On two-component Bose-Einstein condensates in a ring

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Abstract

A Bose-Einstein condensate is a type of gas consisting of one or more types of particles called bosons which are cooled to a temperature very close the absolute zero. Under these conditions the particles all start to occupy their lowest quantum state. Once in these states it is necessary to use quantum physics to describe the behavior of the particles by the means of a wave function which describes the probability of finding the particles in different locations. In this thesis we will study a gas consisting of two components. The wave function satisfies the Schrödinger equation, which is a linear partial differential equation. The gas will be considered to be confined within a thin ring with a cross section small enough to treat it as one dimensional. By using the mean field approximation we're able to consider a much simpler model than the individual particle interaction and reduce the many-body problem to a one-body problem. However, the linear Schrödinger equation is replaced by the non-linear Schrödinger equation. We will also investigate the mean-field yrast spectrum, where these states are the ones with minimum energy for a given angular momentum. The existence of a minimum gives the possibility of having persistent currents as argued in previous research. In order to identify the yrast states, we first look for critical points of the energy under the constraints of total probability mass and angular momentum using a special Ansatz. We then try to determine if they are minimizers using analytic and numerical methods.

Populärvetenskaplig sammanfattning

Vi har studerat en matematisk modell för ett Bose-Einstein kondensat bestående av två komponenter. Gasen är uppbyggd av bosoner, t.ex. fotoner eller atomer som har en jämn summa av protoner, neutroner och elektroner. Denna typ av gas uppvisar en del intressanta egenskaper vid väldigt låga temperaturer nära den absoluta nollpunkten, främst att den är superflytande, vilket upptäcktes redan 1938 i flytande helium. Detta innebär att gasen påverkas väldigt lite av friktion, och skulle kunna ge ett alternativ till att transportera information liknande ett superledande material. Allt eftersom temperaturen sänks så antar atomerna sina lägsta tillåtna kvantmekaniska tillstånd. Genom att placera gasen i ett magnetfält kan man fånga atomerna i en ring-potential där man sedan kan försätta kondensatet i rotation och studera superfluiditeten. I kvantmekaniken beskrivs gasen med hjälp av en vågfunktion som anger sannolikheten för att hitta partiklarna i olika positioner. Vi använder medelfältsapproximationen där flerkroppsproblemet ersätts med ett enkroppsproblem. Vågfunktionen uppfyller då en icke-linjär differential ekvation. Vi löser denna och undersöker vilka av lösningarna som ger lägst energi.

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1 Introduction

A Bose-Einstein condensate is a gas which is cooled to a very low temperature close to absolute zero. At this point all particles try to occupy the lowest quantum state possible. Here the wave like properties of the particles makes it possible to describe the interaction between them by the means of wave functions as solutions to a nonlinear differential equation (the nonlinear Schrödinger equation). In this thesis we consider a gas consisting of two different components of equal particle mass contained in a thin ring. We also assume that the interaction strengths are equal. We can then describe the system using only one wave function for each component, satisfying the equation (see [7])

$$i\frac{\partial u_s}{\partial t} = -\frac{\partial^2 u_s}{\partial \theta^2} + 2\pi\gamma(x_A|u_A|^2 + x_B|u_B|^2)u_s, \quad s = A, B. \tag{1}$$

We denote our two species using the index $s = A, B$. N is the total number of atoms, N_s the number of atoms of each component and $x_s = \frac{N_s}{N}$. The ring geometry gives rise to the periodic boundary condition $u(\theta + 2\pi, t) = u(\theta, t)$. We restrict our attention to standing-wave solutions

$$u_s(\theta, t) = e^{-i\mu_s t}\Psi_s(\theta - \Omega t) \tag{2}$$

where $\Psi(\theta) = (\Psi_A(\theta), \Psi_B(\theta))$ is periodic and $\mu_s, \Omega \in \mathbb{R}$.

The standing waves are critical points of the energy subject to constraints of fixed mass and momentum. The energy functional is given by

$$E[\Psi] = \sum_s x_s \int_0^{2\pi} |\Psi_s'|^2 d\theta + \pi\gamma \int_0^{2\pi} \rho^2(\theta) d\theta \tag{3}$$

where γ is the scaled effective interaction between the two components taken to be positive since we're considering the case with a repulsive interaction between particles and

$$\rho(\theta) = x_A|\Psi_A|^2 + x_B|\Psi_B|^2 \tag{4}$$

is the normalized number density. Note that

$$x_A + x_B = 1 \tag{5}$$

and we choose $x_B \leq x_A$, without loss of generality. The probability mass of each component is normalized to unity

$$m_s[\Psi_s] = \int_0^{2\pi} |\Psi_s|^2 d\theta = 1, \quad s = A, B, \tag{6}$$

and the angular momentum is given by some fixed value l_0:

$$l[\Psi] = \frac{1}{i}\sum_s x_s \int_0^{2\pi} \bar{\Psi}_s \Psi_s' d\theta = l_0. \tag{7}$$

In order to obtain a well-defined mathematical problem, we introduce the space

$$X = \{\Psi \in C^2(\mathbb{R}; \mathbb{C}^2) : \Psi(\theta + 2\pi) = \Psi(\theta) \quad \text{for all} \quad \theta \in \mathbb{R}\} \tag{8}$$

and for $l_0 \in \mathbb{R}$ the subset

$$X_{l_0} = \{\Psi \in X : m_A[\Psi_A] = m_B[\Psi_B] = 1, l[\Psi] = l_0\}. \tag{9}$$

We wish to find $\tilde{\Psi} \in X_{l_0}$ such that

$$E[\tilde{\Psi}] \leq E[\Psi]$$

for all $\Psi \in X_{l_0}$. Such a minimizer must satisfy the Euler-Lagrange equation

$$-\Psi_s'' + i\Omega\Psi_s' + 2\pi\gamma\rho(\theta)\Psi_s = \mu_s\Psi_s, \quad s = A, B \tag{10}$$

where Ω and μ_s, $s = A, B$, are real numbers known as Lagrange multipliers. This equation can also be derived by substituting the Ansatz (2) into (1). Not all solutions of the Euler-Lagrange equation are minimizers, however. In general, they are only constrained critical points of the energy. The existence of a minimizer is not completely obvious, but follows by the direct methods of the calculus of variations - see e.g. the discussion on p. 2 of [8].

The main focus will be to find the critical functions that solve (1) similarly to what was done in [12], [1] with slightly more mathematical outlook and motivation. When they are found we make sure that these do indeed minimize the energy functional (3) both analytically and numerically. Recent studies have examined a Bose-Einstein condensate consisting of ^{87}Rb atoms with two different spins and how the spin would influence persistent currents of the gas and its stability [2].

2 Solving the two component case

We now try to solve the Euler-Lagrange equation (10) by introducing polar coordinates, $\Psi_s(\theta)=r_s(\theta)e^{i\varphi_s(\theta)}$. By doing so we must have the following boundary conditions

$$r_s(\theta + 2\pi) - r_s(\theta) = 0 \tag{11}$$

$$\varphi_s(\theta + 2\pi) - \varphi_s(\theta) = 2\pi J_s, \quad \text{for some integer} \quad J_s \in \mathbb{Z} \tag{12}$$

where the integers J_s are known as winding numbers. Now we can rewrite our Euler-Lagrange equation (10) and separate the real and imaginary parts:

$$-(r_s'' - r_s\varphi_s'^2) + i\Omega(r_s i\varphi_s') + 2\pi\gamma\rho(\theta)r_s = \mu_s r_s, \tag{13}$$

$$-(2r_s'\varphi_s' + r_s\varphi_s'') + \Omega r_s' = 0. \tag{14}$$

Equation (14) can be integrated to

$$\varphi_s'(\theta) = \frac{W_s}{2r_s^2} + \frac{\Omega}{2}, \tag{15}$$

where W_s is some integration constant to be determined. From (6) we get that

$$\int_0^{2\pi} r_s^2 d\theta = 1, \quad s = A, B \tag{16}$$

and by using this together with (7) and (15) we get the total angular momentum

$$l_0 = x_A l_A + x_B l_B \tag{17}$$

with

$$l_s = \pi W_s + \frac{\Omega}{2}. \tag{18}$$

Next we substitute (15) into (13) to solve for the real part of our system. After simplification and multiplication with r_s^3 we obtain

$$r_s''r_s^3 - \frac{W_s^2}{4} + \tilde{\mu}_s r_s^4 - 2\pi\gamma\rho(\theta)r_s^4 = 0, \tag{19}$$

where $\tilde{\mu}_s = \mu_s + \frac{\Omega^2}{4}$. By making the following Ansatz

$$r_B^2 = c_B(1 + c_A^{-1}r_A^2) \tag{20}$$

we're able write this in terms of only one component, where the coefficients c_s are to be determined. Substituting this into (4) we obtain

$$\rho = \frac{c_A x_A + c_B x_B}{c_A} r_A^2 + x_B c_B \tag{21}$$

Due to (16), c_B and c_A have to satisfy

$$1 = c_B(2\pi + c_A^{-1}). \tag{22}$$

This Ansatz is made to simplify the system. However we can't ensure that all possible solutions of (19) fulfill (21). If we now apply our new expression for $\rho(\theta)$ in (19) we arrive at

$$r_s''r_s^3 - \frac{W_s^2}{4} + \bar{\mu}_s r_s^4 - 2\pi\gamma_s r_s^6 = 0, \tag{23}$$

$$\gamma_s = c_s^{-1}(c_A x_A + c_B x_B)\gamma, \quad \bar{\mu}_s = \tilde{\mu}_s \mp 2\pi\gamma x_{\bar{s}} c_{\bar{s}}, \tag{24}$$

where $\bar{s}$ refers to the complementary component of our system. At this point we can already give an expression for our coefficients c_s through the means of the ratio $r = \gamma_A/\gamma_B = c_B/c_A$. Substituting this in (22), we get

$$c_A = \frac{1-r}{2\pi r}, \quad c_B = \frac{1-r}{2\pi}. \tag{25}$$

Here we realize that there are two different classes of solutions to equation (23). We distinguish between the two cases $c_A x_A + c_B x_B = 0$ and $c_A x_A + c_B x_B \neq 0$.

2.1 Class (i) solutions

First we look at when $c_A x_A + c_B x_B = 0$ which simplifies equation (23) by the vanishing interaction factor $\gamma_s = 0$. Next we use a suitable variable substitution, $r_s = \sqrt{\rho_s}$, to simplify the calculations. Equation (23) transforms into

$$\frac{1}{2}\rho_s\rho_s'' - \frac{1}{4}\rho_s'^2 + \bar{\mu}_s\rho_s^2 - \frac{W_s^2}{4} = 0. \tag{26}$$

Differentiation gives

$$\frac{1}{2}\rho_s\rho_s''' + \bar{\mu}_s 2\rho_s\rho_s' = 0. \tag{27}$$

Assuming that $\rho_s \neq 0$, we get

$$\rho_s''' + 4\bar{\mu}_s\rho_s' = 0. \tag{28}$$

This equation has three linearly independent solutions 1, $\cos\sqrt{4\bar{\mu}_s}\theta$, $\sin\sqrt{4\bar{\mu}_s}\theta$. Using the boundary conditions (11) and the normalization (16) we obtain

$$\rho_s = \frac{1}{2\pi}(1 + d_s \cos j_s(\theta - \theta_{0s})), \tag{29}$$

where $j_s = 1, 2...$, and we have introduced a phase constant θ_{0s} to avoid the sine term. Here we must take $|d_s| \leq 1$ in order to have $\rho_s \geq 0$. The parameters j_s and $\bar{\mu}_s$ are related by the formula

$$\bar{\mu}_s = \frac{j_s^2}{4}. \tag{30}$$

In the two component case there are a number of parameters, some of which are free and others fixed depending on the Bose-Einstein condensate's properties. First we have $r = \frac{-x_A}{x_B}$, due to the assumption $c_A x_A + c_B x_B = 0$. Moreover, $r \leq -1$ since $x_B \leq x_A$. Next we have the relation

$$\rho_B = \frac{1-r}{2\pi} + r\rho_A, \tag{31}$$

due to (21). Looking back at equation (26) with the solution to the density of states (29), we can find an expression for the integration constant W_s:

$$W_s^2 = \frac{j_s^2}{4\pi^2}(1 - d_s^2). \tag{32}$$

Furthermore we obtain the following relations by inserting the solutions (29) into (31):

$$j_A = j_B = j, \quad \theta_{0A} = \theta_{0B} = \theta_0, \quad d_B = r d_A. \tag{33}$$

In summary, we are free to choose d_B, with $|d_B| < 1$, $j = 1, 2, ...$ and θ_0 as we please and then the other parameters will be determined. For convenience we choose $d_B \geq 0$ which implies that $d_A \leq 0$.

Once this has been established we seek to find the phase functions $\varphi_s(\theta)$. Through integrating equation (15) with respect to θ we arrive at

$$\varphi_s(\theta) - \varphi_s(\theta_0) = \frac{\Omega}{2}(\theta - \theta_0) + \frac{W_s}{2}\int_{\theta_0}^{\theta} \frac{d\theta'}{\rho_s(\theta')}. \tag{34}$$

By making the series of variable substitutions $\phi' = j(\theta' - \theta_0)$ and then $u' = \tan(\frac{\phi'}{2})$, one finds that

$$\frac{W_s}{2}\int_{\theta_0}^{\theta} \frac{d\theta'}{\rho_s(\theta')} = \frac{\text{sgn} W_s}{2}\left\{2n\pi + 2\arctan\left[\sqrt{\frac{1-d_s}{1+d_s}}\tan\frac{\bar{\theta}}{2}\right]\right\}, \tag{35}$$

where

$$\text{sgn} W_s = \begin{cases} 1 & \text{if} \quad W_s > 0 \\ 0 & \text{if} \quad W_s = 0 \\ \text{-1} & \text{if} \quad W_s < 0 \end{cases} \tag{36}$$

and

$$j(\theta - \theta_0) = 2n\pi + \bar{\theta}, \quad -\pi < \bar{\theta} < \pi, \quad n \in \mathbb{Z}. \tag{37}$$

as $\bar{\theta} \to \pm\pi$ we find that the right hand side in (35) converges to $\frac{\mathrm{sgn} W_s}{2}(2n \pm 1)\pi$. Setting $\theta = \theta_0 + 2\pi$ in (34) (so that $n = j$ and $\bar{\theta} = 0$ in (37)) and using the boundary condition (12), we arrive at the following equation for Ω:

$$\Omega = 2J_s - j\mathrm{sgn} W_s. \tag{38}$$

It's clear that Ω is not a free parameter since it depends on our chosen value of j, J_s as well as W_s. Equation (34) can also be simplified in the following way using (38) and (37) in order to remove the Ω dependence

$$\varphi_s(\theta) - \varphi_s(\theta_0) = J_s(\theta - \theta_0) - \frac{\mathrm{sgn} W_s}{2}\bar{\theta} + \mathrm{sgn} W_s \arctan\left[\sqrt{\frac{1-d_s}{1+d_s}}\tan\frac{\bar{\theta}}{2}\right]. \tag{39}$$

Taking the difference between (38) with $s = A$ and B, we find that

$$J_A - J_B = \frac{j}{2}(\mathrm{sgn} W_B - \mathrm{sgn} W_A). \tag{40}$$

The final formula for the class (i) solutions is thus

$$\Psi_s(\theta) = \sqrt{\rho_s(\theta)}e^{i\varphi_s(\theta)} \tag{41}$$

with $\rho_s(\theta)$ given by (29) and $\varphi_s(\theta)$ given by (39).

It turns out that there is a much simpler way of expressing these solutions, namely in the form

$$\begin{aligned}\tilde{\Psi}_A(\theta) &= \left(\frac{c_0}{\sqrt{2\pi}} + \frac{c_1}{\sqrt{2\pi}}e^{-ij\mathrm{sgn} W_A(\theta-\theta_0)}\right)e^{iJ_A(\theta-\theta_0)+i\varphi_A(\theta_0)}, \\ \tilde{\Psi}_B(\theta) &= \left(\frac{d_0}{\sqrt{2\pi}} + \frac{d_1}{\sqrt{2\pi}}e^{-ij\mathrm{sgn} W_B(\theta-\theta_0)}\right)e^{iJ_B(\theta-\theta_0)+i\varphi_B(\theta_0)}.\end{aligned} \tag{42}$$

for some real coefficients c_0, c_1, d_0 and d_1.

The normalization (6) implies that

$$\sum_m |c_m|^2 = 1 \quad \sum_m |d_m|^2 = 1. \tag{43}$$

To see that (41) is the same as (42) in polar coordinates for some c_0, c_1, d_0, d_1, consider first the radius squared of $\tilde{\Psi}_A$ in (42):

$$\begin{aligned}|\tilde{\Psi}_A|^2 =& \frac{1}{2\pi} + \frac{c_0c_1}{2\pi}e^{ij(\theta-\theta_0)} + \frac{c_0c_1}{2\pi}e^{-ij(\theta-\theta_0)} \\ =& \frac{1}{2\pi}(1 + d_A\cos j(\theta-\theta_0))\end{aligned}$$

where

$$d_A = 2c_0c_1, \tag{44}$$

Similarly, we find that

$$|\tilde{\Psi}_B|^2 = \frac{1}{2\pi}(1 + d_B\cos j(\theta-\theta_0))$$

where

$$d_B = 2d_0 d_1 \tag{45}$$

This agrees with (29) and (33) if

$$d_0 d_1 = r c_0 c_1, \tag{46}$$

or, equivalently as

$$x_A c_0 c_1 + x_B d_0 d_1 = 0. \tag{47}$$

In order to check that the angle φ_A corresponding to (42) matches (39), it suffices to consider the case $J_s = 0$ and $\varphi_s(\theta_0) = 0$, since the corresponding terms in (39) clearly come from the factor $e^{iJ_s(\theta-\theta_0)+i\varphi_s(\theta_0)}$ in (42).

Equation (39) then yields

$$\varphi_A(\theta) = \left(\frac{\bar{\theta}}{2} - \arctan\left[\sqrt{Q_A}\tan\frac{\bar{\theta}}{2}\right]\right)(-\mathrm{sgn} W_A) \tag{48}$$

and hence

$$\begin{aligned}\tan\varphi_A(\theta) =& \frac{\tan\frac{\bar{\theta}}{2} - \sqrt{Q_A}\tan\frac{\bar{\theta}}{2}}{1+\sqrt{Q_A}\tan^2\frac{\bar{\theta}}{2}}(-\mathrm{sgn} W_A) \\ =& \frac{1/2(1-\sqrt{Q_A})\sin\bar{\theta}}{1/2(1+\sqrt{Q_A}) + 1/2(1-\sqrt{Q_A})\cos\bar{\theta}}(-\mathrm{sgn} W_A)\end{aligned}$$

where

$$Q_A = \frac{1-d_A}{1+d_A}. \tag{49}$$

If we now choose

$$c_0 = \frac{1}{2}(\sqrt{1+d_A} + \sqrt{1-d_A}), \quad c_1 = \frac{1}{2}(\sqrt{1+d_A} - \sqrt{1-d_A})$$

we see that (43) and (44) are satisfied, and that $c_0 > 0$ and $-c_0 \le c_1 \le 0$ (since $-1 \le d_A \le 0$). Moreover

$$1 - \sqrt{Q_A} = \frac{2c_1}{c_0 + c_1}, \quad 1 + \sqrt{Q_A} = \frac{2c_0}{c_0 + c_1}.$$

Thus

$$\tan\varphi_A(\theta) = \frac{c_1 \sin\bar{\theta}}{c_0 + c_1\cos\bar{\theta}}(-\mathrm{sgn} W_A),$$

On the other hand, (42) gives

$$\tan\tilde{\varphi}_A(\theta) = \frac{\mathrm{Im}\,\Psi_A(\theta)}{\mathrm{Re}\,\Psi_A(\theta)} = \frac{c_1\sin(j(\theta-\theta_0))}{c_0 + c_1\cos(j(\theta-\theta_0))}(-\mathrm{sgn} W_A) = \frac{c_1\sin\bar{\theta}}{c_0 + c_1\cos\bar{\theta}}(-\mathrm{sgn} W_A)$$

Since both $\tilde{\Psi}_A$ and $\tilde{\Psi}_B$ are confined to the right half-plane, we obtain that $\Psi_A = \tilde{\Psi}_A$.

Similar computations reveal that $\Psi_B = \tilde{\Psi}_B$ if we choose

$$d_0 = \frac{1}{2}(\sqrt{1+d_B} + \sqrt{1-d_B}), \quad d_1 = \frac{1}{2}(\sqrt{1+d_B} - \sqrt{1-d_B}).$$

This time we have $0 \le d_1 \le d_0$ (since $0 \le d_B \le 1$)

2.2 Class (ii) solutions

In the case where $x_A c_A + x_B c_B \neq 0$ we once again look back at equation (23) and with help from previous research one finds that the solution can be expressed in terms of elliptic functions (see the appendix):

$$r_s(\theta) = \sqrt{s_1 + (s_2 - s_1)\mathrm{sn}^2(\sqrt{\pi\gamma_s(s_3 - s_1)}(\theta - \theta_{0,s}); k_s)} \tag{50}$$

where

$$s_1 = \frac{1}{2\pi} + \frac{K(k_s)j_s^2(E(k_s) - K(k_s))}{\pi^3\gamma_s}, \tag{51}$$

$$s_2 = \frac{1}{2\pi} + \frac{K(k_s)j_s^2(E(k_s) + K(k_s)(k_s^2 - 1))}{\pi^3\gamma_s}, \tag{52}$$

$$s_3 = \frac{1}{2\pi} + \frac{K(k_s)E(k_s)j_s^2}{\pi^3\gamma_s}; \tag{53}$$

can be found in [8, p.24]. There the case $\gamma_s > 0$ is treated, but a direct verification shows that the same formulas are valid when $\gamma_s < 0$. Thus we find

$$r_s(\theta) = \sqrt{A_s - C_s \mathrm{sn}^2(D_s(\theta - \theta_{0,s}); k_s)}, \tag{54}$$

where

$$A_s = s_1, \quad C_s = -\frac{K^2(k_s)j_s^2k_s^2}{\pi^3\gamma_s}, \quad D_s = \frac{K(k_s)j_s}{\pi}. \tag{55}$$

Moreover, it follows from [8, p.7] that

$$\bar{\mu}_s = \pi\gamma_s(s_1 + s_2 + s_3), \quad W_s^2 = 4\pi\gamma_s s_1 s_2 s_3. \tag{56}$$

In order for the solution to be valid the following conditions must be fulfilled:

<u>$\gamma_s > 0$</u>

$$0 < s_1 < s_2 < s_3 \tag{57}$$

<u>$\gamma_s < 0$</u>

$$s_3 < 0 < s_2 < s_1. \tag{58}$$

Just as for the class (i) solutions, we integrate equation (15) which gives equation (34). Next we evaluate the integral

$$\frac{W_s}{2}\int_{\theta_0}^{\theta}\frac{d\theta'}{\rho_s(\theta')} = \frac{W_s}{2A_s}\int_{\theta_0}^{\theta}\frac{d\theta'}{1 - n_s\mathrm{sn}^2(D_s(\theta' - \theta_{0,s}); k_s)}, \tag{59}$$

where

$$n_s = \frac{C_s}{A_s}. \tag{60}$$

The variable substitution

$$u = \frac{j_s K(k_s)}{\pi}(\theta - \theta_{0,s}), \tag{61}$$

gives us that

$$\frac{W_s}{2A_s}\int_{\theta_0}^{\theta}\frac{d\theta'}{1-n_s\mathrm{sn}^2(D_s(\theta'-\theta_{0,s}));k_s)}=\frac{W_s\pi}{2A_sj_sK(k_s)}\int_0^u\frac{du'}{1-n_s\mathrm{sn}^2(u';k_s)}. \tag{62}$$

After expressing $\mathrm{sn}(u;k_s)$ using (108) we use the variable substitution

$$\phi=K^{-1}(u,k_s) \tag{63}$$

and that the integral can be expressed using the incomplete elliptic integral of the third kind

$$\varphi_s(\theta)-\varphi_s(\theta_0)=\frac{\Omega}{2}(\theta-\theta_{0,s})+\frac{W_s\pi}{2A_sj_sK(k_s)}\Pi(n_s;K^{-1}(u,k_s),k_s). \tag{64}$$

The boundary conditions then yield

$$\Omega=2J_s-2\mathcal{M}_s\mathrm{sgn}W_s, \tag{65}$$

where

$$\mathcal{M}_s=\frac{|W_s|}{4A_sj_sK(k_s)}\Pi(n_s;K^{-1}(2j_sK(k_s);k_s),k_s), \tag{66}$$

using the relations

$$K(j_s\pi,k_s)=2j_sK(k_s),\quad j_s\in\mathbb{Z},\quad \text{and}\quad \Pi(n_s;j_s\pi,k_s)=2j_s\Pi(n_s;\frac{\pi}{2},k_s)$$

we can rewrite

$$\mathcal{M}_s=\frac{|W_s|}{2A_sK(k)}\Pi(n_s;\frac{\pi}{2},k_s). \tag{67}$$

where if we apply (54) to (20) we find that

$$k_A=k_B=k,\quad \theta_{0,A}=\theta_{0,B}=\theta_0,\quad j_A=j_B=j \tag{68}$$

Just as for class (i) solutions,

$$J_B-J_A=\mathcal{M}_B\mathrm{sgn}W_B-\mathcal{M}_A\mathrm{sgn}W_A. \tag{69}$$

which will determine for what values of r we obtain a solution given the interval for k. We would also like to remind the reader of the relations obtained from (24)

$$\begin{cases}\gamma_A=(x_A+rx_B)\gamma\\ \gamma_B=(\frac{x_A}{r}+x_B)\gamma\end{cases} \tag{70}$$

which gives the following three equations when trying to find for what value of r, s_1,s_2,s_3 become zero under the condition that $\gamma_A>0$ and $\gamma_B<0$, respectively.

$$r_{0,A}=\frac{-x_A\pi^2\gamma+2j^2K(k)(K(k)-E(k))}{\pi^2\gamma x_B}, \tag{71}$$

$$r_{0,B}=\frac{-x_A\pi^2\gamma}{\pi^2\gamma x_B+2j^2K(k)(E(k)+(k^2-1)K(k))}, \tag{72}$$

$$r_{1,B}=\frac{-x_A\pi^2\gamma}{\pi^2\gamma x_B+2j^2K(k)E(k)}, \tag{73}$$

where $r_{1,B}$ is the maximum value at which we can find a solution and either $r_{0,A},r_{0,B}$ being the minimum. These will determine for each k in what range r we can find solutions.

3 yrast spectrum for two components

We now return to the problem of minimizing the functional (3) under the two constraints (6) and (7). We begin by investigating what happens if we make the change of variable $\psi_s = \Psi_s e^{i\theta}$. We note that ψ_s is 2π-periodic and that

$$m[\psi_s] = m[\Psi_s] = 1. \tag{74}$$

Moreover

$$\begin{aligned} l[\psi] &= \sum_s x_s \left[\frac{1}{i} \int_0^{2\pi} \Psi_s' \bar{\Psi}_s d\theta + \int_0^{2\pi} |\Psi_s|^2 d\theta \right] \\ &= l[\Psi] + x_A + x_B \\ &= l_0 + 1 \end{aligned} \tag{75}$$

due to the facts that

$$\psi_s' \bar{\psi}_s = \Psi_s' \bar{\Psi}_s + i|\Psi_s|^2, \tag{76}$$

$$x_A + x_B = \frac{N_A}{N} + \frac{N_B}{N} = \frac{N_A + N_B}{N_A + N_B} = 1. \tag{77}$$

Once these relations have been established we look at the energy functional assuming $m[\psi_s] = 1$ and $l[\Psi] = l_0$.

$$\begin{aligned} E[\psi] =& \sum_s x_s \int_0^{2\pi} |\Psi_s'|^2 d\theta + \pi\gamma \int_0^{2\pi} \rho(\theta)^2 d\theta \\ &+ \sum_s x_s \left[i \int_0^{2\pi} \Psi_s \bar{\Psi}_s' d\theta - i \int_0^{2\pi} \Psi_s' \bar{\Psi}_s d\theta + \int_0^{2\pi} |\Psi_s|^2 d\theta \right] \\ =& E[\Psi] + 2l[\Psi] + x_A + x_B \\ =& E[\Psi] + 2l_0 + 1 \end{aligned} \tag{78}$$

where we have used partial integration to rewrite

$$i \sum_s x_s \int_0^{2\pi} \Psi_s \bar{\Psi}_s' d\theta = l[\Psi] \tag{79}$$

This can be rewritten as

$$E[\psi] = E[\Psi] + (l_0 + 1)^2 - l_0^2. \tag{80}$$

If we instead set $\psi_s = \bar{\Psi}_s$ similar computations give

$$m[\psi_s] = 1, \quad l[\psi] = -l[\Psi] = -l_0 \quad \text{and} \quad E[\psi] = E[\Psi]. \tag{81}$$

Theorem 1. *Let $e_0(l_0) = E_0(l_0) - l_0^2$. Then $e_0(l_0)$ is a 1-periodic even function.*

Proof. Let ψ with $l[\psi] = l_0 + 1$ be given and set $\Psi_s = \psi_s e^{-i\theta}$. Then $l[\Psi] = l_0$ and (80) implies that

$$e_0(l_0) \leq E[\Psi] - l_0^2 = E[\psi] - (l_0 + 1)^2. \tag{82}$$

Assuming that ψ minimizes E, we obtain

$$e_0(l_0) \leq E_0(l_0+1) - (l_0+1)^2 = e_0(l_0+1). \tag{83}$$

Interchanging the roles of ψ and Ψ, we obtain that

$$e_0(l_0) \geq e_0(l_0+1), \tag{84}$$

and hence

$$e_0(l_0) = e_0(l_0+1), \tag{85}$$

which means that the function $e_0(l_0)$ is 1-periodic. The evenness of e_0 is shown similarly by letting $\psi_s = \bar{\Psi}_s$ and using (81). $\square$

This result is more commonly known as Bloch's theorem [3].

3.1 Energy minimization of class (i) solutions

Once we have established this relation we compute the energy and momentum of class (i) solutions using formulas (42).

If we evaluate the total angular momentum (7) using (42) we find that

$$l[\Psi] = -j(\text{sgn}W_A x_A |c_1|^2 + \text{sgn}W_B x_B |d_1|^2) + J_A x_A + J_B x_B, \tag{86}$$

while evaluating the energy functional (3) is given by

$$\begin{aligned} E[\Psi] =& x_A \int_0^{2\pi} |\Psi_A'|^2 d\theta + x_B \int_0^{2\pi} |\Psi_B'|^2 d\theta + \pi\gamma \int_0^{2\pi} \rho^2 d\theta \\ =& j^2(x_A|c_1|^2 + x_B|d_1|^2) + \frac{\gamma}{2} - 2j(\text{sgn}W_A x_A J_A |c_1|^2 + \text{sgn}W_B x_B J_B |d_1|^2) \\ &+ J_A^2 x_A + J_B^2 x_B \end{aligned} \tag{87}$$

These formulas can be used to plot the relationship between E and l, but they are quite complicated to understand. It turns out that there is an easier approach to find the least energy for $0 \leq l_0 \leq x_B$. Here we follow Anoshkin, Wu and Zaremba [1]; see also Smyrnakis et.al [7]. We replace (42) with $j = 1$ by the expressions

$$\begin{aligned} \tilde{\Psi}_A(\theta) &= \frac{\tilde{c}_0}{\sqrt{2\pi}} + \frac{\tilde{c}_1}{\sqrt{2\pi}} e^{i\theta}, \\ \tilde{\Psi}_B(\theta) &= \frac{\tilde{d}_0}{\sqrt{2\pi}} + \frac{\tilde{d}_1}{\sqrt{2\pi}} e^{i\theta}. \end{aligned} \tag{88}$$

where we no longer require that $|\tilde{c}_1| \leq |\tilde{c}_0|$ and $|\tilde{d}_1| \leq |\tilde{d}_0|$ (notice that we can take $\theta_0 = 0$ and $\varphi_A(0) = \varphi_B(0) = 0$ without loss of generality when computing the energy and momentum). This can be achieved by taking $\text{sgn}W_A = -1$ and $J_A = 0$ or $\text{sgn}W_A = 1$ and $J_A = 1$ (and similarly for the B-component), so that either $\tilde{c}_0 = c_0$ and $\tilde{c}_1 = c_1$ or $\tilde{c}_0 = c_1$ and $\tilde{c}_1 = c_0$ (and similarly for $\tilde{d}_0, \tilde{d}_1$).

Then

$$l[\tilde{\Psi}] = x_A|\tilde{c}_1|^2 + x_B|\tilde{d}_1|^2$$

$$\begin{aligned} E[\tilde{\Psi}] =& x_A|\tilde{c}_1|^2 + x_B|\tilde{d}_1|^2 + \frac{\gamma}{2} \\ =& l[\tilde{\Psi}] + \frac{\gamma}{2} \end{aligned}$$

It remains to be seen which values of l can be attained using the special Ansatz (88).

Note that

$$|\tilde{c}_0|^2 + |\tilde{c}_1|^2 = 1$$

$$|\tilde{d}_0|^2 + |\tilde{d}_1|^2 = 1$$

and

$$x_A|\tilde{c}_0||\tilde{c}_1| = x_B|\tilde{d}_0||\tilde{d}_1|.$$

Using these relations, we find that $l[\tilde{\Psi}] = l_0$ if and only if

$$|\tilde{c}_0|^2 = \frac{(x_A - l_0)(1 - l_0)}{x_A(1 - 2l_0)}, \quad |\tilde{c}_1|^2 = \frac{l_0(x_B - l_0)}{x_A(1 - 2l_0)}, \tag{89}$$

$$|\tilde{d}_0|^2 = \frac{(x_B - l_0)(1 - l_0)}{x_B(1 - 2l_0)}, \quad |\tilde{d}_1|^2 = \frac{l_0(x_A - l_0)}{x_B(1 - 2l_0)}. \tag{90}$$

For these relations to hold l_0 must either be in the interval $0 \le l_0 \le x_B$ or $x_A \le l_0 \le 1$. Specializing to the case $0 \le l_0 \le x_B$ we find that $|\tilde{c}_1| \le |\tilde{c}_0|$, so that $J_A = 0$, while $|\tilde{d}_1| \le |\tilde{d}_0|$ and hence $J_B = 0$ when $0 \le l_0 \le \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2x_B}$, and $|\tilde{d}_0| \le |\tilde{d}_1|$ and hence $J_B = 1$ when $\frac{1}{2} - \frac{1}{2}\sqrt{1 - 2x_B} \le l_0 \le x_B$.

Theorem 2. *Let $0 \le l_0 \le x_B$. Then $E_0(l_0) = l_0 + \frac{\gamma}{2}$, where $\gamma \in \mathbb{R}$.*

Proof. We choose the coefficients in (88) such that $l[\tilde{\Psi}] = l_0$ and add a variation to (88) such that

$$\Psi_A = \tilde{\Psi}_A + \delta\Psi_A, \quad \Psi_B = \tilde{\Psi}_B + \delta\Psi_B \tag{91}$$

where $(\delta\Psi_A, \delta\Psi_B) \in X$ are such that $\Psi \in X_{l_0}$. We expand $\delta\Psi_A$ and $\delta\Psi_B$ in Fourier series:

$$\delta\Psi_A = \sum_m \delta c_m \frac{e^{im\theta}}{\sqrt{2\pi}}, \quad \delta\Psi_B = \sum_m \delta d_m \frac{e^{im\theta}}{\sqrt{2\pi}} \tag{92}$$

If we make use of (91) and the angular momentum constraint (7) we find that

$$x_A(\bar{\tilde{c}}_1 \delta c_1 + \tilde{c}_1 \overline{\delta c_1}) + x_B(\bar{\tilde{d}}_1 \delta d_1 + \tilde{d}_1 \overline{\delta d_1}) = -x_A \sum_m m|\delta c_m|^2 - x_B \sum_m m|\delta d_m|^2. \tag{93}$$

Next we look at how the variation affects the energy and find that

$$E[\Psi] = x_A \int_0^{2\pi} |\tilde{\Psi}'_A + \delta\Psi'_A|^2 d\theta + x_B \int_0^{2\pi} |\tilde{\Psi}'_B + \delta\Psi'_B|^2 d\theta + \pi\gamma \int_0^{2\pi} (\tilde{\rho} + \delta\rho)^2 d\theta \tag{94}$$

where

$$\begin{aligned} \tilde{\rho} &= x_A|\tilde{\Psi}_A|^2 + x_B|\tilde{\Psi}_B|^2 \\ &= \frac{1}{2\pi} \end{aligned} \tag{95}$$

and

$$\delta\rho = \rho - \tilde{\rho}. \tag{96}$$

The constraints (6) imply that

$$\int_0^{2\pi} \delta\rho d\theta = \int_0^{2\pi} \rho d\theta - \int_0^{2\pi} \tilde{\rho} d\theta = 0 \tag{97}$$

and hence

$$\begin{aligned}\int_0^{2\pi} (\tilde{\rho} + \delta\rho)^2 d\theta &= \int_0^{2\pi} \tilde{\rho}^2 d\theta + 2\int_0^{2\pi} \tilde{\rho}\delta\rho d\theta + \int_0^{2\pi} (\delta\rho)^2 d\theta \\ &= \int_0^{2\pi} \tilde{\rho}^2 d\theta + \int_0^{2\pi} (\delta\rho)^2 d\theta.\end{aligned} \tag{98}$$

Expanding the first two term in (94) and using (93), we therefore find that

$$\begin{aligned}E[\Psi_A, \Psi_B] =& E[\tilde{\Psi}] + x_A(\bar{\tilde{c}}_1 \delta c_1 + \tilde{c}_1 \overline{\delta c_1}) + x_B(\bar{\tilde{d}}_1 \delta d_1 + \tilde{d}_1 \overline{\delta d_1}) \\ &+ x_A \sum_{m=-\infty}^{\infty} m^2 |\delta c_m|^2 + x_B \sum_{m=-\infty}^{\infty} m^2 |\delta d_m|^2 + \pi\gamma \int_0^{2\pi} (\delta\rho)^2 d\theta \\ =& E[\tilde{\Psi}] + x_A \sum_{m=-\infty}^{\infty} (m^2 - m)|\delta c_m|^2 + x_B \sum_{m=-\infty}^{\infty} (m^2 - m)|\delta d_m|^2 + \pi\gamma \int_0^{2\pi} (\delta\rho)^2 d\theta.\end{aligned} \tag{99}$$

Since all parts of (99) are positive we can say for certain that $E[\Psi] \geq E[\tilde{\Psi}]$. Hence $E_0(l_0) = E[\Psi] = l_0 + \frac{\gamma}{2}$. □

3.2 Energy minimization of class (ii) solutions

The solutions found belonging to the second kind are rather complicated to work with. Because of this it is not possible to get an explicit result that describes how the energy of the Bose-Einstein condensate will change depending of the angular momentum. Instead one must use numerical tools in order to say anything regarding this part of the yrast spectrum.

While minimum energy in the range $0 \leq l_0 \leq x_B$ was attained by class (i) solutions, the class (ii) solutions minimize the energy when $x_B \leq l_0 \leq 1/2$. As for class (i) we will mainly look at solutions where $j = 1$ as it can be checked numerically that any $j > 1$ would result in a higher energy. The main task will be to determine how the ratio r will change under the elliptic parameter k which is easiest done numerically.

Our ratio can thus be seen as a function $r(m, \mathcal{J})$ where $m = k^2$ and $\mathcal{J} = J_B - J_A$ which is the difference between the winding numbers and is given by (69). By using said equation and evaluating it for different values of m, we are able to determine what allowed values of the ratio $r(m, \mathcal{J})$ there are. Of course this depends on the value of $\mathcal{J}$ and x_B.

Once the values of $r(m, \mathcal{J})$ were determined we started looking at how the energy given by (3) is related to the total angular momentum (17) by evaluating each function at the given r and m value. By combining Theorems 1 and 2 it is clear that the part of the yrast spectrum corresponding to class (i) solutions is fairly simple. Thus the real interest is what occurs when where $l_0 \geq x_B$ and our class (ii) solutions continue.

The following figures were produced numerically utilizing the equations from Section 2.2. We choose to hold certain parameters fixed and find for what interval r for each m we're able to find a solution and for what combination of $\mathrm{sgn}W_A$ and $\mathrm{sgn}W_B$ the function

$$\mathcal{F}(r,m) = \mathcal{M}_B \mathrm{sgn} W_B - \mathcal{M}_A \mathrm{sgn} W_A - \mathcal{J} \tag{100}$$

changes sign. For some combinations and higher values of $\mathcal{J}$, (100) will change sign multiple times leading to multiple solutions. As $m \to 1$ it is necessary to use a finer division to get more values for r. This is needed in order to reach all the way up to $l_0 = 1/2$. The values of x_B determine how close m has to be to 1 in order to reach the limit of l_0. We choose the same division of m for all given values of x_B. This results in l_0 not reaching all the way up to $1/2$ for small values of x_B.

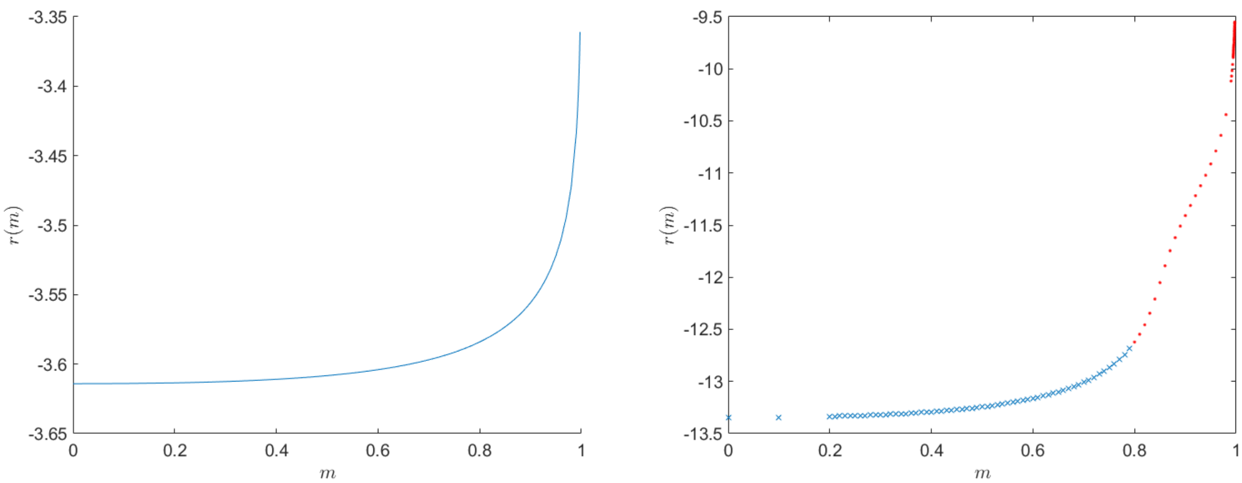

Fig. 1: The ratio $r(m)$ plotted as a function of m for $\gamma = 23$ with $x_B = 0.2$ (left) and $x_B = 0.06$ (right), where the blue crosses and red dots correspond to $\mathcal{J} = 1$ and 2 respectively.

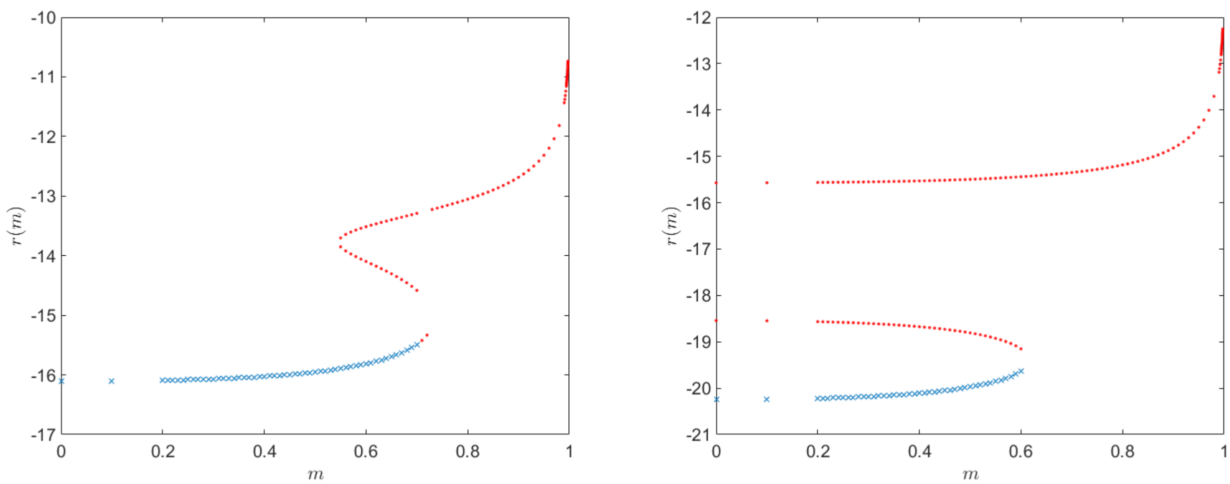

Fig. 2: The ratio $r(m)$ plotted as a function of m for $\gamma = 23$ with $x_B = 0.05$ (left) and $x_B = 0.04$ (right), where the blue crosses and red dots correspond to $\mathcal{J} = 1$ and 2 respectively.

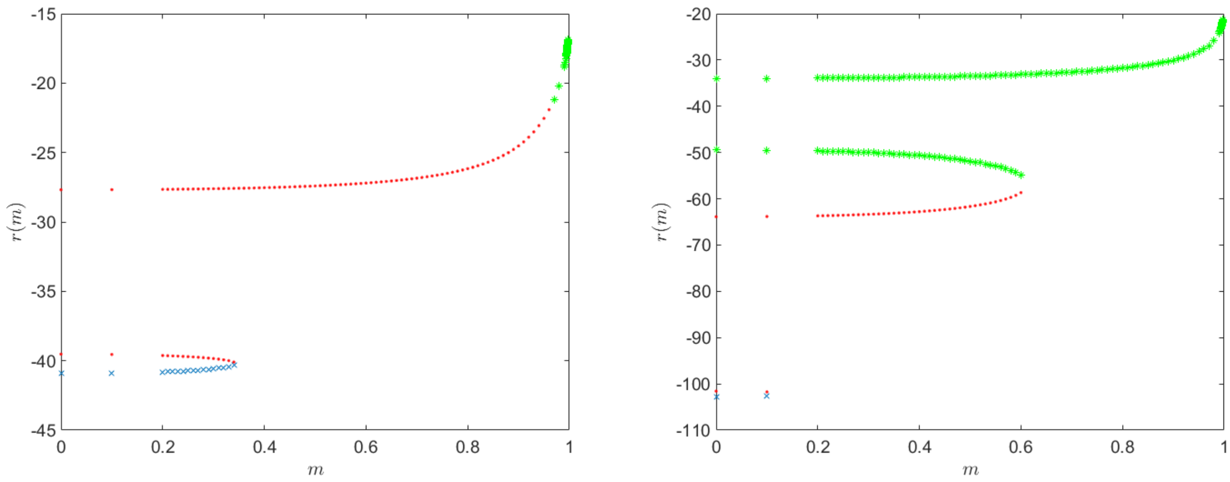

Fig. 3: The ratio $r(m)$ plotted as a function of m for $\gamma = 23$ with $x_B = 0.02$ (left) and $x_B = 0.008$ (right), where the blue crosses, red dots and green asterisks correspond to $\mathcal{J} = 1, 2, 3$ respectively.

For smaller values of x_B the winding number increases and multiple solutions can be found for the same combination of parameters such as sgnW_A and sgnW_B, $\mathcal{J}$ and m. Here it's not enough to only look at the end points of the interval where we can find a solution but instead the interval from r_0 to r_1 (where r_0 is the maximum value of $r_{0,A}$, $r_{0,B}$), is divided into smaller ones. This issue does not come into effect until $\mathcal{J} \geq 3$ for this particular choice of γ. By keeping $J_A = 0$ we allow J_B to determine the value of $\mathcal{J}$ as it jumps for decreasing values of x_B.

What we come to realize is that for decreasing values of x_B (and thus increasing x_A) the number of lobes increases. When examining individual m values it was found that for each x_B there was a singular point for which we would find a possible solution. These singular points all had an end point for which $\mathcal{F} = 0$. Since the code only gave an output if (100) changed sign they're not found in Figures 1 through 3.

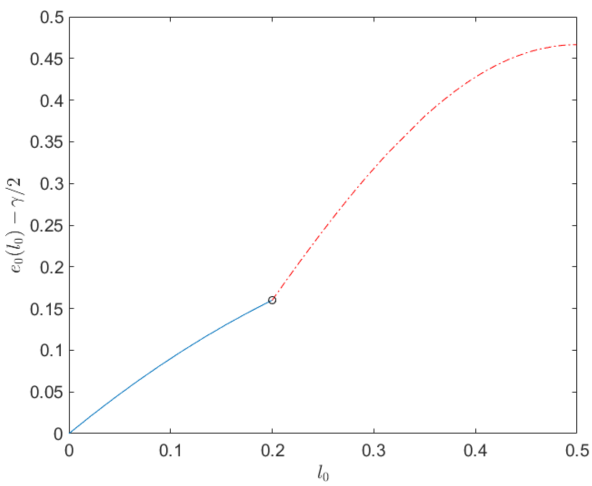

Fig. 4: yrast spectrum $e_0(l)$ as a function of l for $x_B = 0.2$ and $\gamma = 23$. The solid blue part corresponds to the class (i) solutions and the dashed-dotted for class (ii) solutions.

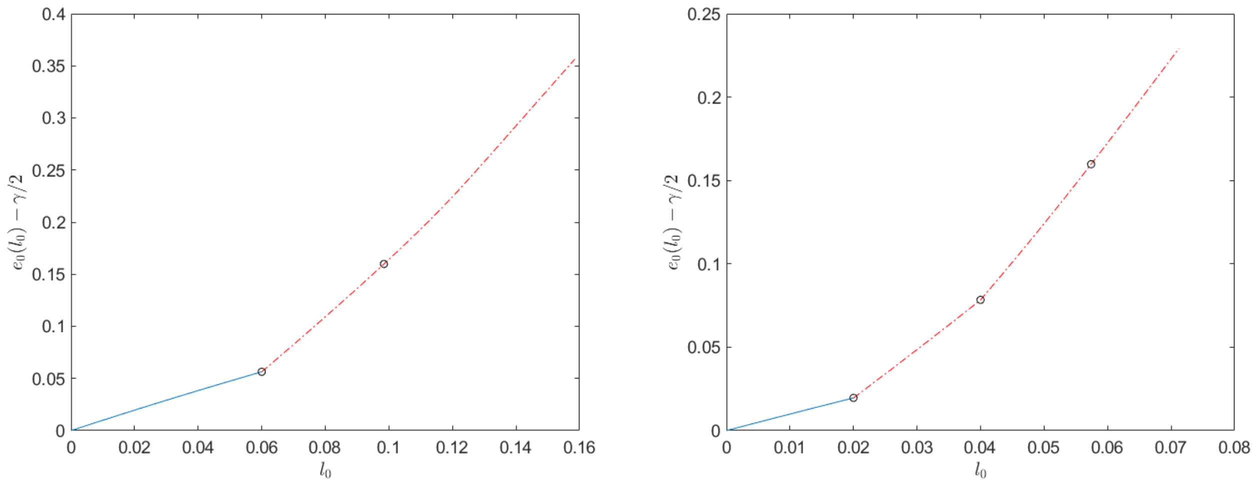

Fig. 5: yrast spectrum $e_0(l_0)$ as a function of l_0 for $x_B = 0.06$ (left) and $x_B = 0.02$ (right) with $\gamma = 23$. The solid blue part corresponds to the class (i) solutions and the dash-dotted red for class (ii) solutions.

In figures 4 through 5 we see how the part corresponding to class (i) solutions breaks off and continues for class (ii). The dark circles represent the point at which $\mathcal{J}$ (and J_B) increases by 1. Each consecutive breaking point indicates a similar jump of J_B. If one were to increase the limit of m closer to 1 it would result in a continuation of the energy function for higher values of l_0.

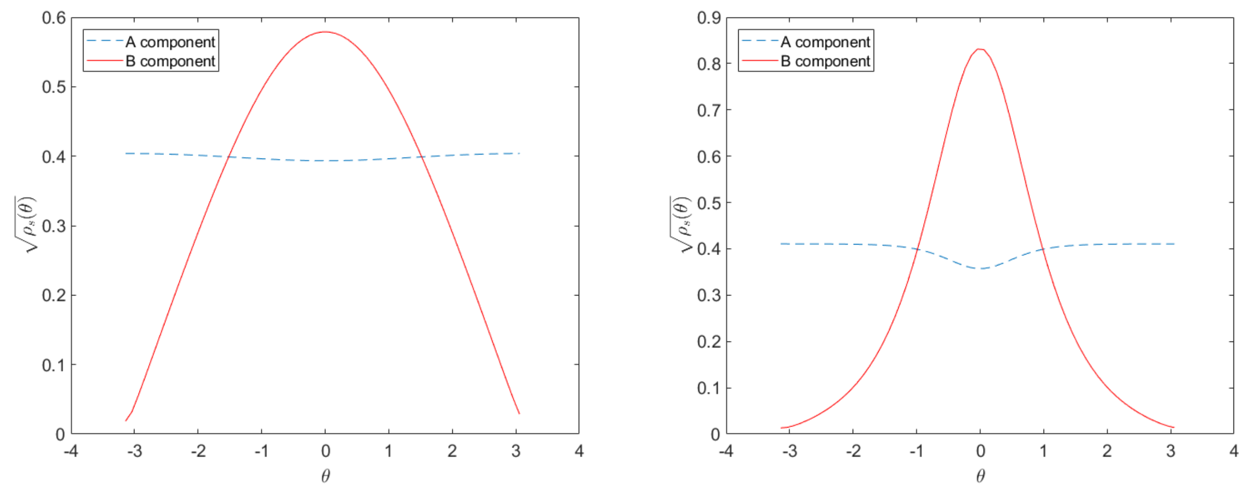

Fig. 6: Amplitude change for increasing values of $l_0 = 0.03, 0.07$ from left to right for $x_B = 0.02$, with the corresponding values of $\Omega = 2.99, 5.19$.

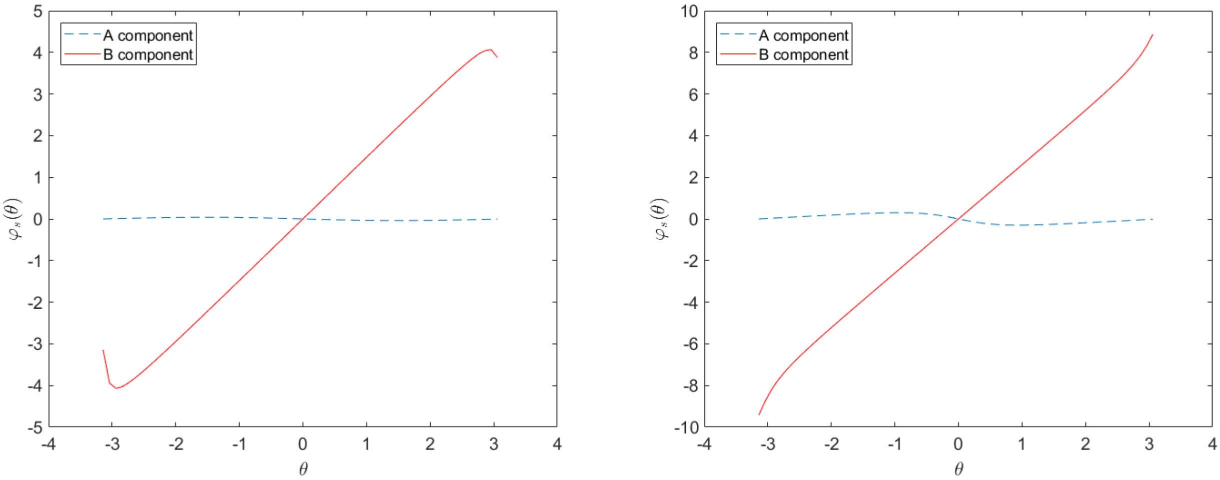

Fig. 7: Angular change for each component over the interval $-\pi \leq \theta \leq \pi$ for the corresponding values mentioned previously in Figure 6

Figure 6 displays how the radius (50) for each component changes with increasing angular momentum. Because of how we choose x_B and x_A we see a more drastic variation over the period 2π for the B-component. Similarly we see how the angular part is almost constant around zero for A while B makes a full rotation.

The special case where $m = 0$ was especially looked at for every x_B treated here and possible combination of $\mathcal{J}$, sgnW_A and sgnW_B for which a solution was found. As one were to expect the radius for both components resulted in $\sqrt{\rho_s} = 1/\sqrt{2\pi}$. For the angle however it was found that due to the choice $J_A = 0$, $\varphi_A = 0$ while for the B-component $-J_B\pi \leq \varphi_B \leq J_B\pi$.

4 Discussion

The following topics were not included in this thesis but would be of interest to examine:

One should have another look at the class (i)-solutions and plot the energy (87) against the angular momentum (86) for different parameters. In particular it would be interesting to see what happens when $l_0 \geq x_B$ for other values of J_A and J_B.

So far we have only proven that the class (i) and class (ii) solutions are critical functions in general. This is a necessary condition, however not enough to ensure that they do in fact minimize the functional (3). In the range $0 \leq l_0 \leq x_B$ we were able to show directly that the class (i) solutions (with same specific parameter values)are minimizers. Another possibility would be to use the second derivative test. This was treated for the one component case in [4].

Through the means of some physical intuition regarding the parameter j provided from [12] we have assumed that $j = 1$ gives the lowest possible energy for the Bose-Einstein condensate. However one should examine what happens when with the energy when $j \geq 1$ in particular for class (ii)-solutions.

One topic which is currently an active area of research is the asymmetric case where at least one of the interaction strengths are different. This case is much more intractable and there are less explicit solutions [6].

Finally we would like to remind the reader that the solutions found comes from Ansatz (20) and that there could be other solutions.

A Appendix

A.1 Elliptic functions

Through out the thesis elliptic functions were used in order to solve the nonlinear Schrödinger equation. Therefore it's important to understand how these functions are defined and how they are related. First we introduce the *incomplete elliptic integral of the first kind*

$$K(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{y} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \tag{101}$$

the *incomplete elliptic integral of the second kind*

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^{y} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt, \tag{102}$$

and the *incomplete elliptic integral of the third kind*

$$\Pi(n; \varphi, k) = \int_0^{\varphi} \frac{d\theta}{(1 - n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{y} \frac{dt}{(1 - nt^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}} \tag{103}$$

where $y = \sin(\varphi)$ with $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $t = \sin(\theta)$ and $k \in [0, 1)$. In the incomplete elliptic integral of the third kind we also have a parameter n which sets it apart from the other two. The *complete* elliptic integrals of first, second and third kind are defined as follows:

$$K \equiv K(k) \equiv K(\pi/2, k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{1} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \tag{104}$$

$$E \equiv E(k) \equiv E(\pi/2, k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^{1} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt, \tag{105}$$

$$\Pi \equiv \Pi(n; k) \equiv \Pi(n; \pi/2, k) = \int_0^{\pi/2} \frac{d\theta}{(1 - n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} \tag{106}$$

$$= \int_0^{1} \frac{dt}{(1 - nt^2)\sqrt{(1 - t^2)(1 - k^2 t^2)}}. \tag{107}$$

The function $\varphi \mapsto K(\varphi, k)$ extends to a strictly increasing function on $\mathbb{R}$. We let $K^{-1}(u, k)$ be its inverse and define

$$\operatorname{sn}(u; k) = \sin K^{-1}(u, k). \tag{108}$$

This is one of the twelve Jacobi elliptic functions though there is only need to define three of them since the rest can be expressed in terms of these. In addition to $\operatorname{sn}(u; k)$, these three are:

$$\begin{cases} \operatorname{cn}(u; k) = \cos K^{-1}(u, k) \\ \operatorname{dn}(u; k) = \sqrt{1 - k^2 \operatorname{sn}^2(u; k)}. \end{cases} \tag{109}$$

The derivatives of these elliptic functions are used in the thesis:

$$\begin{cases} \frac{\partial}{\partial u}\mathrm{sn}(u;k) = \mathrm{cn}(u;k)\mathrm{dn}(u;k) \\ \frac{\partial}{\partial u}\mathrm{cn}(u;k) = -\mathrm{sn}(u;k)\mathrm{dn}(u;k) \\ \frac{\partial}{\partial u}\mathrm{dn}(u;k) = -k^2\mathrm{sn}(u;k)\mathrm{cn}(u;k). \end{cases}$$

We also use the following periodicity and parity properties:

$$\begin{cases} \mathrm{sn}(u;k) = \mathrm{sn}(u+4K;k) \quad \text{(odd)} \\ \mathrm{cn}(u;k) = \mathrm{cn}(u+4K;k) \quad \text{(even)} \\ \mathrm{dn}(u;k) = \mathrm{dn}(u+2K;k) \quad \text{(even).} \end{cases}$$

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