $a, b$ of equal size $N$ , this operation implies that $a$ is element-wise less than $b$ , i.e. $a_i \leq b_i$ for all $1 \leq i \leq N$ .

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# 1

## Introduction

### 1.1 Background and purpose

Resilience—defined as the ability of systems to plan and prepare for, absorb, respond to, and recover from disasters and adapt to new conditions [1]—has become a key aspect in the design of critical infrastructure networks, such as energy or transport systems. As they become ever more complex and interconnected, their response even to small local perturbations may cause major cascading problems throughout the system. This in turn makes the study of modeling and designing these networks of ever more interest and importance.

In this thesis we will analyze traffic networks using a version of Daganzo’s cell transmission model (CTM) [2, 3]. While CTM was used to model highway traffic, later efforts have been made to apply similar models to urban traffic networks. Much of these have focused on the stability of these models, stability of their flow equilibria or the conditions for convergence. In [4] S. Coogan and M. Arcak expand upon the CTM (and other models) to allow for more extensive results relating to equilibria and convergence in highway networks, and in [5] they demonstrate the stability properties of networks with diverging junctions, showing that they exhibit global asymptotic stability and thus allowing for more general topologies. In [6] E. Lovisari, G. Como and K. Salva show how the stability of the equilibrium is dependent on the exogenous inflow and that there is a set of inflows for which there are no equilibria.

The purpose of this thesis then is to instead examine the robustness of a network’s flow equilibrium, in particular their ability to remain below flow capacities following a perturbation. The goal of this examination is to characterize the set of perturbations which retain freeflow in the network, where freeflow refers to the states where no flows are stymied due to capacity issues. While the theoretical results primarily concern the linear system dynamics of the equilibrium, they may still be of use for the study of nonlinear models of traffic networks.

## 1.2 Limitations

This thesis considers a somewhat simplified model in that it is assumed time-invariant. As such, some transient behaviours are not taken into account. In order to have generally applicable results it will also not spend much focus on the particular demand and supply functions and their effects on the equilibria, but for the demand functions' relation to the flow capacities.

## 1.3 Outline

The outline is as follows: in Chapter 2 we will describe the dynamical flow network model analyzed in this thesis along with some basic concepts from graph theory. We will describe the stability conditions and properties for our model and motivate the choice of analyzing the equilibrium's ability to remain in the freeflow region.

In Chapter 3 we will then analyze a flow equilibrium's ability to accept additional exogenous flow while remaining below flow capacity. We will thus only consider non-negative inflow perturbations and we will use the  $\ell_1$ -norm to compare their sizes.

In Chapter 4, we will formulate the properties of the equilibrium flows in response to stochastic exogenous inflows (either Gaussian or exponentially distributed). We will then formulate upper and lower bounds for the probability of the equilibrium flows remaining below flow capacities.

In Chapter 5 we will move on to single-row routing matrix perturbations and find an expression for which we can update the equilibrium matrix. We will then use this to determine for which perturbations the new flow equilibrium is below flow capacity.

In Chapter 6 we will numerically analyze the ability of a simple, cyclic network using FIFO routing to accept various perturbations while being able to either remain or return to freeflow and give some qualitative measures for which perturbations are more pernicious. We will also demonstrate that some of these issues are eliminated by either eliminating all cycles or by using non-FIFO routing.

Finally we will summarize our results in Chapter 7 and suggest future work.



# 2

## Background

This Chapter has four sections: Section 2.1 will describe basic graph theory along with important definitions and concepts, while Sections 2.2 will describe the dynamical flow network model and its properties. Section 2.3 will describe some particulars of the cell transmission model and an expression for the freeflow equilibrium. Section 2.4 will then describe and motivate the explicit problem formulation for this thesis. Section 2.1 is largely based on [7] while Sections 2.2-3 are based on [8] and [6].

### 2.1 Networks as graphs

To model a network one often describes it as a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , consisting of two sets:

- the set of nodes/vertices  $\mathcal{V}$  where each node is assigned a number so that  $\mathcal{V} = \{1, 2, \dots, n\}$ . Here,  $n = |\mathcal{V}|$  is the number of nodes,
- the set of links/edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  which consists of ordered pairs  $(i, j)$  where node  $i$  is linked to node  $j$ .

If the existence of link  $(i, j)$  implies the existence of link  $(j, i)$ , the network is *undirected*; if not, it is *directed*. Links  $(i, i)$ , i.e. links whose head and tail nodes coincide, are called *self-loops*. The *out-neighbors*  $\mathcal{N}_i$  of a node  $i$  is the set of nodes  $j$  for which the links  $(i, j) \in \mathcal{E}$ , while its *in-neighbors*  $\tilde{\mathcal{N}}_i$  is the set of nodes  $j$  for which the links  $(j, i) \in \mathcal{E}$ .

In certain cases it is also useful to associate a positive scalar value  $W_{ij}$  to each link to denote its “strength” or “size”. If so, the graph is instead described as  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$  where the *weight matrix*  $W \in \mathbb{R}_+^{n \times n}$  is a square matrix where  $W_{ij} > 0$  if  $(i, j) \in \mathcal{E}$  and  $W_{ij} = 0$  if  $(i, j) \notin \mathcal{E}$ . If the weights only take the values 0 or 1 the graph is said to be *unweighted*.

A *walk* is a finite sequence of nodes  $\gamma = \{\gamma_0, \gamma_1, \dots, \gamma_l\}$ , which starts in node  $i = \gamma_0$  and ends in node  $j = \gamma_l$ , for which there exists a link  $(\gamma_{h-1}, \gamma_h) \in \mathcal{E}$  for all

$h = 1, \dots, l$ . Here,  $l$  is the *length* of the walk and walks of length 0 is said to go from a node to itself. Relatedly, a walk that does not pass through any node more than once except for possibly the first and last node, i.e.  $\gamma_h \neq \gamma_k$  for any  $0 \leq h < k \leq l$  (except possibly  $\gamma_0 = \gamma_l$ ), is called a *path*. A path of length  $l \geq 2$  which starts and ends in the same node is called a *cycle*. A graph that has no cycles is called *acyclic*; else it is *cyclic*.

If there exists a path from node  $i$  to node  $j$  then it is said that node  $j$  is *reachable* from node  $i$ . If all nodes in a graph are reachable from any other node, that graph is said to be (strongly) *connected*.

For a length- $l$  walk  $\gamma$ , its *weight* is the product of its  $l$  link weights:

$$W_\gamma = \prod_{h=1}^l W_{\gamma_{h-1}\gamma_h} \quad (2.1)$$

where walks of length 0 are said to have unitary weight. Relatedly, the sum of the weights of all length- $l$  walks can be found as the  $(i, j)$ -element of the weight matrix to power  $l$ , i.e.  $(W^l)_{ij}$ . In particular, for unweighted graphs this coincides with the total number of length- $l$  walks from  $i$  to  $j$ . For acyclic graphs, it thus holds that  $W^k = 0$  for some  $1 \leq k \leq n$  (i.e. the weight matrix is nilpotent).

A (row) *sub-stochastic matrix*  $A$  is a matrix with non-negative entries for which all row sums are less than or equal to one:

$$A \mathbf{1} \leq \mathbf{1}. \quad (2.2)$$

If all row sums are equal to one, the matrix is merely said to be stochastic. In compartmental systems with a sub-stochastic weight matrix  $W$ , a node  $i$  is said to be *out-connected* if there exists a path from  $i$  to any node  $j$  for which  $\sum_k W_{jk} < 1$ .

Weight matrices which adhere to (2.2) and for which all nodes are out-connected have a spectral radius which is strictly less than 1, i.e. the largest absolute value of any of the matrix's eigenvalues is strictly less than one. Matrices  $W$  with a spectral radius strictly less than one have the property that as  $k$  approaches infinity,  $W^k \rightarrow 0$ .

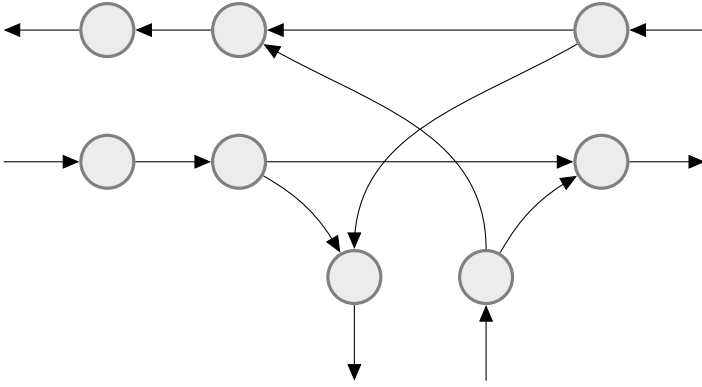
## 2.2 The dynamical flow network model

In our model of a traffic network, we compartmentalize it into discrete stretches of road. These need to be of comparable lengths and not too long. This is to avoid odd results where e.g. mass entering the cell will immediately add to the outflow from that cell several kilometers away. As such, longer stretches of road will be discretized into line sub-graphs in our graph of the entire network.

These cells will be our nodes  $\mathcal{V} = \{1, \dots, n\}$  and our edges  $\mathcal{E}$  are the connections between them. In this model we allow no self-loops, meaning that  $(i, i) \notin \mathcal{E}$  for any  $i \in \mathcal{V}$ . Figure 2.1 shows an example three-way intersection while Figure 2.2 shows how it may be discretized in space.



**Figure 2.1** Example three-way junction (courtesy of Open Street Map).



**Figure 2.2** Discretized graph of the junction in Figure 2.1.

The state  $x_i = x_i(t)$  describes the (non-negative) amount of mass/traffic located in cell  $i$ . The *internal flow matrix*  $F \in \mathbb{R}_+^{n \times n}$  denotes the non-negative flows between the cells where  $F_{ij}$  is the flow from cell  $i$  to cell  $j$ . Due to the lack of self-loops,  $F_{ii} = 0$  for all  $i \in \mathcal{V}$ . Based on simple conservation of mass, the dynamics of the state vector  $x = x(t) \in \mathbb{R}_+^n$  are

$$\dot{x} = u + F^T \mathbb{1} - F \mathbb{1} - w \quad (2.3)$$

where  $u \in \mathbb{R}_+^n$  is exogenous input/inflow,  $F^T \mathbb{1} \in \mathbb{R}_+^n$  is the flow *into* each cell from its neighboring cells,  $F \mathbb{1} \in \mathbb{R}_+^n$  is the flow *out from* each cell to its neighboring cells and  $w \in \mathbb{R}_+^n$  is the exogenous outflow.

We define *source cells*  $\mathcal{R} \subseteq \mathcal{V}$  as the subset of cells which accepts exogenous inflow (e.g. from highway on-ramps): if  $i \notin \mathcal{R}$  then  $u_i = 0$ . In turn, *drain cells*  $\mathcal{S} \subseteq \mathcal{V}$  is the subset of cells with outflow going out of the system (e.g. to highway off-ramps): if  $w_i > 0$  (for any time  $t$ ), then  $i \in \mathcal{S}$ . If  $i \in \mathcal{R}$  or there exists a path starting in a cell  $j \in \mathcal{R}$  and ending in  $i$ , then the cell  $i$  is *inflow-connected*. Similarly, if  $i \in \mathcal{S}$  or there exist a path starting in a cell  $i$  and ending in a cell  $j \in \mathcal{S}$ , then the cell  $i$  is *outflow-connected*. If all cells are outflow-connected (inflow-connected) then the network topology  $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mathcal{R}, \mathcal{S})$  is called outflow-connected (inflow-connected).

## Revised notation

To better describe the flow dynamics, we introduce the notation

$$z = F\mathbb{1} + w \quad (2.4)$$

for the *total outflows* for all cells. We introduce an additional condition which says that if there is no mass in a cell, it can not have any outflow:

$$x_i = 0 \implies z_i = \sum_{j \in \mathcal{V}} F_{ij} + w_i = 0 \quad (2.5)$$

This condition also ensures that  $x(t) \geq 0$  for all times  $t \geq 0$ . Using this notation also allows us to rewrite internal flows and the external outflows as in (2.6).

$$F_{ij} = R_{ij}z_i, \quad w_i = (1 - \sum_{l \in \mathcal{V}} R_{il})z_i, \quad i, j \in \mathcal{V} \quad (2.6)$$

where  $R_{ij}$  describes what fraction of the outflow  $z_i$  goes to cell  $j$ . As such,  $R_{ii} = 0$  since we have no self-loops and  $\sum_{j \in \mathcal{V}} R_{ij} \leq 1$  for all  $i \in \mathcal{V}$  since the sum of all outflows (internal and external) from a cell has to equal  $z_i$ . Relating all this new notation to (2.3) yields

$$\dot{x} = u - (I - R^T)z \quad (2.7)$$

where  $R \in \mathbb{R}_+^{n \times n}$  is a *routing matrix*. In this notation, a cell  $i \in \mathcal{S}$  necessarily has  $\sum_{j \in \mathcal{V}} R_{ij} < 1$  or  $w_i = 0$  at all times (as per (2.6)). Rather than keeping track of the external outflows  $w$ , this notation allows us to focus on the internal system flows and cells in  $\mathcal{S}$  (and by extension the system) will lose mass continuously.

