A UTILITY APPROACH: STRATEGY ANALYSIS AND OPTIMIZATION

Master's thesis in Mathematical Statistics

By

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Abstract

Utility theory and Monte Carlo simulations are used to calculate optimal allocation for long term as well as, risk averse investors with a portfolio consisting of one risky asset and one risk-free bank account. The problems solved in this thesis are divided into two types, static and dynamic. A strategy is given for a static problem, such as Buy and Hold (B&H) or Constant Weights (CW) and optimal weights are calculated and analyzed. Optimal allocations for different static strategies can be compared with Bootstrap. Analysis is done with four different equity models. The four models are constructed so that the two first moments are almost equal in order for them to be comparable. The main conclusion is that when stochastic jumps are introduced (the Merton model), it does not affect the optimal weight in the risky asset to decrease as much as when stochastic volatility is introduced (the Heston model). A volatility control (VC) model was analyzed and compare with CW and it was found that VC wasn't a better option for a long term investor.

Dynamic programming is used to solve the Bellman-equation for optimal dynamic strategies. Validation is used by comparing the results with problems where theoretical result are known. It was found when solving a consumption and investment problem, that the consumption result are much more robust compared to the optimal weights. The error when using 10^5 simulations for the validation problems is only $\sim \pm 10^{-9}$ for consumption but $\sim \pm 0.01$ for optimal weight. Underlying processes are introduced e.g., stochastic volatility, and an algorithm that uses a proxy function on the maximum argument instead of the objective function is presented. The main reason for this is that the algorithm is much more flexible compared to the using regression on the objective function i.e., the same algorithm can be used for any underlying asset without changing the algorithm much. The downside is that many trajectories are needed to get the same confidence interval. There is left to prove theoretically that the algorithm converges to the right solution in a general case but several validation tests hint at convergence.

Keywords: Utility optimization, Portfolio analysis, Dynamic programming, Bellman equation.

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Chapter 1

Introduction

1.1 Background

Utility theory dates back to the nineteenth century and is still applied today in e.g., economics, philosophy and AI. The general idea is that preference for a rational agent can be represented with a so called utility function. Some typical utility functions are assumed in this thesis and the results from those are presented. It is not claimed that all agents or even any agent can be represented completely with a relatively simple utility function. The philosophy behind this thesis is that utility theory should be used as a tool to analyze how an agent could act intelligently when taking financial decisions and has little knowledge about the economics behind the financial products.

A big topic today, in the finance industry, is "Robo-advisor" and to do that accurately must an agent's personal preference be taken into account. Those preferences are approximated by a utility function. Utility theory is used to analyze long-term investments with and without consumption during a time period. There are infinite utility functions that can be studied, that needs therefore to be heavily limited. It was concluded to focus on HARA and CRRA utility function through literature analysis of mainly Hult et al. (2012), Danthine and Donaldson (2015) and Campbell and Viceira (2001). Some special features which some people may have are added to CRRA/HARA and studied.

To optimize allocations for an rational agent are models for the financial assets needed. During the last few decades have several models be presented for both equities and interest rates. For equities is the Black and Scholes (B-S), Merton, Heston and Bates model used. Effects of stochastic interest rates are studied, when the Vacicek model is used.

1.2 Financial assets and stylized facts

The stylized facts are given from empirical observations from time series of stock/index returns and similar financial products. For a more details, see Cont (2001) and Lindström et al. (2015). In this thesis, the most important stylized facts will be presented.

Autocorrelation in returns and absolute returns: There seem to be no (or very little) autocorrelation for returns. However there are some autocorrelation for absolute returns ($Cov(|r_t|, |r_s|) \neq 0, \quad s \neq t$) and the difference $|t - s|$ is not "to large".

Volatility clustering: The volatility is considered to be constant for the simple B-S model. But time series of returns show that the volatility of a financial asset seem to "cluster", meaning that the returns are bigger (both negative and positive) for high volatility time periods. Volatility clustering can explain some of the effect of positive autocorrelation in absolute returns. The Heston model (and extension) takes this into account.

Gain/loss asymmetry and unconditional heavy tails: The highest gains are smaller than the biggest losses. These can be models with a Hidden Markov model where one have two underlying models, such as in Nystrup et al. (2018). The Merton model is used in this thesis to simulated this property. This model has stochastic jumps from a Poisson process and the jump size is assumed to be log-normal i.e., the process is a Lévy Process. The unconditional heavy tails show that the returns have heavier tails then a typical Gaussian. When both the effects are combined i.e., Lévy process and stochastic volatility is the model called the Bates model. This will be explored in more detail later on.

1.3 Objective and limitations

The goal is to establish a framework using Monte-Carlo simulations, convex optimization and dynamic programming to solve optimal allocation problems, for long term investments such as pension plans. There will only be two assets in the portfolio, one risky asset and one risk-free. The risky-free account will be considered to be a bank account and not a bond.

Since the paper Black and Scholes (1973) where the so called B-S model for equities (and even indexes) has more advanced models been presented such as Merton, Heston and Bates model. These models should, in theory, take more of the stylized facts into account and should be a better representation of the "real-world". One of the questions behind this thesis is how these added components to the models effect optimal allocation for an investor. Optimal allocation in this thesis means that the expected utility is maximized.

It was deiced that the parameters for the B-S would be the approximated average for S&P-500 index. To be able to make the models comparable will the parameter not be estimation through statistical analysis of a financial time series, but reasonable parameters will be found in other ways. This is to guarantee approximately equal two first moments. Another reason for this choice is because it would take too much time from the work to make a reliable parameter estimator for example for the Bates model. Even if one would find good methods for the parameter estimation would a very long time-series be needed so that the models would have equal two first moments.

It would, even with data, not be certain to get the models to be comparable. That would be another interesting problem to study in the future i.e., the difference in the two first moment from the same time series for the different models for different parameter estimation techniques. Notice that the comparison will be made compared to B-S and not to each other. The reason for this is that the parameters are not estimated from time series and it is not certain that the parameters selected for example Heston and Merton could be from the same time-series. Since B-S only takes the two first moments into account will the models be comparable to the extended models. But it is a a clear limitation of this thesis that no parameter estimation is done for the models, especially for practical implementations.

The main optimization problem will be done for a case where the investor only consumes at the final time point. There will be four different strategies considered but mainly will the focus be on two of them. Those two strategies are Buy and Hold (B&H) and constant weights (CW). The strategies are as simple as they sound, for B&H does one allocate today and does not change the investment until the final time point and for CW the assets are rebalanced regularly so that the weights are the same at the beginning of each time period. The weights for these strategies will be optimized and compared for the four models (B-S, Merton, Heston and Bates). The other two strategies are volatility control and CPPI, these strategies will be defined in detail later. Those models will not be optimized in any way. For volatility control, feasible risk control products will be used. CPPI will only be considered in the theory chapter and not be analyzed.

Dynamic models will be considered in the second part. 'Dynamic models' means that the investor can consume and/or invest during the investment horizon in almost any way. The theory behind this is stochastic optimal control and to solve these problems is Dynamic Programming used. The general problem is resolved by solving the Bellman-equation for a certain model.

1.4 Literature

A lot of studies on optimal consumption and allocation have been done. A good summary of the continuous case can be found in Björk (2009) chapter $19-20$. The classical Merton problem was solved for a B-S and constant rate model in Merton (1969). The problem for optimal consumption and investment in a two asset portfolio for a CRRA investor for a continuous model and the HJB-equation is solved. The equation related to the 'optimal control rule' and 'value function' is in the continuous case called HJB-equation and Bellman-equation for the discrete case. The solution shows that the optimal weight is constant and the optimal proportional consumption is independent of the current wealth. Since this article have many articles been published with extension from the easy B-S, constant rate, CRRA and complete market assumptions. Some examples are Duffie et al. (1997) and Musiela and Zariphopoulou (2006). In those articles are mainly the definition of the utility function extended to HARA and including stochastic income. Income, such as labour payments, cannot be replicated in the financial market and therefor the model is considered "incomplete" in this respect.

Several papers have solved problems with stochastic volatility for CRRA investors, see Fleming and Hernández-Hernández (2003) and Chacko and Viceira (2005). Both these papers use a Heston model. The most recent article on this topic and the most advanced one is Wang et al. (2017) where a closed form solution is found for a Heston model and HARA-utility function. The Heston model used in this article is of a special form. The Brownian motions in the stock price process and the volatility process are assumed to be equal. This is a big downside for the solution since the correlation between returns and volatility is usually negative.

All the literature mention above is for the continuous case, but this thesis is in the discrete time. There have been many papers published for the discrete problem also. The general approaches in these papers are to use MC-methods. The main article that inspired the methods in this thesis are Brandt et al. (2005) and van Binsbergen and Brandt (2007). The later article shows that the performance increases when the optimal weights are solved recursively instead of on the value function. This is used in the method here. In this thesis, every result limited to one state variable such as volatility, interest rate or portfolio value (when that effects the solution).

Chapter 2

Theory

This chapter provides the theory that is needed for this thesis. The goal is to give the reader a good overview of the theoretical framework that lies as a foundation of this work.

The first part starts with the general probability theory and discusses the measures $\mathbb{P}$ and $\mathbb{Q}$ in financial theory and it is used to define models for the financial valuations. The following section goes quite deep into utility theory and specifies the assumed preferences of the investors. The part about utility ends with the difficulties about path and draw-down dependent utility functions. The first section about path dependent utility function is considered to be memory-less and only takes the current portfolio value into account. The last part of this chapter derives a special utility function with draw-down. Consumption is introduced during the investment time-horizon and the problems with discount functions is discussed.

The last part of the theory chapter is about finding the optimal 'control rule' for a stochastic control problem. This is done with dynamic programming. Theory start with the continuous problem and that is used to formulate the discrete problem.

2.1 Financial models

First of all, it is needed to define a probability space for the dynamics in a financial market. The definitions in this part is heavily inspired by Björk (2009). Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the sample space, $\mathcal{F}$ is a σ-algebra generated on Ω and $\mathbb{P}$ is a probability measure on the σ-algebra and the sample space $(\Omega, \mathcal{F})$ such that $\mathbb{P}(\Omega) = 1$. The filtration $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ contains all information and prices in the market until t. From definition is $\{\mathcal{F}_t\}_{t\geq 0} \subseteq \mathcal{F}$ and $s \leq t \implies \mathcal{F}_s \subseteq \mathcal{F}_t$. If a stochastic process $X_t \in \mathcal{F}_t$, then it is said that X is adapted to filtration $\underline{\mathcal{F}}$ until time t.

The 'real-world' measure $\mathbb{P}$ can be used to model e.g., stock price. A risk neutral measure $\mathbb{Q}$ can instead be constructed by using arbitrage arguments. One of the assumptions for the construction of a risk-neutral measure is that the market is complete. The risk-neutral measure will be used for the interest rate models since the price of a bond is not unique under $\mathbb{P}$-measure, Björk (2009).

2.1.1 Risk-free asset

The risk-free asset will be considered to be a guaranteed asset that has zero probability to default. There can still be a stochastic risk in the price of a risk-free asset i.e., a bound. This means that the short rate can be stochastic and unknown in the future. Assume that the risk-free short rate (or simply short rate) is $r_f(t)$, the dynamic of a continuous risk-free bank account is

$$dB_t = r_f(t)B(t)dt, \quad B(0) = 1.$$

For simplicity does the bank account start with 1 unit of money. The simplest model is to assume that $r_f(t)$ is constant. The volatility of the short-rate is so small, at least compared to other assets such as stocks, that this approximation can be good enough for some financial applications. This assumption is done e.g., in the classic Black–Scholes formula for pricing derivative, Black and Scholes (1973). The bank account at time $t > 0$ will be $B(t) = B_0 e^{r_f t}$ if the rate is constant. If the risk-free rate is time varying then will the account have the amount

$$B(t) = e^{\int_0^t r_f(s)ds}.$$

If the integral is calculated for a constant rate is the same result found. In fact this is strictly linked to the price of a risk-free zero-coupon bond (ZCB) i.e., bond that only pay out the nominal amount at maturity date T, since the bank-account is used as a discount factor. The price of a ZCB is

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right], \tag{2.1}$$

if it is assumed that the nominal amount is 1, $P(T,T) = 1$. The short rate is not an actual rate observed in the market since it is considered to be the rate for an incremental time span. The rates that are observable in the market are the forward rate, and the 'risk-free' forward rates are given from state bonds prices. The notation of forward rates is $f(t,T)$ where t is the current time and T is the time to maturity. The forward rate is calculated from the current price of ZCB with maturity T.

In this thesis, the risk-free asset will be considered to be a 'regular bank account' and get interest rate payments every Δt. However this is the same as considering buying a risk-free bond with maturity date Δt and reinvesting.

Stochastic models are used to calculate density functions of the future rates. A generalization of the short rate models can be defined as

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{Q}}(t),$$

where $\mu(t, r(t))$ is the drift and $\sigma(t, r(t))$ is the diffusion term. The simplest model is that the short rate is constant i.e., $\mu(t, r(t)) = 0$ and $\sigma(t, r(t)) = 0$.

The short rate models that are stochastic will assume to be characterized by an affine term structure (ATS). Define the pricing function as $F(r(t), t, T) = p(t, T)$. The argument for ATS comes from pricing of ZCB. The price of a ZCB, assumed ATS and $P(T,T) = 1$, is

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)}. \tag{2.2}$$

A property of F is $\log(F) = A(t,T) - B(t,T)r(t)$.

There are clear relations between the short rate in Equation (2.1) and the ATS in (2.2). To assure that the short rate model have ATS should the dynamic function ($\mu(\cdot)$ and $\sigma(\cdot)$) be such that

$$\begin{aligned}
\mu(t, r(t)) &= \alpha(t)r(t) + \beta(t), \\
\sigma(t, r(t)) &= \sqrt{\gamma(t)r(t) + \delta(t)}, \\
\frac{\partial B(t,T)}{\partial t} + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) &= -1, \\
B(T,T) &= 0, \\
\frac{\partial A(t,T)}{\partial t} &= \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\
A(T,T) &= 0.
\end{aligned}$$

The detailed mathematical reasons for why these relations are true can be found in e.g., Lindström et al. (2015) page 234 and Björk (2009) pages $377 - 379$. The rates are not directly observable in the real world but the actual prices of the bonds are. The forward rate can be found since $f(t,T) = \frac{\partial \log(p(t,T))}{\partial t}$. It is from these equations that the short rate can be calculated, the mathematical proofs are omitted, for full details see Lindström et al. (2015) chapter 11.

The theory so far can be used to find forward and short rates and from this, it can be used to find good models for the rates. One of the first relatively simple ATS-model to be use it is the Vasicek model, Vasicek (1976). The model for the short rate is

$$dr(t) = (b - ar(t))dt + \sigma dW^{\mathbb{Q}}(t). \tag{2.3}$$

The Vasicek model is a model on short rate using the Ornstein-Uhlenbeck process. The parameter b/a is the expected long term mean of $r(t)$, a is 'the speed a process returns to b/a' and σ is the volatility of the short rate. One property of this model is that the short rate can become negative and that is the case in many countries today, including Sweden. This was a criticism of the model before the financial crises in 2008.

It is a positive attribute for risk-free short rate that the Vasicek model can take negative numbers, but for corporate bond is this a still problem. It seems quite unrealistic that an investor would lend money to a corporation that maybe will default for a negative rate. To solve this problem one can use the Cox-Ingersoll-Ross model (CIR). The dynamics are

$$dr(t) = (b - ar(t))dt + \sigma\sqrt{r(t)}.dW^{\mathbb{Q}}(t)$$

In fact both Vasicek and CIR model are a special case of the CKLS-model with the dynamic $dr(t) = (b - ar(t))dt + \sigma r(t)^{\gamma} dW^{Q}(t)$ where for Vasicek is $\gamma = 0$ and CIR is $\gamma = 1/2$, Lindström et al. (2015) page 270. In this thesis, only risk-free rates will be considered and the Vasicek model will be used.

2.1.2 Equity models

From research about the return from risky assets have some 'stylized facts' been found. The facts are categorized into eight different facts in Lindström et al. (2015), they can also be found in Rydén et al. (1998) and Cont (2001). The important thing with a model that is used to simulate a stock price or return (basically the same thing) is that the model takes these facts into account. However, the model becomes very advanced if it takes everything into account. The most famous model is to use a Geometric Brownian Motion (GBM). Let the price at time t be $S(t)$, the model is

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^{\mathbb{P}}(t)$$

where μ is the drift and σ is the volatility. Notice that the Brownian motion and the dynamics are in real world measure $\mathbb{P}$. This is because, in this thesis, return is the only thing of interest. The models are not used to price e.g., derivatives, where the risk-neutral measure $\mathbb{Q}$ is needed.

This model is presented in Black and Scholes (1973) and is called the Black-Scholes model (B-S). Under $\mathbb{Q}$-measure is the drift term the risk-free to ensure no arbitrage. Volatility clustering and heavy tails are not considered in this model. This is a clear down-side for the model and need ad extenuation. The B-S has a close form solution. The log-stock price is given by $\log(S(t)) \sim N(S(0) + (\mu - \sigma^2/2)t, t\sigma^2)$, the returns are therefore log-normal.

Extended models

In Heston (1993) was a model presented, simply called the Heston model. This model considers the volatility to be stochastic and correlated with the stock price. By adding stochastic to the volatility is the stylized fact of volatility clustering considered. The dynamics of the stock price and the volatility is

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{V(t)S(t)}dW_1^{\mathbb{P}}(t), \\ dV(t) &= \kappa(\theta - V(t))dt + \sigma_V\sqrt{V(t)}dW_2^{\mathbb{P}}(t), \\ dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t) &= \rho dt, \end{aligned}$$

where $V(t)$ is the variance. The correlation between $S(t)$ and $V(t)$ means that the auto-correlation of the squared return (or equivalently auto-correlation to absolute returns) is not zero, as is found in the statistical analysis of financial return series. In fact is this also true already when adding stochastic volatility but the correlation is less if $Cov(dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t)) = 0$

Another approach to solve the lack of coherence with the stylized fact and the B-S model is to add stochastic jumps to the model. The dynamics is

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^{\mathbb{P}}(t) + S(t_-)dJ(t), \tag{2.4}$$

where $S(t_-)$ means the price before the jump i.e., the price at time t is the price of $S(t_-)$ + 'Jump'. This model is called The Merton Jump Diffusion Model (MJD), Merton (1976). The process $J(t)$ is a stochastic process with two underlying randomness. First of all is the stochastic process if a jump occurs, this is considered to be a Poisson process with intensity λ_J and the number of jumps until t is $N(t)$. The amplitude of the jump is the second stochastic property. The jumps are considered to have a log-normal distribution with mean μ_J and standard deviation of σ_J. The jumps are denoted Y_i where $i \in (0, N(t))$. This means that the total jump process is at time t is $J(t) = \exp\big(\sum_{i=1}^{N(t)} Y_i\big)$ and if $N(t) = 0$ is the jump process defined as $J(t) = 1$. This extended model takes the stylized fact of heavy tails and gain/loss asymmetry into account through the jumps. The expected value of the jumps at time t is

$$\begin{aligned} E\Big[\exp(\sum_{i=1}^{N(t)} Y_i)\Big] &= E\Big[E\big[\exp(\sum_{i=1}^{N(t)} Y_i)|N(t)\big]\Big] \\ &= E\Big[\exp(N(\mu_J + \sigma_J^2/2))\Big] = \exp(\lambda_J t(\exp(\mu_J + \sigma_J^2/2) - 1)). \end{aligned}$$

The Heston model and the MJD takes care of different stylized facts. A natural approach is to combine the two models and this was done in Bates (1996). The Bates model is used, in this thesis, when trying to simulate the real world asset as realistic as possible. The dynamics of Bates model is

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{V(t)}S(t)dW_1^{\mathbb{P}}(t) + S(t_-)dJ(t), \\ dV(t) &= \kappa(\theta - V(t))dt + \sigma_V\sqrt{V(t)}dW_2^{\mathbb{P}}(t), \\ dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t) &= \rho dt, \end{aligned} \tag{2.5}$$

where all of the parameters and dynamics are as in the Merton model and MJD. It is clear that the Bates model has both the positive aspects of the Heston model and MJD. Since the model is more complex does the difficulty working with the model increase e.g., simulation and parameter estimation.

2.2 Preference and Utility theory

Assume that an agent (A) needs to make decisions in the action set χ i.e., all possible actions. The actions in χ can have deterministic or stochastic outcomes but the density function of the outcome generated by action $x \in \chi$ is considered to be known. The actions are considered to be able to be represented with real-value numbers so $\chi \subseteq \mathbb{R}^M$ where M is the dimension of χ.

A rational agent will be interested in being able to decide which action is the best. Consider now two possible actions x and $y \in \chi$. The notation $x \succeq y$ means that A prefers x over y and $x \sim y$ means that A is indifferent, that happens if and only if $x \succeq y$ and $y \succeq x$. Here are four axioms that will be considered to be true for all sets of the actions $x, y \in \chi$.

1. **Completeness:** For all possible $x, y \in \chi$ the actions can be orders such that $x \succeq y$, $x \preceq y$ or $x \sim y$. This means that all actions in χ can be ranked.

2. **Transitivity** Let $x, y, z \in \chi$. If $x \succeq y$ and $y \succeq z$ than $x \succeq z$.

3. **Monotonicity** Let $x, y \in \chi \subseteq \mathbb{R}^M$ and $x_i \geq y_i \forall i \in (1, M)$ with at least one strict inequality. than is $x \succ y$.

4. **Convexity** Let $x, y, z \in \chi$ and $x \succeq z$ and $y \succeq z$ but $x \neq y$. than for any $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \succ z$.

These four axioms are a version of axioms used in the Von Neumann–Morgenstern (VNM) utility theorem. The goal is to find a function $u(\cdot)$ that orders the actions in χ in the correct order, where $u(\cdot) : \mathbb{R}^M \to \mathbb{R}$. This means that $x \succeq y \iff u(x) \geq u(y) \quad \forall x, y \in \chi$. An important theorem for utility theorem is VNM's theorem:

Theorem 1. *(VNM Utility Theorem) The four axioms above are considered to be true. Let's now consider two random outcomes for two actions with known density functions, $x, y \in \chi$, than there exist a utility function $u(\cdot)$ such that*

$$x \succeq y \qquad \iff \qquad \mathbb{E}[u(x)] \geq \mathbb{E}[u(y)].$$

This theorem will not be proven for the general case but some specific utility functions will be derived. For financial and economic application does x denote the consumption. The utility function measures 'utils' of consumption. The term agent agent is used in general utility theory but the term investor will used instead in this thesis. For more specific and more advanced theory and proofs of existence of utility functions see Barbera et al. (1998) chapter 1.

2.2.1 Risk aversion and measure of risk

Risk is almost always a part of decision making and it is assumed that two investor can react differently to the same risk. Risk is a broad term but the focus will be on an investor's financial risk behavior. The goal is to be able to quantify and analyze risk preferences. This report will focus on risk averse investors:

Definition 1. *(Risk aversion). An investors preferences can be measured with utility function $u(\cdot)$. Let a risky asset $\hat{z}$ described as a random variable with $\mathbb{E}[\hat{z}] = 0$. The investor is said to be risk averse if and only if $\mathbb{E}[u(x+z)] \leq u(x)$ for any $x > 0$. For the special case $\mathbb{E}[u(x+z)] = u(x)$ is the investor considered to be risk neutral. It will only be considerd (strictly) risk averse investors i.e, $\mathbb{E}[u(x+z)] < u(x)$.*

This means that a risk averse investor will not add risk to his current investment x if the expected pay-off is zero. A general risky asset can be decomposed to $z = \mathbb{E}[z] + \hat{z}$ where $\mathbb{E}[z] \neq 0$ and $\mathbb{E}[\hat{z}] = 0$. From Definition 1 $\mathbb{E}[u(X+z)] < u(X + \mathbb{E}[z])$ (for the special case when $E(z) = 0$ is simply $z = \hat{z}$). It is known from Jensen's inequality that u has to be a concave function. The fact that u is a concave function can be argued straight from economic arguments:

- $u'(x) > 0$: This means 'more is better'. It is assumed that more consumption is better than less. The first derivative of u is called the marginal utility function.

- $u''(x) < 0$: The marginal utility is decreasing i.e., an extra unit of wealth increases the utility less when the wealth is bigger. This is equivalent to risk aversion if $u \in C^2(\mathbb{R}_+)$.

A risk averse investor dislikes assets that are expected to pay zero but this does not mean that the investors does not like or prefer risky assets that have positive expected value. The risk-premium that an investor is willing to get paid (or pay) for a security z is defined

$$\mathbb{E}[u(X+z)] = u(X + \mathbb{E}[z] - \Pi(X,z)). \tag{2.6}$$

This means that the investor is indifferent to choose between, $X + z \sim X + \mathbb{E}[z] - \Pi(X,z)$. Assume now that $\mathbb{E}[z] = 0$. By looking at the Taylor series expansion of the utility function $u(\cdot)$ of the LHS of Equation (2.6)

$$\mathbb{E}[u(X+z)] \approx \mathbb{E}[u(X) + z \cdot u'(X) + \frac{1}{2}z^2 u''(X)] = u(X) + \frac{1}{2}\sigma_z^2 u''(X),$$

since $\mathbb{E}[z] = 0$ and where $\mathbb{E}[z^2] = \sigma_z^2$. From Equation (2.6) we also get from the RHS,

$$u(X) + \frac{1}{2}\sigma_z^2 u''(X) \approx u(X) - \Pi(X,z)u'(X).$$

$\Pi(X,z)$ can be found and Arrow-Pratt risk aversion ($\Pi_{A-P}(X,z)$) is defined as

$$\Pi_{A-P}(X,z) = -\frac{1}{2}\sigma_z^2 \frac{u''(X)}{u'(X)} = \frac{1}{2}\sigma_z^2 ARA(X).$$

$ARA(X)$ is called absolute risk aversion. For a risk averse investor $ARA(X) > 0$, risk neutral investor $ARA(X) = 0$ and for a risk loving investor $ARA(X) < 0$. Relative risk aversion (RRA) is than $RRA(X) = X \cdot ARA(X)$. These measures are only approximations since it is only derived from first and second order Taylor expansions. Generally is X the investors current wealth.

2.2.2 Prudence and temperance

In this section lotteries will be used. The notation for a lottery with two outcomes is $[A, p; B, 1-p]$ which means that A will happen with probability p and B will happen with probability $1-p$. If the lottery is denoted by $[A; B]$ than A and B will happen with equal probabilities, $p = 1/2$.

Definition 2. *(Prudence) Let an investor be represented with a utility function $u(\cdot)$. If the marginal utility function is strictly convex then the investor is said to be 'prudent'. When an investor is prudent then the following are equivalent:*

1. *The investors marginal utility function is strictly convex i.e, $\lambda u(x) + (1-\lambda)u(y) > u(\lambda x + (1-\lambda)y), \forall \lambda \in [0,1]$.*

2. $u'''(x) > 0$.

The fact that the third derivative is strictly positive comes from the basic properties of strictly convex functions.

The economic interpretation of a convex marginal utility is that an investor will have 'aversion to downside risk'.

Example 1. *Consider that an investor with an initial wealth X_0 and that he owns a risky asset Y with $\mathbb{E}[Y] < 0$ and $Y = [-k, p; 0, 1-p]$ where $k > 0$ i.e., it is an insurance asset. