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A Conservative Discretization of a Hamiltonian System

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Abstract

The reaction–diffusion equation (RDE) is a natural way of describing a system where there is not only diffusion but also interaction with its surroundings. The RDE has been the topic of interest in previous papers for its usefulness of explaining pattern formations in nature. The static RDE has, in one dimension, the form of a Newton equation. We study a discretized version of this equation, with applications in biology (where a cell can be seen a natural discretization), economics, and computer simulations. In particular we consider a class of discretized Newton equations that allows for a conserved quantity. Using this we manage to find a conservative discretization of the Φ^4 system.

Popular Abstract

Almost anywhere you look in nature, you will see some sort of pattern. In the artichokes from your local supermarkets vegetable aisle, in the horns of wild deer or in the snowflakes falling on your face during winter, patterns arise everywhere. But what exactly is a pattern? Most humans have an instinctual affinity for patterns, we find them beautiful. But disregarding aesthetics, why do physicists find patterns so useful?
Patterns are a discernible regularities. These regularity can be very complex and to this day scientists and mathematicians haven't been able to explain exactly how patterns arise in nature. It was as late as the 80's when we understood how snowflakes pattern get their shapes, even though they had been a topic of interest for several hundred years.
The formation of a pattern is just one small step away from total chaos. When working with patterns, it feels like a small miracle that they even form in nature, considering how hard they are to reproduce mathematically. The difficulty of describing patterns in nature can be overcome by using so-called discrete time, since computers must use discrete time. What is meant by this, is that time is not continuous and you jump from one "time box" to the other, like you do when playing hopscotch. To illustrate this point let us look at how populations are studied. Populations, of course, change all the time but you can only measure them discretely. So it's only natural to describe population using a discrete time model. It can also be used in economics where the same situation as population applies. Finding and understanding patterns in models like this can be the difference between a recession and a economic boom, or how growth regulators for plants diffuses in cells.

One would believe that for patterns to arise there should be some sort of conserved quantity. Usually in physics the energy or momentum is conserved. And the conserved quantity for the discrete time case could be used to approximate the continuous time case, and therefore bridging the discrete time case of the computer to the continuous time world that we live in.

Contents

1 Introduction

Conservation laws and hence conserved quantities have a deeply rooted place in physics. Such quantities could also be useful e.g. in the dynamics for pattern formation in plants, involving the plant growth hormone auxin. Such dynamics can be modeled by the so called reaction–diffusion equation (RDE). The RDE is a normal diffusion equation with a extra term that depends on the current variable value. This term is very useful e.g. for describing chemicals that react with each other on a local level. By requiring stationary solutions, the one-dimensional RDE becomes equivalent to Newtons equation, so we expect a conserved quantity.

To be able to use a computer's full potential it is particularly useful to have your mathematical model discretized. This has natural applications in biology[6] and economics[3]. We consider a stationary RDE, which fortunately can be discretized. The properties of this system was studied by Nils H. Truedsson in 2014 in particular a set of special cases allowing for a conserved quantity. Being free from chaos, these systems form a reasonable starting points for approximating classical Newtonian systems.[1]

2 The mathematical background

The mathematical background is outlined in N. H. Truedsson bachelor's thesis (2014)[1].

We consider the function $u(x,t)$, which represents the concentration of some pattern-forming substance as a function of position x and time t. The RDE is a normal diffusion equation with an extra $u(x,t)$-dependent term.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \tag{2.1}$$

where $F(u)$ is some function representing the local interactions.

We seek stationery patterns, $u(x,t) \to u(x)$. Then we can rewrite equation (2.1) as

$$0 = \frac{\partial^2 u}{\partial x^2} + F(u). \tag{2.2}$$

This analogous to Newton equation with an acceleration term, $\frac{\partial^2 u}{\partial x^2}$, and a force term $F(u)$.

In a discrete space the second derivative at a point can be approximated to be proportional to the sum of differences to neighbouring points, i.e. the sum of the two neighbour values minus twice the current value.[7] Hence, we can approximate equation (2.2) by

$$\sum_{j \in N(i)} u(x_j) = f(u_i), \tag{2.3}$$

where $N(i)$ is the set of nearest neighbours to the the point i and $f(u_i)$ is related to $F(u)$, and has the form (as a consequence of approximating the second derivative to a discrete case) $f(u_i) = 2u_i + \delta^2 F(u_i)$, where δ is the spatial separation between two points and for any $F(u_i)$.

In one dimension, equation (2.3) becomes $0 = u_{i+1} + u_{i-1} - f(u_i)$, or more compactly,

$$u_+ + u_- = f(u), \tag{2.4}$$

where u is the value at a point on a one dimensional lattice and $u_\pm$ are the neighbouring values. We note that this equation is translation and reflection symmetric.

Equation (2.4) corresponds to a map T in the phase space $(u, v) = (u, u-)$. It can be seen as representing a discrete Hamilton dynamics.

$$T : \begin{cases} u_+ = f(u) - v, \\ v_+ = u. \end{cases} \tag{2.5}$$

Trajectories in this phase space represent static patterns, and in order to display a conserved quantity, H, this must fulfill the requirement $H(u_+, u) = H(u, u_-) = H(u_-, u)$, where the last equality reflects the assumed reflection symmetry of H. Using this criterion on H it was found that the function $f(u)$ in equation (2.4) must have the form[1]

$$f(u) = \frac{Bu^2 - Du + E}{Au^2 - Bu + C}, \tag{2.6}$$

where A, B, C, D and E are real parameters, and A, B, and C cannot be zero simultaneously.

The conserved quantity then looks like

$$H(u, v) = Au^2v^2 - Buv(u + v) + C(u^2 + v^2) + Duv - E(u + v). \tag{2.7}$$

3 Theoretical analysis

The naïve discretization of Newton's equation,

$$u'' = -V'(u) \xrightarrow{naive} u_+ + u_- = 2u - V'(u), \tag{3.1}$$

has been considered for a Φ^4 system,[2] where the potential $V(u)$ is given by $V(u) = -au^2 + bu^4$, with a and b real positive constants.

Naïvely discretized, its equation of motion (EoM) showed chaotic behavior, indicating the absence of a conserved quantity. A more careful approach, based on the form in equation (2.6), could yield a discrete conservative system that well approximates the continuous Φ^4-system, at least in some parts of parameter space.

3.1 Choosing a $f(u)$

The EoM for Φ^4 has three stationary solutions, which are equidistant, which means that our equation (2.4) must have three so called fixpoints u^*, i.e when equation (2.6) becomes $u_+ = u_- = u = u^*$. Equation (2.6) gives

$$2u^* = \frac{Bu^{*2} - Du^* + E}{Au^{*2} - Bu^* + C}, \tag{3.2}$$

This equation indeed has three fixpoints, assuming $A \neq 0$.

Since the Φ^4-potential is antisymmetric, we require that $f(u)$ be an odd function, which can only be the case if $B = E = 0$. Equation (3.2), after rewriting, becomes

$$u^*(2Au^{*2} + 2C + D) = 0. \tag{3.3}$$

By rescaling our variable (see Appendix A), u, we can put $2C + D = -2A$, and we end up with the simple expression

$$u^*(u^{*2} - 1) = 0, \tag{3.4}$$

which has the solutions $u^* = 0, \pm 1$, and this is where our fixpoints will be located.

Our $f(u)$ now contains a single independent parameter, D/C, which we can replace by -2μ. The final expression for f will be

$$f(u) = \frac{2\mu u}{(\mu - 1)u^2 + 1}, \tag{3.5}$$

where μ is the only remaining parameter. We observe that for $\mu \to 1$ we have $f(u) \to 2u$ which is consistent with the continuum limit.

3.2 Linearizing $f(u)$

To determine the behavior of $f(u)$ near a fixpoint, we analyze the map T (2.5). Finding the eigenvalues, λ, of its Jacobian will give us the characteristics of the fixpoints. The Jacobian is

$$J = \left.\begin{bmatrix} \frac{\partial f(u) - v}{\partial u} & \frac{\partial f(u) - v}{\partial v} \\ \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \end{bmatrix}\right|_{(u^*, v^*)} = \begin{bmatrix} f'(u^*) & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.6}$$

The determinant is indeed 1, reflecting the area preservation, and the trace is $f'(u^*)$.[3] Thus, a fixpoint u^* can be characterized by a stability parameter $\eta = \frac{1}{2} f'(u^*)$, that defines the eigenvalues by

$$\lambda^2 - 2\eta\lambda + 1 = 0. \tag{3.7}$$

Depending on the value of η there will be three different scenarios, with different behaviors,

- $\eta > 1$, with $\lambda_{1,2} = e^{\pm\theta}$,
- $\eta < -1$, with $\lambda_{1,2} = -e^{\pm\theta}$,
- $-1 < \eta < 1$, with $\lambda_{1,2} = e^{\pm i\phi}$,

for some parameters θ and ϕ related to η, and where i is the imaginary unit.[4]

Since the sum of the eigenvalues is the trace of the matrix, η must be a $\cos\phi$ for $|\eta| < 1$ and $\pm\cosh\theta$ for $|\eta| > 1$.

Thus from $f'(u^*)$ one can readily determine if the fixpoint u^* is elliptic, i.e $|\eta| < 1$, or hyperbolic, i.e $|\eta| > 1$.

3.3 Behavior of $f(u)$

3.3.1 one-cycle (i.e. fixpoint)

For our chosen $f(u)$, we have fixpoints at $u = 0$ and at $u = \pm 1$, with the respective stability parameters η_0, $\eta_\pm$. For $u^* = 0$, we have $\eta_0 = \mu$, and for $u^* = \pm 1$ the value $\eta_\pm = -1 + \frac{2}{\mu}$.

We plot these to see when they are smaller or larger than one.

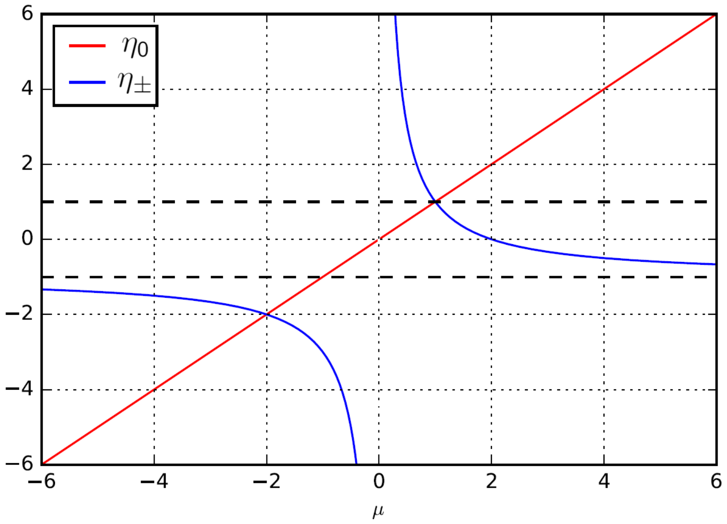

Figure 1: Plot of $\eta_0(\mu)$ and $\eta_\pm(\mu)$. The dotted lines mark the boundaries between the hyperbolic and elliptic cases.

From figure 1 we can easily see how the three different fixpoints will behave for different μ. Thus, when $\mu < -1$, the fixpoints will all be unstable and hyperbolic. In the range $\mu \in (-1, 1)$, the fixpoint at $u^* = 0$ will be stable and elliptic and the other two will be unstable and hyperbolic. For $\mu > 1$ the fixpoint at $u^* = 0$ will be unstable and hyperbolic and the other two will be stable and elliptic. The boundary cases are (i) when $\mu = 0$, resulting in a trivial situation $f(u) = 0$, and (ii) $\mu = 1$, yielding a straight line trajectory.

We are interested in a scenario where the fixpoint at $u = 0$ is unstable and the other two are stable. Looking at figure 1 we see that that is the case when $\mu > 1$.

3.3.2 two-cycle

Another interesting feature is the two-cycle, that is to say when the system oscillates between two points.

To have a two-cycle, equation (2.4) yields

$$\begin{cases} 2x &= f(y), \\ 2y &= f(x). \end{cases} \tag{3.8}$$

where x and y are the two points between which the variable u should be oscillating. To have a two-cycle we require $x \neq y$, which implies $x = -y$ and the relation (See appendix B)

$$xy = \frac{1+\mu}{\mu - 1} \Rightarrow x = \pm\sqrt{\frac{-1-\mu}{\mu-1}}. \tag{3.9}$$

This means that we will have a two-cycle at $(x, -x)$ and $(-x, x)$, where $x = \sqrt{\frac{-1-\mu}{\mu-1}}$ will be real only for $-1 < \mu < 1$.