## Monotone dynamical flow networks and their stability

Early work regarding the CTM assumed linear relations between the state of the system and the flows. To better describe real traffic behaviours, we thus have to expand the class of dynamical flow networks. A *monotone* dynamical flow network is of the form

$$\dot{x} = u + f(x) \quad (2.8)$$

where  $f(x)$  is Lipschitz-continuous (the absolute value of the rate of change is bounded) and  $(\nabla f(x))^T$  is a compartmental matrix. Monotone systems have some useful stability properties, as described in [8]:

**Theorem 2.1.** *For a monotone dynamical flow network*

$$\dot{x} = u + F^T(x)\mathbb{1} - F(x)\mathbb{1} - w(x),$$

- (i) *the partial order between external inflows and initial states is preserved by the state at any time;*
- (ii) *the  $\ell_1$ -distance between any two solutions is never increasing*

Moreover, if the external inflows  $u$  are constant in time, then:

- (iii) *every equilibrium  $x^*$  is stable;*
- (iv) *an equilibrium  $x^*$  is globally asymptotically stable iff it is locally asymptotically stable;*
- (v) *an equilibrium  $x^*$  is asymptotically stable if the compartmental matrix  $(\nabla f(x^*))^T$  is outflow-connected;*
- (vi) *the trajectory started from  $x(0) = 0$  is always entry-wise monotonically non-decreasing in time, hence convergent to a (possibly infinite) limit.*

Here,  $\ell_1$ -distance refers to the  $\ell_1$  norm where a distance  $d_1$  between two vectors  $p$  and  $q$  is calculated as

$$d_1(p, q) = \|p - q\|_1 = \sum_{j=1}^n |p_j - q_j|. \quad (2.9)$$

In other words, statement (ii) from Theorem (2.1) says that for any two solutions  $x$ ,  $\tilde{x}$  and times  $t, s \geq 0$  it holds that

$$\|x(t+s) - \tilde{x}(t+s)\|_1 \leq \|x(t) - \tilde{x}(t)\|_1. \quad (2.10)$$

## 2.3 Cell transmission model

In this section we will explore the particulars of a class of dynamical flow network models known as the cell transmission model. This will serve to support our choice in limitations for this thesis.

### Demand and supply functions

To account for congestion effects in a network we first introduce *demand functions*  $\varphi_i(x_i)$  that give an upper bound for the outflow  $z_i$  given the current mass  $x_i$ , i.e.  $0 \leq z_i \leq \varphi_i(x_i)$ . These functions are assumed to be

- Lipschitz-continuous,

- non-decreasing (the more mass, the more outflow can be provided),
- concave (as the current mass grows larger, outflow increments less or equally as much from additional mass),
- $\varphi_i(0) = 0$  (in accordance to (2.5)).

To model the back propagation of congestion down-stream, where the current mass of a cell impacts its ability to accept more mass, we introduce (non-negative) *supply functions*  $\sigma_i(x_i)$  which are assumed to be

- Lipschitz-continuous,
- non-increasing (the more mass, the less inflow can be accepted),
- concave (larger the current mass, the more inflow decrements from additional mass),
- $\sigma_i(x_i) = 0$  for all  $x_i \geq \bar{x}_i$  where  $\bar{x}_i \geq 0$  is a possibly infinite constant representing cell  $i$ 's *buffer capacity*.

The supply functions can then describe the supply constraints

$$u + R^T z \leq \sigma(x) \quad (2.11)$$

which say that, at all times, the sum of all exogenous and endogenous inflows to a cell has to be less than or equal to the value of that cell's supply function for the current mass in that cell.

The demand and supply functions interact to determine the flows in the network given the current state of the system. The effect of this interaction can be demonstrated by plotting the demand function of a cell  $i$  alongside its supply function, as in Figure 2.3. The throughput of cell  $i$  is initially dependent only on the demand function. However, at some point the demand and supply intersect and  $\varphi_i(x_i) = \sigma_i(x_i)$ . Past this point the flow is instead dependent on the supply function. This is a variant of the *fundamental diagram of traffic flows* which illustrates that as the mass of a cell increases the resulting outflow will only increase up to a point, after which congestion effects will impede the throughput.

Motivated by this behaviour, we define a cell's *flow capacity* as

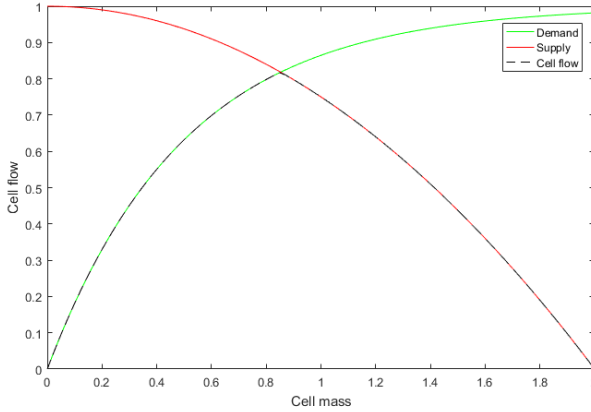
$$C_i = \max_{x_i \geq 0} (\min [\varphi_i(x_i), \sigma_i(x_i)]) \quad (2.12)$$

i.e. as the largest value of the minimum of the two for any non-negative value of  $x_i$ . As we will see later, this puts a constraint on any potential flow equilibria.

### Outside the freeflow region

The *freeflow region*  $\mathcal{F}$  is the set of system states which do not violate the supply constraints (2.11):

$$\mathcal{F} = \{x \in \mathbb{R}_+^n : u + R^T \varphi(x) \leq \sigma(x)\} \quad (2.13)$$



**Figure 2.3** The demand and supply functions of a cell, alongside the resulting cell flow.

In other words: in the freeflow region, there are no (upstream) congestion effects at any cell.

But how do we deal with the situations where we are outside the freeflow region, i.e. where the current state of the system violate the supply constraints? Remember that the demand function only provides an upper bound for a cell's outflow:

$$0 \leq z_i \leq \varphi_i(x_i) \quad (2.14)$$

To ensure that we always adhere to supply constraints we introduce *flow control parameters*  $\gamma$  which stymie the flows whenever necessary, in effect slowing down traffic. For example, if we were to stymie the total outflow of a cell  $i$  whenever that cell violates the supply constraints:

$$\begin{aligned} z_i &= \gamma_i \varphi_i(x_i), \quad 0 \leq \gamma_i \leq 1 \\ 0 &\leq \gamma_i \varphi_i(x_i) \leq \varphi_i(x_i) \end{aligned} \quad (2.15)$$

This means that whenever we are in the freeflow region  $\gamma_i = 1$  and we simply have that our outflow is  $z_i = \varphi_i(x_i)$ . How we choose the flow parameters depend on the *routing rules* that apply to the cell. These routing rules can apply to the total outflow or to each individual outgoing link of the cell, and as such the flow dynamics may be much more complex outside the freeflow region than they are within it.

In the original (real-time) cell transmission model, the flow control parameters are chosen according to a first-in-first-out (FIFO) routing rule which stymies the

total outflow of a cell if any of its out-neighbors are congested:

$$z = \Gamma(x)\varphi(x), \quad \Gamma(x) = \text{diag}(\gamma(x))$$

$$\gamma_i(x) = \sup \left\{ \xi \in [0, 1] : \max_{k \in \mathcal{N}_i} \left( \xi \sum_{h \in \mathcal{V}} R_{hk} \varphi_h(x_h) - \sigma_k(x_k) \right) \leq 0 \right\} \quad (2.16)$$

where  $\mathcal{N}_i$  is the set of out-neighbors for cell  $i$ . As we can see, this has no effect on our network flows while in the freeflow region. This routing strategy is not necessarily monotone outside the freeflow region however (page 14 of [8]) and as such Theorem 2.1 may not apply in all such cases. As we will see in Chapter 6, by using other routing strategies one can have a globally monotone network.

### The freeflow equilibrium

In the freeflow region, assuming a constant inflow  $u$  and a constant routing matrix  $R$ , we can find the *freeflow equilibrium flows* by rearranging (2.7):

$$z^* = (I - R^T)^{-1}u = Hu (\leq C) \quad (2.17)$$

where the notation  $H = (I - R^T)^{-1}$  for the *equilibrium matrix* has been introduced for future benefit. Here we have also stated the necessary condition that these equilibrium flows be at or below their respective flow capacities. By definition, all cell masses are constant at equilibrium and as such the flow *into* and flow *out of* a cell must be equal. Any net difference would give rise to a change in cell mass.

Since  $R$  is sub-stochastic ( $R\mathbb{1} \leq \mathbb{1}$ ), then if the network is out-connected the matrix  $(I - R^T)$  has a spectral radius less than 1 which allows us to re-write (2.17) as

$$z^* = (I - R^T)^{-1}u = (I + R^T + (R^T)^2 + (R^T)^3 + \dots)u \quad (2.18)$$

analogous to a geometric series. The rightmost side of (2.18) illustrates that such an equilibrium equals the direct exogenous inflow  $u_i$ , plus the weighted sum  $\sum_{j \in \mathcal{V}} R_{ji}u_j$  of the external inflows to cells one step upstream from  $i$ , plus the weighted sum  $\sum_{j \in \mathcal{V}} (R^2)_{ji}u_j$  of the external inflows to cells two steps upstream from  $i$ , and so on. This shows that  $h_{ij} = H_{ij}$  is the sum of the weights of all walks which start in cell  $j$  and end in cell  $i$ , and as such is a measure of the influence/gain of inflow  $j$  on the equilibrium flow of cell  $i$ .

A set of particular interest related to these freeflow equilibria is the *feasibility region*  $\Lambda_F$  which is the set of inflows  $u$  for which the equilibrium flows  $z^*$  are all below cell flow capacities  $C$ :

$$\Lambda_F = \{u \in \Lambda : z_i^* \leq C_i \forall i \in \mathcal{V}; z^* = (I - R^T)^{-1}u\} \quad (2.19)$$

In this region, the freeflow equilibrium flows are all allowable given the network's flow capacities. Outside of this region the flow equilibrium is instead dependent on the routing rules in use. Due to the wide range of options and permutations of these



routing rules, this region is particularly interesting if we are to make any generally applicable observations.

Finally, we will state that all freeflow equilibria have an unambiguous corresponding mass equilibrium found using the inverse demand function:

$$x_i^* = \varphi_i^{-1}(z_i^*). \quad (2.20)$$

### Properties of equilibria

We can make use Theorem 2.1 to make further statements regarding the stability of any equilibria in monotone systems:

**Corollary 2.1.1.** *Consider a transportation network with topology  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, R)$ , demand functions  $\varphi_i$ , supply functions  $\sigma_i$ , and capacities  $C_i$ . For a constant inflow vector  $u$  and a constant, out-connected routing matrix  $R$ , let  $H = (I - R^T)^{-1}$ . Then,*

- (i) *if  $Hu \leq C$  then  $x^* = \varphi^{-1}(z^*)$  is a locally asymptotically stable equilibrium for the (original) CTM;*
- (ii) *if  $Hu \not\leq C$ , then the CTM admits no equilibrium.*

We will prove this corollary:

- (i) the condition guarantees that  $x^*$  is a free-flow equilibrium for the original CTM. Local asymptotic stability follows from Theorem 2.1.
- (ii) Assume by contradiction that there exists such an equilibrium  $x^*$  for the CTM. Let  $z_i$  be the outflow from cell  $i$  at such an equilibrium. Then, flow balance implies that  $z_i = \sum_j R_{ji} z_j + u_i$  in each cell  $i$ . Hence  $(I - R^T)z = u$  so that  $z = Hu$ . On the other hand, the supply and demand constraints imply that  $z_i \leq \varphi_i(x_i^*)$  and  $z_i \leq \sigma_i(x_i^*)$  so that  $z_i \leq C_i$  for every  $i$ .

## 2.4 Problem formulation

Our goal for this thesis is to characterize the set of perturbations which retain freeflow in a network with a constant, out-connected routing matrix  $R$ . In short, we want to find the cases where the new, perturbed equilibrium  $\tilde{z}^*$  is a freeflow equilibrium and therefore abides by

$$\tilde{z}^* \leq C \quad (2.21)$$

We will do this for a few reasons, the primary one being that any freeflow equilibrium is asymptotically stable: globally so for a globally monotone network and locally otherwise. As such it is of great interest to know which perturbations cause the network to leave the freeflow region as we are guaranteed to find a (stable) equilibrium following any perturbations not in that critical set.

Another reason is that we have an explicit expression for such equilibria in (2.17) (and (2.18)) which allows us to analyze them further than we might other equilibria which depend more directly on the routing rules as well as the supply and demand functions in the network. For the purpose of this thesis's theoretical results (Chapters 3–5) we will therefore only take into account the flow capacities  $C$  relating to the equilibrium flows and we will ignore any congestion due to the supply constraints.

