A Weak Formulation of Constraints to Couple Rigid and Elastic Bodies - a Study with FEniCS

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Abstract

Many complex mechanical systems are simplified by considering them as multibody systems. In recent years, flexible multibody dynamics has become more and more in demand. In [1], a weakly constrained joint model was presented, which in a well-defined way models the interconnection of an elastic and a rigid body by a massless rigid joint. However, the joint model assumes that the orientation of the joint-elastic body interface is unaffected by the displacement field of the elastic body. The aim of this study is to highlight the limitations of the joint model due to this assumption.

In this study, the joint model was applied for two-body system of an elastic and a rigid body, connected by a small rigid joint. During deformation the joint-body interface was expected to rotate due to the displacement field of the elastic body. However, due to the assumption the interface stayed fixed which distorted the displacement field of the elastic body. This assumption could be avoided if the orientation of the interface during deformation was predicted, which would be possible by the use of observer points.

Keywords — flexible multibody dynamics, differential-algebraic equations, interconnecting rigid joints

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Chapter 1

Introduction

1.1 Background and Aim

Many mechanical systems can naturally be treated as multibody systems. A multibody system is defined as a set of bodies and interconnecting elements. In multibody dynamics, the motion of the overall system is typically of main interest, rather than the deformations of the individual bodies. By treating the bodies as rigid, they can be modeled as point masses, which greatly reduces the complexity of the overall system. Therefore, multibody dynamics has mainly been modeled by rigid bodies.

The interconnecting elements play a key role in multibody dynamics. Typical interconnecting elements are springs, dampers, actuators, and joints. The former three, referred to as force elements, serve as additional interconnecting forces between the bodies. They are incorporated in the model by extending the force term in Newton's equations. Joints constrain the relative position of two bodies, by an algebraic relation, a geometric constraint. In the presence of joints, constrained rigid multibody dynamics is retrieved, which is described as a system of both ordinary differential equations (ODEs) and algebraic equations, a system of differential-algebraic equations (DAEs). System of DAEs are in general much more difficult to solve numerically in time, and are needed to be handled with great care.

A flexible multibody system refers to a multibody system which contains both rigid and elastic bodies. In recent years, due to an increased interest in light-weight and high-precision mechanics, flexible multibody dynamics has become more and more in demand [1]. For flexible multibody systems, the deformation of the system is a heterogeneous combination of continuous displacement fields and discrete rigid body motions.

Consider a constrained two-body system where a small rigid joint, modeled as massless, interconnects an elastic and a rigid body. The rigid joint-elastic body interface should perform rigid body motions, to conform with both the continuous elasticity model of the elastic body, as well as the discrete rigid body motion of the rigid joint. These requirements hold for a weakly constrained joint model presented in [1]. The weakly constrained modeling technique is employed for a slider-crank example, as well as a pantograph and catenary example (the interconnecting wires and cables transmitting electrical current from electrical cables to high-speed trains), with promising results. However, in both those examples, rigid and elastic bodies were connected directly, without any physical joints.

To apply the joint model in practice, it is crucial to understand its limitations, to avoid misuse. Therefore, the aim of this study is to highlight the limitations of the weakly constrained joint model, in presence of physical joints. A drawback with the joint model is that it assumes that the orientation of the rigid joint-elastic body interface is unaffected by the displacement field of the elastic body. This assumption distorts the displacement field, and thereby the structural analysis, if the true displacement field, without the assumption, would render in a rotated interface.

In this study, the limitation of the weakly constrained model in presence of a physical joint, is highlighted by implementing the joint model for an elastic-rigid body system with a massless rigid joint, where the true dynamics requires the joint body to rotate due to the displacement field of the elastic body.

1.2 Overview of Content

This report is divided into a theory chapter, an implementation chapter, a numerical experiments chapter, and lastly, summary and conclusions. The main objective in the theory chapter is to present constrained flexible multibody dynamics, which is needed to highlight the limitations of the weakly constrained joint model.

The theory chapter begins with a section which introduces Lagrangian mechanics, for unconstrained and constrained dynamics for the translational motion of point masses. With Lagrangian mechanics, the equations of motion (EOM) is derived from a variational problem (VP) with fixed endpoints in time. The constrained dynamics is expressed by a Lagrange multiplier technique.

In the second section, the EOM for a constrained rigid multibody system is derived. In the two following sections, the EOM is derived, for an unconstrained and a constrained, elastic body, respectively. Thereafter, in Section 2.5, the VP for a constrained flexible multibody system is finally presented. Section 2.5 also includes presentations of three massless rigid joint models from [1], including the weakly constrained joint model.

The EOM for an elastic body has to be discretized, in both time and space, before the dynamics can be solved numerically on a machine. Finite element (FE) discretization is a conventional technique for discretizing elastic bodies in space. FE discretization is presented in Section 2.6. There it is shown that FE discretization of constrained elastic bodies generate systems of DAEs. In Section 2.7, two time-integrating schemes are presented shortly, the backwards differentiation formula (BDF) methods and Newmark's method.

The implementation chapter mainly focuses on how to discretize EOM in space by FE discretization. A FE discretization Python module called FEniCS is employed. A work-around, to discretize constrained dynamics, is presented.

The numerical experiments consists of two parts. In the first experiment, an elastic body attached to a rigid wall was considered. The attachment was implemented both with a Dirichlet boundary condition and as a geometric constraint. The numerical solutions for the unconstrained and constrained dynamics were compared to validate the implementations.

In the second experiment, a rigid body was connected to the elastic body, through a massless rigid joint, modeled by the weakly constrained joint model. For the second experiment, the limitations of the weakly constrained joint model is highlighted.

Chapter 2

Theory

The main objective of the theory chapter is to introduce the EOM for a constrained flexible multibody system, which is needed to implement the numerical experiments.

In this study, Lagrangian mechanics is followed. For Lagrangian mechanics, a weak formulation of the EOM in both time and space is derived from a VP with fixed endpoints in time. In the first section, Lagrangian mechanics is introduced, both for unconstrained and constrained dynamics, for the translational motion of point masses. In the second section, the EOM is first derived for an unconstrained rigid body, where both translational and rotational motions of point masses have to treated. The VP is thereafter extended to constrained rigid multibody systems.

In the third section, the dynamics of an unconstrained elastic body is presented. The deformation of elastic bodies is, as opposed to rigid bodies, continuous in space, which requires a more thorough analysis. In Section 2.1.1, the model is extended to constrained elastic body dynamics.

In Section 2.5, the VP for constrained flexible multibody systems is presented. In the multibody setting, the deformation composes of both rigid body motions and an elastic displacement field. The section also includes presentations of three massless rigid joint models from [1], including the weakly constrained joint model.

For elastic bodies, the EOMs, which are weak in space and strong in time (weak-strong forms), are derived from the VPs. In Section 2.6, FE discretization is briefly presented to discretize the system in space. In the last section, the time-integrators BDF and Newmark's methods are presented to discretize the system in time.

2.1 Introduction to Lagrangian Mechanics and Constrained Dynamics

In this section, the translational motion of point masses is considered. In the first subsection, Euler-Lagrange's equation is derived from Newton's second law. In the next subsection, Euler-Lagrange's equation is shown to describe the dynamics of an extremal for a variational problem (VP) with fixed endpoints in time. The VP considered in this study is Lagrangian mechanics. The VP is thereafter extended to model dynamics in presence of holonomic constraints. The dynamics of the extremal of the extended VP is shown to be described by a system of DAEs, in classical mechanics. Under a set of assumptions, an initial value problem (IVP) based on the extremal is shown to be unique. Lastly, the sensitivity to perturbations for systems of DAEs are discussed.

2.1.1 Connection Between Classical and Lagrangian Mechanics

In this subsection, Euler-Lagrange's equation for a specific Lagrangian is derived from Newton's second law, for the translational motion of point masses. Thereafter, it is shown that the specific Euler-Lagrange's equation, for a closed, autonomous (time-invariant) system, implies that the total energy of the system is conserved. Moreover, a motion which fulfills Euler-Lagrange's equation defines an extremal of a VP known as Hamilton's principle of least action. In the next subsection, Euler-Lagrange's equation is derived by solving the VP.

In this and the next subsection, the connection between classical and Lagrangian mechanics is presented, for the translational motion of point masses, for a closed, autonomous system. In classical mechanics, such a system is expressed by Newton's second law,

$$M\ddot{\varphi}_*(t) = f(\varphi_*(t)), \quad t_0 \leq t \leq t_1, \tag{2.1}$$

where $M \in \mathbb{R}^{n_\varphi \times n_\varphi}$, $\varphi_* \in C_2([t_0, t_1], \mathbb{R}^{n_\varphi})$, f, and t denote the mass matrix, the motion, the applied resultant force, and the time between time points t_0 and t_1, respectively. $\mathbb{R}^{n_\varphi}$ denotes the degrees of freedom of the system. By introducing additional velocity variables $v = \dot{\varphi}_* \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi})$, Eq. 2.1 is rewritten as a first order system,

$$\dot{\varphi}_*(t) = v(t), \tag{2.2a}$$
$$M\dot{v}(t) = f(\varphi_*(t)). \tag{2.2b}$$

The notation $C_1([t_0, t_1], \mathbb{R}^{n_\varphi})$ refers to the space $C^1((t_0, t_1), \mathbb{R}^{n_\varphi})$, with continuity requirements at the endpoints in time, where derivatives are defined as one-sided derivatives. Eq. 2.2, accompanied with initial conditions, $\varphi_*(t_0) = \varphi_0$ and $\dot{\varphi}_*(t_0) = \dot{\varphi}_0$, constitute an initial value problem (IVP). Under the assumptions that M is invertible, and that v and f are uniformly Lipschitz continuous in time, the IVP has a unique solution, for a sufficiently small interval in time, according to Picard-Lindelöf's theorem [2]. Furthermore, the time interval can be extended by successively applying the theorem.

Under the assumption that f is conservative, it can be expressed as $f(\varphi_*) = -\frac{\mathrm{d}V(\varphi_*)}{\mathrm{d}\varphi_*}$, for a potential energy $V(\varphi_*)$. Thus, Eq. 2.1 can be expressed as

$$0 = -\frac{\mathrm{d}V(\varphi_*)}{\mathrm{d}\varphi_*} - M\ddot{\varphi}_*(t). \tag{2.3}$$

Consider the Lagrangian

$$L(\varphi_*, \dot{\varphi}_*) = T(\dot{\varphi}_*) - V(\varphi_*), \tag{2.4}$$

where $T(\dot{\varphi}_*) = \frac{\dot{\varphi}_*^T M \dot{\varphi}_*}{2}$ is the kinetic energy. Since $\frac{\partial L}{\partial \dot{\varphi}_*} = \frac{\mathrm{d}T}{\mathrm{d}\dot{\varphi}_*} = \frac{\mathrm{d}T}{\mathrm{d}\dot{\varphi}_*^T} = \frac{M\dot{\varphi}_* + M^T\dot{\varphi}_*}{2}$ and $\frac{\partial L}{\partial \varphi_*} = -\frac{\mathrm{d}V}{\mathrm{d}\varphi_*}$, Eq. 2.3 can, under the assumption that M is symmetric, be reformulated to Euler-Lagrange's equation,

$$0 = \frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}. \tag{2.5}$$

Thus, closed, autonomous, conservative mechanical systems can, equivalently to Eq. 2.1 , be modeled by Euler-Lagrange's equation, for $L = T - V$. Furthermore, a φ_* fulfilling Eq. 2.5 defines an extremal of a corresponding VP, known as Hamilton's principle of least action. Retrieving the EOM, from the VP, is the approach taken in Lagrangian mechanics.

Pre-multiplying Euler-Lagrange's equation with $\dot{\varphi}_*^T$ generates

$$0 = \dot{\varphi}_*^T\left(\frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}\right), \tag{2.6}$$

which for an autonomous system, $L = L(\varphi_*(t), \dot{\varphi}_*(t))$, can be rewritten as

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(L - \dot{\varphi}_*^T \frac{\partial L}{\partial \dot{\varphi}_*}). \tag{2.7}$$

The Hamiltonian H is defined as

$$H := \dot{\varphi}_*^T \frac{\partial L}{\partial \dot{\varphi}_*} - L. \tag{2.8}$$

Then, Eq. 2.7 is restated as

$$0 = -\frac{\mathrm{d}H}{\mathrm{d}t}. \tag{2.9}$$

Since, closed, autonomous mechanical systems relies on the conservation of the total energy in time, the Hamiltonian is interpreted as the total energy. $L = T - V$, considered for relating Newton's second law to Euler-Lagrange's equation, inserted into Eq. 2.8 gives

$$H = \dot{\varphi}^T M \dot{\varphi}_* + V - \frac{\dot{\varphi}_*^T M \dot{\varphi}_*}{2} = T + V, \tag{2.10}$$

which is the total energy for unconstrained mechanical systems. Thus, the choice $L = T - V$ should be used for modeling closed, autonomous, unconstrained mechanical systems.

Similarly to Lagrangian mechanics, Hamiltonian mechanics relies on solving a corresponding VP. Furthermore, after regarding the momentum as a variable, the VP can be reformulated as a decoupled system of first order differential equations. While the physical interpretation of the Hamiltonian is clear, there is a drawback compared to Lagrangian mechanics. By the premultiplication of $\dot{\varphi}_*$ in Eq. 2.6, false solutions corresponding to stationary systems ($\dot{\varphi}_* = 0$) are introduced. Hamiltonian mechanics is not pursued any further in this study. For a presentation of Hamiltonian mechanics, see an introductory textbook on calculus of variations, eg. [3]. Lastly, Lagrangian and Hamiltonian mechanics are not restricted to autonomous systems.

2.1.2 Lagrangian Mechanics

Lagrangian mechanics stems from a VP with fix endpoints in time, Hamilton's principle of least action. For the VP, an extremal, embedded in a space of admissible motions, is assumed to exist. The extremal is retrieved, by applying a stationarity condition over the space of admissible motions. Euler-Lagrange's equation is shown to hold for an extremal. Therefore, an extremal is defined as the motion for which Euler-Lagrange's equation holds.

Consider the space V_φ of all motions $\varphi(t) \in C_2([t_0, t_1], \mathbb{R}^{n_\varphi})$, for $t_0 \leq t \leq t_1$, which are known at the time points t_0 and t_1, $\varphi(t_0) = \varphi_0$ and $\varphi(t_1) = \varphi_1$,

$$V_\varphi = \{\varphi(t) \mid \varphi(t) \in C_2([t_0, t_1],\, \mathbb{R}^{n_\varphi}),\, \varphi(t_0) = \varphi_0,\, \varphi(t_1) = \varphi_1\}. \tag{2.11}$$

For $\varphi(t_0) = \varphi_0$ and $\varphi(t_1) = \varphi_1$ to be well-defined, t_0, t_1, φ_0, and φ_1 are all assumed to be finite. To be able to categorize spaces by their codomain, the notation $V_\varphi = V_\varphi(\mathbb{R}^{n_\varphi})$ is adopted. Assume that an extremal $\varphi_*(t) \in C_2([t_0, t_1], \mathbb{R}^{n_\varphi})$ is embedded in V_φ. An extremal is defined as a motion which solves Euler-Lagrange's equation, Eq. 2.5. The space of a single extremal is defined as

$$V_{\varphi_*} := \{\varphi_*(t) \mid V_\varphi \ni \varphi_*(t)\}. \tag{2.12}$$

$\ni$ refers to a single element embedded in a space, whereas $\in$ refers to any element embedded in a space. By linearizing the problem around φ_*, an affine space $V_{\tilde{\varphi}}$ is retrieved, which can be expressed as

$$V_{\tilde{\varphi}} = V_{\varphi_*} \oplus V_{\theta\eta}, \tag{2.13}$$

where $V_{\theta\eta}$ is a linear space, in $\theta \in V_\theta$ for each $\eta \in V_\eta$, of admissible variations,

$$V_{\theta\eta} = V_\theta \otimes V_\eta, \tag{2.14}$$

with

$$V_\theta = \{\theta \mid \theta \in \mathbb{R}\}, \tag{2.15a}$$
$$V_\eta = \{\eta(t) \mid \eta(t) \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi}),\ \eta(t_0) = 0,\ \eta(t_1) = 0\}. \tag{2.15b}$$

$\eta \in V_\eta$ denotes the direction, and the relative magnitude, of an admissible variation in time, while $\theta \in V_\theta$ denotes a magnitude which is constant in time for the admissible variation. Since the endpoints in $\varphi \in V_\varphi$ are prescribed, only admissible variations which vanish at the endpoints are sought. This requirement is retrieved by forcing all $\eta \in V_\eta$ to vanish at the endpoints. The affine space $V_{\tilde{\varphi}}$ is referred to as the trial function space, and $\tilde{\varphi} \in V_{\tilde{\varphi}}$ as an admissible motion. According to Eq. 2.13, an admissible motion is element-wise expressed as the sum of the extremal and an admissible variation

$$\tilde{\varphi}(\theta, t) = \varphi_*(t) + \theta\eta(t), \quad t_0 \le t \le t_1. \tag{2.16}$$

The linearization was performed to enable to use the mathematical machinery for linear theory. However, if η is a nonlinear function, then $\tilde{\varphi} \in V_{\tilde{\varphi}}$ is only a valid approximation of $\varphi \in V_\varphi$ for small θ.

To retrieve Hamilton's principle of least action, the action integral is first introduced

$$j(\theta) = \int_{t_0}^{t_1} L(\tilde{\varphi}(\theta, t), \dot{\tilde{\varphi}}(\theta, t))\mathrm{d}t, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}, \tag{2.17}$$

where $L(\tilde{\varphi}, \dot{\tilde{\varphi}}) : \mathbb{R}^{2n_\varphi} \to \mathbb{R}$ is a bilinear functional. Hamilton's principle of least action is stated as

Variational problem 1 (Unconstrained problem)**.** *For a given pair of tuples (t_0, φ_0) and (t_1, φ_1) find an extremal $\varphi_*(t) \in V_\varphi(\mathbb{R}^{n_\varphi})$, for $t_0 \le t \le t_1$, under the assumption that $\varphi_*(t)$ exists, such that*

$$0 = j'(0), \quad \forall \eta \in V_\eta(\mathbb{R}^{n_\varphi}). \tag{2.18}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L(\tilde{\varphi}(\theta, t), \dot{\tilde{\varphi}}(\theta, t))\mathrm{d}t, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}(\mathbb{R}^{n_\varphi}), \tag{2.19}$$

where $M(t) = (\frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*})(\varphi_*(t), \dot{\varphi}_*(t)) \in C([t_0, t_1], \mathbb{R}^{n_\varphi})$ is a continuous function.

Eq. 2.18 denotes the stationarity condition. To find $\varphi_* \in V_\varphi(\mathbb{R}^{n_\varphi})$, under the assumption that it exists, the action integral is differentiated

$$\begin{aligned} j'(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{t_0}^{t_1} L(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta})\mathrm{d}t \\ &= \int_{t_0}^{t_1} \big(\eta^T \frac{\partial L}{\partial \tilde{\varphi}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta}) + \dot{\eta}^T \frac{\partial L}{\partial \dot{\tilde{\varphi}}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta})\big)\mathrm{d}t, \end{aligned} \tag{2.20}$$

Insertion of Eq. 2.20, into the stationarity condition, Eq. 2.18, gives

$$0 = j'(0) = \int_{t_0}^{t_1} \big(\eta^T \frac{\partial L}{\partial \varphi_*}(\varphi_*, \dot{\varphi}_*) + \dot{\eta}^T \frac{\partial L}{\partial \dot{\varphi}_*}(\varphi_*, \dot{\varphi}_*)\big)\mathrm{d}t, \quad \forall \eta \in V_\eta, \tag{2.21}$$

where the notation

$$\frac{\partial L}{\partial \varphi_*}(\varphi_*, \dot{\varphi}_*) := \frac{\partial L}{\partial \tilde{\varphi}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta})\big|_{\theta=0}, \quad \frac{\partial L}{\partial \dot{\varphi}_*}(\varphi_*, \dot{\varphi}_*) := \frac{\partial L}{\partial \dot{\tilde{\varphi}}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta})\big|_{\theta=0}, \tag{2.22}$$

is employed. Performing integration by parts on the last term generates

$$\int_{t_0}^{t_1} \dot{\eta}^T \frac{\partial L}{\partial \dot{\varphi}_*}\mathrm{d}t = \big[\eta^T \frac{\partial L}{\partial \dot{\varphi}_*}\big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \eta^T \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}\mathrm{d}t = -\int_{t_0}^{t_1} \eta^T \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}\mathrm{d}t. \tag{2.23}$$

Insertion into Eq. 2.21 gives the EOM in weak form

$$0 = \int_{t_0}^{t_1} \eta^T \big(\frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}\big)\mathrm{d}t, \quad \forall \eta \in V_\eta. \tag{2.24}$$

To continue, consider the fundamental lemma of calculus of variations,

Lemma 1 (Fundamental lemma of calculus of variations in time)**.** *Let $N(t) \in C([t_0, t_1], \mathbb{R}^{n_\varphi})$. If*

$$\int_{t_0}^{t_1} h(t)^T N(t)\mathrm{d}t = 0, \quad \forall h \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi}), \tag{2.25}$$

with $h(t_0) = h(t_1) = 0$, then $N(t) = 0$ for $t_0 \leq t \leq t_1$.

The lemma is proved by contradiction, as shown in Appendix A.1. The unconstrained dynamics, $M(t) = \frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}$, is required to be a continuous function in time by VP 1. Since also $\eta \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi})$, the prerequisites for the fundamental lemma of calculus of variations are met. By applying the lemma, Euler-Lagrange's equation, Eq. 2.5, is retrieved for φ_*, which was the definition for φ_* being an extremal. VP 1 is the process of finding a local extremum for an unconstrained system in calculus of variations [3].

Consider the specific Lagrangian $L = T - V$. For the specific Lagrangian, with $\varphi_* \in C_2([t_0, t_1], \mathbb{R}^{n_\varphi})$, $M(t)$ is a continuous function in time. Hence, all requirements for VP 1 are fulfilled. Applying the lemma gives Euler-Lagrange's equation

$$\frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*} = 0, \quad \text{with } L = T - V. \tag{2.26}$$

By applying the lemma, the formulation changes from a weak to a strong formulation in time. A classical theorem in functional analysis is that a strong solution implies a weak solution, but not vice versa [4]. Since the endpoints in time $(\varphi_*(t_0), \dot{\varphi}_*(t_1)) = (\varphi_0, \dot{\varphi}_0)$ are given, a boundary value problem (BVP) is retrieved,

Boundary value problem 1. *For a given pair of endpoint conditions, $(\varphi_*(t_0), \dot{\varphi}_*(t_1)) = (\varphi_0, \dot{\varphi}_0)$, find a motion path $\varphi_* \in C_2([t_0, t_1], \mathbb{R}^{n_\varphi})$ for which*

$$\frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*} = 0, \tag{2.27a}$$

where

$$L(\varphi_*, \dot{\varphi}_*) = T(\dot{\varphi}_*) - V(\varphi_*). \tag{2.27b}$$

In the previous subsection, Euler-Lagrange's equation, with $L = T - V$, was related to Newton's second law, written as a first order system, Eq. 2.2. Moreover, under the assumptions of uniformly Lipschitz continuity of f and v, and M being invertible, an IVP based on Eq. 2.2 provides a unique motion $\varphi_*(t)$, for $t_0 \leq t \leq t_1$. Therefore, the IVP for a specific pair of initial conditions corresponds to the BVP 1 for a specific pair of endpoint conditions. Thus existence and uniqueness of the motion for the IVP implies that the motion for the corresponding BVP exists and is unique. As shown above, BVP 1 is reformulation in strong form of VP 1, which is a problem in weak form. The existence and uniqueness of a solution for BVP 1 implies existence, but not uniqueness, of a solution for VP 1.

In the integration by part step in the above proof, as well as for the connection to classical mechanics (see Section 2.1.1), the requirement $\varphi_* \in C_2([t_0, t_1], \mathbb{R}^{n_\varphi})$ is needed. However, the EOM in weak form, Eq. 2.24, can be proved under the weaker assumptions that $\varphi_* \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi})$, and furthermore that $\varphi_* \in \mathcal{D}_1([t_0, t_1], \mathbb{R}^{n_\varphi})$, where $\mathcal{D}_1$ denotes the function space of piecewise differentiable functions with continuous derivates, for finite many pieces. For $\varphi_* \in \mathcal{D}_1([t_0, t_1], \mathbb{R}^{n_\varphi})$ it is possible to model multibody systems in presence of actuators, which can change the characteristics of the system in an instant. Proofs under the weaker assumptions are provided in introductory textbooks on calculus of variations, eg. [3].

2.1.3 Extension to Constrained Dynamics

In a multibody setting, the presence of joints serves as holonomic constraints, which generates constrained dynamics. Consider $n_\lambda < n_\varphi$ holonomic, or geometric, constraint equations,

$$g(\varphi_*(t)) = 0, \quad \varphi_* \in V_{\varphi_*}, \tag{2.28}$$

where $g \in C_1([t_0, t_1], \mathbb{R}^{n_\lambda})$. Time-differentiation gives

$$G(\varphi_*(t))\dot{\varphi}_*(t) = 0, \tag{2.29}$$

where $G := \frac{\mathrm{d}g(\varphi_*)}{\mathrm{d}\varphi_*} \in C([t_0, t_1], \mathbb{R}^{n_\lambda \times n_\varphi})$ denotes the constraint Jacobian. Assume that G has full row rank, which means that the rows in G are linearly independent. Then, the constraint restricts the degrees of freedom of the system, from the space $\mathbb{R}^{n_\varphi}$ to a manifold of n_s dimensions, where $n_s := n_\varphi - n_\lambda$.

For constrained dynamics, Hamilton's principle of least action is solved subject to Eq. 2.28,

Variational problem 2 (Constrained problem)**.** *For a given pair of tuples* (t_0, φ_0) *and* (t_1, φ_1)*, where* φ_0 *and* φ_1 *fulfill Eq. 2.28, find an extremal* $\varphi_*(t) \in V_{\varphi_*}(\mathbb{R}^{n_\varphi})$*, under the assumption that* $\varphi_*(t)$ *exists, such that*

$$0 = j'(0), \quad \forall \eta \in V_\eta(\mathbb{R}^{n_\varphi}), \tag{2.30a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L(\tilde{\varphi}(\theta, t), \dot{\tilde{\varphi}}(\theta, t))\mathrm{d}t, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}(\mathbb{R}^{n_\varphi}), \tag{2.30b}$$

subject to

$$g(\tilde{\varphi}(t)) = 0, \quad g \in C_1([t_0, t_1], \mathbb{R}^{n_\lambda}), \tag{2.30c}$$

with $G(\varphi_*(t))$ *having full row rank.*

Eq. 2.30c is stated as a strong formulation in time, whereas the stationarity condition, Eq. 2.30a, enforces stationarity of the extremal in a weak sense in time. This inconsistency in the formulations can be circumvented in two different ways, either by solely considering motions on the constrained manifold, for which Eq. 2.30c automatically fulfilled. Alternatively Eq. 2.30c is incorporated into Eq. 2.30b. Thus, with by reformulations, Eq. 2.30c is avoided.

Consider a parametrization $s(t) \in C_2([t_0, t_1], \mathbb{R}^{n_s})$ spanning the constrained manifold. After applying a coordinate transformation to the minimal coordinates spanning the parametrization, $\varphi = \varphi(s(t)) \in C_2([t_0, t_1], \mathbb{R}^{n_s})$, the n_λ dimensions related to the constraints do not have to be considered. Thus, the dynamics can be described by an unconstrained VP on the manifold,

Variational problem 3 (Minimal coordinates)**.** *For a given pair of tuples (t_0, φ_0) and (t_1, φ_1), and a parametrization $s(t) \in C_2([t_0, t_1], \mathbb{R}^{n_s})$ of the constrained manifold, find an extremal $\varphi_*(s(t)) \in V_\varphi(\mathbb{R}^{n_s})$, for $t_0 \leq t \leq t_1$, under the assumption that $\varphi_*(s(t))$ exists, such that*

$$0 = j'(0), \quad \forall \eta(s(t)) \in V_\eta(\mathbb{R}^{n_s}), \tag{2.31a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L(\tilde{\varphi}(\theta, s(t)), \dot{\tilde{\varphi}}(\theta, s(t)))\mathrm{d}t, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}(\mathbb{R}^{n_s}), \tag{2.31b}$$

Under the assumption that a given $s(t)$ is non-singular for all admissible motions, VP 3 is solved just as a reduced unconstrained VP 1. The derived EOM in strong form are known as the state space form. The state space form has the advantage that the number of unknowns are reduced by n_λ equations. However, its main drawback is that a prescribed parametrization is required to employ the coordinate transformation.

A typical example, when the state space form is handy, is to model the dynamics on a sphere in three dimensions. With Canonical coordinates, the VP is constrained. However, by introduction of spherical coordinates, the problem can be reformulated as a two-dimensional unconstrained VP for the two angles spanning the sphere. For many complex structures it is not possible to prescribe the constrained manifold. Then, another approach is needed.

The alternative approach is based on a Lagrange multiplier technique which follows from Lagrange's multiplier theorem,

Theorem 1 (Lagrange's multiplier theorem)**.** *Under the assumption that $\varphi_* \in C_2([t_0, t_1], \mathbb{R}^{n_\varphi})$ is an extremal of the constrained VP 2, there exists a specific Lagrange multiplier $\lambda_* \in C([t_0, t_1], \mathbb{R}^{n_\lambda})$ such that φ_* is an extremal of the unconstrained VP 1 for Lagrangian P,*

$$P(\tilde{\varphi}, \dot{\tilde{\varphi}}) = L(\tilde{\varphi}, \dot{\tilde{\varphi}}) - g(\tilde{\varphi})^T \lambda_*, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}, \tag{2.32}$$

with $g \in C_1([t_0, t_1], \mathbb{R}^{n_\lambda})$, ie. $0 = \frac{\partial P}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial P}{\partial \dot{\varphi}_*}$.

First, note that $P = P(\tilde{\varphi}, \dot{\tilde{\varphi}})$ is a bilinear functional in $\tilde{\varphi}$ and $\dot{\tilde{\varphi}}$, which only holds for a specific Lagrange multiplier λ_*. For the proof of the theorem, Eq. 2.30c is considered

$$0 = g(\tilde{\varphi}) = g(\varphi_* + \theta\eta). \tag{2.33}$$

Performing differentiation with respect to θ generates

$$\frac{\mathrm{d}}{\mathrm{d}\theta} g(\tilde{\varphi}) = \frac{\mathrm{d}g(\tilde{\varphi})}{\mathrm{d}\tilde{\varphi}}\eta = 0. \tag{2.34}$$

Thereafter, setting $\theta = 0$ gives

$$G(\varphi_*(t))\eta = 0, \tag{2.35}$$

with $G(\varphi_*(t)) = \frac{\mathrm{d}g(\varphi_*)}{\mathrm{d}\varphi_*} = \frac{\mathrm{d}g(\tilde{\varphi})}{\mathrm{d}\tilde{\varphi}}|_{\theta=0}$. Since G denotes the constraint Jacobian, η is restricted to the tangent plane of the constraints. Thus, for VP 2, η is restricted to the space

$$V_{\hat{\eta}} = \{\eta(t) \mid \eta(t) \in V_\eta,\ G(\varphi_*(t))\eta = 0\}. \tag{2.36}$$

The derivations steps from VP 1, which render in the EOM in weak form in time, Eq. 2.24, can be employed for VP 2, with V_η replaced by $V_{\hat{\eta}}$. The updated EOM in weak form is then stated as

$$0 = \int_{t_0}^{t_1} \hat{\eta}^T M \mathrm{d}t, \quad \forall \hat{\eta} \in V_{\hat{\eta}}(\mathbb{R}^{n_\varphi}), \tag{2.37}$$

where $M(t) = \frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*} \in C([t_0, t_1], \mathbb{R}^{n_\varphi})$ is the uconstrained dynamics. Since $\hat{\eta} \in V_{\hat{\eta}}(\mathbb{R}^{n_\varphi})$ is restricted to the tangent plane of the constraint, the fundamental lemma of calculus of variations, Lemma 1, cannot be employed for Eq. 2.37. However, by projecting $M(t)$ onto the tangent plane of the constraints, for each time point t, the weak form is reduced to only be required to hold on the tangent plane, with $\hat{\eta} \in V_{\hat{\eta}}(\mathbb{R}^{n_\varphi})$ reduced to $\eta \in V_\eta(\mathbb{R}^{n_s})$,

$$0 = \int_{t_0}^{t_1} \eta^T \Pi M \mathrm{d}t, \quad \forall \eta \in V_\eta(\mathbb{R}^{n_s}). \tag{2.38}$$

Π denotes the complement to the orthogonal projector onto the range of $G(\varphi_*)^T$

$$\Pi := I - G^T(GG^T)^{-1}G, \tag{2.39}$$

where I denotes the identity matrix. Invertibility of GG^T follows from that G has full row rank, as shown in Appendix A.2. Due to Eq. 2.35, projection onto the tangent plane of the constraints, and projection onto the complement of $G(\varphi_*)^T$, are equivalent projections. For a presentation of orthogonal projectors, see a textbook on numerical linear algebra, eg. [5, p. 46]. Applying the fundamental lemma of calculus to Eq. 2.38 generates

$$0 = \Pi M = M - G^T(GG^T)^{-1}GM. \tag{2.40}$$

Set the Lagrange multiplier $\lambda_* = (GG^T)^{-1}GM$. That $(GG^T)^{-1}$ is continuous in time follows from that G are continuous in time, and that an invertible square matrix is continuous, as proved in Appendix A.3. Since also M is continuous in time, $\lambda_* \in C([t_0, t_1], \mathbb{R}^{n_\lambda})$. Since $M(t) = \frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}$, Eq. 2.40 corresponds to

$$0 = \frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*} - G(\varphi_*)^T\lambda_*, \tag{2.41}$$

which is equivalent to

$$0 = \frac{\partial P}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial P}{\partial \dot{\varphi}_*}, \tag{2.42}$$

for $P(\varphi_*, \dot{\varphi}_*) = L(\varphi_*, \dot{\varphi}_*) - g(\varphi_*)^T\lambda_*$. Since Eq. 2.42 is Euler-Lagrange's equation for the Lagrangian P, φ_* is an extremal of the unconstrained VP 1 for Lagrangian P. Hence, the theorem is proved.