To get the behavior near this two-cycle we need the Jacobian after a complete cycle which is the matrix (3.6) evaluated at one of the point multiplied with the same matrix but evaluated at the other point. The trace of this Jacobian, $2\eta_{2c}$, will give us the characteristics of this two-cycle, similar to the one-cycle scenario, and yields $\eta_{2c} = \frac{\mu^2+8\mu+8}{\mu^2}$. Plotting η_{2c} as function of μ we get figure 2. (See appendix C)

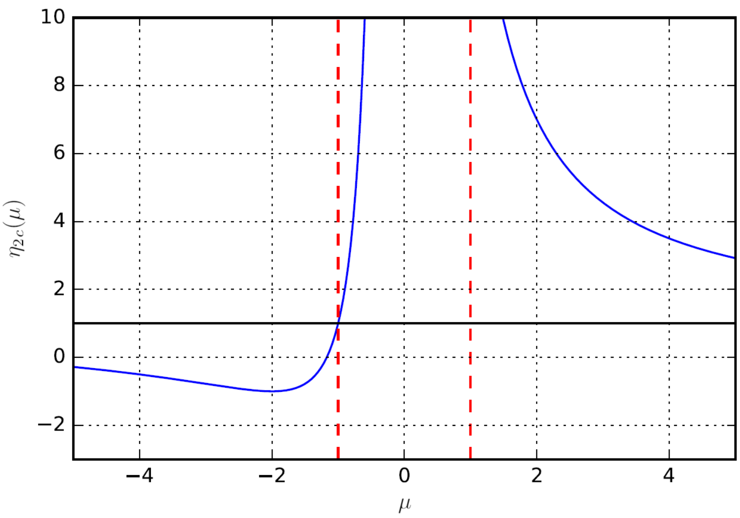

Figure 2: Plot of $\eta_{2c}(\mu)$. The dotted lines are $\mu = -1, 1$, between which the two-cycle is real. The straight black line is $\eta_{2c}(\mu) = 1$.

As we can see in the region were η_{2c} is real, the value of η_{2c} will always be greater than one, which means that when we have a two-cycle it will always be hyperbolic, except for $\eta_{2c} = -1$ where no conclusions can be drawn, from a linear analysis.

3.4 Φ^4 and the continuum limit

It is convenient to change our phase space coordinates as follows. We define a pseudo-position, q, as the average of u and v, and a pseudo-momentum, p, proportional to their difference. To get a proper continuum limit we define

$$\begin{cases} q = \frac{u+v}{2}, \\ p = \frac{u-v}{\delta}. \end{cases} \tag{3.10}$$

where $\delta > 0$ is the lattice distance, where we must have $\delta \to 0$ in the continuum limit. This implies for the next step

$$\begin{cases} q_+ = \frac{u_+ + u}{2}, \\ p_+ = \frac{u_+ - u}{\delta}. \end{cases} \tag{3.11}$$

We get that the map T in terms of q, p becomes

$$T : \begin{cases} q_+ = \frac{f\left(\frac{\delta p}{2}+q\right)+\delta p}{2}, \\ p_+ = \frac{f\left(\frac{\delta p}{2}+q\right)-2q}{\delta}. \end{cases} \tag{3.12}$$

In the continuum limit, we also must have $\mu \to 1$ in order to approach a continuous time Hamilton system with a Φ^4-potential. (See Appendix D)

To that end we rewrite μ as $\mu = 1 + \varepsilon$, where ε is some small positive number coupled to δ. Our $f(u)$ then becomes

$$f(u) = \frac{2u + 2u\varepsilon}{\varepsilon u^2 + 1}. \tag{3.13}$$

Plugging this into equation (3.12) we have the map

$$\begin{cases} q_+ = \frac{\delta p}{2} + \frac{1}{2}\left[\frac{(\frac{\delta p}{2}+q)+(\frac{\delta p}{2}+q)\varepsilon}{\varepsilon(\frac{\delta p}{2}+q)^2+1}\right], \\ p_+ = \frac{-2q}{\delta} + \frac{2}{\delta}\left[\frac{(\frac{\delta p}{2}+q)+(\frac{\delta p}{2}+q)\varepsilon}{\varepsilon(\frac{\delta p}{2}+q)^2+1}\right]. \end{cases} \tag{3.14}$$

Simplifying and neglecting higher order terms, we see that to get a reasonable continuum limit we must set $\varepsilon \propto \delta^2$ (which means that $\varepsilon \ll \delta$). Setting $\varepsilon = \alpha\delta^2$, we get

$$\begin{cases} q_+ = q + \delta p + \varepsilon(q - q^3) \approx q + \delta p, \\ p_+ = p + \frac{2\varepsilon}{\delta}\left(q - q^3\right) = p + 2\alpha\delta\left(q - q^3\right). \end{cases} \tag{3.15}$$

This can be seen as an approximation of the continuous time Hamilton system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = 2\alpha\left(q - q^3\right). \end{cases} \tag{3.16}$$

where the dot is the derivative with respect to time. The corresponding Hamiltonian is $\mathcal{H} = \frac{p^2}{2} - \alpha q^2 + \frac{\alpha}{2}q^4$, with the Φ^4-potential $V(q) = -\alpha q^2 + \frac{\alpha}{2}q^4$.

4 Plotting and results

Combining equations (2.5) and (3.5), we plot some resulting trajectories in phase space for different μ. The fixpoints will lie on the line $u = v$, which will be plotted as a dotted line. The initial values will be randomized in a box called initbox. The box is centered at the origin and will have side lengths equal to the value of initbox.

The number of trajectories will be 100 and the number of iterations, for each trajectory, will be 10000 for all figures in this section, unless otherwise stated.

The trivial cases $\mu = 0, 1$ are not plotted.

4.1 Trajectory portraits

4.1.1 $\mu < -1$

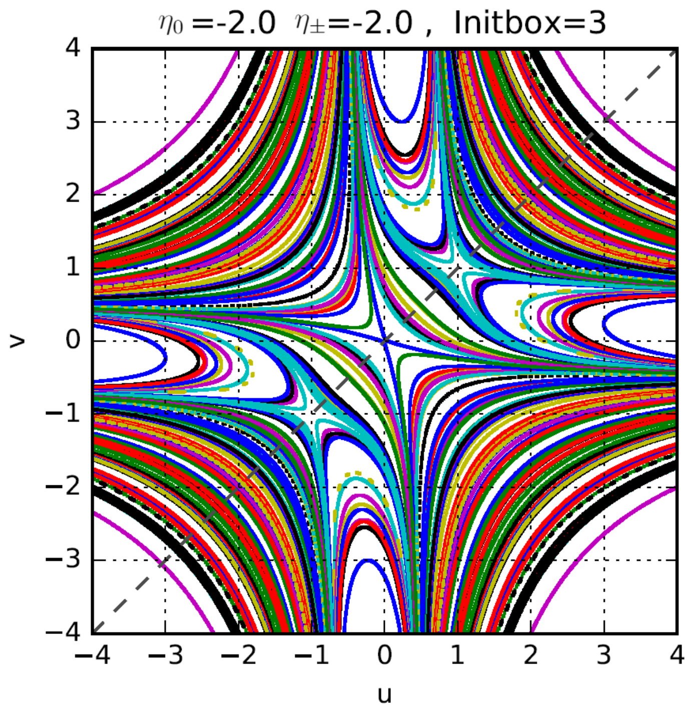

Figure 3: Trajectory for $\mu = -2$ as an example of $\mu < -1$.

In figure 3 it is easy to distinguish the fixpoints which lie, in agreement with theory, on $-1, 0$ and 1 and are all hyperbolic since $\mu < -1$. Another consequence of $\mu < -1$ is that there is no sign of a two-cycle.

4.1.2 $\mu = -1$

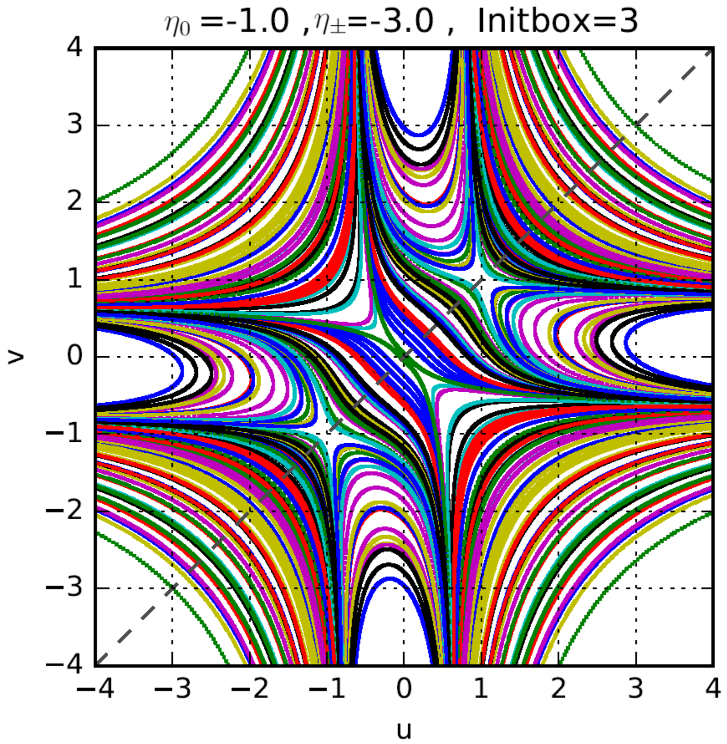

Figure 4: Trajectory for $\mu = -1$ is an interesting boundary case since neither our fixpoint at the origin, nor the two-cycle will behave predictably from a linear analysis.

In figure 4 the fixpoints at -1 and 1 are still hyperbolic, in agreement with theory. The fixpoint at the origin looks hyperbolic, which is something we could not predict without including higher order terms. The two-cycle should be at the origin, which is why we cannot see any trace of it.

4.1.3 $\mu \in (-1, 1)$

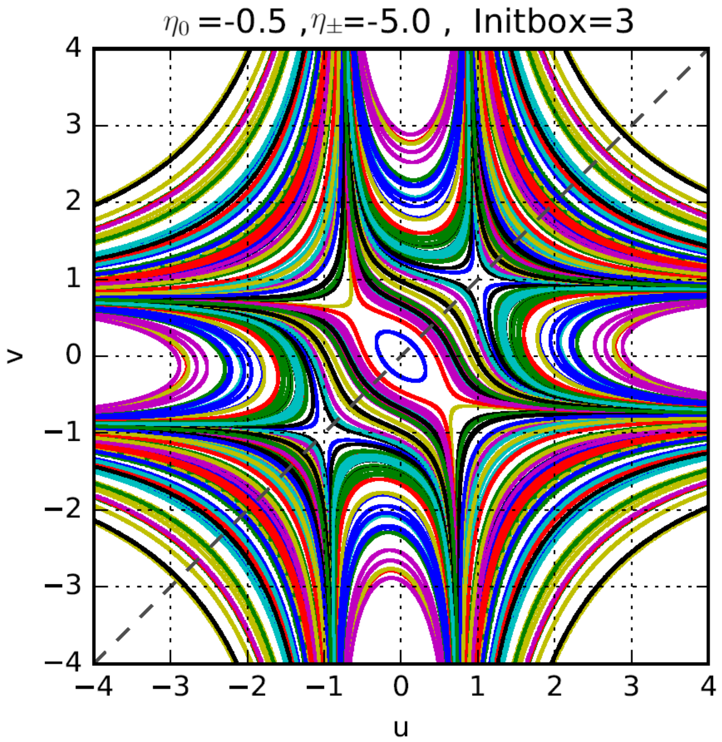

Figure 5: Trajectory plot for $\mu = -0.5$, as an example of a number in the range $\mu \in (-1, 1)$.

In figure 5 we again see that the fixpoints are all in agreement with theory. The two-cycle now appears at roughly (-0.5,0.5) and (0.5,-0.5) which is in perfect agreement with theory which says that they should appear at $\pm 1/\sqrt{3} \approx \pm 0.577$.

4.1.4 $\mu > 1$

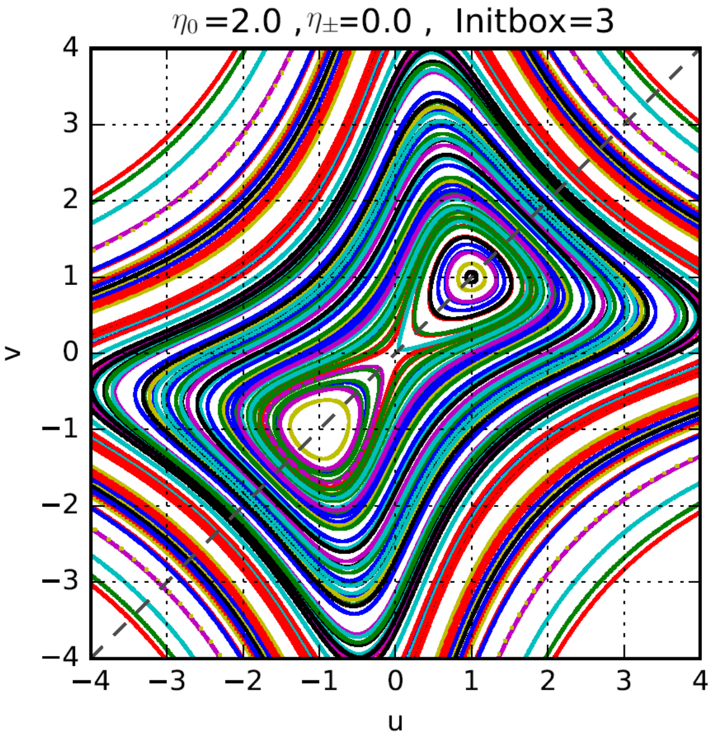

Figure 6: Trajectory plot for $\mu = 2$.

In figure 6 all the fixpoints are again in agreement with theory. The fixpoints at $\mu = -1, 1$ are now elliptic and our fixpoint at $\mu = 0$ is hyperbolic. There is no sign of a two-cycle since μ is not in the range $\mu \in (-1, 1)$.

4.1.5 Approaching Φ^4: $\mu = 1 + \varepsilon$

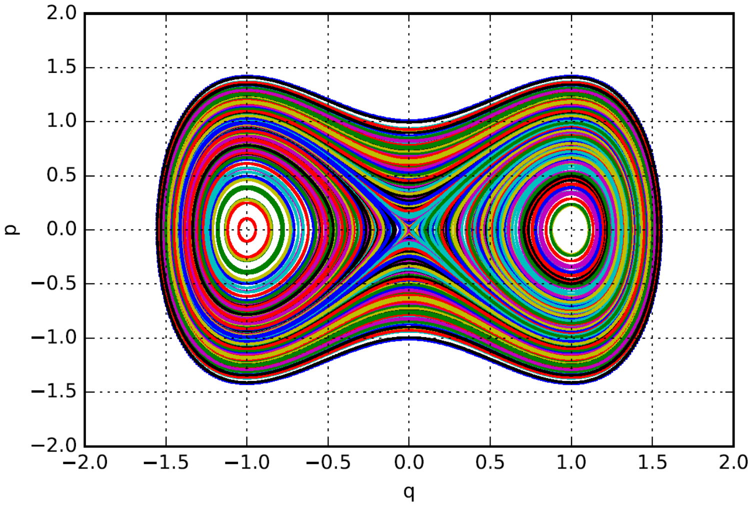

Figure 7: Trajectory plot in the (q, p)-space. The initial values are in a box of size 2, with $\alpha = 1$, $\delta = 0.01$, $\varepsilon = 0.0001$ and $\mu = 1.0001$.

In figure 7 this is plotted using the map (3.15). Visually, the trajectories show strong similarity to those of a continuous time Hamiltonian with Φ^4-potential. Since figure 7 is in q, p-space our fixpoints are no longer on the diagonal line $u = v$ but instead on the q-axis. The number of trajectories is 500 and the number of iterations is 50000 for each.

5 Summary

The topic of this report was to investigate the behavior of a certain subclass of area-preserving maps with a conserved quantity, related to the discrete RDE. In particular we studied a symmetric special case with three equidistant fixpoints, necessary for approximating the dynamics in a Hamiltonian system with a Φ^4-potential, as used e.g. in the Higgs mechanics[5]. We analyze the system with respect to the properties of the fixpoints and two-cycles, confirmed by comparing to trajectory plots from computer simulations.

In suitable phase space coordinates the symmetric case above was shown to possess a continuum limit in terms of a continuous time Hamiltonian system with a Φ^4-potential, which was also confirmed by simulations.

Possible extensions of this project could be **i**) To investigate higher cycles to see if more are analytically attainable. **ii**) To find other classical systems that could be approximated using this method. **iii**) The Φ^4-potential is used in the Higgs mechanism.

A Appendix A

We have the function (from equation (3.2))

$$f(u) = \frac{-Du}{Au^2 + C}. \tag{A.1}$$

$$\tag{A.2}$$

Rewriting this we have

$$f(u) = \frac{-\frac{D}{C}u}{\frac{Au^2}{C} + 1} = \Big/ \mu \equiv -\frac{D}{2C} \Big/ = \frac{2\mu u}{\frac{u^2}{C} + 1} \tag{A.3}$$

By rescaling u with some rescaling factor λ, $u \to \lambda u'$, we have

$$\lambda(u'_+ + u'_-) = \frac{2\mu u' \lambda}{\frac{Au^2\lambda^2}{C} + 1} \Leftrightarrow (u'_+ + u'_-) = \frac{2\mu u'}{\frac{Au'^2\lambda^2}{C} + 1} \tag{A.4}$$

We can now choose $\lambda > 0$ freely. By relating it to the value μ as $\frac{A\lambda^2}{C} = \mu - 1$, we get

$$f(u) = \frac{2\mu u}{(\mu - 1)u^2 + 1} \tag{A.5}$$

B Appendix B

For a two-cycle, $(a, b) \leftrightarrow (b, a)$ we must have that

$$\begin{cases} 2a & = f(b) = \frac{2\mu b}{(\mu-1)b^2+1} \\ 2b & = f(a) = \frac{2\mu a}{(\mu-1)a^2+1} \end{cases} \tag{B.1}$$

Subtract the LHS from both sides of both equations and then put the expressions on a common denominator:

$$\begin{cases} 0 & = -ab^2 + a\mu b^2 - \mu b + a \\ 0 & = -ba^2 + b\mu a^2 - \mu a + b \end{cases} \Leftrightarrow \begin{cases} \mu(ab^2 - b) & = a(b^2 - 1) \\ \mu(a^2 b - a) & = b(a^2 - 1) \end{cases} \Leftrightarrow \begin{cases} \mu & = \frac{a(b^2-1)}{ab^2-b} \\ \mu & = \frac{b(a^2-1)}{ba^2-a} \end{cases} \tag{B.2}$$

By taking the difference between these two expressions we end up with

$$\frac{(b-a)(a+b)}{ab(ab-1)} = 0 \tag{B.3}$$

We must have $a \neq b$ since we want a two-cycle, yielding $a = -b$.

Plugging this into our previous relations we get

$$\begin{cases} 0 & = -a(-a)^2 + a\mu(-a)^2 - \mu(-a) + a \\ 0 & = -(-a)a^2 + (-a)\mu a^2 - \mu a + (-a) \end{cases} \tag{B.4}$$

This gives the equation

$$(\mu - 1)a^2 + \mu + 1 = 0 \Leftrightarrow a^2 = \frac{-\mu - 1}{\mu - 1} \tag{B.5}$$

Taking the square root we get

$$a = \pm\sqrt{\frac{-\mu - 1}{\mu - 1}} \tag{B.6}$$

C Appendix C

To determine the stability of the two-cycle we first need to have the Jacobi matrix of our map (2.5).

The Jacobi matrix is as follows

$$\begin{bmatrix} \frac{\partial f(u)-v}{\partial u} & \frac{\partial f(u)-v}{\partial v} \\ \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \end{bmatrix} = \begin{bmatrix} f'(u) & -1 \\ 1 & 0 \end{bmatrix} \tag{C.7}$$

The stability will of course be determined after one complete cycle in the two-cycle. This means that we will have to matrix multiply, one Jacobian at one of the points on the two-cycle and one from the other remaining point.

$$\begin{bmatrix} f'(x) & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f'(y) & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f'(x)f'(y) - 1 & -f'(x) \\ f'(y) & -1 \end{bmatrix} \tag{C.8}$$

The determinant is of course 1 and trace is $f'(x)f'(y) - 2$. Defining the characteristic η_{2c} by $2\eta_{2c} = f'(x)f'(y) - 2$, or in terms of the parameter μ, $\eta_{2c} = \frac{\mu^2+8\mu+8}{\mu^2} \geq -1$. We can write the characteristic polynomial $\lambda^2 - 2\eta_{2c}\lambda + 1$, where λ is the eigenvalues of the matrix. If $|\eta_{2c}| < 1$, η_{2c} must be $\cos\phi$, where ϕ is some parameter, and we have elliptic two-cycle. If $\eta_{2c} > 1$, η_{2c} must be $\cosh\theta$, where θ is some parameter, and we have hyperbolic two-cycle.

D Appendix D

We have the relation $f(u) = 2u + \delta^2 F(u)$. $F(u)$ can be seen as the force which means that we can write it as $-V'(u) = F(u)$ where $V(u)$ is a potential. In our case the potential $V(u) = -u^2 + u^4$ will be used. Another way to write $f(u)$ is $f(u) = \frac{2\mu u}{(\mu-1)u^2+1}$. For a good approximation of a potential like $V(u) = -u^2 + u^4$, we need these two expressions of $f(u)$ to be approximately equal to each other. We get

$$2u + \delta^2(u - u^3) \approx \frac{2\mu u}{(\mu - 1)u^2 + 1} \tag{D.1}$$

$$\delta^2(u - u^3) \approx \frac{2(\mu - 1)(u - u^3)}{(\mu - 1)u^2 + 1} \tag{D.2}$$

It's quite clear that in order for both sides to be approximately the same μ must be near 1, that is $\mu = 1 + \epsilon$, where ϵ is a small positive number.

We get

$$\delta^2(u - u^3) \approx \frac{2\epsilon(u - u^3)}{\epsilon u^2 + 1} \tag{D.3}$$

Taylor expanding ϵ around we get

$$\delta^2(u - u^3) \approx 2\epsilon(u - u^3) + \mathcal{O}(\epsilon^2) \tag{D.4}$$

We see that for $\mu = 1+\epsilon$ we get a good approximation and we also get the relation $\epsilon \propto \delta^2$.

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