# 3

## Deterministic inflow perturbations

In this Chapter, a network equilibrium's tolerance to deterministic inflow disturbances is examined. We will assume that the routing matrix  $R$  is constant and we will ignore the effects of any supply constraints in the analysis. The perturbed exogenous inflow is given by

$$\tilde{u} = u + \hat{u}, \quad \hat{u} \geq 0 \quad (3.1)$$

where  $u \in \Lambda_F$  are the constant inflows causing some original freeflow equilibrium  $z^*$ . At some time the (non-negative) disturbances  $\hat{u}$  are applied and assumed to be in effect sufficiently long for a new equilibrium to be reached. Our goal is to find the smallest additional inflow  $\hat{u}_{min}$  (measured in the  $\ell_1$  norm) necessary to force the new inflow vector outside the feasibility region  $\Lambda_F$ , i.e. so that it violates the condition in (2.21):

$$\tilde{z}^* = H\tilde{u} \leq C.$$

This is found in Section 3.1. In Section 3.2 we will generalize this to where all inflows are given disturbance costs  $\alpha_i$  and the minimal disturbance cost  $A_{min}$  necessary to violate (2.21) is determined.

### 3.1 Smallest critical perturbation

For this section we want to find the smallest inflow as measured in  $\ell_1$  norm. In  $\ell_1$  norm, the size of a vector  $p$  is found as

$$\|p\|_1 = \sum_{j=1}^n |p_j|. \quad (3.2)$$

Since we assume that the perturbations  $\hat{u}$  are non-negative, we may re-write this into a simpler expression:

$$\|\hat{u}\|_1 = \sum_{j \in \mathcal{R}} \hat{u}_j. \quad (3.3)$$

The perturbed inflow  $\tilde{u}$  is assumed to be applied for a sufficiently long time that a new equilibrium is reached. This new equilibrium  $\tilde{z}^*$  is given as

$$\tilde{z}^* = H\tilde{u} = H(u + \hat{u}) = z^* + H\hat{u}. \quad (3.4)$$

Combining equations (2.21) and (3.4) gives us a condition for the disturbance

$$\tilde{z}^* = z^* + H\hat{u} \leq C \iff H\hat{u} \leq C - z^*. \quad (3.5)$$

In other words, the inflow perturbation needs to adhere a condition similar to (2.21), only with more restrictive *residual* capacities. For each cell  $i$  it holds that

$$(H\hat{u})_i \leq (C - z^*)_i \iff \sum_{j \in \mathcal{V}} h_{ij}\hat{u}_j \leq C_i - z_i^*. \quad (3.6)$$

A critical disturbance causes equality in (3.6) for at least one cell. To find the smallest disturbance necessary to violate (3.5), it is first necessary to determine the smallest disturbance necessary to cause this equality for each individual cell. In order to determine this disturbance, we will first make a proposition:

**Proposition 3.1.** *For any cell  $i \in \mathcal{V}$ : if there exists a set of cells  $k, l, \dots \in \mathcal{R}$  such that  $h_{ik} = h_{il} = \dots > h_{ij}$  for all  $j \neq k, l, \dots \in \mathcal{R}$  the smallest disturbance (in  $\ell_1$ -norm) which causes equality in (3.6) for the chosen cell is distributed among the inflows  $k, l, \dots$  or can be focused on any of the single inflows.*

To prove Proposition 3.1, consider two disturbances which cause equality in (3.6) for a cell  $i$ :  $\hat{u}^{(1)} \geq 0$  which has its disturbance solely affecting inflow  $k$  (chosen as in Proposition 3.1), and  $\hat{u}^{(2)} \geq 0$  which has its disturbance divided along two inflows  $k$  and  $m$  (where  $h_{im} \leq h_{ik}$ ). Thus

$$\begin{aligned} (H\hat{u}^{(1)})_i &= C_i - z_i^*, & (H\hat{u}^{(2)})_i &= C_i - z_i^* \\ (H\hat{u}^{(1)})_i &= (H\hat{u}^{(2)})_i \iff h_{ik}\hat{u}_k^{(1)} = h_{ik}\hat{u}_k^{(2)} + h_{im}\hat{u}_m^{(2)} \\ & & [\hat{u}_k &= \hat{u}_k^{(1)} - \hat{u}_k^{(2)}] \\ h_{ik}(\hat{u}_k^{(2)} + \hat{u}_k) &= h_{ik}\hat{u}_k^{(2)} + h_{im}\hat{u}_m^{(2)} \\ h_{ik}\hat{u}_k &= h_{im}\hat{u}_m^{(2)} \\ \hat{u}_k &= \frac{h_{im}}{h_{ik}}\hat{u}_m^{(2)}. \end{aligned} \quad (3.7)$$

Here  $\frac{h_{im}}{h_{ik}} \leq 1$  (the special case where  $h_{ik} = 0$  coincides with cell  $i$  being disconnected, making it impervious to inflow perturbations) which allows us to finally determine that

$$\|\hat{u}^{(1)}\| = \hat{u}_k^{(2)} + \hat{u}_k \leq \hat{u}_k^{(2)} + \hat{u}_m^{(2)} = \|\hat{u}^{(2)}\| \quad (3.8)$$

Analogous arguments can be made when the disturbances of  $\hat{u}^{(2)}$  are divided among more than two inflows or entirely on other inflows than  $k$ .  $\square$

Proposition 3.1 finally allows us to find the size of the smallest disturbance  $\delta_i$  which causes equality in (3.6) for cell  $i$ :

$$\delta_i = \frac{C_i - z_i^*}{\max_{j \in \mathcal{R}}(h_{ij})} \quad (3.9)$$

In other words, it is found by dividing the residual capacity by the largest inflow weight in row  $i$  of  $H$  corresponding to a source cell  $j \in \mathcal{R}$ . Finally, the smallest disturbance  $\hat{u}_{min}$  necessary to violate (3.5) is given by:

$$\hat{u}_{min} = \min_{i \in \mathcal{V}}(\delta_i) = \min_{i \in \mathcal{V}, j \in \mathcal{R}} \left( \frac{C_i - z_i^*}{h_{ij}} \right). \quad (3.10)$$

A perturbation any larger than this quotient focused on the exogenous inflow  $j$  corresponding to the minimum would thus violate the condition.

We have thus found the smallest necessary perturbation to force the new equilibrium outside of freeflow. Alternatively, we have found an upper bound for the size of a perturbation that can arbitrarily affect the inflows with no risk of forcing the inflow vector out of the feasibility region. As we can see in (3.10) it is not sufficient to merely see which residual capacity is the smallest to find our bound. We also have to consider how reachable they are from the inflows; since  $h_{ij}$  is the sum of the weights of all walks which start in cell  $j$  and end in cell  $i$ , the size of  $h_{ij}$  is proportional to how reachable cell  $i$  is from cell  $j$ . Our bound  $\hat{u}_{min}$  is thus dependent on the interaction between the original equilibrium  $z^*$  and the general network topology.

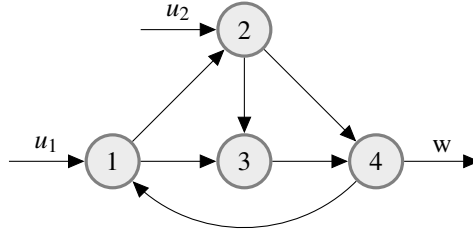
### Example

To illustrate the use of the expression found, we will consider a simple cyclic network of four cells as shown in Figure 3.1. This network has two source cells (cells 1 and 2) and only one drain cell (cell 4). The routing matrix, capacity vector and unperturbed exogenous inflows for this network are given in (3.11).

$$R = \begin{pmatrix} 0 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 4 \\ 3.2 \\ 3 \\ 5 \end{pmatrix}, \quad u = \begin{pmatrix} 2.3 \\ 0.5 \\ 0 \\ 0 \end{pmatrix} \quad (3.11)$$

The equilibrium matrix and the resulting flow equilibrium are

$$H = \begin{pmatrix} 1.25 & 0.25 & 0.25 & 0.25 \\ 0.75 & 1.15 & 0.15 & 0.15 \\ 0.80 & 0.56 & 1.16 & 0.16 \\ 1.25 & 1.25 & 1.25 & 1.25 \end{pmatrix}, \quad z^* = \begin{pmatrix} 3.00 \\ 2.30 \\ 2.12 \\ 3.50 \end{pmatrix} \quad (3.12)$$



**Figure 3.1** Simple cyclic network

Next we will calculate the residual capacities and disturbances  $\delta_{ij}$  for  $j = 1, 2$  according to (3.9).

$$C - z^* = \begin{pmatrix} 1.00 \\ 0.90 \\ 0.88 \\ 1.50 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0.80 & 4.00 \\ 1.20 & 0.78 \\ 1.10 & 1.57 \\ 1.20 & 1.20 \end{pmatrix} \quad (3.13)$$

Here we can see that  $\hat{u}_{max} = 0.78$  (applied to the exogenous inflow to cell 2) and that while cell 3 has the smallest residual capacity, it can manage perturbations larger than 1 to either inflow while remaining below capacity. This illustrates that it is insufficient to merely see which residual capacity is the smallest to find a network equilibrium's tolerance to additional exogenous inflow.

## 3.2 Cost of attack

The previous section assumes that the associated cost of disturbing any inflow is equal. In this section, the case where each inflow has a positive cost associated with it is considered. For a disturbance vector  $\hat{u}$  and a cost vector  $\alpha$  where  $\alpha_i > 0$  for all  $i \in \mathcal{R}$  we introduce a cost function as:

$$A(\hat{u}; \alpha) = \alpha^T \hat{u} = \sum_{j \in \mathcal{R}} \alpha_j \hat{u}_j \quad (3.14)$$

The previous section explored the case where all the weights were equal (and unitary), i.e.  $\alpha = \mathbb{1}$ . Now the goal is instead to find the minimal cost  $A_{min}$  necessary for any disturbance vector  $\hat{u}$  to violate the condition in (3.5) for some cost vector  $\alpha$ . These costs could correspond to e.g. the probability of increased flows or some infrastructural cost related to the current inflows. No matter how these costs are chosen, the first step is to define the disturbance to inflow  $j \in \mathcal{R}$  which causes equality for cell  $i$  in (3.5):

$$\delta_{ij} = \frac{C_i - z_i^*}{h_{ij}}. \quad (3.15)$$

If we apply any of these disturbances to their corresponding exogenous inflows, the remaining residual capacity for cell  $i$  will be zero. The cost of applying any single perturbation  $\delta_{ij}$  is found as

$$A(\delta_{ij}, \alpha) = \alpha_j \delta_{ij}. \quad (3.16)$$

There is no benefit in partitioning a disturbance along several exogenous inflows. To illustrate this, imagine that we divide our perturbation  $\hat{u}^{ex}$  along inflows  $k$  and  $l$ . To ensure that the remaining residual capacity of cell  $i$  equals zero, it will have to be a convex combination:

$$\hat{u}_j^{ex} = \begin{cases} \lambda \delta_{ik}, & j = k \\ (1 - \lambda) \delta_{il}, & j = l \\ 0, & \text{otherwise.} \end{cases} \quad (3.17)$$

From this we can discern that the cost of this disturbance will be:

$$\begin{aligned} A(\hat{u}^{ex}, \alpha) &= \alpha_k (\lambda \delta_{ik}) + \alpha_l ((1 - \lambda) \delta_{il}) \\ &[\alpha_k \delta_{ik} = a, \alpha_l \delta_{il} = b] \\ A(\hat{u}^{ex}, \alpha) &= \lambda a + (1 - \lambda) b \end{aligned} \quad (3.18)$$

Here we can see that we can minimize our cost further by changing the value of  $\lambda$ : if  $a > b$  the cost is minimized for  $\lambda = 0$  and if  $a < b$  the cost is minimized for  $\lambda = 1$ . Only in the special case  $a = b$  does it not matter what  $\lambda$  we choose, the cost will remain constant no matter our choice. Similar arguments can be made against partitioning a disturbance along 3 or more inflows. We have thus shown that there is no benefit in partitioning a disturbance along several inflows.

All that remains is to see which product between the size of an individual perturbation and the cost for the corresponding inflow is the smallest for us to determine  $A_{min}$ :

$$A_{min} = \min_{i \in \mathcal{V}, j \in \mathcal{R}} (\alpha_j \delta_{ij}) = \min_{i \in \mathcal{V}, j \in \mathcal{R}} \left( \alpha_j \frac{C_i - z_i^*}{h_{ij}} \right) \quad (3.19)$$

This result is (not very surprisingly) quite similar to the one found in Chapter 3.1, only we also take into consideration the cost of applying a perturbation on the associated exogenous inflow  $j$ .

### Example

Let us use the same example as from Section 3.1, only this time we will have different costs for the two inflows. We will choose the costs so that a greater cost correspond to an increased likelihood of perturbations along that inflow. Let us say that while inflow  $u_1$  is known to be rather stable, inflow  $u_2$  is known to fluctuate. We will therefore choose  $\alpha_2 > \alpha_1$ , in particular we will choose the cost vector:

$$\alpha = (1 \quad 1.25 \quad 0 \quad 0)^T \quad (3.20)$$

For this network, the resulting perturbations  $\delta_{ij}$  for  $j = 1, 2$  are:

$$\delta = \begin{pmatrix} 0.80 & 4.00 \\ 1.20 & 0.78 \\ 1.10 & 1.57 \\ 1.20 & 1.20 \end{pmatrix} \quad (3.21)$$

To find the associated costs of the perturbations in  $\delta$ , we multiply the first column by  $\alpha_1 = 1$  and the second column by  $\alpha_2 = 1.25$ . This gives us the costed perturbations  $\delta^\alpha$ :

$$\delta^\alpha = \begin{pmatrix} 0.80 & 5.00 \\ 1.20 & 0.98 \\ 1.10 & 1.96 \\ 1.20 & 1.50 \end{pmatrix} \quad (3.22)$$

From this we can see that  $A_{min} = 0.80$  is the minimal cost necessary to force the new equilibrium out of the feasibility region. And unlike the example in Section 3.1, we have changed both which cell  $i$  which is at risk as well as which exogenous inflow  $j$  which causes the most issue.

### 3.3 Addendum - Capacity perturbations

While we have focused on inflow perturbations in this Chapter, the results we have gathered can be used to address the issue of flow capacity perturbations:

$$\tilde{C} = C - \hat{C}, \quad \hat{C} \geq 0. \quad (3.23)$$

In particular we can give an upper bound  $\hat{C}_{max}$  on the largest reduction in flow capacities (measured in  $\ell_1$ -norm) for which the equilibrium remains a freeflow equilibrium:

$$\hat{C}_{max} = \min_{i \in \mathcal{V}} (C_i - z_i^*). \quad (3.24)$$

In short,  $\hat{C}_{max}$  equals the smallest marginal capacity of any cell. Only perturbations greater than this are able to force the equilibrium out of freeflow.



# 4

## Stochastic inflow perturbations

In this Chapter, we will explore a network's resilience to stochastic perturbations in the exogenous inflows. Since exogenous inflows are unlikely to be entirely deterministic and predictable, this analysis may be of better use in some cases. Like in Chapter 3, we will again limit our analysis to the cases where the routing matrix  $R$  is constant and where there are no upstream congestion effects, allowing us to ignore the effects of any supply functions.

We assume that the exogenous inflows are random variables whose expected values who are within the feasibility region  $\Lambda_F$ , and that they change slowly enough for the network to reach a new equilibrium before changing again. We will then formulate some bounds for the probability for the resulting equilibrium flows to exceed the flow capacities:

$$P(\bar{z}^* \not\leq C) = 1 - P(\bar{z}^* \leq C) = 1 - P(C - \bar{z}^* \geq 0) \quad (4.1)$$

In Section 4.1 we will give the general expressions for our lower and upper probability bounds. In Section 4.2 we will assume that the inflows are normally distributed while in Section 4.3 they are assumed independently, exponentially distributed. Whether it is reasonable for the exogenous inflows to have these distributions will be noted in their respective sections. Finally in Section 4.3 we will do a short example to show the validity of these bounds.

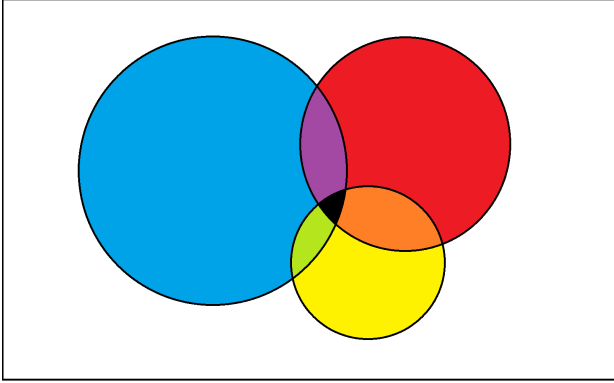
### 4.1 Probability bounds

Due to the complex connections of a real-world network and the possible interdependencies of its exogenous inflows, it is difficult to formulate very strict bounds for any probabilities with regard to the resulting network flows. As such the bounds described in this thesis are quite lenient and make as few assumptions of the characteristics of either the network or the exogenous inflows as possible. In short, the range of values between our lower and upper bounds may be quite large.

For our lower bound of the probability that the equilibrium flows will exceed their capacities, we will choose the largest probability for any single cell to exceed its capacity (independent of the other cells):

$$b_L = \max_{j \in \mathcal{V}} [P(\tilde{z}_j^* > C_j)] = \max_{j \in \mathcal{V}} [1 - P(\tilde{z}_j^* \leq C_j)] \quad (4.2)$$

That this is a lower bound is quite intuitive: the probability that *at least* one capacity is violated is necessarily greater than or equal to the largest likelihood that any *single* capacity is violated. To illustrate the validity of this bound, consider the Venn diagram in Figure 4.1 where the three circles correspond to the probability of cell 1–3 to exceed its capacity. The probability that the equilibrium flows will exceed capacities is thus represented by the area enclosed by those three circles. This total area is by necessity greater than or equal to the largest area of any individual circle, with equality if one circle encloses the others or if the circles (and thus the events) coincide entirely.



**Figure 4.1** A Venn diagram portraying three events and their interdependencies.

For our upper probability bound we will make use of Boole's inequality (also known as union bound) which states that if we denote  $E_1, E_2, \dots, E_n$  as the events where cell  $1, 2, \dots, n$  exceeds its capacity, it holds that

$$P\left(\bigcup_{j=1}^n E_j\right) \leq \sum_{j=1}^n P(E_j). \quad (4.3)$$

In other words: the probability that *at least* one capacity is violated is smaller than or equal to the sum of the individual probabilities that a capacity is violated (with equality if all these events are independent). If we again consider Figure 4.1 we can see that this seems reasonable. The total area enclosed by the three circles must be

smaller than the sum of the individual areas since there is some overlap. If there was no overlap, the complementary (independent) result becomes apparent.

As such, our upper bound  $b_U$  is

$$b_U = \min \left[ 1, \sum_{j \in \mathcal{V}} P(\tilde{z}_j^* > C_j) \right] = \min \left[ 1, n - \sum_{j \in \mathcal{V}} P(\tilde{z}_j^* \leq C_j) \right] \quad (4.4)$$

where we have taken into account that we sum over  $n$  cells and that a probability can not be larger than 1. In conclusion, the probability that the equilibrium leaves the feasibility region is bounded according to

$$b_L \leq P(\tilde{z}^* \not\leq C) \leq b_U. \quad (4.5)$$

## 4.2 Normally distributed inflows

In this section we will assume that the exogenous inflows  $u$  are Gaussian, or normally distributed. This assumption has some issues though in that Gaussian variables are supported for all  $u \in \mathbb{R}$ , which causes an issue when we consider that we only allow non-negative inflows  $u \geq 0$ . During the rest of this section we will assume that the probability of negative inflows is sufficiently small for the effects of this possibility to be negligible to the final results.

We start off by declaring some basic notation for the distribution of any exogenous inflow  $j \in \mathcal{R}$ :

$$u_j \sim \mathcal{N}(\bar{u}_j, \sigma_{jj}^2), \quad \bar{u}_j = \mathbb{E}[u_j], \quad \sigma_{jj}^2 = \text{Var}[u_j] = \mathbb{E}[(u_j - \bar{u}_j)(u_j - \bar{u}_j)^T] \quad (4.6)$$

i.e.  $\bar{u}_j$  is the mean of  $u_j$  and  $\sigma_{jj}^2$  is its variance. This can be rewritten as

$$u_j = \bar{u}_j + \varepsilon_j, \quad \mathbb{E}[\varepsilon_j] = 0, \quad \text{Var}[\varepsilon_j] = \mathbb{E}[\varepsilon_j \varepsilon_j^T] = \sigma_{jj}^2 \quad (4.7)$$

which “reimagines”  $u_j$  as being a constant inflow  $\bar{u}$  perturbed by Gaussian noise  $\varepsilon_j$  with mean 0 and variance  $\sigma_{jj}^2$ . The benefit of this second convention will prove itself shortly, where we will determine the properties of the (random) equilibrium  $\tilde{z}^*$ :

$$\begin{aligned} \tilde{z}^* &= H\tilde{u} = H(\bar{u} + \varepsilon), \quad \mathbb{E}[\tilde{z}^*] = H\bar{u} = \bar{z}^* \leq C, \\ \text{Var}[\tilde{z}^*] &= \mathbb{E}[(H\varepsilon)(H\varepsilon)^T] = H\mathbb{E}[\varepsilon\varepsilon^T]H^T = H\Sigma H^T \end{aligned} \quad (4.8)$$

where  $\bar{u}$  is the mean vector of  $u$  and  $\Sigma$  is the (pseudo-) covariance matrix of  $\varepsilon$ . We also introduce  $\bar{z}^*$  as the expectation of the equilibrium flows, for which all flows are at or below their capacity. For any single cell  $i \in \mathcal{V}$  we can further determine:

$$\begin{aligned} \tilde{z}_i^* &= \sum_{j \in \mathcal{V}} h_{ij} \tilde{u}_j = \sum_{j \in \mathcal{V}} h_{ij} (\bar{u}_j + \varepsilon_j), \quad \mathbb{E}[\tilde{z}_i^*] = \sum_{j \in \mathcal{V}} h_{ij} \bar{u}_j = \bar{z}_i^* \\ \text{Var}[\tilde{z}_i^*] &= \mathbb{E} \left[ \sum_{j \in \mathcal{R}} h_{ij} \varepsilon_j \left( \sum_{k \in \mathcal{R}} h_{ik} \varepsilon_k \right) \right] = \sum_{j \in \mathcal{R}} \sum_{k \in \mathcal{R}} h_{ij} h_{ik} \mathbb{E}[\varepsilon_j \varepsilon_k] = \sum_{j, k \in \mathcal{R}} h_{ij} h_{ik} \sigma_{jk}^2 \end{aligned} \quad (4.9)$$

where  $\sigma_{jk}^2 = \Sigma_{jk}$  is the covariance of inflow noises  $j$  and  $k$ . Using these results, we can give an expression for the probability that equilibrium flow  $i$  will be greater than its capacity:

$$\mathbf{P}(z_i^* > C_i) = 1 - \mathbf{P}(z_i^* \leq C_i) = 1 - \Phi \left( \frac{C_i - \bar{z}_i^*}{\sqrt{\sum_{j,k \in \mathcal{R}} h_{ij} h_{ik} \sigma_{jk}^2}} \right) \quad (4.10)$$

where  $\Phi(x)$  is the cumulative distribution function for the standard normal distribution (i.e. one with mean 0 and standard deviation 1). Now we finally have all we need to give some bounds for the probability that the equilibrium will be in the feasibility region: as the lower bound  $b_L$  we have the largest probability for any single cell to exceed their capacity:

$$b_L = \max_{i \in \mathcal{V}} \mathbf{P}(z_i^* > C_i) = \max_{i \in \mathcal{V}} \left[ 1 - \Phi \left( \frac{C_i - \bar{z}_i^*}{\sqrt{\sum_{j,k \in \mathcal{R}} h_{ij} h_{ik} \sigma_{jk}^2}} \right) \right]. \quad (4.11)$$

As our upper bound  $b_U$  we have the sum of the individual cells' probabilities of exceeding their capacities:

$$b_U = \min \left[ 1, \sum_{i \in \mathcal{V}} \mathbf{P}(z_i^* > C_i) \right] = \min \left[ 1, n - \sum_{i \in \mathcal{V}} \Phi \left( \frac{C_i - \bar{z}_i^*}{\sqrt{\sum_{j,k \in \mathcal{R}} h_{ij} h_{ik} \sigma_{jk}^2}} \right) \right]. \quad (4.12)$$

### 4.3 Exponentially distributed inflows

For this section we assume that the inflows are independent and exponentially distributed. One benefit of this assumption is that exponential distributions are only supported for non-negative values, meaning that there are no concerns with applicability as there were for Gaussian inflows. This analysis is instead somewhat limited by requiring the inflows to be independent, but we find results that may be useful even for cases where the dependencies between the inflows are sufficiently weak.

In any case, we assume to have independent, exponentially distributed inflows according to:

$$\begin{aligned} u_j &\sim \text{Exp}(a_j), \quad f_{u_j}(y) = a_j \exp(-a_j y), \quad y \geq 0 \\ \mathbf{E}[u_i] &= \frac{1}{a_j}, \quad \text{Var}[u_i] = \frac{1}{a_j^2}, \quad \forall j \in \mathcal{R} \end{aligned} \quad (4.13)$$

where  $\exp(x) = e^x$ , the exponential function. The equilibrium outflow for cell  $i$  is found as

$$\bar{z}_i^* = \sum_j h_{ij} u_j = \sum_j z_{ij} \quad (4.14)$$

where we have introduced the notation  $z_{ij} = h_{ij} u_j$ . The distribution of each  $z_{ij}$  where  $j \in \mathcal{R}$  is found as

$$z_{ij} = h_{ij} u_j \sim \text{Exp}\left(\frac{a_j}{h_{ij}}\right) = \text{Exp}(\beta_{ij}) \quad (4.15)$$

where  $\beta_{ij} = \frac{a_j}{h_{ij}}$ . If there is only one exogenous inflow ( $|\mathcal{R}| = 1$ ) then we know the probability density function for  $\bar{z}_i^*$  as

$$f_{\bar{z}_i^*}(z_i) = \beta_{ij} \exp(-\beta_{ij} z_i). \quad (4.16)$$

For two or more exogenous inflows ( $|\mathcal{R}| \geq 2$ ), assuming that the rates  $\beta_{ij}$  are all pairwise distinct (i.e.  $\beta_{ij} \neq \beta_{ik}$  for all  $j \neq k \in \mathcal{R}$ ), [9] states that the equilibrium flow  $\bar{z}_i^*$  gets the probability density function

$$f_{\bar{z}_i^*}(z_i) = \left[ \prod_{j \in \mathcal{R}} \beta_{ij} \right] \sum_{k \in \mathcal{R}} \left( \frac{\exp(-\beta_{ik} z_i)}{\prod_{\substack{l \in \mathcal{R} \\ l \neq k}} (\beta_{ik} - \beta_{il})} \right). \quad (4.17)$$

The limitation that the rates need to be pairwise distinct is a rather lenient, though not insignificant one. As such, we will be satisfied by saying that “for two or more independent, exponential inflows it holds that the equilibrium *generally* has the probability distribution function seen in (4.17).”

The cumulative density function for  $\bar{z}_i^*$  is found as

$$F_{\bar{z}_i^*}(z_i) = \int_0^{z_i} f_{\bar{z}_i^*}(x) dx = P(\bar{z}_i^* \leq z_i). \quad (4.18)$$

Further, the probability that the equilibrium flow  $\bar{z}_i^*$  will violate its capacity  $C_i$  is:

$$P(\bar{z}_i^* > C_i) = 1 - P(\bar{z}_i^* \leq C_i) = 1 - F_{\bar{z}_i^*}(C_i) \quad (4.19)$$

Now we finally have all that we need to find our probability bounds for the equilibrium  $\bar{z}^*$  to violate at least one flow capacity: the lower bound  $b_L$  is again found as the largest probability for any single inflow to exceed its capacity:

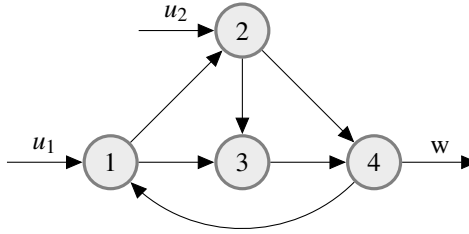
$$b_L = \max_{i \in \mathcal{V}} P(\bar{z}_i^* > C_i) = \max_{i \in \mathcal{V}} \left( 1 - F_{\bar{z}_i^*}(C_i) \right). \quad (4.20)$$

And for the upper bound  $b_U$  we again use Boole’s inequality:

$$b_U = \min \left[ 1, \sum_{i \in \mathcal{V}} P(\bar{z}_i^* > C_i) \right] = \min \left[ 1, n - \sum_{i \in \mathcal{V}} F_{\bar{z}_i^*}(C_i) \right]. \quad (4.21)$$

### Example

To illustrate these results we will again make use of our simple cyclic network from Chapter 3, only this time we will add another exogenous inflow to cell 2.



**Figure 4.2** Simple cyclic network with two source cells.

$$C = \begin{pmatrix} 4 \\ 3 \\ 3 \\ 5 \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} 2.3 \\ 0.5 \\ 0 \\ 0 \end{pmatrix} \quad (4.22)$$

For the Gaussian inflows, we will say that their standard deviations equal a fourth of their mean values, rendering it very unlikely (less than 1 in 30 000) for any of the flows to be negative. We will also say that the correlation coefficient for the two inflows is  $\rho = 0.2$ . Under these specifications, the resulting probability bounds  $b^G$  for the probability of the Gaussian inflow being outside the feasibility region is presented in (4.23).

$$b_L^G = 0.0841 \leq P(\bar{z}^* \notin C) \leq 0.1729 = b_U^G \quad (4.23)$$

In order to numerically test these values, we will approximate the true probability using simple cumulative frequency analysis from one million simulations in Matlab. Our estimate is presented in (4.24).

$$P(\bar{z}^* < C) \approx 0.0842 \quad (4.24)$$

The exponential inflows  $u_1$  and  $u_2$  are assumed to have means 2.3 and 0.5 respectively, meaning that  $a_1 = \frac{1}{2.3} \approx 0.43$  and  $a_2 = \frac{1}{0.5} = 2$ . The resulting probability bounds  $b^E$  are shown in (4.25).

$$b_L^E = 0.2601 \leq P(\bar{z}^* \notin C) \leq 0.9842 = b_U^E \quad (4.25)$$

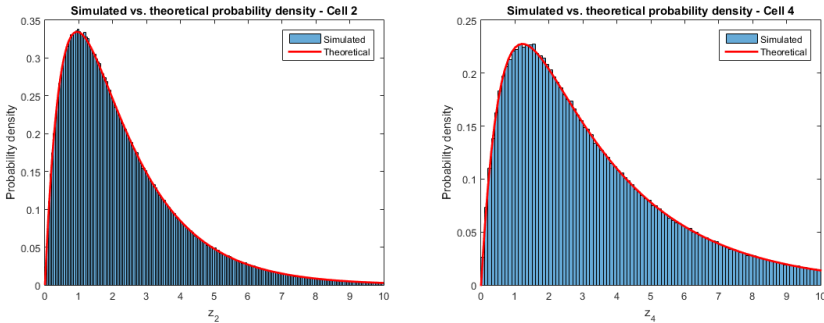
We will approximate the true probability in the same manner as for the Gaussian case, though for two cases: one where the inflows are independent and one where

they are correlated with a coefficient  $\rho = 0.2$ . This allows us to see how valid our bounds are even in cases where the assumption of independent inflows is untrue. The resulting probabilities are shown in (4.26).

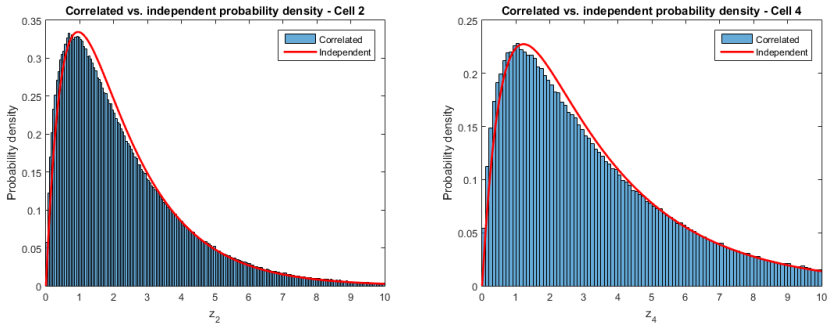
$$P(\tilde{z}_{indep}^* < C) \approx 0.2822, \quad P(\tilde{z}_{corr}^* < C) \approx 0.2835 \quad (4.26)$$

We can see that for this particular example both choices of inflow distributions gave estimated probabilities that skirted toward their respective lower bounds, even for the correlated exponentially distributed inflows. We can also see that in our example, our theoretical results for exponentially distributed inflows were of use even when those inflows were somewhat correlated.

If we compare our theoretical probability distributions for the equilibrium flows (4.17) to that of our simulation, we can see some difference between them. Figure 4.3 shows the simulated and theoretical probability distributions of cells 2 and 4 for the case of independent exogenous inflows, where we see that the simulated distribution follows the theoretical one quite closely (i.e. the red line follows the blue bars). In turn, Figure 4.4 shows the difference between the correlated distributions and our independent distributions for cell 2 and 4. There is a noticeable difference between the area enclosed by the independent distribution and the area enclosed by the (simulated) correlated distribution. As earlier claimed however, for a sufficiently “small” correlation of  $\rho = 0.2$  the difference is not very big and our probability bounds still hold in this case. This serves to demonstrate that our results can still be of use even when the inflows are not entirely independent.



**Figure 4.3** Comparison between the simulated and theoretical probability density functions of cells 2 and 4 for independent exogenous inflows.



**Figure 4.4** Comparison between the probability density functions of cells 2 and 4 for the cases of correlated and independent exogenous inflows .



# 5

## Routing matrix perturbations

In this section, we will explore the effects of routing matrix perturbations:

$$\tilde{R} = R + \hat{R} \tag{5.1}$$

where  $\hat{R}$  is the perturbation matrix. In particular, the effect of single-row perturbations on the equilibrium matrix will be analyzed in Section 5.2. Section 5.3 will find a condition for whether the resulting, new equilibrium is a freeflow equilibrium following such a routing matrix perturbation. We will also have a short discussion of some rules that our perturbations must follow for them to have reasonable real-world interpretations in Section 5.1.

A real-world analogue of these perturbations could for instance be wide-spread use of GPS-routing, which may have a noticeable effect on the effective routing ratios in a network. Further causes of altered traffic behaviour could be sobriety checkpoints or faulty beliefs of reduced capacities among drivers.

### 5.1 Perturbation rules

For this section, we will remind ourselves that a network is interpreted as a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, R)$  where  $\mathcal{V}$  is our nodes/cells,  $\mathcal{E}$  is our links and  $R$  is our (sub-stochastic) routing matrix. To ensure that perturbations have sensible, real-world analogues we should introduce some rules for perturbations  $\hat{R}$  to  $R$ . First some necessary conditions for  $\tilde{R} = R + \hat{R}$ :

- (i) preserves non-negativity of  $\tilde{R}$

This ensures that there are no negative flows. Also, since  $R$  describes stochastic fractions, negative entries have no reasonable interpretation.

- (ii) preserves sub-unitary row sums:  $\tilde{R}\mathbb{1} \leq \mathbb{1}$

This has two effects: it ensures that no cell generates mass out of nowhere (since  $\sum_{j \in \mathcal{V}} R_{ij} > 1$  would imply that we can route an excess of the total outflow for cell  $i$ ) and also that the power series interpretation of  $(I - \hat{R}^T)^{-1}$  still holds (since the spectral radius remains less than one).

(iii) adheres to link topology

This is to ensure that no links are introduced that are not allowed with respect to the topology. E.g. allowing one to move from one side of town to the other without passing *through* town, or to move over water where there is no bridge. This also ensures that the network remains in/out-connected.

There is one final rule which we may want to introduce, but which is less apparent in its necessity or interpretation:

(iv) does not change row sums:  $\hat{R}\mathbb{1} = 0$

This means that the fraction of a cell's outflow that leaves the system remains unchanged. This has the somewhat odd interpretation that all flows always redirect within the network, meaning that we assume that no-one's response to being unable to use a link would be to (however temporarily) leave the system. Or indeed the opposite, where increased ability to use a link would cause one to remain in the system rather than leave it.

## 5.2 The new equilibrium matrix

In this section we will explore the effects of single-row perturbations to the routing matrix, corresponding to perturbing how the outflow from a single cell is distributed. This will give us an insight into the wider effects of such perturbations on the equilibrium. This could in turn be used to quickly or in real time assert whether e.g. a re-routing due to construction or an accident will cause issues for the equilibrium. The perturbations described in this section are assumed to adhere to at least rules (i)–(iii) in Section 5.1.

The Sherman-Morrison formula states that for an invertible square matrix  $A \in \mathbb{R}^{n \times n}$  and two column vectors  $v, w \in \mathbb{R}^n$ , the matrix  $A + vw^T$  is invertible iff  $1 + w^T A^{-1} v \neq 0$  and its inverse is given by

$$(A + vw^T)^{-1} = A^{-1} - \frac{A^{-1}vw^T A^{-1}}{1 + w^T A^{-1}v} \quad (5.2)$$

Our first step is thus to match our notation for a perturbation on row  $i$  to the one

described in the formula:

$$v = \begin{pmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \vdots \\ \hat{r}_{n-1} \\ \hat{r}_n \end{pmatrix}, w_j = \begin{cases} -1, & j = i \\ 0, & \text{otherwise} \end{cases}, wv^T = -\hat{R} = - \begin{pmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \end{pmatrix} \quad (5.3)$$

$$A = I - R^T, \quad A^{-1} = H = \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix}, \quad A + wv^T = I - R^T - \hat{R}^T \quad (5.4)$$

where  $H_j$  denotes row  $j$  of  $H$ ,  $\hat{r}_j$  is the perturbation to routing factor  $r_{ij}$  and where  $\hat{R}$  is our perturbation matrix. As noted above, for the new matrix inverse to even exist, we have to ensure that

$$1 + w^T H v = 1 - H_i v = 1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j \neq 0. \quad (5.5)$$

For the inverse to exist, the sum of the products of the equilibrium weights and the routing changes to the corresponding cell can not equal 1. Using basic linear algebra (and the invertible matrix theorem in particular), we can surmise that this coincides with making the rows and columns of  $I - R^T - \hat{R}^T$  linearly dependent, rendering the matrix rank deficient. This may also coincide with making one or more cells of  $\tilde{R}$  out-disconnected, which would mean that a subset of one or more cells merely accumulate any mass that they receive. As such, the calculation (5.5) may serve as a quick check to see whether a perturbation is likely to cause major issues throughout the network.

The numerator  $A^{-1} w v^T A^{-1}$  of (5.2) is calculated as:

$$H(-\hat{R}^T)H = - \begin{pmatrix} 0 & \cdots & H_{1v} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & H_{nv} & \cdots & 0 \end{pmatrix} H = - \begin{pmatrix} h_{i1}H_{1v} & h_{i2}H_{1v} & \cdots & h_{in}H_{1v} \\ h_{i1}H_{2v} & h_{i2}H_{2v} & & \vdots \\ \vdots & & \ddots & \\ h_{i1}H_{nv} & \cdots & & h_{in}H_{nv} \end{pmatrix} \quad (5.6)$$

where we in the first step perform the multiplication of  $H(-\hat{R}^T)$ , before finally multiplying by  $H$  from the right. Collecting the results from (5.2)–(5.6) allows us

to write an explicit formulation of the new equilibrium matrix:

$$\tilde{H} = (I - R^T - \hat{R}^T)^{-1} = H + \frac{1}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j} \begin{pmatrix} h_{i1} H_{1v} & h_{i2} H_{1v} & \cdots & h_{in} H_{1v} \\ h_{i1} H_{2v} & h_{i2} H_{2v} & & \vdots \\ \vdots & & \ddots & \\ h_{i1} H_{nv} & \cdots & & h_{in} H_{nv} \end{pmatrix} \quad (5.7)$$

where the subtraction in (5.2) is cancelled by the sign of the numerator as seen in (5.6). Before fully analyzing this result, we will write the expression for a single element  $\tilde{h}_{kl}$  of the new equilibrium matrix following a perturbation to the routing of cell  $i$ :

$$\tilde{h}_{kl} = h_{kl} + \frac{h_{il} \sum_{m \in \mathcal{V}} h_{km} \hat{r}_m}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j}. \quad (5.8)$$

From this we see that the overall effect of the perturbation is dependent on the term  $\frac{1}{1 - \sum_j h_{ij} \hat{r}_j}$ . If  $\sum_j h_{ij} \hat{r}_j \approx 1$  the overall effect of the perturbation is greatly increased, while the overall effect is lessened the further away from 1 the sum gets. We can also see that if  $\sum_j h_{ij} \hat{r}_j < 1$  the overall sign of the perturbation corrections will be positive; if  $\sum_j h_{ij} \hat{r}_j > 1$  the sign is instead overall negative.

Beyond that, the change of each row of the equilibrium matrix is primarily proportional to the term  $H_{kv}$ . Redirecting the outflow from  $i$  towards cells  $m$  whose exogenous inflow most affect the stationary outflow  $z_k^*$  will increase all elements of the row (since  $h_{il}$  is non-negative as  $H$  is a non-negative matrix) and thus increase  $z_k^*$ . Redirecting flow away from these cells will instead decrease all elements of the row, reducing the stationary outflow  $z_k^*$ . This is quite intuitive if one considers that  $h_{km}$  is in some sense a measure of how reachable cell  $k$  is from cell  $m$ ; increasing the inflow to cell  $m$  should increase its overall mass, which in turn increases its outflow which we have concluded is very able to reach cell  $k$ .

Finally, the change of each column of  $\tilde{H}$  is proportional to  $h_{il}$ . As such, the change in effect of exogenous inflow  $l$  on the equilibrium flows  $z^*$  is in large dependent on the degree to which  $z_i^*$  is dependent on the exogenous inflow to cell  $l$ . As  $h_{il}$  is a measure of this, it is natural that changing the outflow distribution of a cell which does not depend much on the exogenous inflow to cell  $l$  should not have much of an effect on the influence of that inflow for all other stationary flows.

### 5.3 The new equilibrium flows

Now that we know the effect of a perturbation on the equilibrium matrix, let us explore the effect on the equilibrium flows. In particular, whether they are still below capacities. Another way to frame the issue: changing the routing matrix may deform the feasibility region (compare (2.19)). Our goal is to see whether an exogenous

inflow  $u$  which was in the feasibility region before the perturbation still remains there after the perturbation.

For the purposes of this analysis, we will again assume that the routing of cell  $i$  has been perturbed by the perturbation vector  $v = (\hat{r}_1 \dots \hat{r}_n)^T$ . We will begin by writing the new perturbed equilibrium matrix in full:

$$\tilde{H} = H + \hat{H}, \quad \hat{H} = \frac{H\hat{R}^T H}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j}. \quad (5.9)$$

In other words  $\hat{H} = \tilde{H} - H$  is the difference between the new and old equilibrium matrices. Going forward it will prove useful to not complete the calculations in the numerator. After the perturbation we get the new equilibrium:

$$\tilde{z}^* = \tilde{H}u = (H + \hat{H})u = z^* + \hat{z}^* \leq C \iff \hat{z}^* \leq C - z^* \quad (5.10)$$

where we have introduced  $\hat{z}^* = \tilde{z}^* - z^*$  as the difference between the new and old equilibrium flows (note that  $\hat{z}^*$  is not necessarily non-negative). Like in Chapters 3 and 4, we again arrive at an expression which needs to adhere to some (residual) capacity constraints. We can however make some further analysis:

$$\hat{z}^* = \frac{H\hat{R}^T H}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j} u = \frac{H\hat{R}^T}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j} z^* = \frac{Hvz_i^*}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j} \leq C - z^* \quad (5.11)$$

where we first perform the operation  $Hu = z^*$ , and then see that due to the structure of  $H\hat{R}^T$  (as seen in (5.6)),  $H\hat{R}^T z^* = H(-vw^T)z^* = Hv(-w^T z^*) = Hvz_i^*$ . From this result we can see for each element  $k$  of  $\hat{z}^*$ :

$$\hat{z}_k^* = \frac{H_k v z_i^*}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j} \leq C_k - z_k^* \quad (5.12)$$

where  $H_k$  is again row  $k$  of  $H$ . Now we have all that we need to give a condition for whether the new equilibrium will remain below capacity (or alternately: whether an inflow vector  $u$  will remain in the feasibility region) after a (single row) routing matrix perturbation. We will collect our result in a proposition:

**Proposition 5.1.** *If  $u \in \Lambda_F$ : Following a perturbation to row  $i$  of the routing matrix (as described in (5.1) and (5.3)) which adheres to (at least) rules (i)–(iii) in Section 5.1, the new equilibrium  $\tilde{z}^*$  will be a freeflow equilibrium if*

$$\tilde{z}^* = \tilde{H}u \leq C \iff \frac{\sum_{l \in \mathcal{V}} h_{kl} \hat{r}_l}{1 - \sum_{j \in \mathcal{V}} h_{ij} \hat{r}_j} \leq \frac{C_k - z_k^*}{z_i^*}, \quad \forall k \in \mathcal{V}. \quad (5.13)$$

Note that we have divided both sides of (5.12) with  $z_i^*$  to have the result in (5.13). In the case where  $z_i^* = 0$  we can see that we can make any change we wish

to row  $i$  of  $R$  without risk of violating capacity. Since this coincides with cell  $i$  being inflow-disconnected (no exogenous flow reaches the cell) it stands to reason that changing its routing has no bearing on the equilibrium.

In all other cases where  $z_i^* > 0$  the effective residual capacity  $\frac{C_k - z_k^*}{z_i^*}$  is inversely proportional to  $z_i^*$ . In order to rationalize this, consider that we are changing how we are routing the outflow of cell  $i$  at all times, including when at equilibrium. From this perspective, it stands to reason that the larger the initial equilibrium flow (that we are re-routing), the less margin we have for ill-considered perturbations.

# 6

## Dynamical perturbations

In this section, we will explore a network's ability to withstand various dynamical perturbations and either remain or return to the feasibility region  $\Lambda_F$  and/or the freeflow region  $\mathcal{F}$ . This will be done by simulating the dynamics of a simple, cyclic network. For this network we will explore the performance of a first-in-first-out (FIFO) routing strategy. This routing will ensure that the network flows adhere to the supply constraints (2.11):

$$u + R^T z \leq \sigma(x)$$

From this analysis we hope to gain a better understanding of which perturbations are particularly disruptive, and how we can relate those results to the network topology. The simulations will be performed in Matlab where the dynamics are discretized with zero-order hold in time steps of size  $h = 0.1$ . In Section 6.1 we will describe the routing strategy alongside a more resilient alternative. In Section 6.2 we will present our network topology alongside an acyclic alternative. In Section 6.3 we will see which periodic exogenous inflows cause congestion deadlock. In Section 6.4 we will examine cell mass perturbations and how the network's ability to withstand them depend on which cell is perturbed.

### 6.1 Routing rules

As noted in Chapter 2, routing rules determine how flows are stymied within the network to the supply constraints outside the freeflow region. In this thesis we will primarily analyze first-in-first-out (FIFO) routing as described in (2.16):

$$z = \Gamma(x)\varphi(x), \quad \Gamma(x) = \text{diag}(\gamma(x))$$
$$\gamma_i(x) = \sup \left\{ \xi \in [0, 1] : \max_{k \in \mathcal{N}_i} \left( \xi \sum_{h \in \mathcal{V}} R_{hk} \varphi_h(x_h) - \sigma_k(x_k) \right) \leq 0 \right\}$$

In words, FIFO routing will retain the routing proportions to a cell's out-neighbors at all times by limiting the total outflow of that cell. It does this by determining

the largest value of  $\xi \in [0, 1]$  for which the supply constraints hold for all of cell  $i$ 's out-neighbors and stymieing the total outflow by that factor. This means that if any of  $i$ 's out-neighbors are congested, the flow to all non-congested are stymied as well. That makes this routing quite vulnerable to cause cascading congestion issues, where the buildup of mass in cell  $i$  due to the inability of traffic to leave it will cause cells upstream from  $i$  to have to stymie their flow, and so on.

To lessen the risk of these cascading congestion issues, one might instead consider non-FIFO routing:

$$F_{ij} = R_{ij} \gamma_{ij}(x) \varphi_i(x_i), \quad \gamma_{ij}(x) = \sup \left\{ \xi \in [0, 1] : \xi \sum_{h \in \mathcal{V}} R_{hj} \varphi_h(x_h) \leq \sigma_j(x_j) \right\} \quad (6.1)$$

In effect, non-FIFO routing is FIFO routing applied on each individual outgoing link of a cell  $i$  rather than that cell's total outflow. This significantly lessens the risk of cascading congestion issues at the cost of changing the effective routing proportions in the network whenever there are congestion issues. A network wherein all cells make use of non-FIFO routing can be proved to be globally monotone which (per Theorem 2.1) makes any equilibrium globally stable.

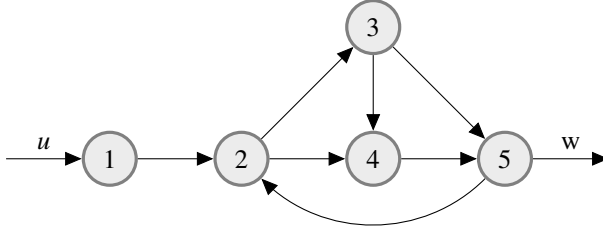
These routing rules can be thought to model different real-world situations. The situation where the total outflow of a cell is stymied (FIFO) can be imagined to model a single-lane stretch of road: if the vehicle in front can not enter the next cell (e.g. it can not turn right onto another road), it will have to stop and wait for an opportunity. This will cause all vehicles behind it—even those who are not going the same direction—to have to wait as well (this may go some way to explain the name first-in-first-out). Non-FIFO routing instead models a multi-lane stretch of road, where there are individual lanes dedicated to all vehicles going a certain direction, e.g. a two-lane road where vehicles can either turn right or go straight. As one can imagine, real-world networks would use some combination of the two (or entirely different rules) where some cells are best described by FIFO and some by non-FIFO routing. For the purpose of this thesis however, we will only consider the cases where all cells in the network share the same routing rule.

## 6.2 Network topologies

To test the performance of FIFO routing in response to various perturbations we need some network to test it in. For this purpose we will yet again use the simple network that we have used in the examples of Chapters 3 and 4 for the basic topology. Due to the formulation of FIFO (and non-FIFO) routing—which does not take into account exogenous inflows—we have to introduce an intermediate cell between the exogenous inflow and the earlier cell 1. This is analogous to e.g. making the on-ramp of a highway our intermediate cell. Implicit in this convention is the assumption that we can not hinder any exogenous inflow; it will enter the system

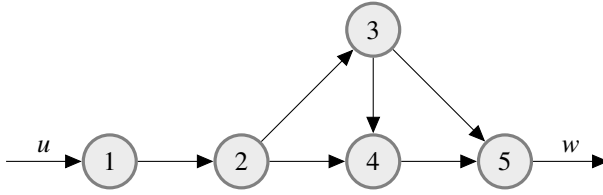


unhindered no matter how congested the intermediate cell is. Figure 6.1 shows the new topology where we have renumbered the cells in order to introduce the intermediate cell.



**Figure 6.1** Simple cyclic network.

We also want to examine how cycles in a network can impact the network's response to various perturbations. By simply removing the link from cell 5 to cell 2 we will remove all cycles present. Figure 6.2 show the acyclic network we will use to contrast our cyclic network to.



**Figure 6.2** Simple acyclic network.

Now that we have described the basic topology of our networks, we will define the other features of our networks. Starting with our routing matrices  $R_{cyc}$  and  $R_{acyc}$  for the cyclic and acyclic case, respectively.

$$R_{cyc} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0.1 & 0 & 0 & 0 \end{pmatrix}, \quad R_{acyc} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.2)$$

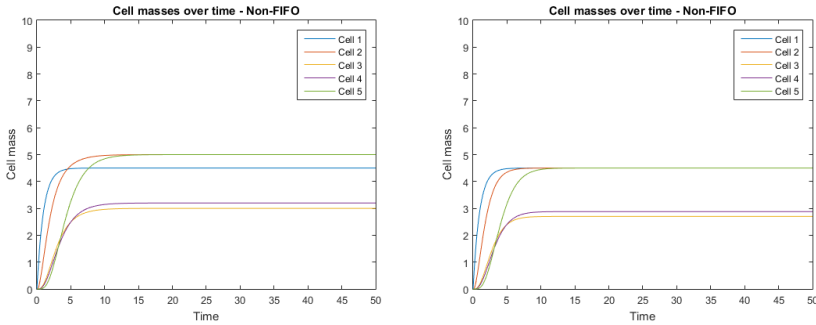
Next we will define our demand and supply functions. For the purposes of this thesis, we are not overly interested in how the structure and distribution of these functions affects a network's dynamics. We will therefore assume that they are both linear in structure and with relatively large flow and buffer capacities. We will further assume that the demand and supply functions are the same in structure for all

cells. In (6.3) we give the explicit expressions for the demand and supply functions of all cells.

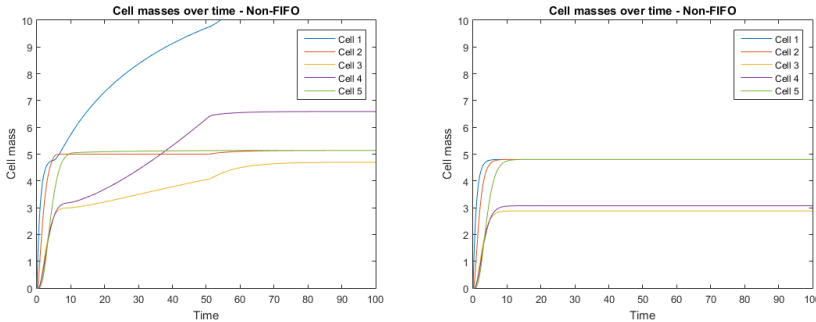
$$\varphi_i(x_i) = \begin{cases} x_i, & x_i \leq 10 \\ 10, & x_i > 10 \end{cases}, \quad \sigma_i(x_i) = \begin{cases} 10 - x_i, & x_i \leq 10 \\ 0 & x_i > 10 \end{cases}, \quad i \in \mathcal{V} \quad (6.3)$$

From this we can discern that the flow and buffer capacities for all cells are  $C_i = 5$  and  $\bar{x}_i = 10$ .

Before continuing on to expose our two networks to various perturbations, we will make one final note; our two networks have different sets  $\Lambda_F$ :  $\Lambda_{cyc} \neq \Lambda_{acyc}$ . We will illustrate this by comparing the non-FIFO cases for a constant exogenous inflow of  $u = 4.5$  or  $u = 4.8$ . Figures 6.3 and 6.4 show the cell masses over time, starting from empty cells, in response to these inflows.



**Figure 6.3** Cell masses resulting from an inflow of  $u = 4.5$  for the cyclic (left) and acyclic (right) network.



**Figure 6.4** Cell masses resulting from an inflow of  $u = 4.8$  for the cyclic (left) and acyclic (right) network.

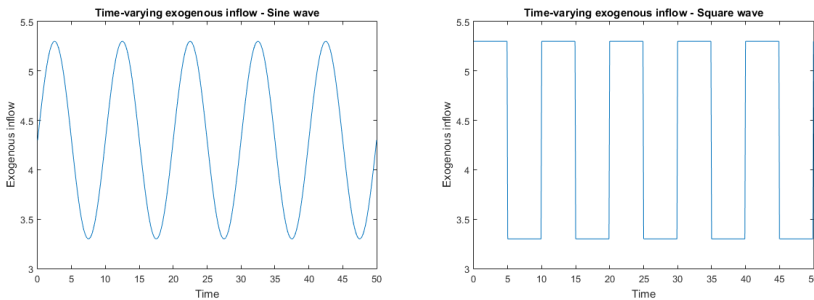
In Figure 6.3 we can see that the equilibria are different for the two networks; all masses are greater in the cyclic network than they are in the acyclic network. This is explained by considering that any mass which enters the cyclic network is able to remain in it for longer than it is able to in the acyclic network. For this particular example, that effect is coupled with the fact that we have redirected a fraction of cell 5's total outflow to feed back into the network, rather than have it all leave the system directly (as in the acyclic case). In Figure 6.4 we can see that these effects causes issues for an exogenous inflow  $u = 4.8$  where the mass of cell 1 will increase beyond  $x_1 = 10$ , even after the other cells in the network have arrived at their "equilibrium" masses. In the acyclic case however, all cells are able to find equilibrium masses which are even below  $x_i = 5$ .

### 6.3 Exogenous inflow perturbations

In this section we will explore the influence of periodic exogenous inflows on the cell masses. In particular we will explore the network's ability to withstand inflow which periodically exceeds the largest constant inflow in the feasibility region:  $u = 4.5$ . We will model this by having a constant inflow perturbed by either a sine wave or a square wave, as described in (6.4).

$$u_{sine}(t) = 4.3 + A \sin \frac{2\pi t}{T}, \quad u_{square}(t) = \begin{cases} 4.3 + A, & t \bmod T < \frac{T}{2} \\ 4.3 - A, & t \bmod T > \frac{T}{2} \end{cases}, \quad t \geq 0 \quad (6.4)$$

where 4.3 is the constant inflow,  $A$  is the amplitude of the variation and  $T$  is its period. We will make special note of the fact that the average value of these inflows (i.e. the average exogenous inflow) are both  $\bar{u} = 4.3$ , which is strictly within the feasibility region. The exogenous inflows for a constant inflow  $\bar{u} = 4.3$ ,  $A = 1$  and  $T = 10$  are shown in 6.5 for both perturbations.



**Figure 6.5** Exogenous inflows perturbed by periodic variations.

For these two perturbed inflows, we will see what pairings of perturbation amplitudes and periods will not cause any cell masses to grow unbounded. This was

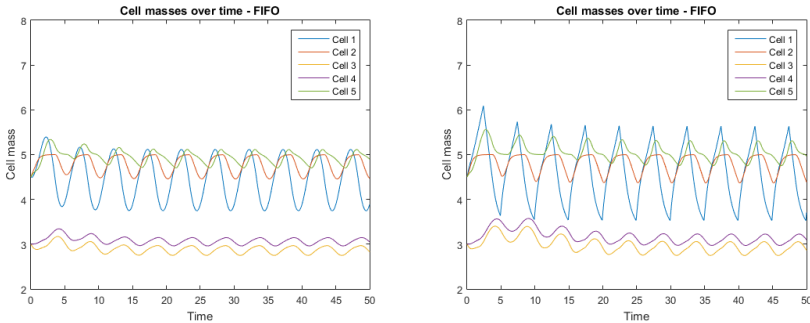
done by simply choosing e.g. the amplitude as a set value and then incrementing the period until any cell masses diverge. Due to the time cost of simulating the network dynamics, we will limit our analysis to only handle values determined to two decimal places.

The fact that there exists a limit to these pairings for which no cell masses will grow unbounded demonstrates that there the network is unable to unload the additional mass introduced during the peaks of the inflow during the valleys. Since the network is cyclic this remaining mass will then cause additional congestion issues during the next period, further exacerbating the buildup of mass in the network.

Our results are presented in Table 6.1 where increasing either the period or the amplitude of the perturbation by 0.01 would cause a cell mass to increase unbounded. Figure 6.6 shows the cell masses in response to the two perturbed inflows with amplitude  $A = 1$  and period  $T = 5$ , which corresponds to no cell masses growing unbounded. Figure 6.7 conversely shows the cell masses in response to the two perturbed inflows with amplitude  $A = 2$  and period  $T = 5$ , which corresponds to at least one cell mass growing unbounded.

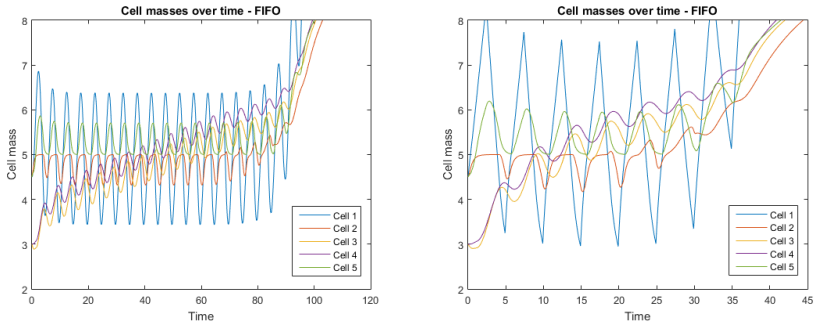
Sine wave		Square wave	
Amplitude	Period	Amplitude	Period
1	8.75	0.86	8
1.08	8	1.13	6
2	4.28	1.74	4
2.15	4	3.73	2

**Table 6.1** Pairings of wave amplitudes and periods for which no cell mass will grow unbounded.



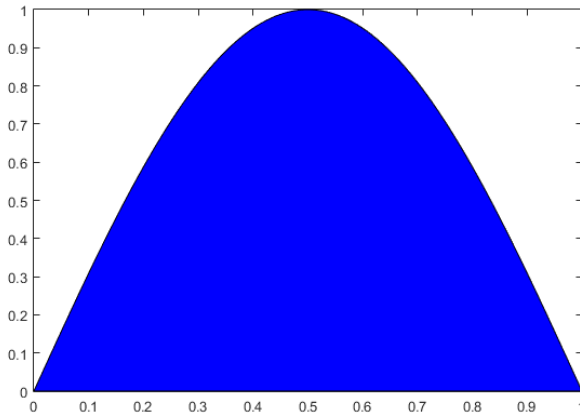
**Figure 6.6** Cell masses resulting from sine (left) and square (right) wave perturbations with amplitude 1 and period 5.

From these results we can discern some near-inverse proportionality between the amplitude and periods: if we double the period of our perturbations, we have



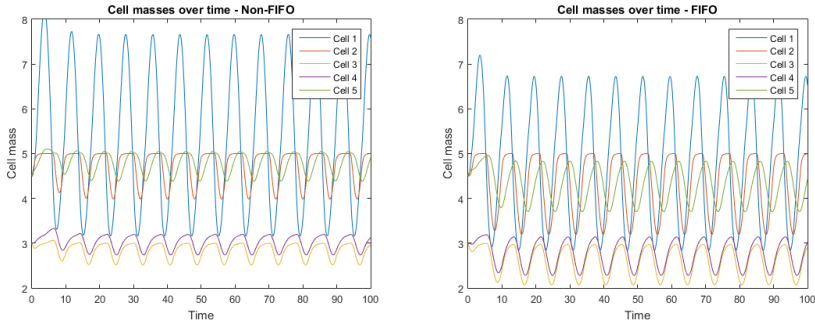
**Figure 6.7** Cell masses resulting from sine (left) and square (right) wave perturbations with amplitude 2 and period 5.

to (approximately) halve the amplitude. We can also see that for a given period, the amplitude of a sine wave can be greater than that of a square wave without causing issues. This is likely due to the fact that the total mass flowing into the network during every inflow peak is greater for the square wave than for the sine wave. Figure 6.8 shows the area during a peak for a sine wave of amplitude 1 and period 2, enclosed in the area of a square wave of the same amplitude and period. All the white space in the figure represents the additional mass introduced by a square wave during a single peak compared to a similar sine wave. This additional mass will cause more congestion issues which will negatively impact the network's ability to shed that additional mass.



**Figure 6.8** Comparison of the areas of a sine and a square wave of the same amplitudes and periods.

Finally, we will demonstrate that either removing the cycles or changing to non-FIFO routing will entirely remove the risk of unbounded cell mass growth in our network. Figure 6.9 shows the responses of non-FIFO routing and the acyclic network when the inflow is perturbed by a sine wave of amplitude 2 and period 8.



**Figure 6.9** Cell masses resulting from sine wave perturbation of amplitude 2 and period 8 for non-FIFO routing (left) and the acyclic network (right).

**Cell mass perturbations** In this section we will explore the network’s resilience in response to cell mass perturbations. We will model this by increasing the mass of a single node at time  $t = 10$  by some set amount and analyzing whether any cell masses are caused to grow unbounded. We will test this for two different constant inflows:  $u = 4.3$  which is strictly within the feasibility region and  $u = 4.5$  which is the largest inflow in the feasibility region.

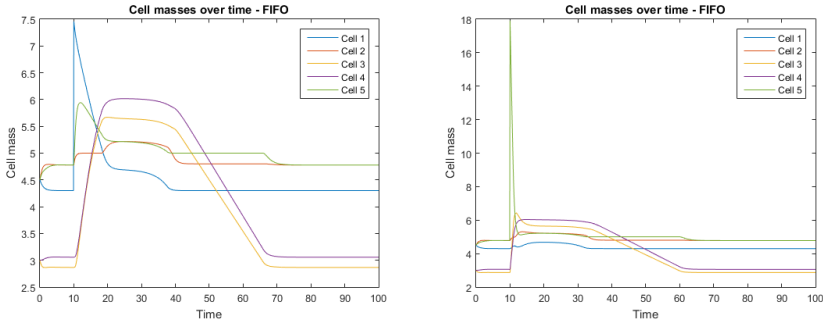
In the first case, we will find the cases where perturbations will cause congestion issues which deadlock and become unable to solve themselves. In the latter, we will find that although the constant exogenous inflow is as large as it can be, the network can still accommodate more mass without causing any congestion deadlocks.

The results of these trials are presented in Table 6.2. In Figure 6.10 we show the cell masses following mass increments of 3.16 to cell 1 and 13.14 to cell 5 for  $u = 4.3$ . Figure 6.11 shows the cell masses following mass increments of 0.50 to cell 1 and 5.08 to cell 5 for  $u = 4.5$ . Figure 6.12 shows the cell masses diverging following perturbations of size 5 on cell 1 and of size 15 on cell 5 for a constant inflow  $u = 4.3$ .

From these results we can see that the further away from a drain cell the perturbed cell is, the less the mass of that cell can be increased. This should not be very surprising: the fewer steps necessary for the additional mass introduced to leave the system, the less effect it should have on the network. But why can cell 3 withstand larger perturbations than cell 4, even though all the outflow of cell 4 goes to cell 5 (thus making it very close to a drain cell)? That might be explained by considering how cell 4 acts as a buffer for the additional mass in the network: while the

$u = 4.3$		$u = 4.5$	
Cell	Mass increment	Cell	Mass increment
1	3.16	1	0.50
2	4.57	2	0.91
3	6.76	3	4.46
4	6.13	4	4.06
5	13.14	5	5.08

**Table 6.2** Largest impulse for which no cell mass will grow unbounded.

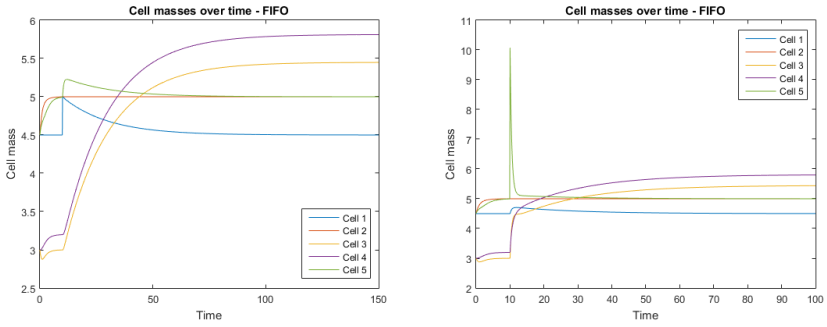


**Figure 6.10** Cell masses resulting from cell mass perturbations on cell 1 (left) and cell 5 (right) for a constant inflow  $u = 4.3$ .

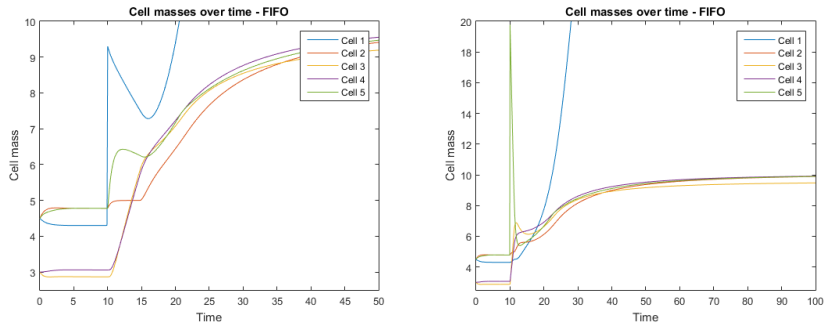
mass remains in the network for a longer time when cell 3 is perturbed, a portion of that mass goes directly to cell 4. This increase in the mass of cell 4 does not cause any additional congestion issues and merely acts as a temporary buffer until the perturbation mass has had time to leave the network.

These results show that it is of greater import to watch for sudden influxes of cell mass in cells which are far from drain nodes. This seems to be true no matter how far into the feasibility region the constant inflow is, though the margins obviously decrease as the exogenous inflow increases.

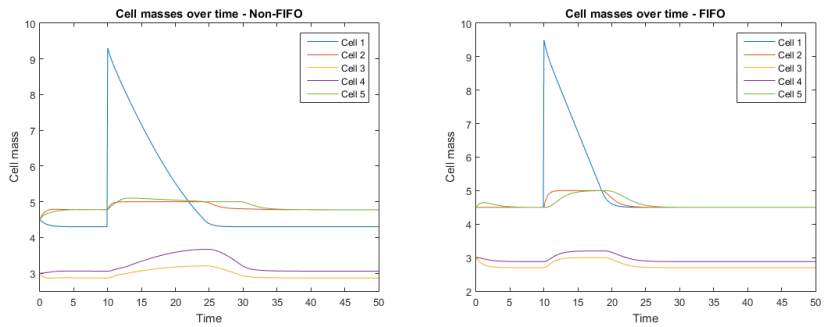
Finally, we will show that these effects can be entirely counteracted by either introducing non-FIFO routing or by removing all cycles. Figure 6.13 shows the cell masses in response to a mass increase of size 5 to cell 1 for a constant inflow of  $u = 4.3$  in those two cases.



**Figure 6.11** Cell masses resulting from cell mass perturbations on cell 1 (left) and cell 5 (right) for a constant inflow  $u = 4.5$ .



**Figure 6.12** Diverging cell masses resulting from cell mass perturbations on cell 1 (left) and cell 5 (right) for a constant inflow  $u = 4.3$ .



**Figure 6.13** Cell masses resulting from cell mass perturbations on cell 1 for a constant inflow  $u = 4.3$  for non-FIFO routing (left) and the acyclic network (right).



# 7

## Conclusions

In this final Chapter, we will summarize and explicitly state our results. We will hearken back to the introduction in Chapter 1 and show that we have fulfilled our stated purpose and goal. We will then use these results and their limitations to suggest refinements or approaches which may prove interesting to explore going forward.

### 7.1 Summary

The purpose of this thesis has been to examine the robustness of a network's flow equilibria in response to various perturbations. Since the dynamics at or near maximum flow capacities are highly dependent on the routing rules as well as demand and supply functions in use, we defined this as having all equilibrium flows below capacities following the perturbation(s) for Chapters 3–5. The goal of this analysis was to characterize the set of perturbations which retain freeflow within the system.

In Chapter 3 we examined the effects of deterministic perturbations on the exogenous inflows, defined as non-negative additions to one or more exogenous inflow. We compared these perturbations using the  $\ell_1$ -norm so that we can make qualitative statements when comparing different perturbations. Using this measure, we found a lower bound on the size of a deterministic perturbation necessary to force at least one equilibrium flow beyond its capacity. We found that it is insufficient to merely consider the smallest residual capacity; one has to also consider that the residual capacities are affected to different degrees by increments to the different exogenous inflows. We later generalize the measure of perturbation size to any affine cost function and find the smallest cost necessary to force at least one equilibrium flow beyond its capacity.

In Chapter 4 we explored the effects of stochastic exogenous inflows on the equilibrium flows. We find the probability distributions for each equilibrium flow resulting from either normally or (independent) exponentially distributed inflows. We also provided a short discussion on the issue of applicability with the two approaches, such as their support of negative values or their demand of inflow inde-

pendence. For these two cases we then formulate upper and lower bounds for the probability that these exogenous inflows will cause one or more equilibrium flow to exceed its capacity. We find that our results are consistent with simulations, and that our results regarding independent, exponentially distributed inflows are of use even when the inflows are weakly correlated.

In Chapter 5 we calculated the effects of single-row perturbations to the routing matrix on the equilibrium matrix. We find how the effect of such perturbations depend on the structure of the initial equilibrium matrix and provide an alternative to calculating the new matrix inverse for the perturbed routing matrix wholesale. We then examine how these perturbations deform the feasibility region and find a test to determine which exogenous inflows remain in the region following its deformation.

Finally in Chapter 6 we simulate a simple, cyclic network using FIFO routing to explore the effects of periodic inflows and cell mass perturbations. We find that exogenous inflows that periodically exceed the feasibility region can cause the network to leave the freeflow region, even when the average of those inflows is strictly within the feasibility region. For cell mass perturbations, we find that their ability to accept additional mass without causing cascading congestion issues is proportional to their distance to a drain node. We also find that other cells in the network may act as buffers, where a cell that is further away from a drain cell than its out-neighbor may still accept larger perturbations than the out-neighbor does. Finally, we find that we can make our network impervious to both inflow and cell mass perturbations by either removing all cycles or by introducing non-FIFO routing.

## 7.2 Future work

Going forward, one of the most interesting approaches to refine our results would be to relate them to the supply constraints as well as the flow capacities. Using (2.11) one could likely refine our results to put stricter limitations on any perturbations.

Another interesting exploration would be to examine the cases of time- and/or state-dependent routing matrices. While we touched on non-FIFO routing (which can be considered a form of state-dependent routing matrix) we did not take into account the effects of e.g. traffic lights or vehicles choosing to alter their route in response to down-stream congestion.

One avenue to explore would be to use the results from Chapter 5 to formulate a bound for the necessary size of a routing matrix perturbation to violate flow capacities. Some initial efforts proved promising, but time constraints caused this line of inquiry to be cut from the thesis.

The results from Chapter 6 are somewhat limited from the fact that we only tested the one network topology and routing strategy. Doing similar trials for more complex topologies (e.g. with several exogenous inflows), mixed routing strategies or for different, non-linear demand and supply functions may shed more light on the interaction between network structure and equilibrium robustness.

Finally, it would also be of great interest to apply these results to real-world topologies and see their applicability. Factors such as whether perturbations to exogenous inflows are likely to stay constant for sufficiently long for our results to be of use or the applicability of Gaussian or exponential modeling of exogenous inflows may affect the use of our results.

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<i>Title and subtitle</i> <b>On Robustness of Equilibria in Transportation Networks</b>			
<i>Abstract</i> <p>As infrastructural systems become ever more complex and interconnected, they may also become ever more vulnerable to system-wide faults due to local disturbances. As such it is of great importance to design these system to be resilient, i.e. to be able to withstand and recover from disturbances or new conditions. In the case of traffic networks, while much work has been done to analyze the stability of these systems, there is still little work to analyze their resilience.</p> <p>This thesis analyzes a variant of Daganzo's Cell Transmisson Model to explore the robustness of equilibria in dynamical flow networks in response to various perturbations. In particular it tries to characterize the set of perturbations which force a freeflow equilibrium out of freeflow. Since any such equilibrium is locally asymptotically stable, the retention of freeflow would thus ensure a retention of stability.</p> <p>The report first finds the smallest necessary size (in <math>\ell_1</math>-norm and for arbitrary affine cost functions) of any deterministic perturbations to the exogenous inflows to violate freeflow. Second it finds bounds for the probability of the equilibrium flows to violate freeflow due to stochastic exogenous inflows; either normally or independent, exponentially distributed. Third it finds the new equilibrium matrix and a condition for the retention of freeflow following a single-cell routing perturbation. Finally it simulates a simple network's performance in response to periodic exogenous inflows and cell mass increments, where it is shown that exogenous inflows with feasible averages may still cause system-wide faults and that mass increments are more disruptive the further away the affected cell is from a drain cell.</p>			
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