Remark 1. $\lambda_*(t) = 0$, *for* λ_* *in Theorem 1, implies that the motion* φ_* *for unconstrained and constrained dynamics are exactly the same.*

If $\lambda_* = 0$, then $P = L$, implying that φ_* is an extremal of both VP 1 and VP 2.

According to Lagrange's multiplier theorem, an extremal φ_* of the constrained VP 2 is also an extremal of the unconstrained VP 1 for the Lagrangian P, for a specific Lagrange multiplier $\lambda_*(t) \in C([t_0, t_1], \mathbb{R}^{n_\lambda})$. Thus, as an expense for reformulating the constrained VP as an unconstrained VP, a specific Lagrange multiplier $\lambda_*(t) \in C([t_0, t_1], \mathbb{R}^{n_\lambda})$ has to be prescribed.

This last obstacle is circumvented by extending the VP to finding an extremal $(\varphi_*(t), \lambda_*(t))$, for $t_0 \leq t \leq t_1$, embedded in a space (V_φ, V_λ) with given endpoint conditions, $(\varphi(t_0), \lambda(t_0)) = (\varphi_0, \lambda_0)$ and $(\varphi(t_1), \lambda(t_1)) = (\varphi_1, \lambda_1)$. For Theorem 1 to hold, the given (λ_0, λ_1) must equal $(\lambda_*(t_0), \lambda_*(t_1))$. The tuples $(t_0, \varphi_0, \lambda_0)$ and $(t_1, \varphi_1, \lambda_1)$ are denoted as consistent endpoint conditions, if φ_0 and φ_1 fulfill Eq. 2.28, and $(\lambda_0, \lambda_1) = (\lambda_*(t_0), \lambda_*(t_1))$.

Under the assumption that φ_* is an extremal of VP 2, the extremal component $\lambda_*(t) \in C([t_0, t_1], \mathbb{R}^{n_\lambda})$ exists, according to the theorem. Moreover, it is embedded in the function space

$$V_\lambda = \{\lambda(t) \mid \lambda(t) \in C([t_0, t_1], \mathbb{R}^{n_\lambda}),\ \lambda(t_0) = \lambda_0,\ \lambda(t_1) = \lambda_1\}, \tag{2.43}$$

where the space of an extremal component λ_* is expressed by,

$$V_{\lambda_*} = \{\lambda_*(t) \mid V_\lambda \ni \lambda_*(t)\}. \tag{2.44}$$

Define the affine space $V_{\tilde{\lambda}}$, by linearizing the space around $\lambda_* \in V_\lambda$, as

$$V_{\tilde{\lambda}} := V_{\lambda_*} \oplus V_{\theta\vartheta}, \tag{2.45}$$

where the space of admissible variations, with respect to λ_*, is defined as

$$V_{\theta\vartheta} := V_\theta \otimes V_\vartheta, \tag{2.46}$$

with $V_{\theta\eta}$ from Eq. 2.15a and

$$V_\vartheta := \{\vartheta(t) \mid \vartheta(t) \in C_1([t_0, t_1], \mathbb{R}^{n_\lambda}),\ \vartheta(t_0) = 0,\ \vartheta(t_1) = 0\}. \tag{2.47}$$

The requirement for $\vartheta \in V_\vartheta$ to vanish at the endpoints is included since $\lambda_*(t_0)$ and $\lambda_*(t_1)$ are assumed to be known.

The updated Lagrangian $L_{\text{C}} : \mathbb{R}^{2n_\varphi + n_\lambda} \to \mathbb{R}$ is a trilinear functional in $(\tilde{\varphi}, \dot{\tilde{\varphi}}, \tilde{\lambda})$, extended from the bilinear functional P, Eq. 2.32,

$$L_{\text{C}}(\tilde{\varphi}, \dot{\tilde{\varphi}}, \tilde{\lambda}) = L(\tilde{\varphi}, \dot{\tilde{\varphi}}) - g(\tilde{\varphi})^T \tilde{\lambda}, \quad (\tilde{\varphi}, \tilde{\lambda}) \in (V_{\tilde{\varphi}}, V_{\tilde{\lambda}}). \tag{2.48}$$

L_{C} is referred to as the constrained Lagrangian. Hamilton's principle of least action, for finding an extremal $(\varphi_*(t), \lambda_*(t)) \in (V_{\varphi_*}, V_{\lambda_*})$, is stated as

Variational problem 4 (Lagrange multiplier technique)**.** *For a given pair of consistent endpoint conditions* $(t_0, \varphi_0, \lambda_0)$ *and* $(t_1, \varphi_1, \lambda_1)$*, find an extremal* $(\varphi_*(t), \lambda_*(t)) \in (V_{\varphi_*}, V_{\lambda_*})$*, for* $t_0 \leq t \leq t_1$*, under the assumption that* $(\varphi_*(t), \lambda_*(t))$ *exists, such that*

$$0 = j'(0), \quad \forall (\eta, \vartheta) \in (V_\eta, V_\vartheta), \tag{2.49a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L_{\text{C}}(\tilde{\varphi}(\theta, t), \dot{\tilde{\varphi}}(\theta, t), \tilde{\lambda}(\theta, t)) \mathrm{d}t, \quad (\tilde{\varphi}, \tilde{\lambda}) \in (V_{\tilde{\varphi}}, V_{\tilde{\lambda}}). \tag{2.49b}$$

Assume that φ_* is an extremal of VP 2. The existence of λ_* follows from Theorem 1. To find (φ_*, λ_*), the action integral is differentiated

$$\begin{aligned} j'(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{t_0}^{t_1} L_{\mathrm{C}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta}, \lambda_* + \theta\vartheta)\mathrm{d}t \qquad (2.50)\\ &= \int_{t_0}^{t_1} \big(\eta^T \frac{\partial L_{\mathrm{C}}}{\partial\tilde{\varphi}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta}, \lambda_* + \theta\vartheta) + \dot{\eta}^T \frac{\partial L_{\mathrm{C}}}{\partial\dot{\tilde{\varphi}}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta}, \lambda_* + \theta\vartheta)\\ &+ \frac{\partial L_{\mathrm{C}}}{\partial\tilde{\lambda}}(\varphi_* + \theta\eta, \dot{\varphi}_* + \theta\dot{\eta}, \lambda_* + \theta\vartheta)\vartheta\big)\mathrm{d}t, \end{aligned}$$

Insertion of Eq. 2.50 into the stationarity condition, Eq. 2.49a, gives

$$0 = j'(0) = \int_{t_0}^{t_1} \big(\eta^T \frac{\partial L_{\mathrm{C}}}{\partial\varphi_*}(\varphi_*, \dot{\varphi}_*, \lambda_*) + \dot{\eta}^T \frac{\partial L_{\mathrm{C}}}{\partial\dot{\varphi}_*}(\varphi_*, \dot{\varphi}_*, \lambda_*) + \frac{\partial L_{\mathrm{C}}}{\partial\lambda_*}(\varphi_*, \dot{\varphi}_*, \lambda_*)\vartheta\big)\mathrm{d}t, \qquad (2.51)$$

Performing integration by parts in time on the term including $\dot{\eta}$, see Eq. 2.23, generates

$$0 = \int_{t_0}^{t_1} \eta^T \big(\frac{\partial L_{\mathrm{C}}}{\partial\varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\mathrm{C}}}{\partial\dot{\varphi}_*}\big)\mathrm{d}t + \int_{t_0}^{t_1} \frac{\partial L_{\mathrm{C}}}{\partial\lambda_*}\vartheta\mathrm{d}t, \quad \forall(\eta, \vartheta) \in (V_\eta, V_\vartheta). \qquad (2.52)$$

Insertion of Eq. 2.48 into Eq. 2.52 leads to the equations of constrained motion (EOCM) in weak form

$$0 = \int_{t_0}^{t_1} \eta^T \big(\frac{\partial L}{\partial\varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}_*} - G(\varphi_*)^T\lambda_*\big)\mathrm{d}t + \int_{t_0}^{t_1} \vartheta^T g(\varphi_*)\mathrm{d}t, \quad \forall(\eta, \vartheta) \in (V_\eta, V_\vartheta). \qquad (2.53)$$

Setting $\eta = 0$ gives

$$0 = \int_{t_0}^{t_1} \vartheta^T g(\varphi_*)\mathrm{d}t, \quad \forall\vartheta \in V_\vartheta. \qquad (2.54)$$

Eq. 2.54, and thereby also VP 4, only requires that the holonomic constraint equations hold for the extremal component φ_* in a weak sense in time. This is a relaxation on the restrictions to the constraints provided in VP 2, which was strong in time and was also forced to hold for all admissible motions $\tilde{\varphi} \in V_{\tilde{\varphi}}$. Therefore, the assumption that $\varphi_*(t)$ is an extremal of VP 2 can be relaxed to that there exists an extremal (φ_*, λ_*) of VP 4.

For Eq. 2.54 all the prerequisites are met to employ the fundamental lemma of calculus of variations, Lemma 1. By instead setting $\vartheta = 0$ generates

$$0 = \int_{t_0}^{t_1} \eta^T \big(\frac{\partial L}{\partial\varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}_*} - G(\varphi_*)^T\lambda_*\big)\mathrm{d}t, \quad \forall\eta \in V_\eta. \qquad (2.55)$$

Consider $M_{\mathrm{C}}(t) = M(t) - G(\varphi_*)^T\lambda_*$. Since $M(t) \in C([t_0, t_1], \mathbb{R}^{n_\varphi})$ (see Section 2.1.2) and $G(\varphi)_*^T\lambda_*(t)$ are continuous in time, so is M_{C}. As a consequence, all the prerequisites are met to employ the fundamental lemma of calculus of variations. By employing Lemma 1 for Eqs. 2.54 and 2.55, a BVP for a system of second order differential-algebraic equations (DAEs) is retrieved,

Boundary value problem 2. *For a given pair of consistent endpoint conditions $(t_0, \varphi_0, \lambda_0)$ and $(t_1, \varphi_1, \lambda_1)$, find a $(\varphi_*, \lambda_*) \in (V_{\varphi_*}, V_{\lambda_*})$ path such that*

$$\frac{\partial L}{\partial\varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}_*} - G(\varphi_*)^T\lambda_* = 0, \qquad (2.56a)$$

$$g(\varphi_*) = 0. \qquad (2.56b)$$

By performing the derivation steps from Eq. 2.1 to Eq. 2.5 backwards on Eq. 2.56a, Eq. 2.56 is rewritten as Lagrange's equations of the first kind, or the EOCM in strong form,

$$M\ddot{\varphi}_*(t) = f(\varphi_*(t)) - G(\varphi_*(t))^T\lambda_*(t), \tag{2.57a}$$
$$g(\varphi_*(t)) = 0, \tag{2.57b}$$

where $M \in \mathbb{R}^{n_\varphi \times n_\varphi}$ and $f \in C([t_0, t_1], \mathbb{R}^{n_\varphi})$ denote a symmetric mass matrix and a conservative force, respectively.

The system of DAEs, Eq. 2.57, should be understood as the system

$$M\ddot{\varphi}_*(t) = f(\varphi_*(t)), \tag{2.58}$$

restricted to move along $g(\varphi_*(t)) = 0$. The restriction is provided by the constraint component $-G(\varphi_*)^T\lambda_*$, which can be interpreted as a force pulling the motion back to a path constrained by $g(\varphi_*) = 0$.

To model Eq. 2.57 numerically, the system is discretized in time by a time-integration method (see Section 2.7). To enforce the motion to follow the constrained path, the constraint equations and the constraint term should hold for the new time-update t_{new}, $g(\varphi_*(t_{\text{new}})) = 0$ and $-G(\varphi_*(t_{\text{new}}))^T\lambda_*(t_{\text{new}})$. Thus, with an explicit time-integrating scheme for the acceleration, a system would still be required to be solved at each time-update. Hence, implicit time-integrators are employed for systems of DAEs.

By introducing additional velocity variables $v \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi})$, Eq. 2.57 is rewritten as a system of first order DAEs,

$$\dot{\varphi}_* = v, \tag{2.59a}$$
$$M\dot{v} = f(\varphi_*) - G(\varphi_*)^T\lambda_*, \tag{2.59b}$$
$$0 = g(\varphi_*). \tag{2.59c}$$

Compared to IVPs based on systems of ODEs, IVPs based on systems of DAEs are more difficult to solve numerically. This is mainly due to that the latter is more sensitive to perturbations of the system, and that the initial conditions $(\varphi_0, v_0, \lambda_0)$ are forced to satisfy

$$M\dot{v}_0 = f(\varphi_0) - G(\varphi_0)^T\lambda_0, \tag{2.60a}$$
$$g(\varphi_0) = 0, \tag{2.60b}$$
$$G(\varphi_0)v_0 = 0. \tag{2.60c}$$

where the Eq. 2.60c stems from Eq. 2.29.

Initial conditions, which fulfill Eq. 2.60, are referred to as consistent initial conditions. In practice, for complex structures, finding consistent initial values is a challenging procedure [1]. In the next subsection, the sensitivity to perturbations is discussed for IVPs based on systems of DAEs, by introducing the perturbation and differentiation indices.

2.1.4 Sensitivity to Perturbations for Systems of DAEs

Consider a general system of first order differential equations,

$$F(x(t), \dot{x}(t), t) = 0, \tag{2.61}$$

with a solution $x(t)$ for $t_0 \leq t \leq t_1$. When solving the system numerically in time, a slightly perturbed system has to be considered. The perturbation arises mainly due to the time-discretization scheme, but also from the limitation on how precisely a value is evaluated on a machine. Consider the perturbed system

$$F(\hat{x}(t), \dot{\hat{x}}(t), t) = \delta(t), \tag{2.62}$$

where $\delta(t)$ is a perturbation. A way to estimate the sensitivity to perturbations for a system is by its perturbation index. If the perturbation index of Eq. 2.61 is $k \geq 1$, then there exists an estimate, on the norm of the global error induced by the perturbation $\|x(t) - \hat{x}(t)\|$, which is dependent on the norm of the initial error $\|x(t_0) - \hat{x}(t_0)\|$, and the norm of the perturbation and its derivatives,

$$\|x(t) - \hat{x}(t)\| \leq \|x(t_0) - \hat{x}(t_0)\| + \max_{t_0 \leq \epsilon \leq t} \|\delta(\epsilon)\| + ... + \max_{t_0 \leq \epsilon \leq t} \|\delta^{k-1}(\epsilon)\|, \tag{2.63}$$

if the perturbation $\delta(t)$ is sufficiently small [1, p. 34]. Systems of ODEs have perturbation index $k = 0$, with the estimate [1, p. 34]

$$\|x(t) - \hat{x}(t)\| \leq \|x(t_0) - \hat{x}(t_0)\| + \max_{t_0 \leq \epsilon \leq t} \| \int_{t_0}^{\epsilon} \delta(\tau) \mathrm{d}\tau \|. \tag{2.64}$$

For systems with perturbation indices $k \geq 2$, the magnitude, of the maximal oscillatory behavior at any time point of the perturbation ($\max_{t_0 \leq \epsilon \leq t} \|\delta^i(\epsilon)\|$ for $i \in [1, k-1]$), amplifies the bound on $\|x(t) - \hat{x}(t)\|$. Specifically, highly oscillatory perturbations with tiny magnitudes generate strict bounds for systems with perturbation indices $k \leq 1$, but loose bounds for systems with perturbation indices $k \geq 2$. Note, the perturbation induced by the machine error ϵ_{mac} (the error induced by that the evaluations of two values approaching each other eventually become indistinguishable on a machine) is tiny, $\epsilon_{\text{mac}} \sim 10^{-16}$, but random, and therefore naturally oscillatory at some instance in time.

Also, for nonlinear systems, the solutions of discretized systems, at each time step, are required to converge until the residual at that time step is below a specified bound. Thus, the magnitude of the induced perturbation can partly be forced to be small for all time steps. However, the oscillatory behavior of the perturbation is not controlled. Moreover, the oscillatory nature of the two described perturbations increases as the time step size decreases. As a consequence, solving systems numerically, with perturbation indices $k \geq 2$, requires great care.

Consider the system of first order DAEs in Eq. 2.59, the EOCM in strong form. For systems of first order DAEs with holonomic constraints, the perturbation index is exactly the same as the differentiation index [1, p. 35]. The differentiation index defines how many differentiation steps are required to reformulate a well-posed system of first order DAEs, to a system of explicit first order ODEs. In the following paragraph, Eq. 2.59 is shown to have differentiation index three, under the assumptions that G has full row rank and M is symmetric positive definite (SPD). Note, the reformulation, which gives the differentiation index, is completed when a system of explicit ODEs in $(\dot{\varphi}_*, \dot{v}, \dot{\lambda}_*)$ is retrieved.

First, differentiate the constraint equations in time, to retrieve the constraints at velocity level,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(\varphi_*) = G(\varphi_*)\dot{\varphi}_* = G(\varphi_*)v.. \tag{2.65}$$

A second differentiation step in time yields the constraints at acceleration level,

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\varphi_*) = G(\varphi_*)\dot{v} + \kappa(\varphi_*, v), \quad \kappa(\varphi_*, v) = \frac{\mathrm{d}G(\varphi_*)}{\mathrm{d}t} v. \tag{2.66}$$

By combining Eq. 2.59b and Eq. 2.66, a linear system is retrieved,

$$\begin{bmatrix} M & G^T \\ G & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \lambda_* \end{bmatrix} = \begin{bmatrix} f \\ \kappa \end{bmatrix} \tag{2.67}$$

Under the assumptions that G has full row rank and M is SPD,

$$\begin{bmatrix} M & G^T \\ G & 0 \end{bmatrix}, \tag{2.68}$$

is invertible. A proof is provided in Appendix A.4. The matrix is block diagonalized by performing block Gaussian elimination

$$\begin{bmatrix} M & G^T \\ G & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ GM^{-1} & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & -GM^{-1}G^T \end{bmatrix} \begin{bmatrix} I & M^{-1}G^T \\ 0 & I \end{bmatrix}. \tag{2.69}$$

By inverting the three matrices, Eq. 2.67 is rewritten as

$$\dot{v} = M^{-1}(f - G^T\lambda_*), \tag{2.70a}$$

$$\lambda_* = (GM^{-1}G^T)^{-1}(GMf + \kappa) = F_2. \tag{2.70b}$$

By insertion of Eq. 2.70b into λ_* in Eq. 2.70a, a system of explicit ODEs is retrieved in $\dot{v}$,

$$\dot{v} = M^{-1}(f - G^T F_2) = F_1. \tag{2.71}$$

By appedinding Eq. 2.59a, and a time-differentiated Eq. 2.70b, a system of explicit ODEs in $(\dot{\varphi}, \dot{v}, \dot{\lambda})$ is retrieved,

$$\dot{\varphi}_* = v, \tag{2.72a}$$

$$\dot{v} = F_1, \tag{2.72b}$$

$$\dot{\lambda}_* = \dot{F}_2. \tag{2.72c}$$

In total, three differentiation steps were performed to retrieve Eq. 2.72. Therefore, Eq. 2.59 is a system of index-3 first order DAEs. By accompanying Eq. 2.72, with consistent initial conditions $(\varphi_0, v_0, \lambda_0)$, the retrieved IVP, under the assumptions that the right-hand side is uniformly Lipschitz continuous in time, has a unique solution for a sufficiently small interval in time, according to Picard-Lindelöf's theorem [2]. Furthermore, the time interval can be extended to $[t_0, t_1]$ by successively applying the theorem. Thus, for consistent initial conditions, under the set of admissibility assumptions, and that M is SPD, G has full row rank, f is conservative and the the right-hand side of Eq. 2.72 is uniformly Lipschitz continuous in time, the extremal of VP 4 exists (the existence of a solution for the IVP implies the existence of a solution for BVP 2, which furthermore implies the existence of a solution for VP 4).

The sensitivity to perturbations for systems of DAEs can generally be reduced through index reduction. For Eq. 2.59 the simplest way to reduce the index is by replacing the constraint equations by a time-differentiated version, eg. the constraints at velocity level $0 = G(\varphi_*)v$. While this process, reduces the sensitivity to perturbations, it introduces drift-off effects. Drift-off effects refer to that the motion φ_* might deviate away from the constrained path $g(\varphi_*) = 0$ over time. However, there are ways to be able to rewrite systems of index-3 DAEs as systems of index-2 DAEs, while still avoiding drift-off effects. Those methods rely on extending the system, eg. by requiring fulfillment of both $0 = g(\varphi_*)$ and $0 = G(\varphi_*)v$ simultaneously. For details, see [1, p. 46-47]. In this study, only a basic constrained problem is considered, which was successfully solved numerically based on a system of index-3 first order DAEs. An important note, for systems of index-3 DAEs, the constraints at velocity and acceleration level, $0 = G(\varphi_*)v$ and $0 = \frac{\mathrm{d}^2 g(\varphi_*)}{\mathrm{d}t^2}$, serve as hidden constraints, which makes the system more sensitive to perturbations [1, p. 34].

2.2 Rigid Body Dynamics

In this section, planar rigid body dynamics is considered. Rigid bodies are per definition undeformable. The motion of a rigid body is fully described by the translational and rotational motion of a local reference system, placed at its center of mass. In the first subsection, the EOM is derived for unconstrained dynamics. As mentioned previously, interconnecting joints serve as holonomic constraints for multibody systems. In the second subsection, a model for interconnecting massless, rigid, revolute joints is presented. Recall, the aim of this study is to highlight the limitations of a massless, rigid, revolute joint model for interconnecting elastic and rigid bodies.

For completion, a model for interconnecting force elements is presented in the third subsection. Lastly, in the forth subsection, the dynamics of a multibody system, constrained by interconnecting joints, is presented.

2.2.1 Unconstrained Rigid Body Dynamics

In this subsection, an unconstrained rigid body is considered. Rigid bodies are defined as bodies which do not deform due to applied pressure. They may deform due to change in temperature, but the temperature of the bodies are assumed to be constant in this study. The dynamics, at constant temperature, of a rigid body can be fully described, by the motion of a local reference system, placed at its center of mass. For planar motion, the local reference system has two translational and one rotational degree of freedom. Modeling with rigid bodies generates a system of minimal complexity. In a multibody setting, with some bodies being much stiffer than the others, it is often desirable to model the stiffer bodies as rigid bodies.

In Section 2.1.2, the equations of unconstrained motion (EOUM) in weak form was derived from the unconstrained VP 1. To be able to apply the derivation steps to retrieve the rigid body dynamics, the spaces of admissible translational and rotational motions have to be assembled to a space of admissible motions. Retrieving the space of admissible motions is the main new concept in this subsection. Lastly, the specific Lagrangian $L = T - V$ is inserted to retrieve Newton-Euler's EOM in strong form from the EOUM in weak form.

Let a planar rigid body with a bounded domain be considered. Throughout the whole study, only planar motion is treated, to simplify the description of the rotational motion. For a general three-dimensional presentation, see [1]. Assume that the applied forces only vary in the plane of the motion. Then, it is sufficient to model a planar segment $\bar{\Omega} \subset \mathbb{R}^2$ of the rigid body. From here onwards, the rigid body refers to the planar segment of the rigid body.

Place a local, a body-fixed, reference frame at the center of mass of the rigid body. Let $x \in L_2(\bar{\Omega})$ denote the material points of the body with respect to the body-fixed reference frame. Consider the planar motion of the body, with respect to an inertial reference frame. The planar motion is decomposed into a distance from the inertial frame to the center of mass $r(t) \in C_2([t_0, t_1], \mathbb{R}^2)$, as well as material points x mapped to the global reference frame

$$\varphi(r(t), \alpha(t), x) = r(t) + A(\alpha(t))x, \quad A(\alpha(t)) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{2.73}$$

$A(\alpha(t)) \in C_2([t_0, t_1], \mathbb{R}^{2\times 2})$ denotes the rotation matrix from the body-fixed to the inertial reference frame, with $\alpha(t) \in C_2([t_0, t_1], \mathbb{R})$ denoting the angle between the body-fixed reference frame and the inertial reference frame. A schematic is illustrated in Fig. 2.1.

The body is set into motion by body and surface applied forces. The applied forces consist of, a body force density $\beta(x, t) \in L_2(\Omega) \otimes C([t_0, t_1], \mathbb{R}^2)$ applied over the interior domain Ω, and an surface force density $\tau(x, t) \in L_2(\Gamma_{\mathrm{N}}) \otimes C([t_0, t_1], \mathbb{R}^2)$ applied along a bounded Neumann

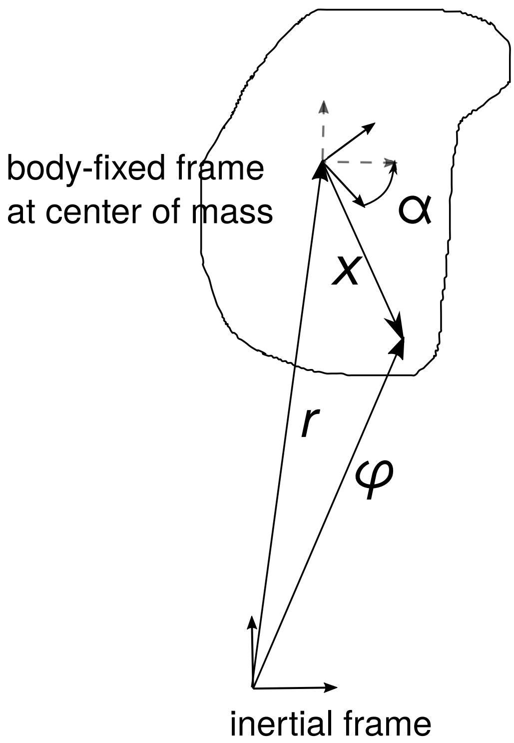

Figure 2.1: A rigid body with a body-fixed frame at the center of mass.

boundary segment Γ_N. Both β and τ are defined with respect to the body-fixed reference frame. The function spaces in space are chosen to guarantee that the forthcoming integrals in space are bounded.

For a given body shape with applied forces, the VP for determining the motion of the rigid body in time, for $t_0 \leq t \leq t_1$, is reduced to determining $r(t)$ and $\alpha(t)$, for given endpoint conditions, $(r(t_0), \alpha(t_0)) = (r_0, \alpha_0)$ and $(r(t_1), \alpha(t_1)) = (r_1, \alpha_1)$. Thus, the spaces of translational V_r and rotational motion V_α of the body-fixed frame is considered,

$$V_r := \{r(t) \mid r(t) \in C_2([t_0, t_1], \mathbb{R}^2), \ \ r(t_0) = r_0, \ r(t_1) = r_1\}, \tag{2.74a}$$

$$V_\alpha := \{\alpha(t) \mid \alpha(t) \in C_2([t_0, t_1], \mathbb{R}), \ \ \alpha(t_0) = \alpha_0, \ \alpha(t_1) = \alpha_1\}. \tag{2.74b}$$

Assume that the extremal components r_* and α_* are embedded in V_r and V_α, respectively,

$$V_{r_*} := \{r_*(t) \mid V_r \ni r_*(t)\}, \tag{2.75a}$$

$$V_{\alpha_*} := \{\alpha_*(t) \mid V_\alpha \ni \alpha_*(t)\}. \tag{2.75b}$$

Recall, $\ni$ refers to a single element embedded in a space, whereas $\in$ refers to any element embedded in a space. By linearizing the problem around r_* and α_*, the affine spaces $V_{\tilde{r}}$ and $V_{\tilde{\alpha}}$ are retrieved. The spaces are expressed as

$$V_{\tilde{r}} := V_{r_*} \oplus V_{\theta z}, \tag{2.76a}$$

$$V_{\tilde{\alpha}} := V_{\alpha_*} \oplus V_{\theta \zeta}, \tag{2.76b}$$

where $V_{\theta z}$ and $V_{\theta\zeta}$ are linear spaces of admissible variations, in $\theta \in V_\theta$ for each $z \in V_z$ and $\zeta \in V_\zeta$, respectively,

$$V_{\theta z} := V_\theta \otimes V_z, \tag{2.77a}$$
$$V_{\theta\zeta} := V_\theta \otimes V_\zeta. \tag{2.77b}$$

z and ζ denote the direction, and the relative magnitude, of the admissible variations in time, for the translational and rotational motion, respectively. Moreover, V_z and V_ζ are defined as

$$V_z := \{z(t) \mid z(t) \in C_1([t_0, t_1], \mathbb{R}^2),\ z(t_0) = 0,\ z(t_1) = 0\}, \tag{2.78a}$$
$$V_\zeta := \{\zeta(t) \mid \zeta(t) \in C_1([t_0, t_1], \mathbb{R}),\ \zeta(t_0) = 0,\ \zeta(t_1) = 0\}. \tag{2.78b}$$

Since r and α are given at the endpoints in time, z and ζ are required vanish there.

Consider the space V_φ based on Eq. 2.73, for admissible translational and rotational motion $V_{\tilde{r}}$, and $V_{\tilde{\alpha}}$,

$$V_\varphi := \{\tilde{r}(t) + A(\tilde{\alpha}(t))x \mid \tilde{r}(t) \in V_{\tilde{r}},\ \tilde{\alpha}(t) \in V_{\tilde{\alpha}},\ x \in L_2(\bar{\Omega}), A \text{ from Eq. 2.73}\}, \tag{2.79}$$

and the corresponding space for the extremal components

$$V_{\varphi_*} := \{r_*(t) + A(\alpha_*(t))x \mid r_*(t) \in V_{r_*},\ \alpha_*(t) \in V_{\alpha_*},\ x \in L_2(\bar{\Omega}),\ A \text{ from Eq. 2.73}\}. \tag{2.80}$$

Consider the difference between $\varphi \in V_\varphi$ and $\varphi_* \in V_{\varphi_*}$,

$$\varphi - \varphi_* = \Big(r_* + \theta z + A(\alpha_* + \theta\zeta)x\Big) - \Big(r_* + A(\alpha_*)x\Big) = \theta z + \Big(A(\alpha_* + \theta\zeta) - A(\alpha_*)\Big)x. \tag{2.81}$$

For each $\zeta(t)$, there is a sufficiently small θ such that the approximation

$$A'(\alpha_*)\theta\zeta \doteq A(\alpha_* + \theta\zeta) - A(\alpha_*), \quad A'(\alpha_*) := \begin{pmatrix} -\sin\alpha_* & -\cos\alpha_* \\ \cos\alpha_* & -\sin\alpha_* \end{pmatrix}, \tag{2.82}$$

is good. Define the space $V_{\theta\eta}$ based on the approximation applied to Eq. 2.81,

$$V_{\theta\eta} := V_\theta \otimes V_\eta \tag{2.83}$$

with V_θ from Eq. 2.15a and,

$$V_\eta := \{z(t) + \zeta(t)A'(\alpha_*(t))x \mid z(t) \in V_z,\ \alpha_*(t) \in V_{\alpha_*},\ \zeta(t) \in V_\zeta,\ x \in L_2(\bar{\Omega}), A' \text{ from Eq. 2.82}\}. \tag{2.84}$$

The affine space $V_{\tilde{\varphi}}$ of admissible motions is defined as

$$V_{\tilde{\varphi}} := \{\varphi_*(x,t) + \theta\eta(x,t) \mid (\varphi_*, \theta\eta)(x,t) \in (V_{\varphi_*}, V_{\theta\eta})\} \tag{2.85}$$

With the space of admissible motions derived, Hamilton's principle of least action for a closed, autonomous system of an unconstrained rigid body is stated as,

Variational problem 5 (Unconstrained rigid body). *For a closed, autonomous system of an unconstrained rigid body with given material points $x \in L_2(\bar{\Omega})$, applied forces with densities $\big(\beta(x,t), \tau(x,t)\big) \in \big(L_2(\Omega) \otimes C([t_0,t_1],\mathbb{R}^2), L_2(\Gamma_N) \otimes C([t_0,t_1],\mathbb{R}^2)\big)$, and tuples (t_0, r_0, α_0) and (t_1, r_1, α_1), find an extremal $\varphi_*(x,t) \in V_\varphi$, under the assumption that $\varphi_*(x,t)$ exists, such that*

$$0 = j'(0), \quad \forall \eta \in V_\eta, \tag{2.86a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L(\tilde{\varphi}(\theta, x, t), \dot{\tilde{\varphi}}(\theta, x, t))\mathrm{d}t, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}, \tag{2.86b}$$

where

$$L(\tilde{\varphi}, \dot{\tilde{\varphi}}) = T(\dot{\tilde{\varphi}}) - V(\tilde{\varphi}), \quad \tilde{\varphi} \in V_{\tilde{\varphi}}. \tag{2.86c}$$

Recall, the Lagrangian $L = T - V$ was, in Section 2.1.1, shown to correspond to the conservation of total energy, for closed, autonomous unconstrained mechanical systems.

In Section 2.1.2, the EOM in weak form in time was derived from the unconstrained VP 1. The same derivation steps are applicable to VP 5. The EOM in weak form is restated here (same as Eq. 2.24)

$$0 = \int_{t_0}^{t_1} \eta^T \big(\frac{\partial L}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_*}\big)\mathrm{d}t, \quad \forall \eta \in V_\eta. \tag{2.87}$$

The kinetic energy of a rigid body is expressed as

$$T(\tilde{\varphi}) = \frac{1}{2}\int_\Omega \rho \dot{\tilde{\varphi}}^T \dot{\tilde{\varphi}} \mathrm{d}x, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}, \tag{2.88}$$

where ρ denotes the density. The potential energy, due to applied work, is expressed as

$$V(\tilde{\varphi}) = -\int_\Omega \tilde{\varphi}^T A\beta \mathrm{d}x - \int_{\Gamma_\mathrm{N}} \tilde{\varphi}^T A\tau \mathrm{d}s, \quad \tilde{\varphi} \in V_{\tilde{\varphi}}, \tag{2.89}$$

Consider the term $\frac{\partial L}{\partial \varphi_*}$, in Eq. 2.87, for the specific Lagrangian $L = T - V$. First, $\frac{\partial L}{\partial \varphi_*} = -\frac{\partial V}{\partial \varphi_*} = -\frac{\partial V}{\partial \varphi_*^T}$. Insertion of Eq. 2.89 generates

$$-\frac{\partial V}{\partial \varphi_*^T} = \int_\Omega A\beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} A\tau \mathrm{d}s. \tag{2.90}$$

Thus,

$$\int_{t_0}^{t_1} \eta^T \frac{\partial L}{\partial \varphi_*}\mathrm{d}t = \int_{t_0}^{t_1} \eta^T A\big(\int_\Omega \beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} \tau \mathrm{d}s\big)\mathrm{d}t \tag{2.91}$$

Insertion of a decoupled $\eta = z + \zeta A'(\alpha_*)x$, according to Eq. 2.84, gives

$$\int_{t_0}^{t_1} \eta^T \frac{\partial L}{\partial \varphi_*}\mathrm{d}t = \int_{t_0}^{t_1} \Big(z^T\big(\int_\Omega A\beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} A\tau \mathrm{d}s\big) + \zeta^T\big(\int_\Omega x^T A'^T A\beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} x^T A'^T A\tau \mathrm{d}s\big)\Big)\mathrm{d}t. \tag{2.92}$$

Since the given x, β, and τ belong to L_2, and Ω and Γ_N are assumed to be bounded, the integrals in space are bounded. Moreover, note that

$$A'^T A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.93}$$

is the matrix format for the signed magnitude of a cross product operation in 2D, ie. for any two vectors a and b, $a^T A'^T A b = n_\alpha^T a \times b$, where n_α is a unit vector in direction of the rotational axis. Eq. 2.92 is shortened to

$$\int_{t_0}^{t_1} \eta^T \frac{\partial L}{\partial \varphi_*}\mathrm{d}t = \int_{t_0}^{t_1} \big(z^T f + \zeta^T \tau_{\mathrm{torque}}\big)\mathrm{d}t, \tag{2.94}$$

where

$$f = \int_\Omega A\beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} A\tau \mathrm{d}s, \quad \tau_{\mathrm{torque}} = \int_\Omega n_\alpha^T x \times \beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} n_\alpha^T x \times \tau \mathrm{d}s \tag{2.95}$$

denote the applied force in the inertial reference frame and the applied torque about the body-fixed frame, respectively.

Thereafter, consider $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\varphi_*} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\varphi_*} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\varphi_*^T}$. Insertion of Eq. 2.88 into $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\varphi_*^T}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot\varphi_*^T} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\dot\varphi_* \mathrm{d}x = \int_\Omega \rho\ddot\varphi_* \mathrm{d}x. \tag{2.96}$$

Thus,

$$\int_{t_0}^{t_1} \eta^T \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\varphi_*}\mathrm{d}t = \int_{t_0}^{t_1}\int_\Omega \eta^T\rho\ddot\varphi_* \mathrm{d}x\mathrm{d}t. \tag{2.97}$$

To continue, the extremal component $\varphi_* \in V_{\varphi_*}$, Eq. 2.80, is decoupled into its translational and rotational components, and thereafter time-differentiated twice

$$\varphi_* = r_*(t) + A(\alpha_*(t))x, \tag{2.98a}$$
$$\dot\varphi_* = \dot r_*(t) + A'(\alpha_*(t))\dot\alpha_*(t)x, \tag{2.98b}$$
$$\ddot\varphi_* = \ddot r_*(t) + A''(\alpha_*(t))\dot\alpha_*^2(t)x + A'(\alpha_*(t))\ddot\alpha_*(t)x. \tag{2.98c}$$

Note, $A'' = -A$, see Eq. 2.73. Insertion of Eq. 2.98c into Eq. 2.97 gives

$$\int_{t_0}^{t_1} \eta^T \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\varphi_*}\mathrm{d}t = \int_{t_0}^{t_1}\int_\Omega \eta^T\rho\big(\ddot r_* - A(\alpha_*)\dot\alpha_*^2 x + A'(\alpha_*)\ddot\alpha_*(t)x\big)\mathrm{d}x\mathrm{d}t. \tag{2.99}$$

Insertion of a decoupled $\eta = z + \zeta A'(\alpha_*)x$, according to Eq. 2.84, gives

$$\begin{aligned}\int_{t_0}^{t_1} \eta^T \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\varphi_*}\mathrm{d}t &= \int_{t_0}^{t_1}\int_\Omega \big(z + \zeta A'x\big)^T\rho\big(\ddot r_* - A\dot\alpha_*^2 x + A'\ddot\alpha_*(t)x\big)\mathrm{d}x\mathrm{d}t \\ &= \int_{t_0}^{t_1}\int_\Omega \rho\big(z^T\ddot r_* - z^T A\dot\alpha_*^2 x + z^T A'\ddot\alpha_* x\big)\mathrm{d}x\mathrm{d}t \\ &+ \int_{t_0}^{t_1}\int_\Omega \rho\big(\zeta^T x^T A'^T\ddot r_* - \zeta^T x^T A'^T A\dot\alpha_*^2 x + \zeta^T\ddot\alpha_* x^T x\big)\mathrm{d}x\mathrm{d}t\end{aligned} \tag{2.100}$$

Recall that the body-fixed frame was placed at the center of mass, Fig. 2.1. As a consequence, $\int_\Omega \rho x\mathrm{d}x = 0$, and Eq. 2.100 is shortened to

$$\begin{aligned}\int_{t_0}^{t_1} \eta^T \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\varphi_*}\mathrm{d}t &= \int_{t_0}^{t_1} z^T m\ddot r_* \mathrm{d}t + \int_{t_0}^{t_1}\Big(\int_\Omega -\zeta^T\rho\dot\alpha_*^2 n_\alpha^T x \times x\mathrm{d}x + \zeta^T J\ddot\alpha_*\Big)\mathrm{d}t \\ &= \int_{t_0}^{t_1}(z^T m\ddot r_* + \zeta^T J\ddot\alpha_*)\mathrm{d}t,\end{aligned} \tag{2.101}$$

where

$$m := \int_\Omega \rho\mathrm{d}x, \quad J := \int_\Omega \rho x^T x\mathrm{d}x, \tag{2.102}$$

denote the mass and moment of inertia about the body-fixed frame, respectively. Note, both m and J are bounded, under the assumption that ρ is bounded. Insertion of, Eq. 2.94 and Eq. 2.101, into the EOM in weak form, Eq. 2.87, gives the decoupled EOM

$$0 = \int_{t_0}^{t_1}(z^T(f - m\ddot r_*) + \zeta^T(\tau_{\mathrm{torque}} - J\ddot\alpha_*))\mathrm{d}t, \quad \forall(z,\zeta)\in(V_z, V_\zeta). \tag{2.103}$$

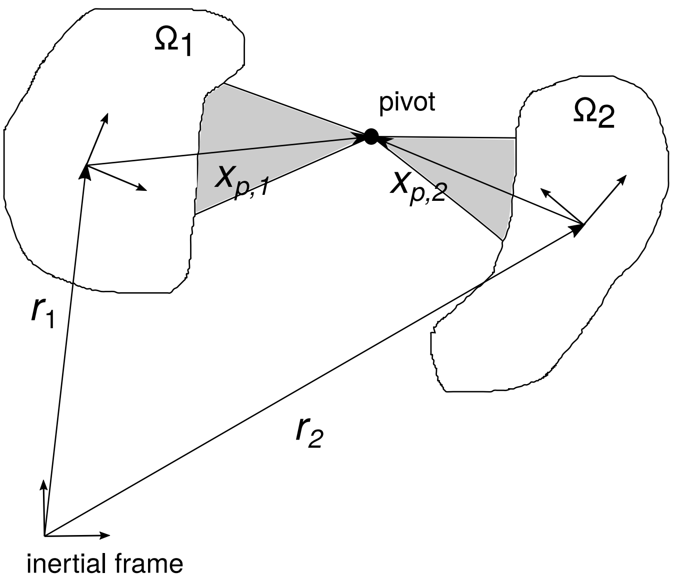

Figure 2.2: Locating the joint of a pivot from two interconnected rigid bodies.

In VP 5, β and τ were given as continuous functions in time. Due to Eq. 2.95, f and τ_{torque}, and furthermore $(f - m\ddot{r}_*)$ and $(\tau_{\text{torque}} - J\ddot{\alpha}_*))$, are continuous in time. Thus, the prerequisites for fundamental lemma of calculus of variations, Lemma 1, are met for each of the two integral terms.

By first setting $z = 0$, and applying the fundamental lemma of calculus of variations for the all admissible variations ζ, and thereafter setting $\zeta = 0$, and applying the fundamental lemma of calculus of variations for the all admissible variations z, Newton-Euler's EOM in strong form is retrieved,

$$m\ddot{r}_* = f, \tag{2.104a}$$

$$J\ddot{\alpha}_* = \tau_{\text{torque}}. \tag{2.104b}$$

Recall, in Section 2.1.1, existence and uniqueness of the dynamics for an IVP based on Newton's second law, under a set of assumptions, was guaranteed by use of Picard-Lindelöf's theorem. The same procedure can reused here to retrieve existence and uniqueness of the dynamics for an IVP based on Newton-Euler's EOM.

2.2.2 Modeling of Joints Between Rigid Bodies

in Section 2.1.3, constrained dynamics was presented in order to enable incorporation of interconnecting joints into models of multibody systems. In this study, massless, rigid, revolute joints are considered. The massless assumption is reasonable since joints are often small, compared to the bodies, in multibody systems. Moreover, the massless assumption leads to less stiff systems of differential equations for the dynamics of multibody systems. In this study, only joints interconnecting pairs of bodies are considered.

For two rigid bodies (referred to by indices 1 and 2) interconnected by a massless, rigid, revolute joint, the constraint is defined by expressing the position of the pivot of the joint

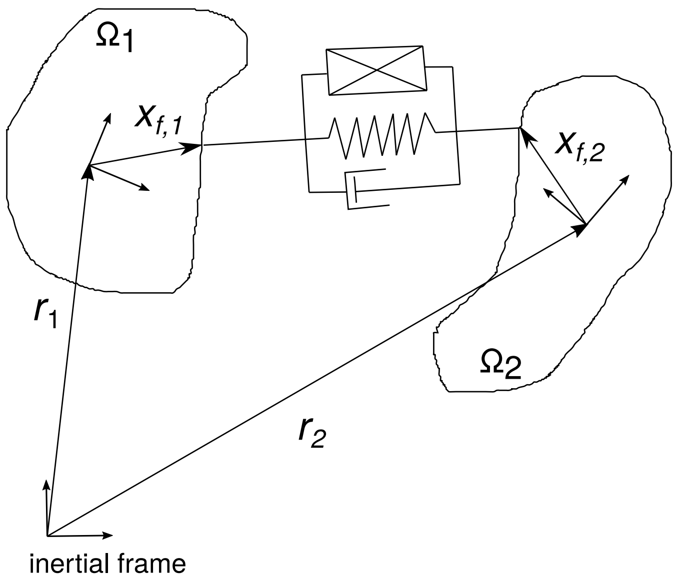

Figure 2.3: A spring, a damper, and an actuator serving as interconnecting force elements between two rigid bodies.

through the two body-fixed frames, $x_{p,1}$ and $x_{p,2}$,

$$g(\varphi_1, \varphi_2) = \varphi_1(x_{p,1}, t) - \varphi_2(x_{p,2}, t) = 0, \tag{2.105}$$

or equivalently,

$$r_1(t) + A(\alpha_1)x_{p,1} - (r_2(t) + A(\alpha_2)x_{p,2}) = 0. \tag{2.106}$$

The model is illustrated in Fig. 2.2. In presence of joints, the constrained Lagrangian, L_{C}, is considered. L_{C} for two interconnected rigid bodies is

$$L_{\mathrm{C}}(\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2) = T_1(\dot{\varphi}_1) + T_2(\dot{\varphi}_2) - \big(V_1(\varphi_1) + V_2(\varphi_2)\big) - g(\varphi_1, \varphi_2)\lambda_{(1,2)}. \tag{2.107}$$

with T and V from Eqs. 2.88 and 2.89.

2.2.3 Force Elements

Interconnecting force elements are typically present in multibody systems. Force elements are characterized by that they affect the applied forces on interconnected bodies. Springs, dampers, and actuators are standard force elements. Two rigid bodies, interconnected by a spring, a damper, and an actuator, are illustrated in Fig. 2.3.

The magnitude of the force, of the three force elements combined, is

$$f_{\text{force elements}}(\xi, \dot{\xi}, t) = k(\xi - \xi_0) + d\dot{\xi} + h(\xi, \dot{\xi}, t), \tag{2.108}$$

where k, ξ_0, d, h denote the spring constant, the spring nominal length, the damping coefficient, and the actuator law, respectively, and where $\xi = c^T c$ with

$$c(\varphi_1, \varphi_2) = \varphi_1(x_{f,1}, t) - \varphi_2(x_{f,2}, t). \tag{2.109}$$

where $x_{f,1}$ and $x_{f,2}$ denote the attachment points for the force element with the two bodies. Dampers and actuators are non-conservative force elements. Moreover, in presence of actuators, switching at prescribed time points, the multibody system is a non-autonomous system. Recall from Section 2.1.1, the Lagrangian $L = T - V$ were retrieved under the assumptions that the applied forces were conservative and that the system was autonomous. For a presentation of non-autonomous multibody systems, in presence of non-conservative forces, see [6].

Springs are incorporated into the model by extending the potential energy of the system. For two interconnected rigid bodies, the potential energy becomes

$$V(\varphi, \xi(\varphi_1, \varphi_2), \dot{\xi}(\dot{\varphi}_1, \dot{\varphi}_2), t) = V_1(\varphi_1) + V_2(\varphi_2) + \frac{k}{2}\big(\xi(\varphi_1, \varphi_2) - \xi_0\big)^2, \tag{2.110}$$

where V_1 and V_2 follow from Eq. 2.89. Just as the applied work in Eq. 2.89 contributed to the force and torque term of Newton-Euler's equation, Eq. 2.104, the extended potential energy would generate extended force and torque terms for Newton-Euler's equation.

Force elements are not of interest in this study and will not be pursued any further.

2.2.4 Constrained Rigid Multibody System

A multibody system is composed of a collection of bodies and interconnecting elements. In this study, only massless, rigid, revolute joints, modeled by Eq. 2.105, are incorporated as interconnecting elements. By reusing derivations from Sections 2.1.3 and 2.2.1, the EOM in strong form, is derived in a few steps.

Consider a system of n_b rigid bodies. The kinetic and potential energies for a multibody system are aggregated versions of the counterparts for a single body,

$$T(\{\dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}) = \sum_{i=1}^{n_b} \frac{1}{2} \int_{\Omega_i} \rho_i \dot{\tilde{\varphi}}_i^T \dot{\tilde{\varphi}}_i \mathrm{d}x, \qquad \tilde{\varphi}_i \in V_{\tilde{\varphi}} \tag{2.111}$$

$$V(\{\tilde{\varphi}_i\}_{i=1}^{n_b}) = \sum_{i=1}^{n_b} (-\int_{\Omega_i} \tilde{\varphi}_i^T A_i \beta_i \mathrm{d}x - \int_{\Gamma_{\mathrm{N}}^i} \tilde{\varphi}_i^T A_i \tau_i \mathrm{d}s), \qquad \tilde{\varphi}_i \in V_{\tilde{\varphi}}. \tag{2.112}$$

The Lagrangian for an unconstrained multibody system follows,

$$L(\{\tilde{\varphi}_i, \dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}) = T(\{\dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}) - V(\{\tilde{\varphi}_i\}_{i=1}^{n_b}), \quad \tilde{\varphi}_i \in V_{\tilde{\varphi}}. \tag{2.113}$$

The collection of all joints is denoted by the set of index pairs $\mathcal{J}$, where the index pairs refer to the indices of the interconnected bodies. Thus, the Lagrangian for a constrained multibody system is

$$L_{\mathrm{C}}(\{\tilde{\varphi}_i, \dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}, \{\tilde{\lambda}_{(j,k)}\}_{(j,k)\in\mathcal{J}}) = L(\{\tilde{\varphi}_i, \dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}) - \sum_{(j,k)\in\mathcal{J}} g(\tilde{\varphi}_j, \tilde{\varphi}_k)^T \tilde{\lambda}_{(j,k)}, \tag{2.114}$$

where $\lambda_{(j,k)}$ denotes the Lagrange multipliers for joint connection (j,k). Hamilton's principle of least action for a rigid multibody system is stated as

Variational problem 6 (Rigid multibody system)**.** *For a closed, autonomous constrained rigid multibody system with given material points* $\{x_i\}_{i=1}^{n_b} \in L_2(\bar{\Omega}_i)$*, applied force densities* $\big(\beta_i(x_i,t), \tau_i(x_i,t)\big) \in \big(L_2(\Omega_i) \otimes C([t_0,t_1],\mathbb{R}^2), L_2(\Gamma_{\mathrm{N}}^i) \otimes C([t_0,t_1],\mathbb{R}^2)\big)$*, and consistent endpoint conditions* $(\{r_0^i, \alpha_0^i\}_{i=1}^{n_b}, \{\lambda_0^{(j,k)}\}_{(j,k)\in\mathcal{J}})$, $(\{r_1^i, \alpha_1^i\}_{i=1}^{n_b}, \{\lambda_1^{(j,k)}\}_{(j,k)\in\mathcal{J}})$*, find an extremal* $(\{\varphi_*^i(x,t)\}_{i=1}^{n_b}, \{\lambda_*^i(t)\}_{(j,k)\in\mathcal{J}})$

where $(\varphi_*^i(x,t), \lambda_*^i(t)) \in (V_{\varphi_*^i}, V_{\lambda_*^i})$, *for* $t_0 \leq t \leq t_1$, *under the assumption that an extremal exists, such that*

$$0 = j'(0), \quad \forall\big(\{\eta_i\}_{i=1}^{n_b}, \{\vartheta_{(j,k)}\}_{(j,k)\in\mathcal{J}}\big), \tag{2.115a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L_{\mathrm{C}}(\{\tilde{\varphi}_i(\theta, x_i, t), \dot{\tilde{\varphi}}_i(\theta, x_i, t)\}_{i=1}^{n_b}, \{\tilde{\lambda}_{(j,k)}(\theta, t)\}_{(j,k)\in\mathcal{J}})\mathrm{d}t, \tag{2.115b}$$

where

$$L_{\mathrm{C}}(\{\tilde{\varphi}_i, \dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}, \{\tilde{\lambda}_{(j,k)}\}_{(j,k)\in\mathcal{J}}) = T(\{\dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}) - V(\{\tilde{\varphi}_i\}_{i=1}^{n_b}) - \sum_{(j,k)\in\mathcal{J}} g(\tilde{\varphi}_j, \tilde{\varphi}_k)^T \tilde{\lambda}_{(j,k)}, \tag{2.115c}$$

and where

$$g(\tilde{\varphi}_j, \tilde{\varphi}_k) = \tilde{\varphi}_j(x_{p,j}, t) - \tilde{\varphi}_k(x_{p,k}, t) = 0, \tag{2.115d}$$

with $\eta_i \in V_{\eta_i}$, $\vartheta_{(j,k)} \in V_{\vartheta_{(j,k)}}$, $\tilde{\varphi}_i \in V_{\tilde{\varphi}_i}$, *and* $\tilde{\lambda}_{(j,k)} \in V_{\tilde{\lambda}_{(j,k)}}$.

The derivations of the EOCM in weak form from VP 6, is an aggregated versions of the derivations pursued from VP 4 to Eq. 2.53. The EOCM in weak form becomes

$$0 = \sum_{i=1}^{n_b} \int_{t_0}^{t_1} \eta_i^T \Big(\frac{\partial(T_i - V_i)}{\partial \varphi_*^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(T_i - V_i)}{\partial \dot{\varphi}_*^i}\Big)\mathrm{d}t \tag{2.116a}$$

$$- \sum_{(j,k)\in\mathcal{J}} \int_{t_0}^{t_1} \eta_{\{j,k\}}^T G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)} \mathrm{d}t \tag{2.116b}$$

$$+ \sum_{(j,k)\in\mathcal{J}} \int_{t_0}^{t_1} \vartheta_{(j,k)}^T g(\varphi_*^j, \varphi_*^k)\mathrm{d}t, \quad \forall(\{\eta_i\}_{i=1}^{n_b}, \{\vartheta_{(j,k)}\}_{(j,k)\in\mathcal{J}}) \in (V_\eta, V_\vartheta), \tag{2.116c}$$

where $G(\varphi_*^{\{j,k\}}) = \frac{\partial g(\tilde{\varphi}^j, \tilde{\varphi}^k)}{\partial \tilde{\varphi}_{\{j,k\}}}|_{\theta=0}$. The subscript $_{\{j,k\}}$, for $\eta_{\{j,k\}}$, denotes that both η_j and η_k are considered. After employing derivation steps from Eq. 2.87 to Eq. 2.103 on Eq. 2.116a, inserting $\eta = z + \zeta A'(\alpha_*)x$, Eq. 2.84, into Eq. 2.116b, and noting that

$$G(\varphi_*^{\{j,k\}})^T = \left\{ \begin{array}{rl} 1, & j \in \{j,k\}, \\ -1, & k \in \{j,k\}, \end{array} \right. , \tag{2.117}$$

the decoupled EOCM in weak form is retrieved

$$0 = \sum_{i=1}^{n_b} \int_{t_0}^{t_1} z_i^T(f_i - m_i \ddot{r}_*^i) + \zeta_i^T(\tau_{\mathrm{torque}}^i - J_i \ddot{\alpha}_*^i)\mathrm{d}t \tag{2.118a}$$

$$- \sum_{(j,k)\in\mathcal{J}} \int_{t_0}^{t_1} \big(z_{\{j,k\}}^T + \zeta_{\{j,k\}}^T x_{p,\{j,k\}}^T A'(\alpha_*^{\{j,k\}})^T\big) G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)} \mathrm{d}t \tag{2.118b}$$

$$+ \sum_{(j,k)\in\mathcal{J}} \int_{t_0}^{t_1} \vartheta_{(j,k)}^T g(\varphi_*^j, \varphi_*^k)\mathrm{d}t, \quad \forall(\{z_i, \zeta_i\}_{i=1}^{n_b}, \{\vartheta_{(j,k)}\}_{(j,k)\in\mathcal{J}}) \in (V_z, V_\zeta, V_\vartheta). \tag{2.118c}$$

By successively setting all admissible variations but one to zero, and employing the fundamental of calculus of variations on the remaining integral, the EOCM in strong form is retrieved,

$$\sum_{i=1}^{n_b} m_i \ddot{r}_*^i + \sum_{(j,k)\in\mathcal{J}} G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)} = \sum_{i=1}^{n_b} f_i, \tag{2.119a}$$

$$\sum_{i=1}^{n_b} J_i \ddot{\alpha}_*^i + \sum_{(j,k)\in\mathcal{J}} x_{p,\{j,k\}}^T A'(\alpha_*^{\{j,k\}})^T G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)} = \sum_{i=1}^{n_b} \tau_{\text{torque}}^i, \tag{2.119b}$$

$$\sum_{(j,k)\in\mathcal{J}} g(\varphi_*^j, \varphi_*^k) = 0. \tag{2.119c}$$

According to Remark 1, for $\lambda_*^{(j,k)} = 0$ the EOUM is retrieved, which, unsurprisingly, is an aggregated version of Newton-Euler's equation, Eq. 2.104.

For $\lambda_*^{(j,k)} \neq 0$, the presence of constraints inhibits a complete decoupling of the EOCM into translational and rotational components. Moreover, due to the presence of $A'(\alpha_*^{\{j,k\}})$, and $A(\alpha_*^{\{j,k\}})$ embedded in Eq. 2.119c (see Eq. 2.106), the EOM is a nonlinear system in time.

Moreover, due to Eq. 2.93,

$$\begin{aligned}
&\sum_{(j,k)\in\mathcal{J}} x_{p,\{j,k\}}^T A'(\alpha_*^{\{j,k\}})^T G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)} = \\
&\sum_{(j,k)\in\mathcal{J}} x_{p,\{j,k\}}^T A'(\alpha_*^{\{j,k\}})^T A(\alpha_*^{\{j,k\}}) A(\alpha_*^{\{j,k\}})^T G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)} = \\
&\sum_{(j,k)\in\mathcal{J}} n_{\alpha,\{j,k\}}^T x_{p,\{j,k\}} \times \left(A(\alpha_*^{\{j,k\}})^T G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)} \right).
\end{aligned} \tag{2.120}$$

Note, $A(\alpha_*^{\{j,k\}})^T G(\varphi_*^{\{j,k\}})^T \lambda_*^{(j,k)}$ are the constraint force on the interconnected bodies mapped to the body-fixed reference frames of the interconnected bodies.

2.3 Unconstrained Elastic Body Dynamics

As opposed to rigid bodies, elastic bodies are deformable. The deformation of an elastic body is a continuous process described by a displacement field. For elastic bodies, Hamilton's principle of least action is a weak formulation in both time and space. From the VP the EOM which is weak in space but strong in time (the weak-strong form) is derived. The EOM in weak-strong form is the starting point for further FE discretization.

In this section, an single elastic body attached by a Dirichlet condition is considered. The derivations, rendering in the EOM in weak-strong form, are performed in the second subsection. In the first subsection, the function spaces in space which the displacement field belongs to, the Sobolev spaces, are briefly introduced. In the third subsection, an IVP is set up from the EOM in weak-strong form. In the last subsection, the EOM in strong-strong form is derived.

In the next section, the attachment to the inertial reference is extended to incorporate attachments to unknown constraint equations, with the aim to incorporate interconnecting joints. Thereafter, to enable flexible multibody dynamics, the elastic body dynamics is decoupled into rigid body motions and elastic displacements by introduction of body-fixed reference frames, in a similar way to Eq. 2.73.

2.3.1 Introduction to Sobolev Spaces

To retrieve the dynamics of an elastic body, an unknown displacement field has to be determined (see Eq. 2.125). For linear elasticity, there exists a unique weak solution for the displacement field over the interior domain Ω, for each time point, if the displacement field belongs to the function space $H_0^1(\Omega)$. Existence and uniqueness proofs, for Dirichlet and Neumann problems for more general linear elliptic differential equations, are presented in textbooks on PDEs, eg. [4, Ch. 9]. In this subsection, only some relevant results are stated.

$H^s(\Omega)$, $s \in \mathbb{R}$, denotes the Sobolev spaces which are also Hilbert spaces (a complete inner product space). Specifically, $H^0(\Omega) = L_2(\Omega)$. The inner product, between $u, v \in H^1(\Omega)$, also includes $(\nabla u, \nabla v)_{L_2(\Omega)}$,

$$(u,v)_{H^1(\Omega)} = (u,v)_{L_2(\Omega)} + (\nabla u, \nabla v)_{L_2(\Omega)} = \int_\Omega v^T u \mathrm{d}x + \sum_{k=1}^{n} \int_\Omega \partial_k v^T \partial_k u \mathrm{d}x, \tag{2.121}$$

with the corresponding norm,

$$\|u\|_{H^1(\Omega)} = \sqrt{(u,u)_{H^1(\Omega)}}. \tag{2.122}$$

By introduction of a multi-index α (an n-tuple of positive integers where n is the number of space dimensions), the inner product between $u, v \in H^k(\Omega)$, $k \in \mathbb{Z}^+$ (positive integers), is stated as

$$(u,v)_{H^k(\Omega)} = (u,v)_{L_2(\Omega)} + \sum_{1 \le k \le |\alpha|} \int_\Omega D^\alpha v^T D^\alpha u \mathrm{d}x, \tag{2.123}$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$, and $x_i \in [1, n]$ denotes a Cartesian coordinate axis. Also, $H^k(\Omega) \subset H^{k-1}(\Omega)$. For $H^s(\Omega)$, $s \in \mathbb{R}$, the inner product is defined by use of Fourier transforms [4, Def. 7.14] (to employ Fourier transforms $H^s(\Omega)$ is first extended to $H^s(\mathbb{R}^d)$, for $\Omega \subset \mathbb{R}^d$). To retrieve a unique displacement field $u \in H^1(\Omega)$, the elastic body is required to be rigidly attached along Dirichlet boundary Γ_D segments, meaning that the displacement field is required to vanish along Γ_D. The requirement on the displacement field u, is included in the notation by the zero subscript $u \in H_0^1(\Omega)$. $H_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$, the space of smooth functions which vanish along Γ_D, in the $H^1(\Omega)$-norm [4, Def. 7.8].

The strong formulation, in both space and time, (strong-strong formulation) of the elastic body EOM contains a second order derivate of the displacement field in space, Eq. 2.161. Therefore, the function space $H^2(\Omega) \cap H_0^1(\Omega)$, which restricts the second order derivatives, is considered [4, Def. 9.10].

In the presence of Neumann Γ_N, or constrained Γ_C boundary segments (see Section 2.4), the displacement field has to be determined along the corresponding boundaries as well. This is enabled by use of the trace operator.

First, the class of C^k domains is defined as,

Definition 1. *A bounded domain Ω is of class C^k, $k \in \mathbb{Z}^+$, if every point on $\partial\Omega$ has a neighborhood N so that $\partial\Omega \cup N$ is a C^k-surface.*

The trace operator γ can then be retrieved by the theorem [4, Th. 7.40]

Theorem 2. *Let $k \in \mathbb{Z}^+$. Assume that the bounded domain Ω is of class C^k, and that also $\partial\Omega$ is bounded. Then, there exists a bounded linear mapping, called the trace operator γ, such that $\gamma : H^k(\Omega) \to H^{k-1/2}(\partial\Omega)$.*

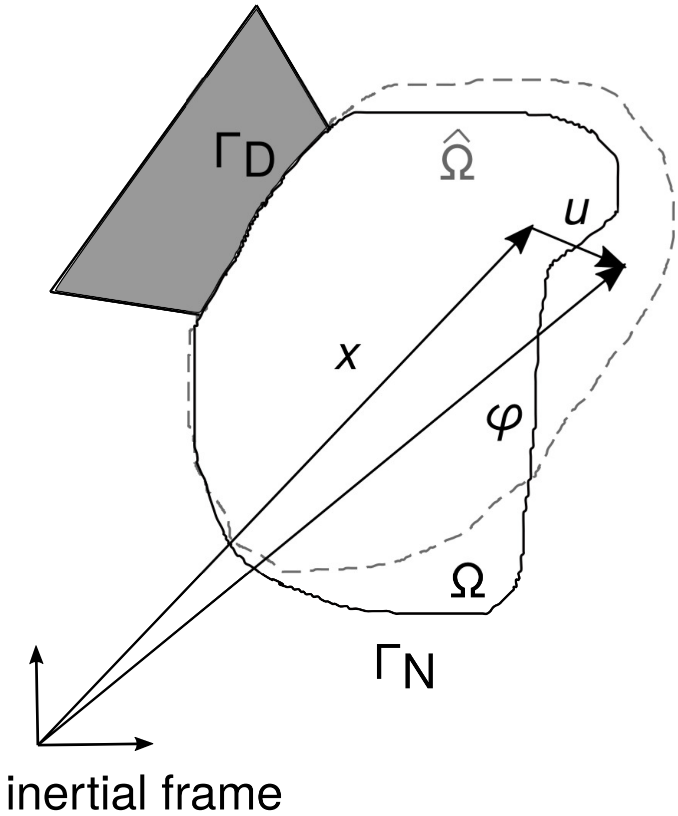

Figure 2.4: Schematic of the deformation of an unconstrained elastic body. The body is not permitted to deform along Γ_{D}.

The boundedness property means that there exists a constant C such that $\|\gamma u\|_{H^{k-1/2}(\partial\Omega)} \leq C\|u\|_{H^k(\Omega)}$, $\forall u \in H^k(\Omega)$. The assumption of Ω being of class C^k can be extended to boundaries which are piecewise of class C^k if the angle between the interconnecting segments at each singularity is nonzero.

Let $H_0^{1/2}(\partial\Omega)$ be defined as the closure of $C_0^\infty(\partial\Omega)$ in the $H^{1/2}(\partial\Omega)$-norm. Then, for $u \in H_0^1(\Omega)$, the boundary displacement field γu is an element of $H_0^{1/2}(\partial\Omega)$. Lastly, $H^s(\Omega)^*$ denotes the dual space of $H^s(\Omega)$.

2.3.2 Derivation of Equations of Unconstrained Motion in Weak-Strong Form

Let a two-dimensional undeformed elastic body over a bounded reference domain $\bar{\Omega} \subset \mathbb{R}^2$, of class C^k, be considered. The body is rigidly attached to an inertial reference frame, along a Dirichlet Γ_{D} boundary segment. $x \in L_2(\bar{\Omega})$ denotes the material points of the undeformed body, with respect to the inertial reference frame. Due to applied force, the body is deformed by a displacement field

$$V_u = \{u(x,t) \mid u(x,t) \in \left(H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)\right) \otimes C_2([t_0,t_1],\mathbb{R}^2),\ u(\cdot,t_0) = u_0,\ u(\cdot,t_1) = u_1\}, \tag{2.124}$$

where u_0 and u_1 denote known initial and final displacement fields. $\varphi(x,t)$ denotes the mapping from the reference to the deformed $\hat{\bar{\Omega}}$ domain,

$$\varphi(u(x,t),x) := x + u(x,t), \quad \varphi : \bar{\Omega} \to \hat{\bar{\Omega}}, \tag{2.125}$$

where $x \in L_2(\bar{\Omega})$ and $u \in V_u$. A schematic of the deformation is illustrated in Fig. 2.4 where Γ_{N} denotes a Neumann boundary segment. Suppose that $x \in L_2(\bar{\Omega})$ is known, then it is sufficient to model the unknown displacement field over time.

The internal displacements affect the internal force of the body. The internal force is expressed by internal stresses and strains. A convenient strain measure for multidimensional use is the Green-Lagrange strain tensor E,

$$E := \frac{1}{2}(\nabla\varphi^T\nabla\varphi - I) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T\nabla u). \tag{2.126}$$

Since only space-differentiated terms are present, the strain tensor is invariant under rigid body motions. Due to the presence of $\nabla u^T\nabla u$, E is nonlinear in ∇u. To retrieve a linear internal force term, a linear strain measure is required. For sufficiently small ∇u, $\nabla u^T\nabla u$ is negligible, and the linear strain tensor ϵ is obtained,

$$\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} := \frac{1}{2}(\nabla u + \nabla u^T). \tag{2.127}$$

The Lagrangian stress tensor σ is retrieved by also defining stresses in the reference domain. The material of the elastic body is assumed to be isotropic and homogeneous, which means that the material properties of the body are rotationally and positionally invariant within the body. As a consequence, $\epsilon_{12} = \epsilon_{21}$, and

$$\epsilon = \frac{1}{2}(\nabla u + \nabla u^T) = \nabla u. \tag{2.128}$$

For an isotropic and homogeneous body, Hooke's law provides a linear relation between σ and ϵ,

$$\sigma := \frac{E}{1-\upsilon^2}\Big((1-\upsilon)\epsilon + \upsilon\,\mathrm{trace}(\epsilon)I\Big) \tag{2.129}$$

where E, υ, and I denote Young's modulus, Poisson's number, and an identity matrix, and $\mathrm{trace}(\epsilon) = \epsilon_{11}+\epsilon_{22}$. The two-dimensional stress model, Eq. 2.129, stems from considering a planar stress assumption for three-dimensional elastic bodies. The assumption is a valid approximation for thin three-dimensional elastic bodies.

Eq. 2.129 can be rewritten in vectorized form as

$$\underline{\sigma} = C\underline{\epsilon}, \tag{2.130}$$

with the vectorized strains and stresses defined as

$$\underline{\epsilon} := (\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12}), \quad \underline{\sigma} := (\sigma_{11}, \sigma_{22}, \sigma_{12}), \tag{2.131}$$

and with the stiffness tensor C defined as

$$C := \frac{\mathrm{E}}{(1-\upsilon)^2}\begin{pmatrix} 1 & \upsilon & \\ \upsilon & 1 & \\ & & (1-\upsilon)/2 \end{pmatrix}. \tag{2.132}$$

In this study, a hyper-elastic material model is considered, which means that an internal energy, as a potential to the internal force, is assumed to exist. The internal energy, or the strain energy, is defined as

$$W(u) := \frac{1}{2}\int_\Omega \sigma(u) : \epsilon(u)\mathrm{d}x, \tag{2.133}$$

where $\sigma : \epsilon = \mathrm{trace}(\sigma\epsilon) = \sigma_{11}\epsilon_{11} + 2\sigma_{12}\epsilon_{12} + \sigma_{22}\epsilon_{22} = \underline{\sigma}^T\underline{\epsilon}$. Metals are commonly modeled as hyper-elastic materials. To retrieve the potential energy of the elastic body, work contributions from the displacements of the body, are included as well,

$$V(u) := \frac{1}{2}\int_\Omega \sigma(u) : \epsilon(u)\mathrm{d}x - \int_\Omega u^T\beta\mathrm{d}x - \int_{\Gamma_\mathrm{N}} u^T\tau\mathrm{d}s. \tag{2.134}$$

where $\beta \in H^1(\Omega)^* \otimes C([t_0, t_1], \mathbb{R}^2)$ and $\tau \in H^{1/2}(\Gamma_\mathrm{N})^* \otimes C([t_0, t_1], \mathbb{R}^2)$ denote the applied body force density and the applied surface force density, the surface traction, respectively. The kinetic energy T is defined as

$$T(\dot{u}) := \frac{1}{2}\int_\Omega \rho\dot{u}^T\dot{u}\mathrm{d}t. \tag{2.135}$$

To eventually state Hamilton's principle of least action for the elastic body, assume there exists an extremal u_*,

$$V_{u_*} := \{u_*(t) \mid V_u \ni u_*(t)\}. \tag{2.136}$$

By linearizing the problem around u_* an affine space $V_{\tilde{u}}$ is retrieved, which is expressed as

$$V_{\tilde{u}} := V_{u_*} \oplus V_{\theta v}, \tag{2.137}$$

where $V_{\theta v}$ is a linear space, in $\theta \in V_\theta$ for each $v \in V_v$, of admissible variations,

$$V_{\theta v} := V_\theta \otimes V_v, \tag{2.138}$$

with

$$V_v := \{v(x,t) \mid v(x,t) \in V_{v_x} \otimes V_{v_t},\ v(\cdot, t_0) = 0,\ v(\cdot, t_1) = 0\}. \tag{2.139}$$

where

$$V_{v_x} := \{v_x(x) \mid v_x(x) \in H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)\}, \tag{2.140a}$$

$$V_{v_t} := \{v_t(t) \mid v_t(t) \in C_1([t_0, t_1],\ \mathbb{R}^2)\}. \tag{2.140b}$$

V_{v_x} denotes the test function space. According to Eq. 2.139, element-wise $v = v_x v_t$, for $v \in V_v$, $v_x \in V_{v_x}$, and $v_t \in V_{v_t}$. With the retrieved space of admissible motions, an unconstrained VP is stated. Note, the resemblance with the unconstrained VP. 1.

Variational problem 7 (Unconstrained elastic body)**.** *For a closed, autonomous system of an unconstrained elastic body with given material points $x \in L_2(\bar{\Omega})$, applied force densities $\big(\beta(x,t), \tau(x,t)\big) \in \big(H^1(\Omega)^* \otimes C([t_0, t_1], \mathbb{R}^2), H^{1/2}(\Gamma_\mathrm{N})^* \otimes C([t_0, t_1], \mathbb{R}^2)\big)$, and tuples (t_0, u_0) and (t_1, u_1), find an extremal $u_*(x,t) \in V_{u_*}$, under the assumption that $u_*(x,t)$ exists, such that*

$$0 = j'(0), \quad \forall v \in V_v, \tag{2.141a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L(\tilde{u}(\theta, x, t), \dot{\tilde{u}}(\theta, x, t))\mathrm{d}t, \quad \tilde{u} \in V_{\tilde{u}}, \tag{2.141b}$$

where

$$L(\tilde{u}, \dot{\tilde{u}}) = T(\dot{\tilde{u}}) - V(\tilde{u}), \quad \tilde{u} \in V_{\tilde{u}}. \tag{2.141c}$$

Insertion of the Lagrangian, Eq. 2.141c, into the action integral, Eq. 2.141b, gives

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \frac{1}{2}\rho(\dot{u}_* + \theta\dot{v})^T(\dot{u}_* + \theta\dot{v})\mathrm{d}x - \int_\Omega \frac{1}{2}\sigma(u_* + \theta v) : \epsilon(u_* + \theta v)\mathrm{d}x \\ + \int_\Omega (u_* + \theta v)^T\beta\mathrm{d}x + \int_{\Gamma_\mathrm{N}} (u_* + \theta v)^T\tau\mathrm{d}s\Big)\mathrm{d}t, \quad \forall\theta v \in V_{\theta v}. \tag{2.142}$$

Applying the stationarity condition, Eq. 2.141a, generates

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \rho\dot{v}^T\dot{u}_*\mathrm{d}x - \int_\Omega \sigma(u_*) : \epsilon(v)\mathrm{d}x + \int_\Omega v^T\beta\mathrm{d}x + \int_{\Gamma_\mathrm{N}} v^T\tau\mathrm{d}s\Big)\mathrm{d}t, \quad \forall v \in V_v. \tag{2.143}$$

Eq. 2.143 is weak in both time and space. Since $v(\cdot, t_0) = v(\cdot, t_1) = 0$, integration by parts of the first term, with respect to time, gives

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \big(\rho v^T\ddot{u}_* + \ \sigma(u_*) : \epsilon(v) - v^T\beta\big)\mathrm{d}x - \int_{\Gamma_\mathrm{N}} v^T\tau\mathrm{d}s\Big)\mathrm{d}t, \quad \forall v \in V_v. \tag{2.144}$$

To retrieve a weak-strong formulation, the admissible variations are separated in space and time, Eq. 2.140,

$$0 = \int_{t_0}^{t_1} v_t^T\Big(\int_\Omega \big(\rho v_x^T\ddot{u}_* + \ \sigma(u_*) : \epsilon(v_x) - v_x^T\beta\big)\mathrm{d}x - \int_{\Gamma_\mathrm{N}} v_x^T\tau\mathrm{d}s\Big)\mathrm{d}t, \quad \forall(v_x, v_t) \in (V_{v_x}, V_{v_t}). \tag{2.145}$$

$\sigma(u) : \epsilon(v) = v_t^T\sigma(u) : \epsilon(v_x)$ since $\epsilon(v) = \nabla v$. For the prescribed function space in time the prerequisites for fundamental lemma of calculus of variations in time, Lemma 1, hold. Applying the lemma generates the weak-strong formulation for the EOUM,

$$\int_\Omega \rho v_x^T\ddot{u}_*\mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x)\mathrm{d}x = \int_\Omega v_x^T\beta\mathrm{d}x + \int_{\Gamma_\mathrm{N}} v_x^T\tau\mathrm{d}s, \quad \forall v_x \in V_{v_x}. \tag{2.146}$$

The conventional FE discretization technique is based on the weak-strong formulation. The conventional FE discretization technique is presented in Section 2.6.

It is important to understand with which measure the integrals should be viewed. Lebesgue measure generates zero measures for contributions from lines and points within the integrals. To enable a FE discretization, which converges to the weak-strong formulation above for increasing spatial resolution, the Lebesgue measure is a natural choice, since the element borders then carries zero measure. For details about Lebesgue measure, see eg. [7].

2.3.3 Compact Notation and Initial Value Problem

In practice, the endpoint condition $u_1 \in H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)$ is presumingly not accessible. Instead a time-differentiated initial interior displacement field $\dot{u}_0 \in H_0^1(\Omega)$ is assumed to be at hand. In this study, an IVP based on the EOM in weak-strong form, Eq. 2.146, is solved numerically. To solve the IVP numerically, discretization schemes, in both space and time, have to be employed. The discretization schemes, in space and time, are introduced in Sections 2.6 and 2.7, respectively.

Before the IVP is presented, compact notations are introduced for the EOM in weak-strong form. The internal force term is written as

$$a(u_*, v_x) := \int_\Omega \sigma(u_*) : \epsilon(v_x)\mathrm{d}x. \tag{2.147}$$

The parentheses denote a bilinear form, for each time point, meaning it is linear in both arguments and maps the arguments to the real numbers. The right-hand side integrals are linear functionals, at each time point, for given pair of applied force densities (β, τ), and are denoted

$$\langle \beta, v_x \rangle := \int_\Omega v_x^T \beta \mathrm{d}x, \tag{2.148a}$$

$$\langle \tau, v_x \rangle := \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s. \tag{2.148b}$$

The inertia term, a bilinear form in $\ddot{u}$ and v, for each time point, is expressed as

$$m(\ddot{u}_*, v) := \int_\Omega \rho v^T \ddot{u}_* \mathrm{d}x. \tag{2.149}$$

With this notation, an IVP, based on the weak-strong formulation, is stated as

Initial value problem 1 (Unconstrained elastic body). *For initial conditions, $u_*(\cdot, t_0) = u_0 \in H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)$ and $\dot{u}_*(\cdot, t_0) = \dot{u}_0 \in H_0^1(\Omega)$, find a displacement field path $u_* \in \big(H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)\big) \otimes C_2([t_0, t_1], \mathbb{R}^2)$ for*

$$m(\ddot{u}_*, v_x) + a(u_*, v_x) = \langle \beta, v_x \rangle + \langle \tau, v_x \rangle, \quad \forall v_x \in V_{v_x}. \tag{2.150}$$

2.3.4 Derivation of Equations of Unconstrained Motion in Strong-Strong Form

In this subsection, the EOUM in strong-strong (both time and space) form is derived. This subsection is included for completion, since the strong-strong form is not used in the numerical experiments in this study.

To retrieve a strong formulation also in space, more restrictive function spaces in space have to be considered, compared to derivations of the weak-strong formulation in the previous section. As stated in Section 2.3.1, the function space for the interior displacement field is updated from $H_0^1(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$. The corresponding boundary displacement field is updated from $H_0^{1/2}(\partial\Omega)$ to $H^{3/2}(\partial\Omega) \cap H_0^{1/2}(\partial\Omega)$,

$$\begin{aligned} V_u = \{ & u(x,t) \mid u(x,t) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \cup \big(H^{3/2}(\partial\Omega) \cap H_0^{1/2}(\partial\Omega)\big) \otimes \\ & C_2([t_0, t_1], \mathbb{R}^2),\ u(\cdot, t_0) = u_0,\ u(\cdot, t_1) = u_1 \}, \end{aligned} \tag{2.151}$$

Moreover, the dual function spaces for β and τ are replaced by L_2-spaces. Thus, $\beta \in L_2(\Omega) \otimes C([t_0, t_1], \mathbb{R}^2)$ and $\tau \in L_2(\Gamma_\mathrm{N}) \otimes C([t_0, t_1], \mathbb{R}^2)$ are considered. During the derivations, the function space $C_0^\infty(\Omega) \cup C_0^\infty(\partial\Omega)$ is employed for the test function space, V_{v_x}. By a density argument, $C_0^\infty(\Omega) \cup C_0^\infty(\partial\Omega)$ is thereafter extended to the closure $H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)$.

The derivation steps to retrieve the EOUM in weak-strong form, Eq. 2.146, is reused under the more restrictive function spaces,

$$\int_\Omega \rho v_x^T \ddot{u}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x) \mathrm{d}x = \int_\Omega v_x^T \beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s, \quad \forall v_x \in V_{v_x}. \tag{2.152}$$

Recall, under the considered assumptions of isotropy and homogeneity, $\epsilon(v_x) = \nabla v_x$, see Eq. 2.128. Also, σ is symmetric. Moreover, in Appendix A.5, the divergence property

$$\nabla \cdot (\sigma v_x) = \mathrm{trace}(\sigma \nabla v_x) + v_x^T \mathrm{div}(\sigma) \tag{2.153}$$

is shown to hold for a symmetric matrix σ and a vector v_x, where $\mathrm{div}(\sigma)$ is defined as the vector whose components are the divergence of the rows of σ,

$$\mathrm{div}(\sigma) := \begin{bmatrix} \nabla \cdot \sigma_{1\cdot}^T \\ \nabla \cdot \sigma_{2\cdot}^T \end{bmatrix}. \tag{2.154}$$

Note, with the restricted function space for the interior displacement field $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $\mathrm{div}(\sigma(u)) \in L_2(\Omega)$. After those observations, and that $\sigma(u) : \epsilon(v_x) = \mathrm{trace}\big(\sigma(u)\epsilon(v_x)\big)$, the inertia term is rewritten as

$$\int_\Omega \sigma(u_*) : \epsilon(v_x)\mathrm{d}x = \int_\Omega \mathrm{trace}(\sigma(u_*)\nabla v_x)\mathrm{d}x = \int_\Omega \nabla \cdot (\sigma v_x)\mathrm{d}x - \int_\Omega v_x^T \mathrm{div}\sigma \mathrm{d}x. \tag{2.155}$$

Applying Gauss's theorem gives

$$\int_\Omega \nabla \cdot (\sigma v_x)\mathrm{d}x - \int_\Omega v_x^T \mathrm{div}\sigma \mathrm{d}x = \int_{\partial\Omega} v_x^T \sigma n \mathrm{d}s - \int_\Omega v_x^T \mathrm{div}\sigma \mathrm{d}x. \tag{2.156}$$

n denotes the outward normal vector on the boundary. Note, $\int_{\partial\Omega} v_x^T \sigma n \mathrm{d}s = \int_{\Gamma_\mathrm{N}} v_x^T \sigma n \mathrm{d}s$ since $v_x \in C_0^\infty(\Omega) \cup C_0^\infty(\partial\Omega)$. Insertion into Eq. 2.152 gives

$$0 = \int_\Omega v_x^T(\rho\ddot{u}_* - \mathrm{div}\sigma - \beta)\mathrm{d}x + \int_{\Gamma_\mathrm{N}} v_x^T(\sigma n - \tau)\mathrm{d}s, \quad \forall v_x \in V_{v_x}. \tag{2.157}$$

To retrieve the strong-strong formulation the fundamental lemma of calculus of variation for space is employed,

Lemma 2 (Fundamental lemma of calculus of variations in space)**.** *Let $\Omega \subset \mathbb{R}^d$ be a open, bounded domain in a d-dimensional space. If $N(x) \in L_1(\Omega)$ satisfies*

$$\int_\Omega h(x)^T N(x)\mathrm{d}x = 0, \quad \forall h \in C_0^\infty(\Omega), \tag{2.158}$$

then, $N = 0$ almost everywhere.

Almost everywhere refers to that the set for which the statement does not hold carries zero measure. The lemma is proved in [8, Lem. 2.21].

Considering all test functions v_x, which vanish along Γ_N, gives

$$0 = \int_\Omega v_x^T(\rho\ddot{u}_* - \mathrm{div}\sigma - \beta)\mathrm{d}x, \quad \forall v_x \in V_{v_x}. \tag{2.159}$$

Under the more restrictive function spaces, $(\rho\ddot{u}_* - \mathrm{div}\sigma - \beta)$ belongs to $L_2(\Omega)$ in space. Since $L_2(\Omega) \subset L_1(\Omega)$, the prerequisites for the lemma are met. Employing the lemma gives the force balance $(\rho\ddot{u}_* - \mathrm{div}\sigma - \beta) = 0$ almost everywhere. By considering all test functions v_x which vanish along Ω, the boundary integral is retrieved,

$$0 = \int_{\Gamma_\mathrm{N}} v_x^T(\sigma n - \tau)\mathrm{d}s, \quad \forall v_x \in V_{v_x}. \tag{2.160}$$

Consider the Lemma 2 for the open, bounded domain Γ_N. $\sigma \in H^{1/2}(\partial\Omega) \cap H_0^{-1/2}(\partial\Omega) \subset L_2(\Gamma_\mathrm{N}) \subset L_1(\Gamma_\mathrm{N})$ along the boundary, follows from that the boundary displacement field belongs to $H^{3/2}(\partial\Omega) \cap H_0^{1/2}(\partial\Omega)$. Since also, τ belongs to $L_2(\Gamma_\mathrm{N}) \subset L_1(\Gamma_\mathrm{N})$ and $v_x \in C_0^\infty(\partial\Omega) =$

$C_0^\infty(\Gamma_{\mathrm{N}})$, the prerequisites for the lemma are met. Employing the lemma yields the Neumann boundary condition. For completion, the Dirichlet boundary condition is added. Summing up, the strong form is retrieved

$$\rho\ddot{u}_*(x,t) = \mathrm{div}\sigma(u_*(x,t)) + \beta(x,t), \qquad \text{in } \Omega, \tag{2.161a}$$

$$u_*(x,t) = 0, \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{2.161b}$$

$$\sigma(u_*(x,t))n(x) = \tau(x,t), \qquad \text{on } \Gamma_{\mathrm{N}}. \tag{2.161c}$$

Note, the equalities in Eqs. 2.161a and 2.161c hold almost everywhere. So far, the EOUM in strong form has been derived for $v_x \in C_0^\infty(\Omega) \cup C_0^\infty(\partial\Omega)$. By a density argument, the test function space can be extended to $v_x \in H_0^1(\Omega) \cup H_0^1(\partial\Omega)$. A proof of the extension for the interior domain is provided in Appendix A.6. A similar technique can be used for the extension along the boundary.

2.4 Constrained Elastic Body Dynamics

In the previous section, the EOM in weak-strong form for an unconstrained elastic body attached to an inertial reference frame was derived. In this section, the model is extended to more general attachments described by geometric constraints along constrained boundaries Γ_{C}. The aim with this extension is to eventually be able to incorporate interconnecting joints into a flexible multibody system, which presented in the next section.

In Section 2.1.3, pointwise geometric constraints were incorporated into the VP by extending the Lagrangian by $-G^T\lambda$. As is shown in the first subsection, the distributed constraints along Γ_{C} is included into the VP by extending the Lagrangian by $-\int_{\Gamma_{\mathrm{C}}} G^T\lambda \mathrm{d}s$. In the second subsection, the EOCM in weak-strong form is derived from the VP. Thereafter, in the third and forth subsections, an IVP based on the EOCM in weak-strong form is set up, and the strong-strong form is derived, respectively.

2.4.1 Variational Problem for a Constrained Elastic Body

In the previous section, the unconstrained dynamics of an elastic body, rigidly attached to an inertial reference frame, along a Dirichlet boundary Γ_{D} segment, was considered. Since the displacement field was known to vanish along Γ_{D}, the displacement field only had to be determined over $\bar{\Omega}\backslash\Gamma_{\mathrm{D}}$.

In this section, more general attachments are considered. The attachments are described as holonomic constraints along a constrained boundary segment Γ_{C},

$$\int_{\Gamma_{\mathrm{C}}} g(u)\mathrm{d}x = 0, \quad u \in V_u. \tag{2.162}$$

with $g \in H^1(\Gamma_{\mathrm{C}}) \otimes C_1([t_0,t_1],\mathbb{R}^{n_\lambda})$. In this subsection, a constrained elastic body, in presence of both Γ_{D} and Γ_{C}, is considered. Then, V_u follows from Eq. 2.124. A schematic for the deformation of the constrained elastic is illustrated in Fig. 2.5.

In Section 2.1.3, a constrained VP 2 was considered for a pointwise holonomic constraint, $g(\tilde{\varphi}) = 0$ for $\tilde{\varphi} \in V_{\tilde{\varphi}}$. Due to Lagrange's multiplier theorem, Theorem 1, the constrained VP 2 could be reformulated as an unconstrained VP with a constrained Lagrangian, VP 4. The same reformulation technique is considered here. First, consider the unconstrained VP 7 subject to Eq. 2.162,

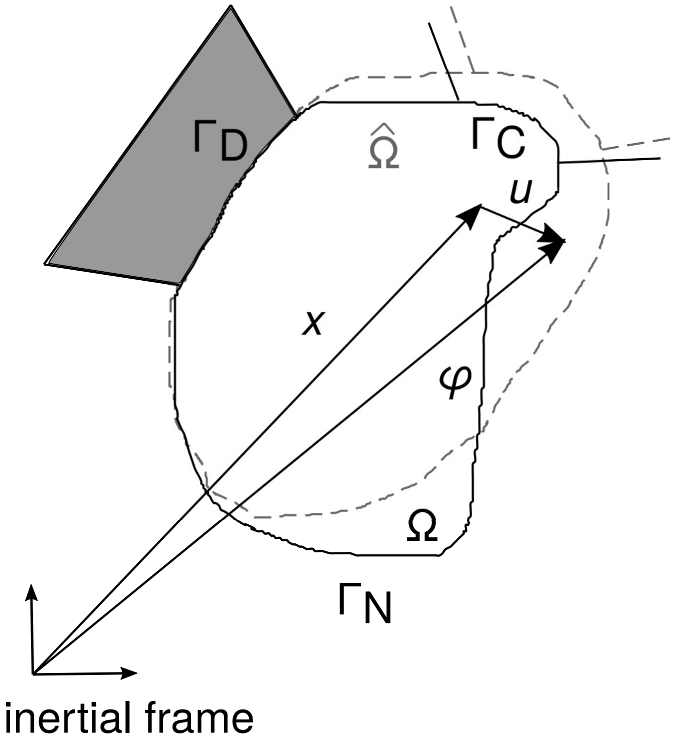

Figure 2.5: Schematic of the deformation of a constrained elastic body. Γ_{C}, unlike Γ_{D}, is permitted to deform.

Variational problem 8. *For a closed, autonomous system of an unconstrained elastic body with given material points* $x \in L_2(\bar{\Omega})$*, applied force densities* $\big(\beta(x,t), \tau(x,t)\big) \in \big(H^1(\Omega)^* \otimes C([t_0,t_1],\mathbb{R}^2), H^{1/2}(\Gamma_{\mathrm{N}})^* \otimes C([t_0,t_1],\mathbb{R}^2)\big)$*, and tuples* (t_0,u_0) *and* (t_1,u_1)*, find an extremal* $u_*(x,t) \in V_{u_*}$*, under the assumption that* $u_*(x,t)$ *exists, such that*

$$0 = j'(0), \quad \forall v \in V_v, \tag{2.163a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L(\tilde{u}(\theta,x,t), \dot{\tilde{u}}(\theta,x,t))\mathrm{d}t, \quad \tilde{u} \in V_{\tilde{u}}, \tag{2.163b}$$

where

$$L(\tilde{u},\dot{\tilde{u}}) = T(\dot{\tilde{u}}) - V(\tilde{u}), \quad \tilde{u} \in V_{\tilde{u}}, \tag{2.163c}$$

subject to

$$\int_{\Gamma_{\mathrm{C}}} g(\tilde{u})\mathrm{d}x = 0, \quad \tilde{u} \in V_{\tilde{u}}, \tag{2.163d}$$

with $g \in H^1(\Gamma_{\mathrm{C}}) \otimes C_1([t_0,t_1],\mathbb{R}^{n_\lambda})$.

$V_{\tilde{u}}$ and V_v are defined in Eqs. 2.137 and 2.139. The constrained VP 8 is reformulated as an unconstrained VP by a Lagrange multiplier technique. First, consider Lagrange's multiplier theorem for the constrained dynamics of an elastic body,

Theorem 3 (Lagrange's multiplier theorem, elastic body). *Under the assumption that $u_* \in V_{u_*}$ is an extremal of the constrained VP 8, there exists a Lagrange multiplier $\lambda_* \in L_2(\Gamma_C) \otimes C([t_0, t_1], \mathbb{R}^{n_\lambda})$ such that u_* is an extremal of the unconstrained VP 7 for Lagrangian P,*

$$P(\tilde{u}, \dot{\tilde{u}}) = L(\tilde{u}, \dot{\tilde{u}}) - \int_{\Gamma_C} g(\tilde{u})^T \lambda_* \mathrm{d}x, \quad \tilde{u} \in V_{\tilde{u}}, \tag{2.164}$$

with $g \in H^1(\Gamma_C) \otimes C_1([t_0, t_1], \mathbb{R}^{n_\lambda})$, ie. $0 = \frac{\partial P}{\partial \varphi_*} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial P}{\partial \dot{\varphi}_*}$.

In this study, a proof is only provided for the restricted function spaces, for which the strong-strong formulation of the EOM holds (see Section 2.3.4), in order to be able to apply Gauss's theorem.

The proof is similar to the one provided in Section 2.1.3. Consider Eq. 2.163d,

$$0 = \int_{\Gamma_C} g(\tilde{u}) \mathrm{d}x = \int_{\Gamma_C} g(u_* + \theta v) \mathrm{d}x \tag{2.165}$$

Differentiating, with respect to θ, and thereafter setting $\theta = 0$, gives

$$\int_{\Gamma_C} G(u_*) v \mathrm{d}x = 0. \tag{2.166}$$

Thus, for VP. 8, v is restricted to the tangent plane of the constraint along Γ_C,

$$V_{\hat{v}} = \{v(t) | v(t) \in V_v, \int_{\Gamma_C} G(u_*) v \mathrm{d}x = 0\}. \tag{2.167}$$

In Section 2.1.1, an extremal was defined as the dynamics for which Euler-Lagrange's equation holds. An equivalent definition is that an extremal is defined as the dynamics for which the fundamental lemma of calculus of variations, in both space and time, holds.

Consider Eq. 2.145,

$$0 = \int_{t_0}^{t_1} v_t^T \Big(\int_\Omega \big(\rho v_x^T \ddot{u}_* + \ \sigma(u_*) : \epsilon(v_x) - v_x^T \beta\big) \mathrm{d}x - \int_{\Gamma_N} v_x^T \tau \mathrm{d}s \Big) \mathrm{d}t, \quad \forall (v_x, v_t) \in (V_{v_x}, V_{v_t}). \tag{2.168}$$

Applying derivation steps from Eq. 2.152 to Eq. 2.145, for the interior space integral, gives the weak-weak formulation of the EOM,

$$\begin{aligned} 0 = &\int_{t_0}^{t_1} \int_\Omega v^T (\rho \ddot{u}_* - \mathrm{div}\sigma - \beta) \mathrm{d}x \mathrm{d}t - \int_{t_0}^{t_1} \int_{\Gamma_N} v^T \tau \mathrm{d}s \mathrm{d}t + \int_{t_0}^{t_1} \int_{\partial\Omega} v^T \sigma n \mathrm{d}s \mathrm{d}t = \\ &\int_{t_0}^{t_1} \int_\Omega v^T (\rho \ddot{u}_* - \mathrm{div}\sigma - \beta) \mathrm{d}x \mathrm{d}t + \int_{t_0}^{t_1} \int_{\Gamma_N} v^T (\sigma n - \tau) \mathrm{d}s \mathrm{d}t + \int_{t_0}^{t_1} \int_{\Gamma_C} v^T \sigma n \mathrm{d}s \mathrm{d}t, \quad \forall v \in V_{\hat{v}}. \end{aligned} \tag{2.169}$$

Note, the additional term $\int_{t_0}^{t_1} \int_{\Gamma_C} v^T \sigma n \mathrm{d}s \mathrm{d}t$, compared to the unconstrained case, due to the presence of holonomic constraints over Γ_C. As in Section 2.3.4, the prerequisites for the fundamental lemma of calculus of variations are met for v:s which vanish over Γ_C, and Ω or Γ_N. Lastly, consider v:s which vanish over Γ_N and Ω,

$$0 = \int_{t_0}^{t_1} \int_{\Gamma_C} v^T \sigma n \mathrm{d}s \mathrm{d}t, \quad \forall v \in V_{\hat{v}}. \tag{2.170}$$

Due to that $v \in V_{\tilde{v}}$ is restricted to the tangent plane of the constraints along Γ_{C}, the fundamental lemma of calculus of variations is not applicable. The problem is circumvented by projecting σn onto the tangent plane of the constraints,

$$0 = \int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T \Pi \sigma n \text{d}s\text{d}t, \quad \forall v \in V_v. \tag{2.171}$$

where Π follows from Eq. 2.39. The functions space V_v, after projection, only span the tangent plane of the constraints of $V_{\tilde{v}}$, and is therefore a lower-dimensional space. Employing the fundamental lemma of calculus of variations to Eq. 2.171 gives

$$0 = \Pi \sigma n = \sigma n - G^T (GG^T)^{-1} G \sigma n \tag{2.172}$$

Set $\lambda_* = (GG^T)^{-1} G \sigma n$. That $\lambda_* \in L_2(\Gamma_{\text{C}}) \otimes C([t_0, t_1], \mathbb{R}^{n_\lambda})$ follows from, that $G \in L_2(\Gamma_{\text{C}}) \otimes C([t_0, t_1], \mathbb{R}^{n_\lambda})$ and $\sigma \in L_2(\Gamma_{\text{C}}) \otimes C([t_0, t_1], \mathbb{R}^{n_\lambda})$. As shown in Appendix A.2, GG^T is invertible under the assumption that G has full row rank. In Appendix A.3, the continuity in time of $(GG^T)^{-1}$ (GG^T is a square matrix) is shown to follow from the continuity in time of GG^T. A similar argument, as in Appendix A.3, $(GG^T)^{-1}$ can be employed to show that $(GG^T)^{-1}$ belongs to $L_2(\Gamma_{\text{C}})$ follows from that GG^T belongs to $L_2(\Gamma_{\text{C}})$.

By replacing $\int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T \sigma n \text{d}s\text{d}t$ with

$$\int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T \Pi \sigma n \text{d}s\text{d}t = \int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T \sigma n \text{d}s\text{d}t - \int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T G^T \lambda_* \text{d}s\text{d}t, \tag{2.173}$$

in Eq. 2.169, the unrestricted space of admissible variations V_v can be employed for the weak-weak formulation of the EOM,

$$\begin{aligned} 0 = &\int_{t_0}^{t_1} \int_{\Omega} v^T (\rho \ddot{u}_* - \text{div}\sigma - \beta) \text{d}x\text{d}t + \int_{t_0}^{t_1} \int_{\Gamma_{\text{N}}} v^T (\sigma n - \tau) \text{d}s\text{d}t \\ &+ \int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T \sigma n \text{d}s\text{d}t - \int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T G^T \lambda_* \text{d}s\text{d}t, \quad \forall v \in V_v. \end{aligned} \tag{2.174}$$

The three first integral terms follows from VP 7. $-\int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} v^T G(u_*)^T \lambda_* \text{d}s\text{d}t$ follows from applying the stationarity condition, Eq. 2.163a, to $-\int_{t_0}^{t_1} \int_{\Gamma_{\text{C}}} g(\tilde{u})^T \lambda_* \text{d}s\text{d}t$. Thus, Eq. 2.174 can be derived from the unconstrained VP 7 for Lagrangian P, Eq. 2.164. Since the fundamental lemma of calculus of variations is applicable for all the integral terms, u in Eq.2.174 is an extremal to VP 7 for Lagrangian P, which concludes the proof.

By Theorem 3, the constrained VP 8 can be reformulated as a unconstrained VP 7 with Lagrangian P, Eq. 2.164. However, the reformulated VP requires that λ_* is prescribed.

As in Section 2.1.3, this obstacle is circumvented by extending the VP to an unconstrained VP for finding an extremal (u_*, λ_*) embedded in a space (V_u, V_λ). V_λ is defined as

$$V_\lambda := \{\lambda(x,t) \mid \lambda(x,t) \in L_2(\Gamma_{\text{C}}) \otimes C([t_0, t_1], \mathbb{R}^{n_\lambda}), \lambda(\cdot, t_0) = \lambda_0, \lambda(\cdot, t_1) = \lambda_1\}. \tag{2.175}$$

For Theorem 3 to hold, the given endpoints (λ_0, λ_1) must equal the endpoints of the specific Lagrange multiplier $(\lambda_*(t_0), \lambda_*(t_1))$. The embedding of λ_* in V_λ is expressed by

$$V_{\lambda_*} = \{\lambda_*(x,t) \mid V_\lambda \ni \lambda_*(x,t)\}. \tag{2.176}$$

The affine space $V_{\tilde{\lambda}}$ is defined by linearizing the space V_λ around $\lambda_* \in V_{\lambda_*}$

$$V_{\tilde{\lambda}} := V_{\lambda_*} \oplus V_{\theta\vartheta}, \tag{2.177}$$

where $V_{\theta\vartheta}$ is a linear space, in $\theta \in V_\theta$, Eq. 2.15a, for each $\vartheta \in V_\vartheta$,

$$V_{\theta\vartheta} = V_\theta \otimes V_\vartheta, \tag{2.178}$$

with

$$V_\vartheta := \{\vartheta(x,t) \mid \vartheta(x,t) \in V_{\vartheta_x} \otimes V_{\vartheta_t},\ \vartheta(\cdot,t_0) = 0,\ \vartheta(\cdot,t_1) = 0\}, \tag{2.179}$$

where

$$V_{\vartheta_x} := \{\vartheta_x(x) \mid \vartheta_x(x) \in H_0^{1/2}(\Gamma_\mathrm{C})^*\}, \tag{2.180a}$$
$$V_{\vartheta_t} := \{\vartheta_t(t) \mid \vartheta_t(t) \in C_1([t_0,t_1],\mathbb{R}^{n_\lambda})\}. \tag{2.180b}$$

With the Lagrange multiplier technique, a constrained Lagrangian $L_\mathrm{C} = L_\mathrm{C}(\tilde{u}, \dot{\tilde{u}}, \tilde{\lambda})$ is considered,

$$L_\mathrm{C}(\tilde{u}, \dot{\tilde{u}}, \tilde{\lambda}) = L(\tilde{u}, \dot{\tilde{u}}) - g(\tilde{u})^T\tilde{\lambda}, \quad (\tilde{u}, \tilde{\lambda}) \in (V_{\tilde{u}}, V_{\tilde{\lambda}}). \tag{2.181}$$

Hamilton's principle of least action, for finding an extremal $(u_*(t), \lambda_*(t)) \in (V_{u_*}, V_{\lambda_*})$, is then stated as

Variational problem 9 (Constrained elastic body)**.** *For a closed, autonomous system of a constrained elastic body with given material points $x \in L_2(\bar{\Omega})$, applied force densities $\big(\beta(x,t), \tau(x,t)\big) \in \big(H^1(\Omega)^* \otimes C([t_0,t_1],\mathbb{R}^2), H^{1/2}(\Gamma_\mathrm{N})^* \otimes C([t_0,t_1],\mathbb{R}^2)\big)$, and consistent endpoint conditions (t_0, u_0, λ_0) and (t_1, u_1, λ_1), find an extremal $(u_*(x,t), \lambda_*(x,t)) \in (V_{u_*}, V_{\lambda_*})$, under the assumption that an extremal exists, such that*

$$0 = j'(0), \quad \forall (v, \vartheta) \in (V_v, V_\vartheta), \tag{2.182a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L_\mathrm{C}(\tilde{u}(\theta,x,t), \dot{\tilde{u}}(\theta,x,t), \tilde{\lambda}(\theta,x,t))\mathrm{d}t, \quad (\tilde{u}, \tilde{\lambda}) \in (V_{\tilde{u}}, V_{\tilde{\lambda}}), \tag{2.182b}$$

where

$$L_\mathrm{C}(\tilde{u}, \dot{\tilde{u}}, \tilde{\lambda}) = T(\dot{\tilde{u}}) - V(\tilde{u}) - \int_{\Gamma_\mathrm{C}} g(\tilde{u})^T\tilde{\lambda}\mathrm{d}s, \quad (\tilde{u}, \tilde{\lambda}) \in (V_{\tilde{u}}, V_{\tilde{\lambda}}), \tag{2.182c}$$

with $g \in H^1(\Gamma_\mathrm{C}) \otimes C_1([t_0,t_1],\mathbb{R}^{n_\lambda})$.

$T(\dot{\tilde{u}})$ and $V(\tilde{u})$ are defined in Eqs. 2.135 and 2.134. In the next subsection, the weak-strong formulation of EOCM is derived from VP 9.

2.4.2 Derivation of Equations of Constrained Motion in Weak-Strong form

Since VP 9 includes the constrained Lagrangian L_C, the derivation steps become an extended version of those presented in Section 2.3.2.

Insertion of the constrained Lagrangian, Eq. 2.182c, into the action integral, Eq. 2.182b, gives

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \frac{1}{2}\rho(\dot{u}_* + \theta\dot{v})^T(\dot{u}_* + \theta\dot{v})\mathrm{d}x - \int_\Omega \frac{1}{2}\sigma(u_* + \theta v) : \epsilon(u_* + \theta v)\mathrm{d}x \tag{2.183}$$
$$+ \int_\Omega (u_* + \theta v)^T\beta\mathrm{d}x + \int_{\Gamma_\mathrm{N}} (u_* + \theta v)^T\tau\mathrm{d}s - \int_{\Gamma_\mathrm{C}} g(u_* + \theta v)^T(\lambda_* + \theta\vartheta)\mathrm{d}s\Big)\mathrm{d}t, \quad \forall(\theta v, \theta\vartheta) \in (V_{\theta v}, V_{\theta\vartheta}),$$

where $V_{\theta v}$ follows from Eq. 2.138. Applying the stationarity condition, Eq. 2.182a, generates

$$\begin{aligned} 0 = \int_{t_0}^{t_1} \Big(\int_\Omega \rho \dot{v}^T \dot{u}_* \mathrm{d}x - \int_\Omega \sigma(u_*) : \epsilon(v) \mathrm{d}x + \int_\Omega v^T \beta \mathrm{d}x \qquad (2.184a) \\ + \int_{\Gamma_\mathrm{N}} v^T \tau \mathrm{d}s - \int_{\Gamma_\mathrm{C}} v^T G(u_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall v \in V_v, \end{aligned}$$

$$0 = \int_{t_0}^{t_1} \int_{\Gamma_\mathrm{C}} \vartheta^T g(u_*) \mathrm{d}s \mathrm{d}t, \quad \forall \vartheta \in V_\vartheta, \qquad (2.184b)$$

where $G(u_*) = \frac{\mathrm{d}g(\tilde{u})}{\mathrm{d}\tilde{u}}|_{\theta=0}$. Eq. 2.184 is weak in both time and space. Since $v(\cdot, t_0) = v(\cdot, t_1) = 0$, performing integration by parts of the inertia term, with respect to time, yields

$$\begin{aligned} 0 = \int_{t_0}^{t_1} \Big(\int_\Omega \rho v^T \ddot{u}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v) \mathrm{d}x - \int_\Omega v^T \beta \mathrm{d}x \qquad (2.185a) \\ - \int_{\Gamma_\mathrm{N}} v^T \tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} v^T G(u_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall v \in V_v, \end{aligned}$$

$$0 = \int_{t_0}^{t_1} \int_{\Gamma_\mathrm{C}} \vartheta^T g(u_*) \mathrm{d}s \mathrm{d}t, \quad \forall \vartheta \in V_\vartheta, \qquad (2.185b)$$

To retrieve a weak-strong formulation, $v \in V_v$ are separated in space and time,

$$\begin{aligned} 0 = \int_{t_0}^{t_1} v_t^T \Big(\int_\Omega \rho v_x^T \ddot{u}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x) \mathrm{d}x - \int_\Omega v_x^T \beta \mathrm{d}x \qquad (2.186a) \\ - \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} v_x^T G(u_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall (v_x, v_t) \in (V_{v_x}, V_{v_t}), \end{aligned}$$

$$0 = \int_{t_0}^{t_1} \vartheta_t^T \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(u_*) \mathrm{d}s \mathrm{d}t, \quad \forall (\vartheta_x, \vartheta_t) \in (V_{\vartheta_x}, V_{\vartheta_t}), \qquad (2.186b)$$

$\sigma(u_*) : \epsilon(v) = v_t^T \sigma(u_*) : \epsilon(v_x)$ since $\epsilon(v) = \nabla v$. Since v_t and ϑ_t belongs to $C_1([t_0, t_1])$ and all other time-dependent functions in Eq. 2.186 belongs to $C([t_0, t_1])$, the prerequisites for fundamental lemma of calculus of variations in time, Lemma 1, are met. Applying the lemma, for both equations, generates the weak-strong formulation for the EOCM,

$$\begin{aligned} 0 = \int_\Omega \rho v_x^T \ddot{u}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x) \mathrm{d}x - \int_\Omega v_x^T \beta \mathrm{d}x \qquad (2.187a) \\ - \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} v_x^T G(u_*)^T \lambda_* \mathrm{d}s, \quad \forall v_x \in V_{v_x}, \end{aligned}$$

$$0 = \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(u_*) \mathrm{d}s, \quad \forall \vartheta_x \in V_{\vartheta_x}. \qquad (2.187b)$$

In comparison to weak-strong formulation for the EOUM, Eq. 2.146, there exists an extra equation which states that displacement field u_* is required to fulfill the constraint equations along Γ_C. Moreover, in the force balance, Eq. 2.187a, there exists an additional term, $\int_{\Gamma_\mathrm{C}} v_x^T G(u_*)^T \lambda_* \mathrm{d}s$, which is interpreted as the force pulling the unconstrained dynamics to the path constrained by Eq. 2.187b.

2.4.3 Compact Notation and Initial Value Problem for $g = u$

In this subsection, an IVP based on Eq. 2.187 is presented in a compact format. IVPs are of interest, since in general, consistent initial conditions, rather than consistent end point conditions, are accessible. In the first numerical experiment in this study, implementations of unconstrained and constrained elastic dynamics are compared, for a constraint which describes a homogeneous Dirichlet condition,

$$g(u_*) = u_* = 0, \quad G(u_*) = 1. \tag{2.188}$$

Compact notations, for the integral terms present in EOUM in weak-strong form, were introduced in Section 2.3.3. For the specific choice of constraints, Eq 2.188, the corresponding integrals in Eq. 2.187 are bilinear functionals,

$$b(u_*, \vartheta_x) := \int_{\Gamma_\mathrm{C}} \vartheta_x^T u_* \mathrm{d}s, \tag{2.189}$$

and

$$b^T(\lambda_*, v_x) := \int_{\Gamma_\mathrm{C}} v_x^T \lambda_* \mathrm{d}s. \tag{2.190}$$

The transpose notation indicates the relation between the two integrals where the integrands consist of the extremal of one of the variables times a test function of the other variable. Furthermore, as shown in Section 2.6.2, the two matrices arising from FE discretization, of Eqs. 2.190 and 2.190, are each others transpose matrix.

The IVP based on Eqs. 2.187 and 2.188 is stated as

Initial value problem 2 (Constrained elastic body)**.** *For consistent initial conditions,* $\big(u_*(x,t_0), \dot{u}_*(x,t_0), \lambda_*(x,t_0)\big) = (u_0, \dot{u}_0, \lambda_0) \in \big(H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega), H_0^1(\Omega), L_2(\Gamma_\mathrm{C})\big)$*, find a solution path* $(u_*, \lambda_*) \in \Big(\big(H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)\big) \otimes C_2([t_0,t_1],\mathbb{R}^2), L_2(\Gamma_\mathrm{C}) \otimes C_2([t_0,t_1],\mathbb{R}^2)\Big)$*, such that*

$$m(\ddot{u}_*, v_x) + a(u_*, v_x) + b^T(\lambda_*, v_x) = \langle \beta, v_x\rangle + \langle \tau, v_x\rangle, \qquad \forall v_x \in V_{v_x}, \tag{2.191a}$$

$$b(u_*, \vartheta_x) = 0, \qquad \forall \vartheta_x \in V_{\vartheta_x}. \tag{2.191b}$$

With $G = 1$, λ_* is interpreted as a negative surface traction along Γ_C.

2.4.4 Derivation of Equations of Constrained Motion in Strong-Strong Form

In this subsection, the the strong-strong formulation of EOCM is derived. This subsection is included for completion, since the strong-strong form is not used in the numerical experiments in this study. The derivation steps follows closely those applied in Section 2.3.4.

To retrieve a strong-strong formulation, more restrictive function spaces in space have to be considered, compared to derivations of the weak-strong formulation in Section 2.4.2. As in Section 2.3.4, the functions space V_u is updated to

$$\begin{aligned} V_u = \{ & u(x,t) \mid u(x,t) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \cup \big(H^{3/2}(\partial\Omega) \cap H_0^{1/2}(\partial\Omega)\big) \otimes \\ & C_2([t_0,t_1],\mathbb{R}^2),\ u(\cdot,t_0) = u_0,\ u(\cdot,t_1) = u_1\}, \end{aligned} \tag{2.192}$$

by replacing $H_0^1(\Omega)$ with $H^2(\Omega) \cap H_0^1(\Omega)$, and $H_0^{1/2}(\partial\Omega)$ with $H^{3/2}(\partial\Omega) \cap H_0^{1/2}(\partial\Omega)$. Moreover, the applied force densities are restricted to L_2-spaces, $\beta \in L_2(\Omega) \otimes C([t_0,t_1],\mathbb{R}^2)$ and $\tau \in L_2(\Gamma_\mathrm{N}) \otimes C([t_0,t_1],\mathbb{R}^2)$. During the derivations, the function space $C_0^\infty(\Omega) \cup C_0^\infty(\partial\Omega)$ is employed for the test function space, V_{v_x}. By a density argument, $C_0^\infty(\Omega) \cup C_0^\infty(\partial\Omega)$ is thereafter extended to the closure $H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)$.

The weak-strong formulation of the EOCM, Eq. 2.187, is considered under the more restrictive function spaces,

$$0 = \int_\Omega \rho v_x^T \ddot{u}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x)\mathrm{d}x - \int_\Omega v_x^T \beta \mathrm{d}x \tag{2.193a}$$
$$- \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} v_x^T G(u_*)^T \lambda_* \mathrm{d}s, \quad \forall v_x \in V_{v_x},$$
$$0 = \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(u_*)\mathrm{d}s, \quad \forall \vartheta_x \in V_{\vartheta_x}. \tag{2.193b}$$

The two integral terms related to the constraint is already written in the format for which the fundamental lemma of calculus of variations is applicable. However, the internal force term is not in the applicable format. The reformulation, as presented in Section 2.3.4, involves applying the divergence property in Appendix A.5 and thereafter Gauss's theorem. The reformulation, for derivations see Eqs. 2.152-2.145 , renders in

$$0 = \int_\Omega v_x^T \left(\rho \ddot{u}_* - \mathrm{div}(\sigma(u_*)) - \beta big\right)\mathrm{d}x \tag{2.194a}$$
$$+ \int_{\Gamma_\mathrm{N}} v_x^T(\sigma n - \tau)\mathrm{d}s + \int_{\Gamma_\mathrm{C}} v_x^T(\sigma n + G(u_*)^T \lambda_*)\mathrm{d}s, \quad \forall v_x \in V_{v_x},$$
$$0 = \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(u_*)\mathrm{d}s, \quad \forall \vartheta_x \in V_{\vartheta_x}. \tag{2.194b}$$

By considering $\vartheta_x = 0$ and $v = 0$ along Γ_C, Eq. 2.145 is retrieved. In Section 2.3.4, the EOUM in strong-strong form, Eq. 2.161, was derived from Eq. 2.145. Considering $\vartheta_x = 0$ and $v = 0$ over $\bar{\Omega}\backslash\Gamma_\mathrm{C}$ gives

$$0 = \int_{\Gamma_\mathrm{C}} v_x^T G(u_*)^T \lambda_* \mathrm{d}s, \quad \forall v_x \in V_{v_x} \tag{2.195}$$

Consider the Lemma 2 for the open, bounded domain Γ_C. Since $G(u_*)^T\lambda_*$ belongs to a subset of $L_1(\Gamma_\mathrm{C})$, and $v_x \in C_0^\infty(\Gamma_\mathrm{C})$ (when $v = 0$ over $\bar{\Omega}\backslash\Gamma_\mathrm{C}$), the prerequisites for the fundamental lemma of calculus of variations in space, Lemma 2, are met. Applying the lemma, gives $\sigma n = -G(u_*)^T\lambda_*$ almost everywhere along Γ_C.

By considering Eq. 2.193 for $v = 0$,

$$0 = \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(u_*)\mathrm{d}s, \quad \forall \vartheta_x \in V_{\vartheta_x}, \tag{2.196}$$

is retrieved. Again, the prerequisites for Lemma 2, are met. Applying the lemma, gives $g(u_*) = 0$ almost everywhere along Γ_C. For $\sigma n = -G(u_*)^T\lambda_*$ and $g(u_*) = 0$, together with the strong-strong formulation of the EOUM, Eq. 2.161, the strong-strong formulation of the EOCM is retrieved,

$$\rho\ddot{u}_*(x,t) = \mathrm{div}\sigma(u_*(x,t)) + \beta(x,t), \quad \text{in } \Omega, \tag{2.197a}$$
$$u_*(x,t) = 0, \quad \text{on } \Gamma_\mathrm{D}, \tag{2.197b}$$
$$\sigma(u_*(x,t))n(x) = \tau(x,t), \quad \text{on } \Gamma_\mathrm{N}, \tag{2.197c}$$
$$\sigma(u_*(x,t))n(x) = -G(u_*(x,t))^T\lambda_*(x,t), \quad \text{on } \Gamma_\mathrm{C}, \tag{2.197d}$$
$$g(u_*(x,t)) = 0, \quad \text{on } \Gamma_\mathrm{C}. \tag{2.197e}$$

Note, Eqs. 2.197a and 2.197c-e hold almost everywhere over their respective domains. By comparing Eqs. 2.197c and 2.197d, $-G(u_*)^T\lambda_*$ is interpreted as a surface traction over Γ_C.

Recall in Eq. 2.59 in Section 2.1.3, $-G(\lambda_*)^T\lambda_*$ was interpreted as the force pulling the unconstrained motion to the path constrained by $g(\varphi_*) = 0$. That $-G(u_*)^T\lambda_*$ in Eq. 2.197 is interpreted as a surface traction, and not as a force, follows from that the constraint is defined over Γ_C, Eq. 2.162, instead of in a pointwise sense, such as $g(\tilde{\varphi}) = 0$.

2.5 Multibody Framework

In the two previous sections, the deformation due to solely elastic displacements have been modeled. Recall that the elasticity model relies on that the elastic displacements are sufficiently small. In a multibody framework, deformations are, in general, large. In Section 2.2, a rigid multibody system was retrieved by equipping each rigid body with a body-fixed reference frame.

By equipping each elastic body with a body-fixed reference frame, the large motions can be described by translations and rotations of the body-fixed reference frame, while the change of the shape of the body is described by an elastic displacement field. In the first subsection, the total deformation field, as a combination of rigid body motions and an elastic displacement field, is presented.

In the second subsection, EOCM in weak-strong form is derived for the full deformation field of a constrained elastic body. In the third subsection, three different constraint models for interconnecting joints between elastic and rigid bodies are presented. In the last subsection, the VP for a flexible multibody system is presented.

2.5.1 Body-fixed Reference Frame

Recall, in Eq. 2.73, a body-fixed reference frame was placed at the center of mass of a rigid body, to enable a decoupling of the motion into translational and rotational components. Here, the body, either elastic or rigid, is also equipped with a body-fixed frame, but this time not necessarily at the center of mass. The motion of the body-fixed frame is, just as in Eq. 2.73, modeled by r, the distance from the inertial to the body-fixed frame, and α, the angle between the body-fixed frame and the inertial frame. The displacement field is defined with respect to the body-fixed frame. The total deformation field φ, with respect to the inertial frame, is expressed as

$$\varphi(r(t), \alpha(t), u(x,t), x) = r(x,t) + A(\alpha(t))(x + u(x,t)), \quad \tilde{\varphi} : \bar{\Omega} \to \hat{\bar{\Omega}}, \tag{2.198}$$

with

$$A(\alpha(t)) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{2.199}$$

where $x \in L_2(\bar{\Omega})$ and A denote the material points, with respect to the body-fixed frame, and the rotation matrix between the body-fixed and inertial reference frame, respectively. A schematic of the deformation is illustrated in Fig. 2.6.

Noteworthy, if $r = 0$ and $\alpha = 0$, the deformation model for an elastic body attached to the inertial frame, is retrieved, Eq. 2.125. If instead $u = 0$, the motion of a rigid body is retrieved, Eq. 2.73. Moreover, if the body-fixed frame is placed at the center of mass, the rigid body motion can be decoupled into translational and rotational components.

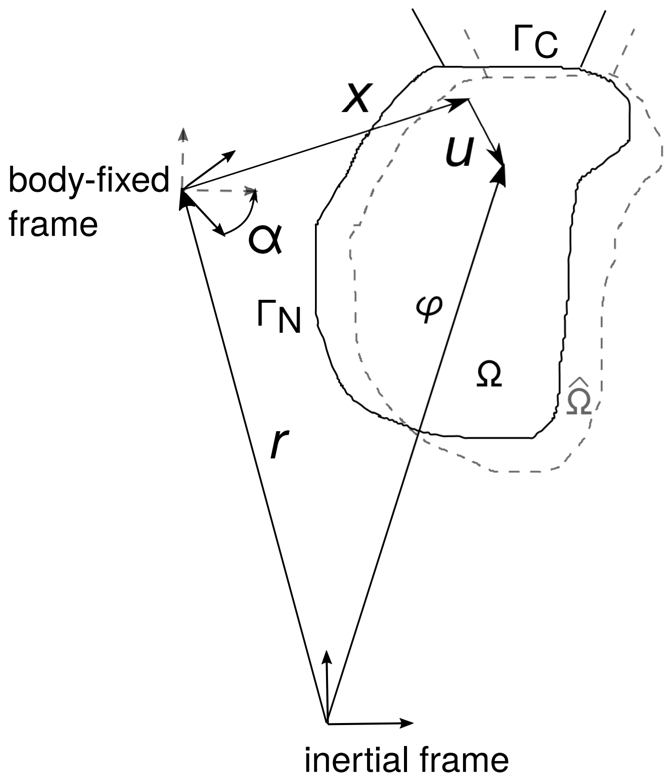

Figure 2.6: Schematic of the full deformation, in a multibody setting, of a constrained elastic body.

2.5.2 Derivation of Equations of Constrained Motion in Weak-Strong Form

In this subsection, the total deformation field for an elastic body, without any Dirichlet boundary conditions but constrained along Γ_{C}, is considered, as illustrated in Fig. 2.6. This is a common situation in multibody settings, since most bodies are interconnected, instead of being directly connected to the inertial reference frame. An important remark, in order to employ a Dirichlet boundary condition, for an elastic body, in a multibody setting, the body-fixed frame is required to be placed along Γ_{D}. With the body-fixed frame placed along Γ_{D}, the condition is fulfilled by setting the boundary displacement field to vanish along Γ_{D}. However, if the body-fixed frame is not placed along Γ_{D}, then the corresponding boundary displacement field is unknown with respect to the body-fixed frame. Naturally, a Dirichlet condition is only possible to set up if the boundary displacement field is known.

To set up a VP based on the total deformation field, Eq. 2.198, the spaces of rigid body translational V_r and rotational V_α motion, elastic displacement field V_u, and Lagrange multipliers

V_λ, are considered,

$$\begin{aligned} V_r =&\{r(t) \mid r(t) \in C_2([t_0,t_1],\, \mathbb{R}^2),\;\; r(t_0) = r_0,\; r(t_1) = r_1\}, &\text{(2.200a)}\\ V_\alpha =&\{\alpha(t) \mid \alpha(t) \in C_2([t_0,t_1],\, \mathbb{R}),\;\; \alpha(t_0) = \alpha_0,\; \alpha(t_1) = \alpha_1\}, &\text{(2.200b)}\\ V_u =&\{u(x,t) \mid u(x,t) \in \big(H^1(\Omega) \cup H^{1/2}(\partial\Omega)\big) \otimes C_2([t_0,t_1],\mathbb{R}^2), &\text{(2.200c)}\\ &\quad u(\cdot,t_0) = u_0,\; u(\cdot,t_1) = u_1\},\\ V_\lambda :=&\{\lambda(x,t) \mid \lambda(x,t) \in L_2(\Gamma_{\mathrm{C}}) \otimes C([t_0,t_1],\mathbb{R}^{n_\lambda}), \lambda(\cdot,t_0) = \lambda_0, \lambda(\cdot,t_1) = \lambda_1\}. &\text{(2.200d)} \end{aligned}$$

Thus, the aim of the VP is to retrieve the constrained motion, between $t_0 \le t \le t_1$, for given consistent endpoint conditions, $(r(t_0), \alpha(t_0), u(\cdot,t_0), \lambda(\cdot,t_0)) = (r_0, \alpha_0, u_0, \lambda_0)$ and $(r(t_1), \alpha(t_1), u(\cdot,t_1), \lambda(\cdot,t_1)) = (r_1, \alpha_1, u_1, \lambda_1)$. Since there does not exist any Dirichlet boundary segments along the body, the zero subscripts for the function spaces in space, in V_u, are not included. The function spaces $V_{r_*}, V_{\alpha_*}, V_{\tilde r}, V_{\tilde\alpha}, V_{\theta z}, V_{\theta\zeta}, V_z$, and V_ζ follows from Section 2.2.1. The function spaces $V_{u_*}, V_{\tilde u}, V_{\theta v}, V_v, V_{v_x}$, and V_{v_t} are adopted from Section 2.3.2, but without any subscript zeros on the function spaces. Lastly, function spaces $V_{\lambda_*}, V_{\tilde\lambda}, V_{\theta\vartheta}, V_\vartheta, V_{\vartheta_x}$, and V_{ϑ_t} are taken from Section 2.4.1.

As in Section 2.2.1, spaces of admissible full deformation $V_{\tilde\varphi}$, and the corresponding variations $V_{\theta\eta}$, are sought from the spaces $V_{\tilde r}$, $V_{\tilde\alpha}$, and $V_{\tilde u}$. First, consider the space V_φ based on applying Eq. 2.198 for admissible displacement fields and rigid body translational and rotational motions,

$$V_\varphi = \{\tilde r(t) + A(\tilde\alpha(t))(x+\tilde u) \,|\, \tilde r(t) \in V_{\tilde r},\, \tilde\alpha(t) \in V_{\tilde\alpha},\, x \in L_2(\bar\Omega),\, \tilde u \in V_{\tilde u},\, A \text{ from Eq. 2.199}\}, \quad (2.201)$$

The corresponding space for the extremal components is

$$\begin{aligned} V_{\varphi_*} =&\{r_*(t) + A(\alpha_*(t))(x+u_*) \mid r_*(t) \in V_{r_*},\\ &\alpha_*(t) \in V_{\alpha_*},\; x \in L_2(\bar\Omega),\; u_* \in V_{u_*},\; A \text{ from Eq. 2.73}\}. \end{aligned} \quad (2.202)$$

Consider the difference between $\varphi \in V_\varphi$ and $\varphi_* \in V_{\varphi_*}$,

$$\begin{aligned} \varphi - \varphi_* &= \Big(r_* + \theta z + A(\alpha_* + \theta\zeta)(x + u_* + \theta v)\Big) - \Big(r_* + A(\alpha_*)(x+u_*)\Big)\\ &= \theta z + \Big(A(\alpha_* + \theta\zeta) - A(\alpha_*)\Big)(x+u_*) + A(\alpha_* + \theta\zeta)\theta v. \end{aligned} \quad (2.203)$$

For each $\zeta(t) \in V_\zeta$, there is a sufficiently small θ such that the approximations

$$A'(\alpha_*)\theta\zeta \doteq A(\alpha_* + \theta\zeta) - A(\alpha_*), \quad A'(\alpha_*) := \begin{pmatrix} -\sin\alpha_* & -\cos\alpha_* \\ \cos\alpha_* & -\sin\alpha_* \end{pmatrix}, \quad (2.204)$$

$$A(\alpha_*) \doteq A(\alpha_* + \theta\zeta) \quad (2.205)$$

are good. With the approximations, Eq. 2.203 becomes

$$\varphi - \varphi_* \doteq \theta z + A'(\alpha_x)\theta\eta(x+u_*) + A(\alpha_*)\theta v. \quad (2.206)$$

Define the space $V_{\theta\eta}$ based on Eq. 2.206,

$$V_{\theta\eta} := V_\theta \otimes V_\eta, \quad (2.207)$$

with V_θ from Eq. 2.15a and,

$$\begin{aligned} V_\eta :=&\{z(t) + \zeta(t)A'(\alpha_*(t))(x+u_*) + A(\alpha_*)v \mid z(t) \in V_z,\; \alpha_*(t) \in V_{\alpha_*},\; \zeta(t) \in V_\zeta,\\ &x \in L_2(\bar\Omega),\; u_* \in V_{u_*},\; v \in V_v, A \text{ and } A' \text{ from Eqs. 2.199 and 2.204}\}. \end{aligned} \quad (2.208)$$

Since V_z, V_ζ, and V_v vanish at the endpoints in time, so thus $\eta(x,t) \in V_\eta$, $\eta(\cdot,t_0) = \eta(\cdot,t_1) = 0$. The affine space $V_{\tilde{\varphi}}$ of admissible motions is defined as

$$V_{\tilde{\varphi}} := \{\varphi_*(x,t) + \theta\eta(x,t) \mid (\varphi_*, \theta\eta)(x,t) \in (V_{\varphi_*}, V_{\theta\eta})\} \tag{2.209}$$

Since the applied forces affect the full deformation, the kinetic and the potential energies, depends on it,

$$T(\dot{\tilde{\varphi}}) = \frac{1}{2}\int_\Omega \rho \dot{\tilde{\varphi}}^T \dot{\tilde{\varphi}} \mathrm{d}x, \tag{2.210}$$

and

$$V(\tilde{\varphi}) = \frac{1}{2}\int_\Omega \sigma(\tilde{u}) : \epsilon(\tilde{u}) \mathrm{d}x - \int_\Omega \tilde{\varphi}^T A\beta \mathrm{d}x - \int_{\Gamma_\mathrm{N}} \tilde{\varphi}^T A\tau \mathrm{d}s, \tag{2.211}$$

with β and τ defined with respect to the body-fixed frame. Since the strain measure is invariant to rigid body motions, $\sigma(\tilde{\varphi}) : \epsilon(\tilde{\varphi}) = \sigma(\tilde{u}) : \epsilon(\tilde{u})$. Hamiltion's principle of least action is stated as

Variational problem 10 (Constrained elastic body, multibody setting)**.** *For a closed, autonomous system of a constrained elastic body with given material points $x \in L_2(\bar{\Omega})$, applied force densities $\big(\beta(x,t), \tau(x,t)\big) \in \big(H^1(\Omega)^* \otimes C([t_0,t_1],\mathbb{R}^2), H^{1/2}(\Gamma_\mathrm{N})^* \otimes C([t_0,t_1],\mathbb{R}^2)\big)$, and consistent endpoint conditions $(t_0, r_0, \alpha_0, u_0, \lambda_0)$ and $(t_1, r_1, \alpha_1, u_1, \lambda_1)$, find an extremal $(\varphi_*(x,t), \lambda_*(x,t)) \in (V_{\varphi_*}, V_{\lambda_*})$, under the assumption that an extremal exists, such that*

$$0 = j'(0), \quad \forall(\eta, \vartheta) \in (V_\eta, V_\vartheta), \tag{2.212a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L_\mathrm{C}(\tilde{\varphi}(\theta,x,t), \dot{\tilde{\varphi}}(\theta,x,t), \tilde{\lambda}(\theta,x,t))\mathrm{d}t, \quad (\tilde{\varphi}, \tilde{\lambda}) \in (V_{\tilde{\varphi}}, V_{\tilde{\lambda}}), \tag{2.212b}$$

where

$$L_\mathrm{C}(\tilde{\varphi}, \dot{\tilde{\varphi}}, \tilde{\lambda}) = T(\dot{\tilde{\varphi}}) - V(\tilde{\varphi}) - \int_{\Gamma_\mathrm{C}} g(\tilde{\varphi})^T \tilde{\lambda} \mathrm{d}s, \quad (\tilde{\varphi}, \tilde{\lambda}) \in (V_{\tilde{u}}, V_{\tilde{\lambda}}), \tag{2.212c}$$

with $g \in H^1(\Gamma_\mathrm{C}) \otimes C_1([t_0,t_1],\mathbb{R}^{n_\lambda})$.

The first derivation steps is closely similar to those performed in Section 2.4.2. Insertion of the constrained Lagrangian, Eq. 2.212c, into the action integral, Eq. 2.212b, gives

$$\begin{aligned} 0 = \int_{t_0}^{t_1} \Big(&\int_\Omega \frac{1}{2}\rho(\dot{\varphi}_* + \theta\dot{\eta})^T(\dot{u}_* + \theta\dot{\eta})\mathrm{d}x - \int_\Omega \frac{1}{2}\sigma(u_* + \theta v) : \epsilon(u_* + \theta v)\mathrm{d}x \\ &+ \int_\Omega (\varphi_* + \theta\eta)^T A(\alpha_*)\beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} (\varphi_* + \theta\eta)^T A(\alpha_*)\tau \mathrm{d}s \\ &- \int_{\Gamma_\mathrm{C}} g(\varphi_* + \theta\eta)^T(\lambda_* + \theta\vartheta)\mathrm{d}s\Big)\mathrm{d}t, \quad \forall(\theta v, \theta\eta, \theta\vartheta) \in (V_{\theta v}, V_{\theta\eta}, V_{\theta\vartheta}). \end{aligned} \tag{2.213}$$

Applying the stationarity condition, Eq. 2.212a, generates

$$\begin{aligned} 0 = \int_{t_0}^{t_1} \Big(&\int_\Omega \rho\dot{\eta}^T \dot{\varphi}_* \mathrm{d}x - \int_\Omega \sigma(u_*) : \epsilon(v)\mathrm{d}x + \int_\Omega \eta^T A(\alpha_*)\beta \mathrm{d}x \\ &+ \int_{\Gamma_\mathrm{N}} \eta^T A(\alpha_*)\tau \mathrm{d}s - \int_{\Gamma_\mathrm{C}} \eta^T G(\varphi_*)^T \lambda_* \mathrm{d}s\Big)\mathrm{d}t, \quad \forall(v, \eta) \in (V_v, V_\eta), \end{aligned} \tag{2.214a}$$

$$0 = \int_{t_0}^{t_1}\int_{\Gamma_\mathrm{C}} \vartheta^T g(\varphi_*)\mathrm{d}s\mathrm{d}t, \quad \forall\vartheta \in V_\vartheta, \tag{2.214b}$$

where $G(\varphi_*) = \frac{\mathrm{d}g(\tilde{\varphi})}{\mathrm{d}\tilde{\varphi}}|_{\theta=0}$. Since $\eta(\cdot, t_0) = \eta(\cdot, t_1) = 0$, performing integration by parts of the inertia term, with respect to time, yields

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \rho \eta^T \ddot{\varphi}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v) \mathrm{d}x - \int_\Omega \eta^T A\beta \mathrm{d}x \tag{2.215a}$$

$$- \int_{\Gamma_\mathrm{N}} \eta^T A\tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} \eta^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall (v, \eta) \in (V_v, V_\eta),$$

$$0 = \int_{t_0}^{t_1} \int_{\Gamma_\mathrm{C}} \vartheta^T g(\varphi_*) \mathrm{d}s \mathrm{d}t, \quad \forall \vartheta \in V_\vartheta. \tag{2.215b}$$

To continue, η is decoupled according to Eq. 2.208, $\eta = z + \zeta A'(\alpha_*(t))x + A(\alpha_*)v$,

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \rho z^T \ddot{\varphi}_* \mathrm{d}x - \int_\Omega z^T A\beta \mathrm{d}x - \int_{\Gamma_\mathrm{N}} z^T A\tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} z^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall z \in V_z, \tag{2.216a}$$

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \rho \zeta (x+u_*)^T A'^T \ddot{\varphi}_* \mathrm{d}x - \int_\Omega \zeta (x+u_*)^T A'^T A\beta \mathrm{d}x - \int_{\Gamma_\mathrm{N}} \zeta (x+u_*)^T A'^T A\tau \mathrm{d}s \tag{2.216b}$$

$$+ \int_{\Gamma_\mathrm{C}} \zeta (x+u_*)^T A'^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall \zeta \in V_\zeta,$$

$$0 = \int_{t_0}^{t_1} \Big(\int_\Omega \rho v^T A^T \ddot{\varphi}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v) \mathrm{d}x - \int_\Omega v^T \beta \mathrm{d}x \tag{2.216c}$$

$$- \int_{\Gamma_\mathrm{N}} v^T \tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} v^T A^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall v \in V_v,$$

$$0 = \int_{t_0}^{t_1} \int_{\Gamma_\mathrm{C}} \vartheta^T g(\varphi_*) \mathrm{d}s \mathrm{d}t, \quad \forall \vartheta \in V_\vartheta, \tag{2.216d}$$

where $A^T A = I$ was used. Recall, in Section 2.2.1, $a^T A'^T Ab$, for any a and b, were rewritten as $n_\alpha^T a \times b$. Also, $v \in V_v$ and $\vartheta \in V_\vartheta$ are separated in space and time,

$$0 = \int_{t_0}^{t_1} z^T \Big(\int_\Omega \rho \ddot{\varphi}_* \mathrm{d}x - \int_\Omega A\beta \mathrm{d}x - \int_{\Gamma_\mathrm{N}} A\tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall z \in V_z, \tag{2.217a}$$

$$0 = \int_{t_0}^{t_1} \zeta \Big(\int_\Omega \rho n_\alpha^T (x+u_*) \times A^T \ddot{\varphi}_* \mathrm{d}x - \int_\Omega n_\alpha^T (x+u_*) \times \beta \mathrm{d}x - \int_{\Gamma_\mathrm{N}} n_\alpha^T (x+u_*) \times \tau \mathrm{d}s \tag{2.217b}$$

$$+ \int_{\Gamma_\mathrm{C}} n_\alpha^T (x+u_*) \times A^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall \zeta \in V_\zeta,$$

$$0 = \int_{t_0}^{t_1} v_t^T \Big(\int_\Omega \rho v_x^T A^T \ddot{\varphi}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x) \mathrm{d}x - \int_\Omega v_x^T \beta \mathrm{d}x \tag{2.217c}$$

$$- \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} v_x^T A^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall (v_x, v_t) \in (V_{v_x}, V_{v_t}),$$

$$0 = \int_{t_0}^{t_1} \vartheta_t^T \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(\varphi_*) \mathrm{d}s \mathrm{d}t, \quad \forall (\vartheta_x, \vartheta_t) \in (V_{\vartheta_x}, V_{\vartheta_t}). \tag{2.217d}$$

Recall the definition of f defined in Eq. 2.95,

$$f = \int_\Omega A\beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} A\tau \mathrm{d}s. \tag{2.218}$$

Moreover, define $\tau^{x+u_*}_{\text{torque}}$ as

$$\tau^{x+u_*}_{\text{torque}} := \int_\Omega n_\alpha^T(x+u_*) \times \beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} n_\alpha^T(x+u_*) \times \tau \mathrm{d}s. \tag{2.219}$$

Insertion of Eqs. 2.218 and 2.219 into Eq. 2.217 gives

$$0 = \int_{t_0}^{t_1} z^T \Big(\int_\Omega \rho \ddot{\varphi}_* \mathrm{d}x - f + \int_{\Gamma_\mathrm{C}} G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall z \in V_z, \tag{2.220a}$$

$$0 = \int_{t_0}^{t_1} \zeta \Big(\int_\Omega \rho n_\alpha^T x \times A^T \ddot{\varphi}_* \mathrm{d}x - \tau^{x+u_*}_{\text{torque}} + \int_{\Gamma_\mathrm{C}} n_\alpha^T x \times A^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall \zeta \in V_\zeta, \tag{2.220b}$$

$$\begin{aligned} 0 = \int_{t_0}^{t_1} v_t^T \Big(& \int_\Omega \rho v_x^T A^T \ddot{\varphi}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x) \mathrm{d}x - \int_\Omega v_x^T \beta \mathrm{d}x \\ & - \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s + \int_{\Gamma_\mathrm{C}} v_x^T A^T G(\varphi_*)^T \lambda_* \mathrm{d}s \Big) \mathrm{d}t, \quad \forall (v_x, v_t) \in (V_{v_x}, V_{v_t}), \end{aligned} \tag{2.220c}$$

$$0 = \int_{t_0}^{t_1} \vartheta_t^T \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(\varphi_*) \mathrm{d}s \mathrm{d}t, \quad \forall (\vartheta_x, \vartheta_t) \in (V_{\vartheta_x}, V_{\vartheta_t}). \tag{2.220d}$$

To retrieve the EOCM in weak-strong form, the fundamental lemma of calculus of variations in time, Lemma 1, is employed. Under the given function spaces, the prerequisites for the lemma are met, for details see Sections 2.2 and 2.4.2. Applying the lemma gives the EOCM in weak-strong form,

$$\int_\Omega \rho \ddot{\varphi}_* \mathrm{d}x = f - \int_{\Gamma_\mathrm{C}} G(\varphi_*)^T \lambda_* \mathrm{d}s, \tag{2.221a}$$

$$\int_\Omega \rho n_\alpha^T (x+u_*) \times A^T \ddot{\varphi}_* \mathrm{d}x = \tau^{x+u_*}_{\text{torque}} - \int_{\Gamma_\mathrm{C}} n_\alpha^T (x+u_*) \times A^T G(\varphi_*)^T \lambda_* \mathrm{d}s, \tag{2.221b}$$

$$\begin{aligned} \int_\Omega \rho v_x^T A^T \ddot{\varphi}_* \mathrm{d}x + \int_\Omega \sigma(u_*) : \epsilon(v_x) \mathrm{d}x &= \int_\Omega v_x^T \beta \mathrm{d}x + \int_{\Gamma_\mathrm{N}} v_x^T \tau \mathrm{d}s \\ & - \int_{\Gamma_\mathrm{C}} v_x^T A^T G(\varphi_*)^T \lambda_* \mathrm{d}s, \quad \forall v_x \in V_{v_x}, \end{aligned} \tag{2.221c}$$

$$0 = \int_{\Gamma_\mathrm{C}} \vartheta_x^T g(\varphi_*) \mathrm{d}s \mathrm{d}t, \quad \forall \vartheta_x \in V_{\vartheta_x}. \tag{2.221d}$$

Since the rotation matrix, and possibly also the constraint equations, are nonlinear, the EOCM is a nonlinear system. Eq. 2.221 is referred to as a weak-strong form even though the first two equations is given in strong form. Eq. 2.221 is not fully decoupled into rigid body translational and rotational motion, and elastic displacements. Especially, the Lagrange multipliers λ_* couple the first three equations. Part of the couplings are hidden in $\ddot{\varphi}_*$. The rest of this subsection is devoted to show how much the inertia terms can be decoupled.

First, $\ddot{\varphi}_*$ is derived,

$$\varphi_* = r_* + A(\alpha_*)(x+u_*), \tag{2.222a}$$

$$\dot{\varphi}_* = \dot{r}_* + \dot{\alpha}_* A'(x+u_*) + A\dot{u}_*, \tag{2.222b}$$

$$\begin{aligned} \ddot{\varphi}_* &= \ddot{r}_* + \ddot{\alpha}_* A'(x+u_*) - \dot{\alpha}_*^2 A''(x+u_*) + A\ddot{u}_* + 2\dot{\alpha}_* A' \dot{u}_*, \\ &= \ddot{r}_* + \ddot{\alpha}_* A'(x+u_*) + \dot{\alpha}_*^2 A(x+u_*) + A\ddot{u}_* + 2\dot{\alpha}_* A' \dot{u}_*. \end{aligned} \tag{2.222c}$$

In the last equality $A'' = -A$ was used. Insertion of Eq. 2.222c into the inertia terms in Eq. 2.221 gives, with the compact notation for the inertia term, Eq. 2.149, used,

$$\int_\Omega \rho\ddot{\varphi}_* \mathrm{d}x = m\ddot{r}_* + A' s_{\text{def.}}\ddot{\alpha}_* + A s_{\text{def.}}\dot{\alpha}_*^2 + \int_\Omega \rho(A\ddot{u}_* + 2\dot{\alpha}A'\dot{u}_*)\mathrm{d}x, \quad (2.223\text{a})$$

$$\begin{aligned}\int_\Omega \rho n_\alpha^T (x+u_*) \times A^T\ddot{\varphi}_*\mathrm{d}x &= \int_\Omega \rho(x+u_*)^T A'^T\ddot{\varphi}_*\mathrm{d}x \quad (2.223\text{b})\\ &= s_{\text{def.}}^T A'^T\ddot{r}_* + J_{\text{def.}}\ddot{\alpha}_* + \int_\Omega \rho(x+u_*)^T\left(A'^T A\ddot{u}_* + 2\dot{u}_*\dot{\alpha}\right)\mathrm{d}x,\\ m(\rho A^T\ddot{\varphi}_*, v_x) &= m(\rho A^T\ddot{r}_*, v_x) + m(\rho A^T A'(x+u_*)\ddot{\alpha}_*, v_x) \quad (2.223\text{c})\\ &\quad + m(\rho(x+u_*)\dot{\alpha}_*^2, v_x) + m(\rho\ddot{u}_*, v_x) + m(\rho\dot{\alpha}_* A^T A'\dot{u}_*, v_x),\end{aligned}$$

with the the mass m defined in Eq. 2.102, the coordinate distance from the center of mass of the deformed body $s(u)$ defined as

$$s_{\text{def.}} = s_{\text{def.}}(u) := \int_\Omega \rho(x+u)\mathrm{d}x, \quad (2.224)$$

and the moment of inertia of the deformed body defined as

$$J_{\text{def.}} = J_{\text{def.}}(u) := \int_\Omega \rho(x+u)^T(x+u)\mathrm{d}x. \quad (2.225)$$

For an undeformed elastic body, with the body-fixed frame placed at the center of mass, Eq. 2.223a and Eq. 2.223b simplify to $m\ddot{r}$ and $J\ddot{\alpha}$ ($J = J_{\text{def.}}(0)$, Eq. 2.102), respectively. $m\ddot{r}$ and $J\ddot{\alpha}$ are also the translational and rotational inertia terms for rigid body dynamics, according to Newton-Euler's equation, Eq. 2.104.

Moreover, if the body-fixed frame and the inertial frame overlap, the inertial term for solely elastic displacements, $m(\rho\ddot{u}, v)$, is retrieved. This highlights the possible simplifications which can be retrieved by a smart choice for placement of the body-fixed reference frame.

2.5.3 Modeling of Interconnecting Joints

As mentioned previously, constrained multibody dynamics was derived in order to enable incorporation of interconnecting joints into the model. In this study, three models for massless, rigid joints, retrieved from [1], are presented to interconnect elastic bodies. As noted in [1], the interconnecting joints, in a multibody context, are often very small compared to the interconnected bodies. Hence, the massless assumption leads to less stiff systems of differential equations, and is therefore employed in this study. For two non-massless joint models, see [1].

For a rigid joint connected to an elastic body, two different modeling approaches meet at the joint-body interface. According to the rigid joint model, the interface is restricted to perform rigid body motions. Whereas, according to the elastic body model, the interface is permitted to perform elastic body motions. Since rigid body motions is a special case of elastic body motions, the interface should be restricted to perform rigid body motions, to conform with both models.

For the first model, where the elasticity model is defined with respect to the interface, the motion of the interface is, in a well-defined way, restricted to rigid body motions. However, the model is only applicable once for each elastic body. The joint models for general use involves model compromises.

In the second model, the constraint along the interface is represented by a point constraint, which violates the elasticity model. The last model is the weakly constrained joint model. Recall,

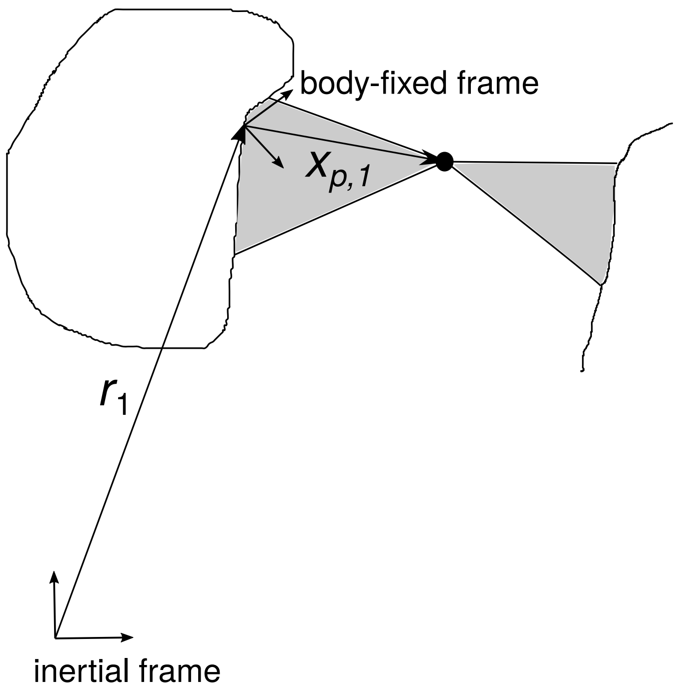

Figure 2.7: Locating the pivot of a joint through a body-fixed reference frame at the joint-body interface.

the aim of this study is to highlight the limitations of the weakly constrained joint model. The last model conforms with the elasticity model, but is restricted to interconnections where one of the bodies is rigid. Also, the model requires that the orientation of the rigid joint-elastic body interface is unaffected by the displacement field of the elastic body.

2.5.3.1 Body-Fixed Frame Attached to Joint

In the first model, the body-fixed frame is placed at the joint-body interface, as illustrated in Fig. 2.7. Since the displacement field along the interface is known, it is possible to apply a Dirichlet condition for the displacement field along the interface. By setting the displacement field to vanish along the interface, the interface is restricted to perform rigid body motions. The rigid motions of the body-fixed frame is constrained just as an interconnected rigid body. An interconnecting joint model for rigid bodies was presented in Section 2.2.2. Thus, with this model, the constraints on the elastic displacement field and the rigid body motions, is decoupled.

However, there is an obvious drawback with this model. It is only applicable once per elastic body. Its use is further limited, for an elastic body, by the fact that, for applying a Dirichlet condition to a rigid attachment, the body-fixed frame is required to be placed along Γ_{D}, as explained in the previous subsection. Moreover, to in a well-defined way include a force element with pointwise attachment, such as those presented in Section 2.2.3, the body-fixed frame has to be placed in its attachment point.

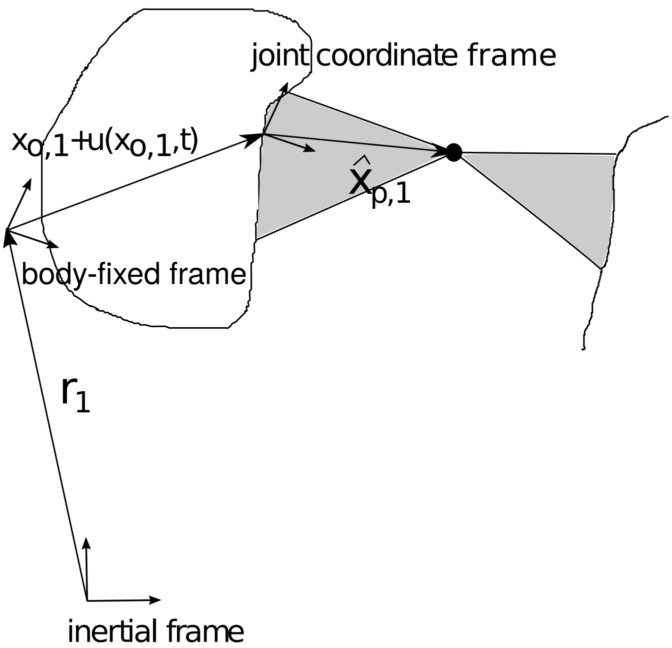

Figure 2.8: Locating the pivot of a joint through an additional coordinate system at the joint-body interface.

2.5.3.2 Intermediate Joint Coordinate Frame

The aim with the second model is to extend the concept from the first model to generic combinations of interconnecting elements. The idea is to equip all rigid interfaces, connected to an elastic body, with local reference systems. Thus, additional joint coordinate reference frames are added to all interfaces, if the body-fixed frame is not placed there. A schematic is illustrated in Fig. 2.8. The distance from the inertial frame to a joint coordinate frame, represented by hats, is defined as

$$\hat{r}_1 := r_1 + A(\alpha_1)(x_{o,1} + u(x_{o,1}, t)). \tag{2.226}$$

The constraint equation for describing interconnected elastic bodies with the second model is expressed as

$$g(\hat{\varphi}_1, \hat{\varphi}_2) = \hat{r}_1 + A(\hat{\alpha}_1)\hat{x}_{p,1} - (\hat{r}_2 + A(\hat{\alpha}_2)\hat{x}_{p,2}) = 0. \tag{2.227}$$

Note the close resemblance to the constraint equation for the rigid-rigid body connection, see Section 2.2.2. However, there is a main drawback with this approach. The constraint equation in Eq. 2.227 is described in a pointwise way, which conforms well with the rigid body motions. However, this model constrains a certain point of the elastic displacement field along the interface, while the remaining parts are treated as unconstrained. This is not a well-defined way to impose a constraint on the elastic displacement field. The model approximates the true dynamics with the assumption that the point constraint is a good approximate for a corresponding well-defined constraint along the rigid joint-elastic body interface.

Moreover, the orientation of the joint coordinate frame, angle $\hat{\alpha}_1$, is defined by use of additional observer points along the boundary. In combination with specialized joint modeling elements, this is a common approach [1, p. 101].

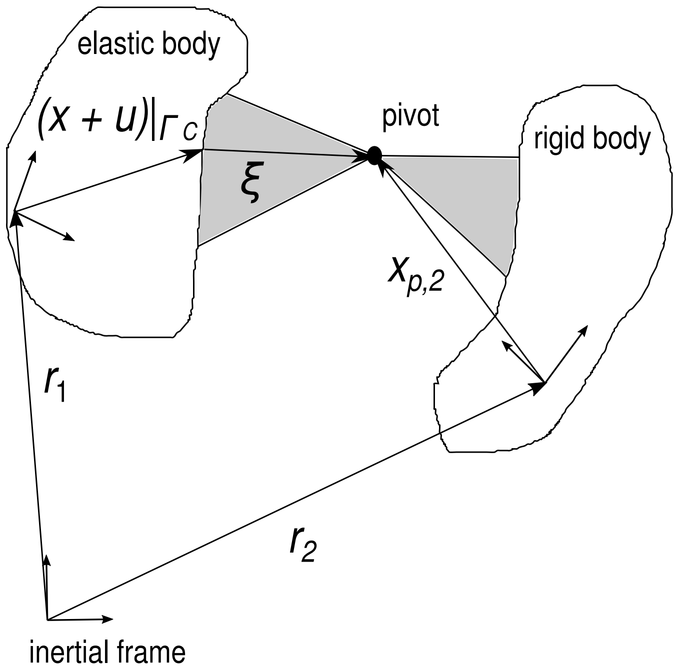

Figure 2.9: Schematic of the constraint equation for the weakly constrained joint model.

2.5.3.3 Weakly Constrained Joint Model

The aim with the third model is to retrieve a generic model, which incorporates elastic displacement fields in a well-defined way. However, the generality has to be restricted to cases where one of the two interconnected bodies is rigid (here body 2). Then, the pivot of the joint can be located through the rigid body. $\xi(x)$ for $x \in L_2(\Gamma_{C,1})$, denotes the distance from the body-joint interface to the pivot. Then, a constraint along $\Gamma_{C,1}$ is expressed as

$$g(\varphi_1, \varphi_2) = \varphi_1(r(t), \alpha(t), u(x,t), x) + \xi(x) - \varphi_2(x_{p,2}, t) = 0, \quad \text{for } x \in L_2(\Gamma_{C,1}), \tag{2.228}$$

with φ_1 from Eq. 2.198 and φ_2 from Eq. 2.73. A schematic is illustrated in Fig. 2.9. By applying Eq. 2.228 over $\Gamma_{C,1}$, as in Eq. 2.162, the constraint is incorporated in a well-defined way, both for the rigid body motions and the elastic displacement field. The aim of this study is to show the limitations of this model. Since $\xi = \xi(x)$, the elastic body-joint interface is assumed to retain its shape in reference to the body-fixed frame of the elastic body. This requirement can severely distort the displacement field, as clarified in the numerical experiments.

2.5.4 Flexible Multibody Dynamics

The last extension step, to enable modeling of the dynamics of a multibody system consisting of both elastic and rigid bodies, flexible multibody dynamics, in presence of interconnecting joints, is simple. The multibody system is retrieved as an aggregation of several single bodies, just as for the rigid multibody system in Section 2.2.4.

Consider a system of n_b bodies. The kinetic and potential energies are retrieved by aggregating Eq. 2.210,

$$T(\{\dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}) = \sum_{i=1}^{n_b} \frac{1}{2} \int_{\Omega_i} \rho_i \dot{\tilde{\varphi}}_i^T \dot{\tilde{\varphi}}_i \mathrm{d}x, \tag{2.229}$$

and Eq. 2.211,

$$V(\{\tilde{\varphi}_i\}_{i=1}^{n_b}) = \sum_{i=1}^{n_b} (\frac{1}{2} \int_{\Omega_i} \sigma_i(\tilde{u}_i) : \epsilon_i(\tilde{u}_i) \mathrm{d}x - \int_{\Omega_i} \tilde{\varphi}_i^T A_i \beta_i \mathrm{d}x - \int_{\Gamma_{\mathrm{N}}^i} \tilde{\varphi}_i^T A_i \tau_i \mathrm{d}s), \tag{2.230}$$

for $\tilde{\varphi}_i \in V_{\tilde{\varphi}}$ from Eq. 2.209. Moreover, the presence of interconnecting joints is identified by a set $\mathcal{J}$ of index pairs, with the indices referring to the indices of the interconnected bodies. The constrained Lagrangian for the constrained flexible multibody system is

$$L_{\mathrm{C}} = T(\{\dot{\tilde{\varphi}}_i\}_{i=1}^{n_b}) - V(\{\tilde{\varphi}\}_{i=1}^{n_b}) - \sum_{(j,k)\in\mathcal{J}} \int_{\Gamma_{\mathrm{C}}^{\{j,k\}}} g(\tilde{\varphi}_j, \tilde{\varphi}_k)^T \tilde{\lambda}_{(j,k)} \mathrm{d}s. \tag{2.231}$$

The superscript $^{\{j,k\}}$, for $\Gamma_{\mathrm{C}}^{\{j,k\}}$, denotes that both Γ_{C}^j and Γ_{C}^k are considered, while the subscript $_{(j,k)}$, for $\tilde{\lambda}_{(j,k)}$, denotes that the same Lagrange multipliers are used for the two interconnected bodies. By inserting the Lagrangian into Hamilton's principle of least action, an aggregated version of EOCM in weak-strong form, can be derived, just as in Section 2.2.4 for rigid multibody systems. To perform the derivations do not provide much further insight, and is therefore omitted. However, in the next subsection, the derivations are performed for a two-body system investigated in the numerical experiments.

2.5.5 Two-Body System with Weakly Constrained Joint Model

In this subsection, the IVP based on the EOM in weak-strong form is presented for a two-body system which is studied in the numerical experiments, see Section 4.2. The two-body system consists of an elastic and a rigid body. The elastic body is, on its left side, attached to a rigid wall, by a Dirichlet boundary condition. On the lower right corner of the elastic block, a rigid revolute joint is attached. The rigid joint is also connected to a rigid body, hanging underneath its pivot. The configuration is illustrated in Fig. 4.4. The only applied force is the gravity, directed downwards in the figure, with body force density $\beta_0 = (0, -9.82)$, with respect to the inertial reference frame.

The inertial reference frame is placed at the lower left corner of the elastic block. Then, the Dirichlet boundary segment is attached to the inertial reference frame (which is a requirement as discussed in Section 2.5), and the elastic body performs solely elastic displacements. The dynamics for the displacement field is modeled as in Section 2.4. The rigid revolute joint is approximated as massless, and modeled with the weakly constrained model, Section 2.5.3.3.

The material of the elastic body is modeled as isotropic and homogeneous. The depth of the two bodies are one percent of their lengths and widths. By approximating the elasticity model for the elastic body by the planar stress model, Eq. 2.129, it is sufficient to model the planar motion of the two-body system.

Recall from Section 2.2.4, the presence of rotations matrices contributed to the EOCM being nonlinear. To retrieve a linear system for the constrained dynamics, only the translational motion r of the rigid body is considered. Then, the constrained Lagrangian for the two-body system becomes

$$L_{\mathrm{C}}(\tilde{u}_1, \tilde{r}_2) = T_1(\dot{\tilde{u}}_1) + T_2(\dot{\tilde{r}}_2) - V_1(\tilde{u}_1) - V_2(\tilde{r}_2) - \int_{\Gamma_{\mathrm{C},1}} g(\tilde{u}_1, \tilde{r}_2)^T \tilde{\lambda} \mathrm{d}s, \tag{2.232}$$

with kinetic energies

$$T_1(\dot{\tilde{u}}_1) := \frac{1}{2}\int_{\Omega_1} \rho \dot{\tilde{u}}_1^T \dot{\tilde{u}}_1 \mathrm{d}x, \tag{2.233a}$$

$$T_2(\dot{\tilde{r}}_2) := \frac{1}{2}\int_{\Omega_2} \rho \dot{\tilde{r}}_2^T \dot{\tilde{r}}_2 \mathrm{d}x, \tag{2.233b}$$

and potential energies

$$V_1(\tilde{\varphi}_1) := \frac{1}{2}\int_{\Omega_1} \sigma(\tilde{u}_1) : \epsilon(\tilde{u}_1)\mathrm{d}x - \int_{\Omega} \tilde{u}_1^T \beta_0 \mathrm{d}x, \tag{2.234a}$$

$$V_2(\tilde{r}_2) := -\int_{\Omega_2} \tilde{r}_2^T A\beta \mathrm{d}x = -\int_{\Omega_2} \tilde{r}_2^T \beta_0 \mathrm{d}x. \tag{2.234b}$$

with $\tilde{u}_1 \in V_{\tilde{u}}$, $\tilde{r}_2 \in V_{\tilde{r}}$, and $\tilde{\lambda} \in V_{\tilde{\lambda}}$ from Eqs. 2.137, 2.76a, and 2.177. Note, the absence of surface work contributions for the potential energies. Also, note that β is defined with respect to the body-fixed frame, whereas β_0 is defined with respect to the inertial frame, $\beta_0 = A\beta$.

The constraint equation for the interconnecting joint becomes, see Eq. 2.228,

$$\begin{aligned} g(u_1, r_2) &= u_1(x,t) + x + (x_{p,1} - x) - (r_2(t) + x_{p,2}) \\ &= u_1(x,t) + x_{p,1} - r_2(t) - x_{p,2} = 0, \quad \text{for } x \in L_2(\Gamma_{\mathrm{C},1}), \end{aligned} \tag{2.235}$$

where $x_{p,1}$ and $x_{p,2}$ are the position of the joint with respect to the two body-fixed frames. Then, Hamilton's principle of least action is stated as

Variational problem 11 (Two-body system)**.** *For a closed, autonomous two-body system of an elastic and a rigid body with given material points $\{x_i\}_{i=1}^2 \in L_2(\bar{\Omega}_i) \cup x_{p,i}$, applied force densities β_0, and consistent endpoint conditions $(t^0, r_2^0, u_1^0, \lambda^0)$ and $(t^1, r_2^1, u_1^1, \lambda^1)$, find an extremal $(u_{*,1}(x,t), r_{*,2}(t), \lambda_*(x,t)) \in (V_{u_*}, V_{r_*}, V_{\lambda_*})$, under the assumption that an extremal exists, such that*

$$0 = j'(0), \quad \forall (v_1, z_2, \vartheta) \in (V_{v_1}, V_{z_2}, V_\vartheta), \tag{2.236a}$$

for

$$j(\theta) = \int_{t_0}^{t_1} L_{\mathrm{C}}(\tilde{u}_1(\theta,x,t), \dot{\tilde{u}}_1(\theta,x,t), \tilde{r}_2(\theta,t), \dot{\tilde{r}}_2(\theta,t), \tilde{\lambda}(\theta,x,t))\mathrm{d}t, \tag{2.236b}$$

where

$$L_{\mathrm{C}}(\tilde{u}_1, \tilde{r}_2) = T_1(\dot{\tilde{u}}_1) + T_2(\dot{\tilde{r}}_2) - V_1(\tilde{u}_1) - V_2(\tilde{r}_2) - \int_{\Gamma_{\mathrm{C},1}} g(\tilde{u}_1, \tilde{r}_2)^T \tilde{\lambda} \mathrm{d}s, \tag{2.236c}$$

for $(\tilde{u}_1, \tilde{r}_2, \tilde{\lambda}) \in (V_{\tilde{u}_1}, V_{\tilde{r}_2}, V_{\tilde{\lambda}})$ and with $g \in H^1(\Gamma_{\mathrm{C}}) \otimes C_1([t_0, t_1], \mathbb{R}^{n_\lambda})$,

$$g(u_1, r_2) = u_1(x,t) + x_{p,1} - r_2(t) - x_{p,2} = 0. \tag{2.236d}$$

The derivations steps to retrieve the EOCM in weak-strong form is a combined version of those performed in Sections 2.1.3 (from VP. 4 to Eq. 2.57) and 2.3.2 (from VP. 8 to Eq. 2.187).

First, the constrained Lagrangian is inserted into the action integral to retrieve

$$
\begin{aligned}
0 = \int_{t_0}^{t_1} \Big(&\int_{\Omega_1} \frac{1}{2}\rho(\dot{u}_{*,1} + \theta\dot{v}_1)^T(\dot{u}_{*,1} + \theta\dot{v}_1)\mathrm{d}x - \int_{\Omega_1} \frac{1}{2}\sigma(u_{*,1} + \theta v_1) : \epsilon(u_{*,1} + \theta v_1)\mathrm{d}x \\
&+ \int_{\Omega_1} (u_{*,1} + \theta v_1)^T\beta\mathrm{d}x + \int_{\Omega_2} \frac{1}{2}\rho(\dot{r}_{*,2} + \theta\dot{z}_2)^T(\dot{r}_{*,2} + \theta\dot{z}_2)\mathrm{d}x + \int_{\Omega_2} (r_{*,2} + \theta z_2)^T\beta\mathrm{d}x \\
&- \int_{\Gamma_{C,1}} g(u_{*,1} + \theta v, r_2 + \theta z_2)^T(\lambda_* + \theta\vartheta)\mathrm{d}s\Big)\mathrm{d}t, \quad \forall(\theta v_1, \theta z_2, \theta\vartheta) \in (V_{\theta v}, V_{\theta z}, V_{\theta\vartheta}),
\end{aligned}
\tag{2.237}
$$

Thereafter, applying the stationarity condition gives

$$
\begin{aligned}
0 = \int_{t_0}^{t_1} \Big(&\int_{\Omega_1} \rho\dot{v}_1^T\dot{u}_{*,1}\mathrm{d}x - \int_{\Omega_1} \sigma(u_{*,1}) : \epsilon(v_1)\mathrm{d}x + \int_{\Omega_1} v_1^T\beta\mathrm{d}x + \int_{\Omega_2} \rho\dot{z}_2^T\dot{r}_{*,2}\mathrm{d}x \\
&+ \int_{\Omega_2} z_2^T\beta\mathrm{d}x - \int_{\Gamma_{C,1}} v_1^T G_{u_{*,1}}^T\lambda_*\mathrm{d}s - \int_{\Gamma_{C,1}} z_2^T G_{r_{*,2}}^T\lambda_*\mathrm{d}s\Big)\mathrm{d}t, \quad \forall(v_1, z_2) \in (V_v, V_z),
\end{aligned}
\tag{2.238a}
$$

$$
0 = \int_{t_0}^{t_1} \int_{\Gamma_{C,1}} \vartheta^T g(u_{*,1}, r_{*,2})\mathrm{d}s\mathrm{d}t, \quad \forall\vartheta \in V_\vartheta,
\tag{2.238b}
$$

where $G_{u_{*,1}} = \frac{\mathrm{d}g(\tilde{u}_1)}{\mathrm{d}\tilde{u}_1}|_{\theta=0}$ and $G_{r_{*,2}} = \frac{\mathrm{d}g(\tilde{r}_2)}{\mathrm{d}\tilde{r}_2}|_{\theta=0}$. Since $v_1(\cdot, t_0) = v_1(\cdot, t_1) = 0$ and $z_2(t_0) = z_2(t_1) = 0$, performing integration by parts on the inertia terms, with respect to time, yields

$$
\begin{aligned}
0 = \int_{t_0}^{t_1} \Big(&\int_{\Omega_1} \rho v_1^T\ddot{u}_{*,1}\mathrm{d}x + \int_{\Omega_1} \sigma(u_{*,1}) : \epsilon(v_1)\mathrm{d}x - \int_{\Omega_1} v_1^T\beta\mathrm{d}x + \int_{\Omega_2} \rho z_2^T\ddot{r}_{*,2}\mathrm{d}x \\
&- \int_{\Omega_2} z_2^T\beta\mathrm{d}x + \int_{\Gamma_{C,1}} v_1^T G_{u_{*,1}}^T\lambda_*\mathrm{d}s + \int_{\Gamma_{C,1}} z_2^T G_{r_{*,2}}^T\lambda_*\mathrm{d}s\Big)\mathrm{d}t, \quad \forall(v_1, z_2) \in (V_v, V_z),
\end{aligned}
\tag{2.239a}
$$

$$
0 = \int_{t_0}^{t_1} \int_{\Gamma_{C,1}} \vartheta^T g(u_{*,1}, r_{*,2})\mathrm{d}s\mathrm{d}t, \quad \forall\vartheta \in V_\vartheta,
\tag{2.239b}
$$

To be able to apply the fundamental lemma of calculus of variations in time, v_1 and ϑ are separated in space and time,

$$
\begin{aligned}
0 = \int_{t_0}^{t_1} v_{t,1}^T\Big(&\int_{\Omega_1} \rho v_{x,1}^T\ddot{u}_*\mathrm{d}x + \int_{\Omega_1} \sigma(u_*) : \epsilon(v_{x,1})\mathrm{d}x - \int_{\Omega_1} v_{x,1}^T\beta\mathrm{d}x + \int_{\Gamma_C} v_{x,1}^T G_{u_{*,1}}^T\lambda_*\mathrm{d}s\Big)\mathrm{d}t \\
&+ \int_{t_0}^{t_1} z_{t,2}^T\Big(\int_{\Omega_2} \rho\ddot{r}_{*,2}\mathrm{d}x - \int_{\Omega_2} \beta\mathrm{d}x + \int_{\Gamma_{C,1}} G_{r_{*,2}}^T\lambda_*\mathrm{d}s\Big)\mathrm{d}t, \quad \forall(v_{x,1}, v_{t,1}, z_2) \in (V_{v_x}, V_{v_t}, V_z),
\end{aligned}
\tag{2.240a}
$$

$$
0 = \int_{t_0}^{t_1} \vartheta_t^T \int_{\Gamma_C} \vartheta_x^T g(u_*)\mathrm{d}s\mathrm{d}t, \quad \forall(\vartheta_x, \vartheta_t) \in (V_{\vartheta_x}, V_{\vartheta_t}),
\tag{2.240b}
$$

Since $v_{t,1}$, z_2, and ϑ_t belongs to $C_1([t_0, t_1])$ and the remaining time-dependent functions belong to $C([t_0, t_1])$, the prerequisites for the fundamental lemma of calculus of variations in time, Lemma 1, are met. Applying the lemma, for the three time-integrals, generates the weak-strong formulation of the EOCM,

$$\int_{\Omega_1} \rho v_{x,1}^T \ddot{u}_{*,1} \mathrm{d}x + \int_{\Omega_1} \sigma(u_{*,1}) : \epsilon(v_{x,1}) \mathrm{d}x + \int_{\Gamma_{\mathrm{C},1}} v_{x,1}^T G_{u_{*,1}}^T \lambda_* \mathrm{d}s = \int_{\Omega_1} v_{x,1}^T \beta \mathrm{d}x, \forall v_{x,1} \in V_{v_x}, \tag{2.241a}$$

$$\int_{\Omega_2} \rho \ddot{r}_{*,2} \mathrm{d}x + \int_{\Gamma_{\mathrm{C},1}} G_{r_{*,2}}^T \lambda_* \mathrm{d}s = \int_{\Omega_2} \beta \mathrm{d}x, \tag{2.241b}$$

$$\int_{\Gamma_{\mathrm{C},1}} \vartheta_x^T g(u_{*,1}, r_{*,2}) \mathrm{d}s = 0, \quad \forall \vartheta_x \in V_{\vartheta_x}. \tag{2.241c}$$

With the constraint equation in Eq. 2.236d, $G_{u_{*,1}} = 1$ and $G_{r_{*,2}} = -1$. Moreover, Eq. 2.241c is rewritten as

$$\int_{\Gamma_{\mathrm{C},1}} \vartheta_x^T u_{*,1} \mathrm{d}s - \int_{\Gamma_{\mathrm{C},1}} \vartheta_x^T r_{*,2} \mathrm{d}s = \int_{\Gamma_{\mathrm{C},1}} \vartheta_x^T (x_{p,2} - x_{p,1}) \mathrm{d}s, \quad \forall \vartheta_x \in V_{\vartheta_x}. \tag{2.242}$$

The right-hand side integral is a linear functional which is compactly denoted as

$$\langle k, \vartheta_x \rangle := \int_{\Gamma_{\mathrm{C},1}} \vartheta_x^T (r_{*,2} + x_{p,2} - x_{p,1}) \mathrm{d}s. \tag{2.243}$$

By denoting $m_2 := \int_{\Omega_2} \rho \mathrm{d}x$, $f_2 := \int_{\Omega_2} \beta \mathrm{d}x$, and recalling the compact notations from Sections 2.3.3 and 2.4.3, Eq. 2.241 is rewritten as

$$m_1(\ddot{u}_{*,1}, v_{x,1}) + a_1(u_{*,1}, v_{x,1}) + b_1^T(\lambda_*, v_{x,1}) = \langle \beta, v_{x,1} \rangle, \qquad \forall v_{x,1} \in V_{v_x}, \tag{2.244a}$$

$$m_2 \ddot{r}_{*,2} - \int_{\Gamma_{\mathrm{C},1}} \lambda_* \mathrm{d}s = f_2, \tag{2.244b}$$

$$b(u_{*,1}, \vartheta_x) - \int_{\Gamma_{\mathrm{C},1}} \vartheta_x^T r_{*,2} \mathrm{d}s = \langle k, \vartheta_x \rangle, \qquad \forall \vartheta_x \in V_{\vartheta_x}. \tag{2.244c}$$

Initial value problem 3 (Two-body system)**.** *For consistent initial conditions,* $\big(u_{*,1}(x,t_0), \dot{u}_{*,1}(x,t_0), r_{*,2}, \dot{r}_{*,2}, \lambda$ $(u_0, \dot{u}_0, r_0, \dot{r}_0, \lambda_0) \in \big(H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega), H_0^1(\Omega), \cdot, \cdot, L_2(\Gamma_{\mathrm{C}})\big)$*, find a solution path* $(u_{*,1}, r_{*,2}, \lambda_*) \in \Big(\big(H_0^1(\Omega) \cup H_0^{1/2}(\partial\Omega)\big) \otimes C_2([t_0, t_1], \mathbb{R}^2), C_2([t_0, t_1], \mathbb{R}^2), L_2(\Gamma_{\mathrm{C}}) \otimes C_2([t_0, t_1], \mathbb{R}^2)\Big)$*, such that Eq. 2.244 holds.*

With $G_{u_{*,1}} = 1$, λ_*, in Eq. 2.244a, is interpreted as a negative surface traction along Γ_{C} with respect to the elastic body. Moreover, with $G_{r_{*,2}} = -1$, $\int_{\Gamma_{\mathrm{C},1}} \lambda_* \mathrm{d}s$, in Eq. 2.244b, is interpreted as a force acting on the rigid body.

2.6 Finite Element Method

Finite element methods (FEM) are the conventional class of methods to discretize the EOM in weak-strong form, of an arbitrarily shaped elastic body. In the first subsection, the methodology is presented for conventional unconstrained dynamics for a single elastic body. In the second subsection, constrained dynamics is considered. The FE discretization gives rise to systems of ODEs and DAEs, for unconstrained and constrained dynamics, respectively. In the next section, time-integrators are discussed to discretize the systems of differential equations in time to retrieve corresponding systems of difference equations. For a thorough introduction to FEM, see an introductory textbook, eg. [9].

2.6.1 Conventional Unconstrained Dynamics

The domain, $\bar{\Omega}$, is discretized, by covering it by small elements Ω^e, with node points at all element vertices. The spatial resolution of the grid is denoted by the subscript h. The displacement over each element is modeled by the EOM in weak-strong form. The homogeneous displacement field over an element $u_*^e(x,t) \in V_{u_*}(\Omega^e)$ is approximated, by a linear combination of discrete displacements $u_*^h(t)$ at the adjacent node points, times local basis functions $N^e(x)$ for $x \in L_2(\Omega)$,

$$u_*^e(x,t) \doteq \sum_{i=1}^{n_h^e} N_i^e(x) u_{*,i}^h(t), \tag{2.245}$$

where n_h^e denotes the number of adjacent vertices. The test functions and applied volume force densities over the elements, v_x^e and β^e, are approximated likewise. τ^e, over the surface elements, is correspondingly approximated, with local basis functions $N_{\partial\Omega}^e(x)$ for $x \in L_2(\Gamma_\mathrm{N})$

$$\tau^e(x,t) \doteq \sum_{i=1}^{n_h^e} N_{\partial\Omega,i}^e(x) \tau_i^h(t), \tag{2.246}$$

The local basis functions are typically polynomials. The Lagrange polynomials are a conventional choice, for details see [9]. The local basis functions are extended to $\bar{\Omega}$ by being set to zero over non-adjacent elements. By the extension, the discretization process over all elements to all vertices can be assembled. Thereby, the EOM in weak-strong form, Eq. 2.150, over the whole domain gets discretized to,

$$v_x^{h,T}\big(M_h \ddot{u}_*^h(t) + A_h u_*^h(t) - \bar{\beta}_h - \bar{\tau}_h\big) = 0, \quad \forall v_x^h \in H_{0,h}^1(\Omega) \cup H_{0,h}^{1/2}(\partial\Omega), \tag{2.247}$$

where $M_h \in \mathbb{R}^{n_{q_h} \times n_{q_h}}$, $A_h \in \mathbb{R}^{n_{q_h} \times n_{q_h}}$, $\bar{\beta}_h \in \mathbb{R}^{n_{q_h}}$, and $\bar{\tau}_h \in \mathbb{R}^{n_{q_h}}$ denote the mass matrix, the stiffness matrix, applied volume forces, and applied surface forces, respectively. n_{q_h} denotes the degrees of freedom of the discretized system. The vertices along the Dirichlet boundary segments are not included, since the displacement field is known to vanish there. The matrices and vectors are expressed as

$$M_h := \sum_{e=1}^{n_e} \sum_{i=1}^{n_h^e} \int_{\Omega^e} \rho N_i^{e,T} N_i^e \mathrm{d}x, \tag{2.248a}$$

$$A_h := \sum_{e=1}^{n_e} \sum_{i=1}^{n_h^e} \int_{\Omega^e} B_i^{e,T} C B_i^e \mathrm{d}x, \tag{2.248b}$$

$$\bar{\beta}_h := \sum_{e=1}^{n_e} \sum_{i=1}^{n_h^e} \int_{\Omega^e} N_i^{e,T} N_i^{e,} \beta_h \mathrm{d}x, \tag{2.248c}$$

$$\bar{\tau}_h := \sum_{e=1}^{n_e} \sum_{i=1}^{n_h^e} \int_{\Gamma_\mathrm{N}^e} N_{\partial\Omega,i}^{e,T} N_{\partial\Omega,i}^e \tau_h \mathrm{d}s, \tag{2.248d}$$

where n_e denotes the number of elements. C denotes the stiffness tensor defined in Eq. 2.132. B^e denotes a the strain-displacement matrix which maps $u_*^h(t)$ to $\underline{\epsilon}(u_*^e)$, such that

$$\underline{\epsilon}(u_*^e) \doteq \sum_{i=1}^{n_h^e} B_i^e(x) u_{*,i}^h(t), \tag{2.249}$$

where $\underline{\epsilon}$ was defined in Eq. 2.131. B^e is retrieved by performing a gradient-like operation to N^e. For a proper definition of B^e, see [9].

The specific test function $\hat{v}_{x,i}^h$ which is one at vertex i and zero at the other vertices can be constructed by discretization of a specific smooth function. Therefore, it belongs to $H_{0,h}^1(\Omega) \cup H_{0,h}^{1/2}(\partial\Omega)$. Since Eq. 2.247 must hold for $\hat{v}_{x,i}^h$ for all vertices i which does not lie in Γ_D,

$$M_h \ddot{u}_*^h(t) + A_h u_*^h(t) = \bar{\beta}_h(t) + \bar{\tau}_h(t), \tag{2.250}$$

follows. Eq. 2.250 is a linear system of second-order ODEs in time. In the following section, time-integrating schemes to discretize systems of ODEs to systems of difference equations are discussed. IVPs based on systems of difference equations can be solved numerically. In the numerical experiments in this study, triangle elements with first-order Lagrange polynomials for N^e are employed. The first-order Lagrange polynomials are continuous within the elements, making B^e (related to N^e by a gradient-like operator) constant within elements. Displacement fields cannot be modeled with zeroth-order Lagrange polynomials, since then stiffness matrix would be zero regardless of u_*^h, since B^e would be zero. Triangle elements with first-order Lagrange polynomials are referred to as 3-node elements, or linear triangle elements in engineering literature.

In this study, the deformations φ, Eqs. 2.125 and 2.198, were decomposed to retrieve homogeneous displacement fields. Eq. 2.247 only holds for a homogeneous displacements field. If a non-homogeneous discretized displacement field w_* would be considered, the first step would be to retrieve a homogeneous discretized displacement field as

$$w_* - \sum_{e=1}^{n_e} \sum_{i=1}^{n_h^e} \int_{\Gamma_\mathrm{D}^e} N_{\partial\Omega,i}^e u_{*,i}^h \mathrm{d}s, \tag{2.251}$$

which would thereafter be discretized according to Eq. 2.247.

2.6.2 Extension to Constrained Dynamics

For constrained dynamics with the Lagrange multiplier technique, the Lagrange multipliers over the surface elements $\lambda_*^e(x,t)$ are approximated by discrete Lagrange multipliers at the adjacent vertices $\lambda_{*,i}^h(t)$, times the local basis function $N_{\partial\Omega}^e(x)$ for $x \in L_2(\Gamma_\mathrm{C})$

$$\lambda_*^e(x,t) \doteq \sum_{i=1}^{n_h^e} N_{\partial\Omega,i}^e(x) \lambda_{*,i}^h(t). \tag{2.252}$$

Since no spatial derivatives of $\lambda_{*,i}$ are included in Eq. 2.169, it is possible to employ zeroth-order Lagrangian polynomials for the local basis function in Eq. 2.252. However, in this study, first order Lagrange polynomials were used.

Consider the EOCM in weak-strong form, Eq. 2.191, for a constrained elastic body with constraint equation $g(u_*) = u_*$. Applying FE discretization to Eq. 2.191 gives

$$v_x^{h,T}(M_h \ddot{u}_*^h + A_h u_*^h + B_h^T \lambda_*^h - \bar{\beta}_h - \bar{\tau}_h) = 0, \qquad \forall v_x^h \in H_{0,h}^1(\Omega) \cup H_{0,h}^{1/2}(\partial\Omega), \tag{2.253a}$$

$$\vartheta_x^{h,T} B_h u_*^h = 0, \qquad \forall \vartheta_x^h \in H_{0,h}^{1/2}(\Gamma_\mathrm{C})^*, \tag{2.253b}$$

with matrix $B_h \in \mathbb{R}^{n_{\lambda_h} \times n_q}$ defined as

$$B_h := \sum_{e=1}^{n_e} \sum_{i=1}^{n_h^e} \int_{\Gamma_\mathrm{C}^e} N_{\partial\Omega,i}^{e,T} N_i^e \mathrm{d}x, \tag{2.254}$$

and where n_{λ_h} denotes the degrees of freedom for vertices along Γ_C. Recall from the previous subsection, the specific test function $\hat{v}^h_{x,i}$ which is one at vertex i, which does not lie in Γ_D, and zero at the other vertices. Consider a corresponding $\hat{\vartheta}^h_{x,i}$. By requiring Eq. 2.253 to hold for $\hat{v}^h_{x,i}$ and $\hat{\vartheta}^h_{x,i}$ for all vertices in $\bar{\Omega}\backslash\Gamma_\text{D}$ and Γ_C respectively, a linear system of second-order DAEs is retrieved,

$$\begin{pmatrix} M_h \ddot{u}^h_*(t) \\ 0 \end{pmatrix} + \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u^h_*(t) \\ \lambda^h_*(t) \end{pmatrix} = \begin{pmatrix} \bar{\beta}_h + \bar{\tau}_h \\ 0 \end{pmatrix}. \tag{2.255}$$

As was shown in Section 2.1.4, Eq. 2.255, after being rewritten as a first order system, belongs to the characteristically stiff systems of index-3 DAEs. A scaling technique is introduced to reduce the stiffness of Eq. 2.255. In the elasticity model, the difference in orders of magnitude between stresses and strains are on the order of Young's modulus, which typically varies between $10^8 - 10^{11}$. The aim of the scaling is to reduce, the difference in order of magnitude, between u^h_* and λ^h_*. To retain the symmetry of the problem, the scaling is applied in a symmetrical way,

$$T \begin{pmatrix} M_h \ddot{u}^h_*(t) \\ 0 \end{pmatrix} + T \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} T \begin{pmatrix} u^h_*(t) \\ \bar{\lambda}^h_*(t) \end{pmatrix} = T \begin{pmatrix} \bar{\beta}_h + \bar{\tau}_h \\ 0 \end{pmatrix}, \tag{2.256}$$

where I and c_scale represent an identity matrix and a scaling coefficient. $\bar{\lambda}^h_*(t)$ is defined as

$$\bar{\lambda}^h_*(t) := \frac{\lambda^h_*(t)}{c_\text{scale}}. \tag{2.257}$$

2.7 Time-Integration Methods

Constrained dynamics is, after FE discretization, modeled as a system of second order DAEs. Time-integration methods, discretize systems of ODEs in time, to generate systems of difference equations. After discretization in time, the combined difference and algebraic equations are recursively updated in time by solving linear systems, if the system of difference and algebraic equations is linear. Otherwise, Newton's method is employed for each time-update.

As argued in Section 2.1.3, implicit time-integrators should be employed for systems of DAEs. There are three main families of implicit time-integrators, the backwards differentiation formula (BDF) methods, Adams-Moulton methods, and the implicit Runge-Kutta methods [10]. In [1, Sect. 7.2], it is argued that implicit Runge-Kutta methods are preferable for the constrained flexible multibody dynamics. However, due to simplicity, BDF methods are considered in this study. Noteworthy, the time-integrators, are based on systems of first order ODEs. Thus, the systems of second order DAEs are rewritten as first order systems, see Eq. 2.59.

In the first subsection, BDF methods are presented. In the second subsection, IDA, a state-of-the-art time-integrator for system of DAEs, is introduced. In the last subsection, Newmark's method is presented, to solve linear systems of first order ODEs.

2.7.1 Backwards Differentiation Formula

The BDF methods are a family of implicit multistep methods. For implicit multistep methods, the time-update is dependent on the current derivative $f(t_{n+k}, y_{n+k})$, as well as the state variables of the current and previous time steps y_{n+i} $(i \in [0, k])$,

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\beta f(t_{n+k}, y_{n+k}), \tag{2.258}$$

where h is time step length and α_i and β are constants, with $\alpha_k \neq 0$. A numerical method is consistent if the property

$$\lim_{h \to 0} \frac{\tau_{\text{error}}}{h} = 0 \tag{2.259}$$

holds, where τ_{error} is the local truncation error. For each k (number of previous time steps), there exists a specific combination of constants α_i and β, such that the BDF method is a consistent method. Moreover, the order of the method, the convergence order of the global error, is k [10]. The consistent BDF methods for one and two previous time steps, BDF-1 and BDF-2, are

$$y_{n+1} - y_n = hf(t_{n+1}, y_{n+1}) \tag{2.260a}$$

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(t_{n+2}, y_{n+2}) \tag{2.260b}$$

BDF-1 is the implicit Euler method. A linear multistep method is called zero-stable if the numerical solution, with test function $f \equiv 0$, is bounded for all time steps. According to Dahlquist Equivalence Theorem, for a consistent linear multistep method, zero-stability is a necessary and sufficient condition for convergence of the system of ODEs [10]. Consistent BDF methods of order $k \leq 6$ are zero-stable [10].

A system is said to be stiff in an interval, if, for any explicit time-integrator, a time step length, which is excessively small in relation to the smoothness of the exact solution within the same interval, is needed to maintain stability. Thus, for stiff systems, consistent BDF methods which permits relatively large time steps are sought. A good indicator is A-stability. To analyze A-stability, Dahlquist's test equation $\dot{y} = \lambda y$, where $\lambda \in \mathbb{C}^-$, $\mathbb{C}^-$ is the set of eigenvalues of the system with non-positive real parts, is set up. Insertion of the test equation into Eq. 2.258 gives

$$\sum_{i=0}^{k} \alpha_i y_{n+i} - h\beta\lambda y_{n+k} = 0, \tag{2.261}$$

with k roots $\xi_1, ..., \xi_k$. The stability region of the BDF methods is defined by the set $\mathcal{S} = \{\mu \in \mathbb{C} : |\xi_i(\mu)| \leq 1, i \in [1, k]\}$, with roots lying on the unit circle being simple. If $\mathbb{C}^- \subset \mathcal{S}$, the method is A-stable. An A-stable method generates, for Dahlquist's test equation with a fixed λ, bounded solutions for arbitrarily large step sizes. Consistent BDF methods are A-stable up to order two [10].

2.7.2 IDA

Implicit Differential-Algebraic (IDA) solver is a state-of-the-art time-integrator, from SUNDIALS, for systems of DAEs [11]. IDA solves non-linear problems formulated as fully implicit systems, $F(q, \dot{q}, t) = 0$, by use of simplified Newton's method at each time step. IDA is based on adaptive step size, adaptive order consistent BDF methods. The included orders for the consistent BDF methods are one to five. IDA is available in Python through the module Assimulo [12]. A thorough explanation of the concepts used by IDA is out of scope for this study. In depth introduction to simulation strategies for systems of DAEs is presented in [13]. It includes explanations about step size and order selection strategies adopted by IDA and DASSL (an alternative state-of-the-art time-integrator for systems of DAEs).

The step size control is based on an error estimate retrieved from a predictor/corrector approach over all state variables [13]. For systems of DAEs, it is advisable to exclude the constraint equations, which are in general unpredictable from the previous time steps, from the predictor/corrector evaluations. Otherwise, there is a risk, of that the step size controller deems that the step size repeatedly should be reduced until a minimum step size length is retrieved, and the program crashes. Recall that BDF methods are based on solving first order systems. Thus, some equations are related to the additional velocity variables, Eq. 2.59. For systems of index-3 DAEs, due to the presence of hidden constraints at velocity and acceleration level (see Section 2.1.4), it is advisable to also exclude equations related to the velocity variables from the predictor/corrector evaluations as well. The exclusion of specified equations is done by the *suppress_alg* command in Assimulo (and equivalently with the *IDASetSuppressAlg* command in SUNDIALS).

2.7.3 Newmark's Method

Newmark's method is a time-integrator for solving linear or nonlinear systems of second order ODEs. In this study, Newmark's method for solving linear systems of second order ODEs is considered. At each iterate, the updated accelerations $\ddot{u}_{n+1}$ is first retrieved by solving a linear system. Thereafter, $\dot{u}_{n+1}$ and u_{n+1} are explicitly updated.

To derive the method, u_{n+1} and $\dot{u}_{n+1}$ are expressed as

$$u_{n+1} = u_n + \int_{t_n}^{t_{n+1}} \dot{u}(s)\mathrm{d}s = u_n + hu_n + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)\ddot{u}(s)\mathrm{d}s, \tag{2.262a}$$

$$\dot{u}_{n+1} = \dot{u}_n + \int_{t_n}^{t_{n+1}} \ddot{u}(s)\mathrm{d}s. \tag{2.262b}$$

In the derivation step in Eq. 2.262a, integration by parts was performed, after inserting the derivative of $s - t_{n+1}$, into the integral. h denotes the time step length. Thereafter, the integrals are approximated by weighting them between the current and updated acceleration,

$$\int_{t_n}^{t_{n+1}} (t_{n+1} - s)\ddot{u}(s)\mathrm{d}s \approx (1-\gamma)h\ddot{u}_n + \gamma h\ddot{u}_{n+1}, \tag{2.263a}$$

$$\int_{t_n}^{t_{n+1}} \ddot{u}(s)\mathrm{d}s \approx (\frac{1}{2} - \beta)h^2\ddot{u}_n + \beta h^2\ddot{u}_{n+1}, \tag{2.263b}$$

with the introduced constants $0 < \gamma < 1$ and $0 < \beta < \frac{1}{2}$. The algorithm is given by inserting Eq. 2.263 into Eq. 2.262, and thereafter the retrieved result, into the linear system of second order ODEs. For a detailed description, see eg. [14]. In this study, the parameters were set at $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, which generates a stable method without dissipation [14].

Chapter 3

FEniCS Implementation

FEniCS is a free and open-source software for solving PDEs with finite element techniques. FEniCS is retrieved either as a Python module or as a C++ package. The aim is to provide the user with automatic mesh generation, assembling, and solving processes, letting the user focus on setting up the weak formulation. In the first subsection, the implementation structure for the unconstrained dynamics for a single elastic body is described. It represents a typical program structure in FEniCS. In the following subsection, obstacles encountered, when extending the framework to constrained dynamics, is presented.

3.1 Unconstrained Elastic Body Dynamics

First, a triangular mesh pattern is retrieved, after the user has specified the geometry, possibly through a CAD file. As of FEniCS v.2017.2, only triangular elements are implemented. Thereafter, discrete function spaces for test and trial functions are defined, in our case as Lagrange polynomials. Then, Dirichlet boundary conditions are provided along boundaries where the displacement field is known.

For time-integrators based on solving systems of second order ODEs, such as Newmark's algorithm, the system of difference equations, which follows from discretization of a system of differential equations in time, can be provided in FEniCS variational format. Then, by successively updating the state variables and calling FEniCS internal solver, time-stepping can be preformed in a few lines, both for linear and nonlinear systems.

However, most time-integrators, such as BDF methods, are based on solving first order ODEs. In this study, rewriting the problem to a first order system was deemed easier to do outside of FEniCS variational format. Therefore, FEniCS were used to assemble the mass and stiffness matrix, and the right-hand side vectors in Eq. 2.250. For time-stepping, both BDF-1 and BDF-2, as well as the IDA solver, were employed. Note, with this approach, the condensation of the stiffness matrix (moving contributions from Dirichlet boundaries from the stiffness matrix to the right-hand side) had to be implemented by the user.

3.2 Constrained Elastic Multibody Dynamics

From an implementation in FEniCS point of view, the biggest obstacle, for extension to constrained dynamics, is that the internal assembler (as of version 2017.2) cannot handle function spaces defined solely along a boundary segment, as is the case for the Lagrange multipliers λ, and the corresponding test functions ϑ_x. This restriction is known to the software developers. The software is under development to handle this issue.

In this study, a work-around was implemented to circumvent this problem. The trial and test function spaces for the Lagrange multipliers were defined over the full domain $\bar{\Omega}$, just as for u and v_x. Due to the extension, the assembled vector and matrices included degrees of freedoms corresponding to Lagrange multipliers on $\bar{\Omega}\backslash\Gamma_{\mathrm{C}}$. The last step of the work-around is to reduce the vector and matrices of all rows and columns related to $\lambda \in \bar{\Omega}\backslash\Gamma_{\mathrm{C}}$ and $\vartheta \in \bar{\Omega}\backslash\Gamma_{\mathrm{C}}$. The same effect can be retrieved by treating $\lambda \in \bar{\Omega}\backslash\Gamma_{\mathrm{C}}$ as homogeneous Dirichlet boundary conditions.

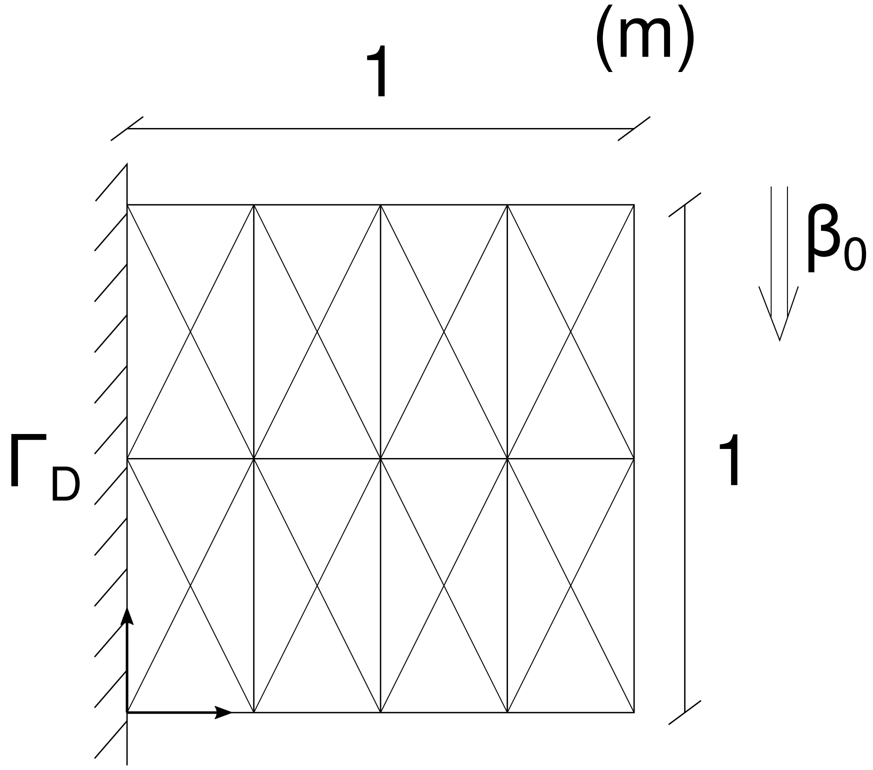

Figure 4.1: Initial configuration for the elastic block problem.

Chapter 4

Numerical Experiments

The numerical experiments constituted of two experiments. In the first test case, an elastic block attached to a rigid wall was modeled both with unconstrained and constrained dynamics. The aim with the first experiment was to validate the implementation of the unconstrained and constrained dynamics. In the second experiment, a rigid block was attached to the elastic block, through a rigid joint. The aim with the second experiment was to highlight the limitations for the usage of interconnecting rigid joints modeled by the weakly constrained joint model, pressented in Section 2.5.3.3.

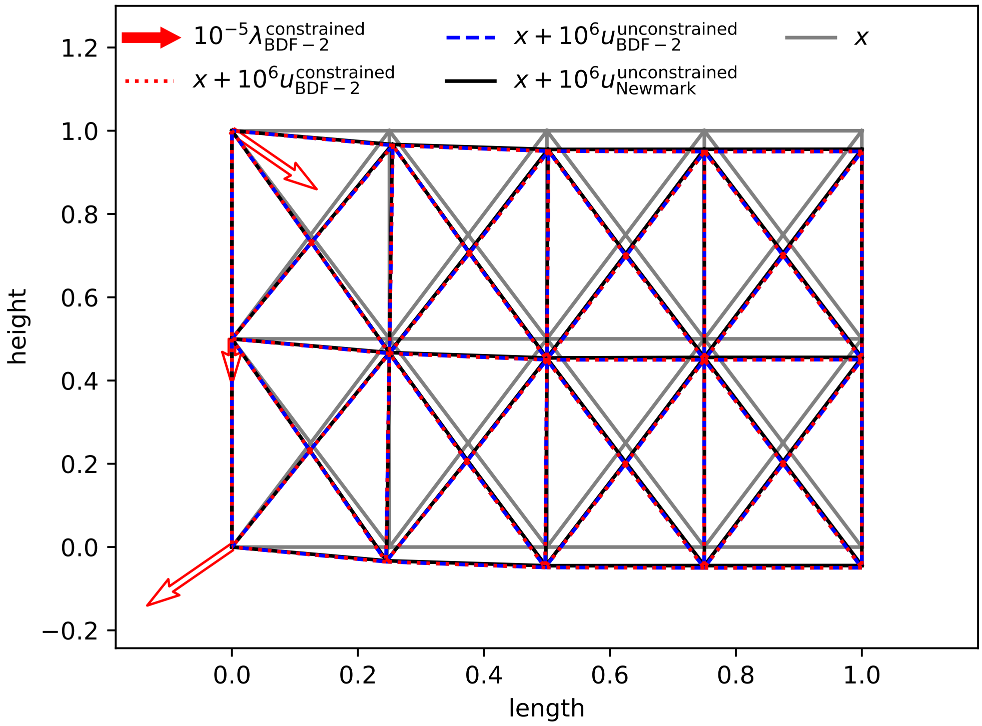

Figure 4.2: Configurations of the elastic block after 10^{-4}s, with step size 10^{-5}, for unconstrained and constrained dynamics.

4.1 Elastic Block

4.1.1 Problem Description

As a first test case, an $1\text{m} \times 1\text{m} \times 0.01\text{m}$ isotropic and homogeneous elastic block was considered. Since the depth of the block was one percent of its length and height, the elastic block was considered thin, and the planar stress model, Eq. 2.129, was applied in the length-height plane. The elastic block was attached to a rigid wall on its left side. The plane of the elastic block was covered by a mesh of 32 linear triangle elements. The inertial reference frame was placed at the lower left corner of the plane of the elastic block. The initial configuration, of the plane of the elastic block, is illustrated in the schematic Fig. 4.1. The elastic block was set to motion due to gravity. The gravitational acceleration with respect to the inertial reference frame β_0 was set to $(0, -9.82)\text{m/s}^2$. A zero surface force was applied on the body along the Neumann boundary segment.

The dynamics was modeled both as unconstrained and constrained elastic body dynamics, by incorporating the attachment to the rigid wall either as a homogeneous Dirichlet boundary condition, or by treating the attachment as a constrained boundary segment Γ_C, for which the constraint equation

$$g(u_*) = u_*, \quad u_* \in V_{u_*} \tag{4.1}$$

is forced to hold. The numerical solutions, for the unconstrained and constrained dynamics, were

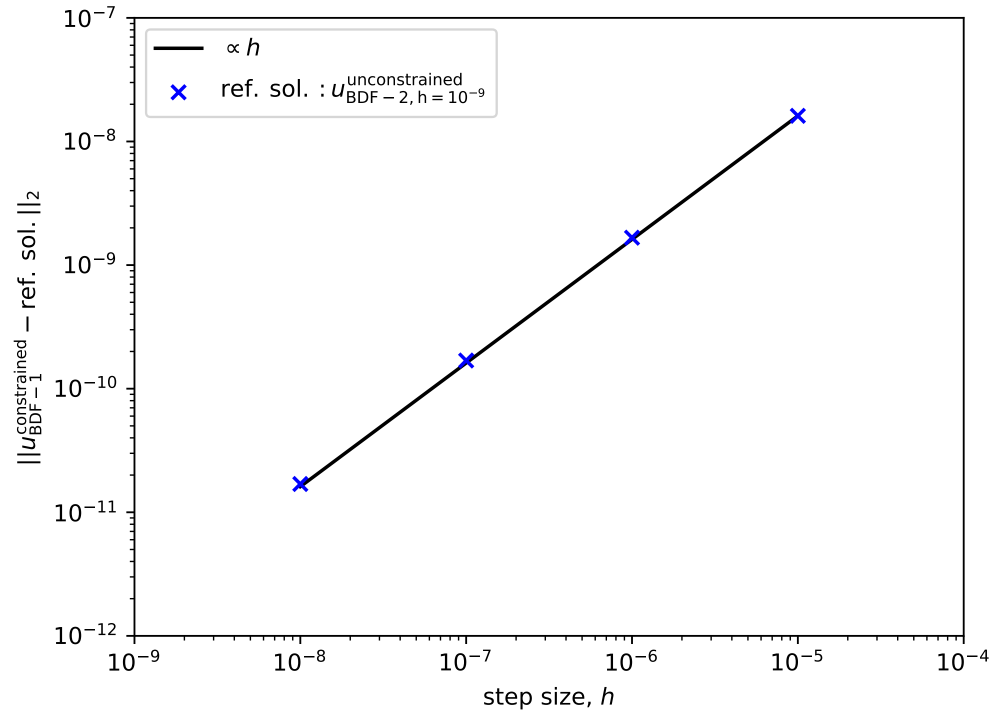

Figure 4.3: Global error convergence in time steps, after 10^{-4}s, for the constrained dynamics with a BDF-1 method, compared to a reference solver; unconstrained dynamics with a BDF-2 method with $h = 10^{-9}$.

retrieved by solving the IVPs generated from discretization, in both time and space, of IVPs 1 and 2, respectively.

For the unconstrained dynamics, FE discretization was performed on the EOUM in weak-strong form, in IVP 1. The discretization rendered in a linear system of second order ODEs in time, see Eq. 2.250. The system of ODEs was discretized in time by both BDF-2 and Newmark's method.

For the constrained dynamics, FE discretization was based on EOCM in weak-strong form in IVP 2. From the discretization, a linear system of second order DAEs arouse, see Eq. 2.255. The differential equations in the system of DAEs were discretized by BDF-1 and BDF-2.

Zero initial displacement $u_0 = 0$ and displacement velocity $\dot{u}_0 = 0$ fields were employed for the unconstrained dynamics. To retrieve consistent initial conditions for the constrained dynamics, the initial Lagrange multiplier field λ_0 was also set to zero.

Both the unconstrained and constrained dynamics were simulated for 10^{-4}s with step sizes ranging from $h = 10^{-5}$s to 10^{-9}s. Young's modulus E and density ρ were set to 180GPa and 2400kg/m^2, to resemble steel. Poisson's ratio υ was set to 0.3.

4.1.2 Results and Discussion

Fig. 4.2 shows the block configuration after 10^{-4}s, for step size $h = 10^{-5}$s, for unconstrained dynamics with BDF-2 and Newmark's method, as well as for constrained dynamics with BDF-

2. The displacement fields are similar for all cases. For the unconstrained and constrained dynamics simulated by BDF-2, the displacement fields were indistinguishable ($\|u_{\text{BDF}-2}^{\text{constrained}} - u_{\text{BDF}-2}^{\text{unconstrained}}\|_2 \sim 10^{-12}$). For the constrained dynamics, the Lagrange multipliers are included in the figure. Recall from Section 2.4.3, the Lagrange multipliers can be interpreted as negative surface traction along Γ_{C}. With this in mind, their orientations seem reasonable. Note, the difference in order of magnitude for displacements and Lagrange multipliers is on the same order of magnitude as E. Thus, also the lengths of the Lagrange multipliers seem reasonable.

A common way to validate an implementation is to check whether a theoretical convergence order is retrieved. BDF-1 is a first order method, meaning that the global error converges linearly with decreasing time step length h. Since BDF-2 is a second order method in time, the first order method cannot distinguish the solution retrieved from the BDF-2 method from the analytical solution. Fig. 4.3 shows the ℓ_2-norm of the differences in displacement field retrieved from, the constrained dynamics with time-integrator BDF-1, for step sizes $h = 10^{-5} - 10^{-8}$s, and, the unconstrained dynamics with time-integrator BDF-2 and $h = 10^{-9}$s, which was employed as a reference solver. As expected, linear convergence is seen in the figure. Together with the close resemblance between the configurations seen in Fig. 4.2, the convergence property indicates that the models have been implemented correctly.

4.2 Two-Body System with Weakly Constrained Joint Model

4.2.1 Problem Description

In the second test case, a rigid joint was attached to the elastic block, furthest down on its right side. The joint body composed of two equilateral triangles with bisection length 0.2m and 0.01m depth. One triangle was attached to the elastic block. The other was hanging down, with a 1m × 1m × 0.01m rigid block attached underneath it. A plane of the structure is illustrated in Fig. 4.4. The mesh of the elastic block was refined to 50 linear triangle elements, so that the rigid joint-elastic body interface $\Gamma_{\text{C},1}$ was covered by a single element facet. The inertial reference frame was kept at the lower left corner of the elastic block. The rotational motion of the rigid block was not included in the model, to retain linearity for the two-body system, see Section 2.5.5. The attachment of rigid wall, to the left side of the elastic block, was treated as a homogeneous Dirichlet boundary condition. The joint was approximated as massless, and modeled by the weakly constrained joint model, Section 2.5.3.3. The constrained dynamics was modeled both with BDF-2 and IDA as time-integrators. The physical parameters for the elastic block were not changed. The density of the rigid block was also set to 2400kg/m^2. A 10^{-3}s simulation was performed with step size 10^{-6}.

4.2.2 Results and Discussion

Fig. 4.5 shows the configuration of the structure after 10^{-4}s simulation with time-integrator BDF-2. The bent shape of the elastic block indicates that its displacements are mainly caused by the load from the rigid block. This is expected as the two blocks have the same weight. The bent shape is in clear contrast to the shape of the single elastic body, as shown in Fig. 4.2.

Even though the unconstrained parts of the boundary on the right side of the elastic block are tilted, the joint-body interface is vertical. The vertical shape of the joint-body interface is explained by that the shape of the interface is unaffected by the displacement field of the elastic body, $\xi(x)$ for $x \in \Gamma_{\text{C},1}$ from Eq. 2.228. To how large extent this orientation distortion affects the displacement field, over the elastic block, is difficult to estimate. However, it can be assumed that the structural analysis close to the interface is invalid.

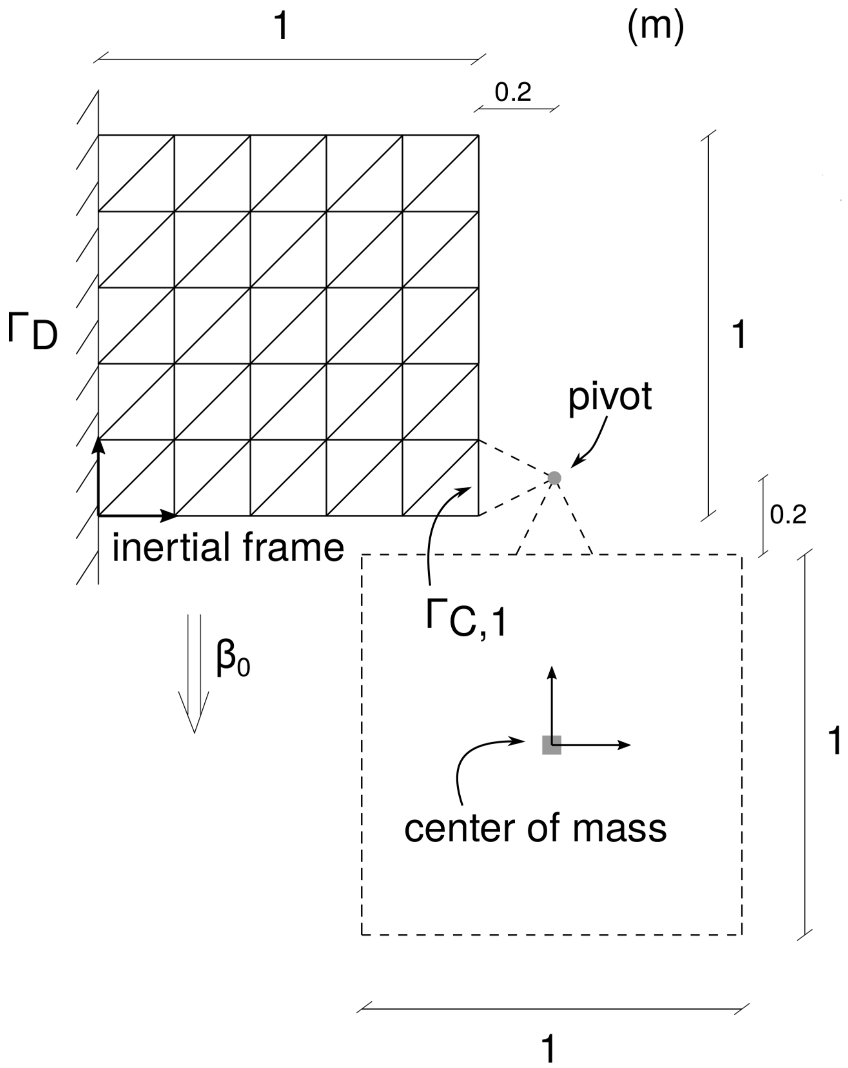

Figure 4.4: Initial configuration of two-body system.

Moreover, the Lagrange multipliers along the interface are dominated by a rotational component, which stems from that the joint-body interface is forced to be vertical. Since the constraints on the interconnected rigid block is transmitted through the Lagrange multipliers, artificially large Lagrange multipliers can also affect the motion of the interconnected rigid body.

Altogether, the joint model seems to distort the dynamics of the interconnected bodies if the deformation includes a rotation of the joint-body interface. The magnitude of the distortions are, in general, difficult to estimate, because it is very problem-dependent. However, the assumption that ξ can be modeled with respect to the reference domain strictly restricts the validity of the model.

To retrieve more accurate results, the weakly constrained joint model could be updated to predict the orientation of the joint-body interface. One alternative is to use observer points, just as for the joint model based on joint coordinate systems. Another alternative is to require that

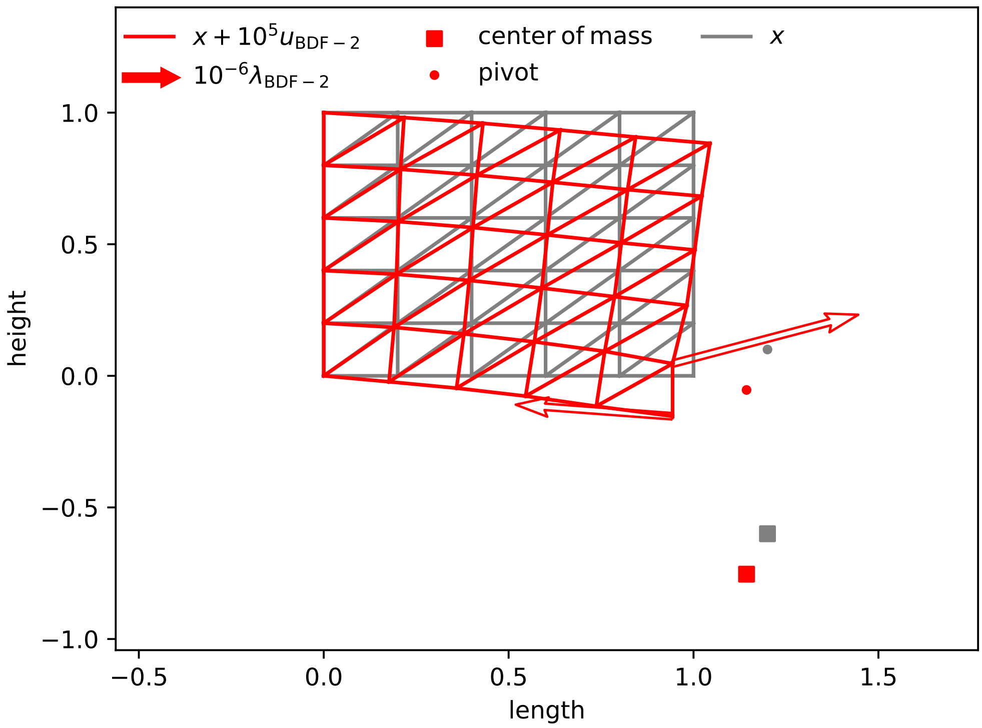

Figure 4.5: Configurations of the two-body system, after 10^{-4}s, with step size 10^{-5}, for unconstrained and constrained dynamics.

the rotational components of the Lagrange multipliers (interpreted as stresses) on average are small enough. The small enough requirement would have to be incorporated as a criteria. The former of the two seems more promising for future studies.

This test case was also tried to solve with IDA. However, with the system formulated as a system of index-3 DAEs, the step size control decreased towards zero until the program crashed. For a similar problem, with the joint covering the whole right side of the elastic block, the simulations succeeded and gave similar results to BDF-2. Index reduction was not performed in this study.

Chapter 5

Summary and Conclusions

5.1 Summary

Many mechanical systems can be treated as multibody systems. Traditionally, rigid multibody dynamics has mainly been considered, since the deformation of the system, rather than the deformation of the individual bodies, are of main interest. In recent years, due to an increased interest in light-weight and high-precision mechanics, flexible multibody dynamics has become more and more in demand. Flexible multibody systems consist of elastic and rigid bodies, interconnected with force elements and joints.

In the theory chapter, Newton-Euler's equations of motion for rigid body dynamics was derived. Thereafter, a weak formulation in space for elastic body dynamics was derived, which is required for performing finite element discretization. The presence of joints constrains the motion of the connected bodies. Therefore, constrained rigid and elastic dynamics were derived, by use of a Lagrange multiplier technique. Finally, the flexible multibody dynamics were retrieved.

In this study, massless rigid joint models for interconnecting elastic and rigid bodies were considered. At the rigid joint-elastic body interface, continuous and discrete deformation models, of the elastic body and the rigid joint, meet. Finding a satisfactory joint model for general usage is still an open topic [1].

In [1], a weakly constrained joint model was presented, for which both the elastic displacements and the rigid body motions are well-defined. Moreover, the model was applied for two numerical experiments with promising results. However, for those examples the elastic and rigid bodies were connected directly, without any physical joint.

To apply the joint model in practice, it is crucial to understand its limitations, to avoid misuse. Therefore, the aim of this study was to highlight the limitations of the weakly constrained joint model, in presence of physical joints. A drawback with the joint model is that it assumes that the orientation of the rigid joint-elastic body interface is unaffected by the displacement field of the elastic body. This assumption distorts the displacement field, and thereby the structural analysis, if the true displacement field, without the assumption, would render in a rotated interface.

The numerical study consisted of two experiments. The objective of the first experiment was to validate the implementation of the constrained and unconstrained dynamics. This was done by modeling an elastic block attached to a rigid wall. The attachment was either incorporated as a homogeneous Dirichlet condition or as a constraint, for the unconstrained and constrained dynamics respectively.

The finite element implementation was performed with a python module called FEniCS. The module provides automatic mesh generation, assembling, and solving processes, making the implementation of the unconstrained elastic dynamics relatively compact. For the constrained dynamics a function space of Lagrange multipliers over solely a constrained boundary had to be implemented. However, to implement that in FEniCS a work-around had to be implemented where, first the Lagrange multipliers were defined over the whole domain. Then, the degrees of freedom that did not correspond to the constrained boundary were reduced away.

In the second experiment, a rigid block was attached to the elastic block by a small rigid joint. For this configuration, the orientation of the interface between the elastic body and rigid joint was expected to change during deformation of the two-body system. The aim with the second experiment was to highlight the limitation of the weakly constrained joint model for a rotating rigid joint-elastic body interface.

As expected, the weakly constrained joint model forced the orientation of the rigid joint-elastic body interface to be fixed during deformation. This clearly distorted the displacement field of the elastic block. To retrieve more accurate results, the joint model could be updated to predict the orientation of the joint-body interface by the use of observer points on the adjacent unconstrained boundary. This is left for future studies.

5.2 Conclusions

The aim of this study was to highlight the limitations of a weakly constrained joint model for connecting elastic and rigid bodies, through massless rigid joints. The joint model required that the orientation of the rigid joint-elastic body interface would not be affected by the displacement field of the elastic body. This assumption distorts the displacement field, and thereby the structural analysis, if the true displacement field would have given a rotated interface.

The limitations of the joint model was shown by applying the joint model for a two-body system of an elastic and a rigid body, connected by a small rigid joint. During deformation the rigid joint-elastic body interface was expected to rotate. However, with the weakly constrained joint model, the orientation of the interface remained fixed. A proposal for circumventing this limitation would be to predict the orientation of the interface by the use of observer points placed on the adjacent unconstrained boundary. This proposal is left for future studies.

Appendix A

Additional Proofs

The appendix contains a handful of proofs used in the theory section.

A.1 Proof of the Fundamental Lemma of Calculus of Variations in Time

In this appendix, the fundamental lemma of calculus of variations to retrieve a strong formulation in time, Lemma 1, is proved. First, the lemma is restated,

Lemma (Fundamental lemma of calculus of variations in time)**.** *Let* $N(t) \in C([t_0, t_1], \mathbb{R}^{n_\varphi})$*. If*

$$\int_{t_0}^{t_1} h(t)^T N(t) \mathrm{d}t = 0, \quad \forall h \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi}) \tag{A.1}$$

with $h(t_0) = h(t_1) = 0$*, then* $N(t) = 0$ *for* $t_0 \leq t \leq t_1$*.*

The proof is a proof by contradiction. In the the proof an arbitrary scalar-valued component of $N(t)$ is of interest, $N_i(t) \in C([t_0, t_1], \mathbb{R})$, where $N(t) = (\{N_i(t)\}_{i=1}^{n_\varphi})$. Assuming that $N(t)$ is not identically zero for $t_0 < t < t_1$, then there exists a time point $t_0 < t_* < t_1$ such that $N(t_*) \neq 0$. Without loss of generality, consider $N_i(t_*) > 0$ and $N_{j \neq i}(t_*) = 0$, for $j \in [1, n_\varphi]$, is assumed to hold. Since N_i is continuous, there exists a sub interval $(\xi_0, \xi_1) \subset [t_0, t_1]$ such that $t_* \in (\xi_0, \xi_1)$ and $N_i(\xi) > N_i(t_*)/2$ for $\xi \in (\xi_0, \xi_1)$. A specific $h \in C_1([t_0, t_1], \mathbb{R}^{n_\varphi})$, with $h(t_0) = h(t_1) = 0$, is defined by

$$h(t) = \begin{cases} (t - \xi_0)^2 (\xi_1 - t)^2 e_i, & \text{if } \xi_0 < t < \xi_1 \\ 0, & \text{otherwise} \end{cases} \tag{A.2}$$

where e_i is the ith unitvector in $\mathbb{R}^{n_\varphi}$. For the specific h, Eq. A.1 becomes

$$0 = \int_{t_0}^{t_1} h(t)^T N(t) \mathrm{d}t \geq \int_{\xi_0}^{\xi_1} \frac{N_i(t_*)}{2} (t - \xi_0)^2 (\xi_1 - t)^2 e_i^T e_i \mathrm{d}t \tag{A.3}$$

$$= \int_{\xi_0}^{\xi_1} \frac{N_i(t_*)}{2} (t - \xi_0)^2 (\xi_1 - t)^2 \mathrm{d}t > 0, \tag{A.4}$$

which is a contradiction. Thus, $N(t) = 0$ for $t_0 < t < t_1$. Since N is continuous, the equality, $N(t) = 0$, can be extended to hold over the closed interval $[t_0, t_1]$. Thereby, the lemma is proved.

A.2 Proof of Invertibility I

In this appendix, that, the matrix $G \in \mathbb{R}^{n_\lambda \times n_\varphi}$ ($n_\lambda < n_\varphi$) has full row rank implies that the square matrix $GG^T \in \mathbb{R}^{n_\varphi \times n_\varphi}$ is invertible, is proved. This property is needed in the proof of Theorem 1.

In the proof of GG^T being invertible, the following lemma is used.

Lemma 3. *For any $m \times n$ matrix A, the null space of A, $Ker(A)$, and the range of A^T, $R(A^T)$ are orthogonal complements, ie.*

$$y^T w = 0, \quad \forall (y, w) \in (R(A^T), Ker(A)) \tag{A.5}$$

Proof: An arbitrary $y \in R(A^T)$ can be expressed as $y = A^T x$ with x being in the preimage of A^T. Consider the inner product with a $w \in Ker(A)$

$$y^T w = (A^T x)^T w = x^T A w = x^T 0 = 0. \tag{A.6}$$

Note, $Aw = 0$ is a property of $w \in Ker(A)$. Since $y^T w = 0$ holds for any $y \in R(A^T)$ and any $w \in Ker(A)$, the lemma is proved.

Consider again, to prove that the matrix $G \in \mathbb{R}^{n_\lambda \times n_\varphi}$ ($n_\lambda < n_\varphi$) has full row rank implies that the square matrix $GG^T \in \mathbb{R}^{n_\varphi \times n_\varphi}$ is invertible. GG^T being invertible is equivalent with that the null space of GG^T only contains the zero vector, $Ker(GG^T) = \{0\}$ [5, Th. 1.3]. Consider an element in $Ker(GG^T)$,

$$GG^T x = 0, \quad x \in Ker(GG^T). \tag{A.7}$$

According to Eq A.7, $G^T x$, an element in the range of G^T, is in $Ker(G)$. Due to Lemma 3, $G^T x = 0$.

Since G has full row rank, $G^T x = 0$ implies that $x = 0$. That any element $x \in Ker(GG^T)$ is required to be zero implies that $Ker(GG^T) = \{0\}$, equivalent with GG^T being invertible.

A.3 Proof that an Invertible Matrix is Continuous

In this appendix, that, an invertible square matrix $A \in \mathbb{R}^{n \times n}$ is continuous, is proved. First, the determinant of A, $det(A)$, is shown to be continuous.

According to Cayley-Hamilton's theorem [15, p. 49], the characteristic polynomial of A is defined as

$$p_A(\lambda) = det(\lambda I - A), \tag{A.8}$$

for a variable λ, where I denotes the identity matrix. Specifically, $-p_A(0) = det(A)$. Since polynomials are continuous, $det(A)$ is continuous.

Consider the notation $det(A) = det(A_1, ..., A_n)$ with the matrix separated into column vectors. Furthermore, according to Cramer's rule [16, Th. 4.1], element j of the solution vector x of a linear system $Ax = b$, where $b \in \mathbb{R}^n$ is the right-hand side and A is an invertible (equivalent with $det(A) \neq 0$) square matrix, is expressed as

$$x_j = det(A_1, ..., A_{j-1}, b, A_{j+1}, ..., A_n)/det(A). \tag{A.9}$$

The solution vector for a linear system, with a basis vector of $e_i \in \mathbb{R}^n$, $i \in [1, n]$, as the right-hand side, is a column vector of A^{-1},

$$Au_i = e_i, u_i = A_i^{-1}. \tag{A.10}$$

Thus, by applying Cramer's rule successively for all basis vectors of $\mathbb{R}^n$, A^{-1} is retrieved with every element expressed as a rational of two determinants. That $det(A)$ is continuous implies that

$$u_{ij} = det(A_1, ..., A_{j-1}, e_i, A_{j+1}, ..., A_n)/det(A), \tag{A.11}$$

any element of A^1, is continuous. Thus, A^{-1} is continuous.

A.4 Proof of Invertibility II

In this appendix, some basic linear algebra definitions and properties are used. They can all be found in an introductory textbook, eg. [5]. The square matrix

$$\begin{bmatrix} M & G^T \\ G & 0 \end{bmatrix}, \tag{A.12}$$

with $M \in \mathbb{R}^{n_\varphi \times n_\varphi}$ and $G \in \mathbb{R}^{n_\lambda \times n_\varphi}$, under the assumptions that G has full row rank and M is symmetric positive definite (SPD), is proved to be invertible. This proof is needed in Section 2.1.4 when a specific system of DAEs is shown to have differentiation index 3. The meaning of all symbols can be retrieved from Section 2.1.

First, perform block Gaussian elimination to retrieve a diagonal matrix

$$\begin{bmatrix} M & G^T \\ G & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ GM^{-1} & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M^{-1}G^T \\ 0 & I \end{bmatrix}, \tag{A.13}$$

where $S := -GM^{-1}G^T$ is the Schur complement matrix. Performing block Gaussian elimination is only possible if M is invertible. That M is invertible follows from the assumption that M is positive definite. Note, positive definite matrices are invertible since their null spaces, per definition, only contains the zero vector. A property of invertible matrices is that their null spaces only contain the zero vector.

Since all diagonal elements of the triangular square matrices are nonzero (they are all one), all the rows of the triangular elements are linearly independent. Therefore, the triangular matrices are invertible. As a consequence, the matrix in Eq. A.12 is invertible if and only if S is invertible.

Next, that M is SPD implies that M^{-1} is SPD is shown. Set $y = Mx$. Since M has full rank (follows from M being invertible), y is nonzero for any nonzero x. M^{-1} is positive definite since

$$y^T M^{-1} y = x^T M^T M^{-1} M x = x^T M^T x = x^T M x > 0, \quad \forall x \neq 0. \tag{A.14}$$

The last inequality, which holds for any nonzero x, is the property that defines a positive definite matrix M. The symmetry of M^{-1} follows from $y = Mx = M^T x$ implies that $M^{-1}y = M^{-T}y = x$, which holds for any x and y.

Lastly, S is shown to be positive definite, which implies invertible. Since G has full row rank, $u = G^T z$ is nonzero for any nonzero z. Since

$$z^T S z = -z^T G M^{-1} G^T z = u^T M^{-1} u, \tag{A.15}$$

positive definiteness of S follows from positive definiteness of M^{-1}.

A.5 Proof of a Divegence Property

In this appendix, the divergence property

$$\nabla \cdot (\sigma v) = \text{trace}(\sigma \nabla v) + v^T \text{div}(\sigma) \tag{A.16}$$

is shown to hold for a symmetric matrix σ and a vector v, where $\mathrm{div}(\sigma)$ is defined as the vector whose components are the divergences of the rows of σ.

$$\mathrm{div}(\sigma) := \begin{bmatrix} \nabla \cdot \sigma_{1\cdot}^T \\ \nabla \cdot \sigma_{2\cdot}^T \end{bmatrix}. \tag{A.17}$$

First, the product rule

$$\nabla \cdot (xy) = (\nabla x)^T y + x \nabla \cdot y \tag{A.18}$$

is proven for a scalar x and a vector y,

$$\nabla \cdot (xy) = \sum \partial_i (x y_i) = x \sum \partial_i y_i + \sum y_i \partial_i x = (\nabla x)^T y + x \nabla \cdot y. \tag{A.19}$$

Then, the product rule, Eq. A.18, is applied twice to Eq. A.16,

$$\nabla \cdot (\sigma v) = \nabla \cdot (v_1 \sigma_{\cdot 1} + v_2 \sigma_{\cdot 2}) = (\nabla v_1)^T \sigma_{\cdot 1} + v_1 \nabla \cdot \sigma_{\cdot 1} + (\nabla v_2)^T \sigma_{\cdot 2} + v_2 \nabla \cdot \sigma_{\cdot 2}. \tag{A.20}$$

Grouping the first and third terms together, and the second and the forth terms, give

$$\nabla \cdot (\sigma v) = \mathrm{trace}(\sigma \nabla v) + v^T \begin{bmatrix} \nabla \cdot \sigma_{\cdot 1} \\ \nabla \cdot \sigma_{\cdot 2} \end{bmatrix}. \tag{A.21}$$

Due to symmetry of σ, $\sigma_{\cdot 1} = \sigma_{1\cdot}^T$ and $\sigma_{\cdot 2} = \sigma_{2\cdot}^T$, and

$$\nabla \cdot (\sigma v) = \mathrm{trace}(\sigma \nabla v) + v^T \mathrm{div}(\sigma). \tag{A.22}$$

Thus, the proof is completed.

A.6 Extension of the Fundamental Lemma of Calculus of Variations in Space

Consider the fundamental lemma of calculus of variations in space,

Lemma (Fundamental lemma of calculus of variations in space)**.** *Let $\Omega \subset \mathbb{R}^d$ be a open, bounded domain in a d-dimensional space. If $N(x) \in L_1(\Omega)$ satisfies*

$$\int_\Omega h(x)^T N(x) \mathrm{d}x = 0, \quad \forall h \in C_0^\infty(\Omega), \tag{A.23}$$

then $h = 0$ almost everywhere.

Almost everywhere refers to that the set for which the statement does not hold carries zero measure. In [8, Lemma 2.21], the lemma is proved. In this appendix the extension to $h \in H_0^1(\Omega)$ is proved.

Since $H_0^1(\Omega)$ was defined as the closure of $C_0^\infty(\Omega)$ in the $H_0^1(\Omega)$-norm, $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. Thus, for every $h_* \in H_0^1(\Omega)$ there exists a sequence $(h_k)_{\mathbb{Z}^+} \in C_0^\infty(\Omega)$ such that it converges strongly to h, ie. $\|h_* - h_k\|_{H_0^1(\Omega)} \to 0$ as $k \to \infty$. Since $H_0^1(\Omega)$ is a Hilbert space, the limit exists.

By evaluating the norm,

$$\|h_* - h_k\|^2_{H_0^1(\Omega)} = \int_\Omega (h_* - h_k)^T (h_* - h_k) \mathrm{d}x + \int_\Omega \nabla (h_* - h_k)^T \nabla (h_* - h_k) \mathrm{d}x \to 0, \tag{A.24}$$

one can conclude that $h_* - h_k \to 0$ almost everywhere. As a consequence

$$\int_\Omega (h_* - h_k)(x)^T N(x) \mathrm{d}x \to 0, \tag{A.25}$$

and therefore Eq. A.23 holds for a specific $h_* \in H_0^1(\Omega)$. Since h was chosen arbitrarily the lemma is proved.

Bibliography

[1] Bernd Simeon. *Computational Flexible Multibody Dynamics*. Springer, 2013.

[2] Hernán G Solari, Mario A Natiello, and Gabriel B Mindlin. *Nonlinear Dynamics: A Two-Way Trip from Physics to Math*. Taylor & Francis, 1996.

[3] Mark Kot. *A First Course in the Calculus of Variations*, volume 72. American Mathematical Society, 2014.

[4] Michael Renardy and Robert C Rogers. *An Introduction to Partial Differential Equations*, volume 13. Springer Science & Business Media, 2006.

[5] Lloyd N Trefethen and David Bau III. *Numerical Linear Algebra*, volume 50. SIAM, 1997.

[6] Ahmed A. Shabana. *Dynamics of Multibody Systems*. Cambridge University Press, 3rd edition, 2005.

[7] Walter Rudin. *Real and Complex Analysis*. Tata McGraw-Hill Education, 2006.

[8] Filip Rindler. Introduction to the modern calculus of variations, 2015.

[9] Niels Saabye Ottosen and Hans Petersson. *Introduction to the Finite Element Method*. Prentice-Hall, 1992.

[10] Arieh Iserles. *A First Course in the Numerical Analysis of Differential Equations*. Cambridge University Press, 2 edition, 2008.

[11] Alan C Hindmarsh, Peter N Brown, Keith E Grant, Steven L Lee, Radu Serban, Dan E Shumaker, and Carol S Woodward. SUNDIALS: Suite of nonlinear and differential/algebraic equation solvers. *ACM Transactions on Mathematical Software (TOMS)*, 31(3):363–396, 2005.

[12] Christian Andersson, Claus Führer, and Johan Åkesson. Assimulo: A unified framework for ODE solvers. *Mathematics and Computers in Simulation*, 116(0):26 – 43, 2015.

[13] Kathryn Eleda Brenan, Stephen L Campbell, and Linda Ruth Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, volume 14. SIAM, 1996.

[14] Steen Krenk. *Non-linear Modeling and Analysis of Solids and Structures*. Cambridge University Press, 2009.

[15] Allen Altman and Steven Kleiman. *A term of commutative algebra*. Worldwide Center of Mathematics, 2013.

[16] Serge Lang. *Introduction to linear algebra*. Springer Science & Business Media, 2012.

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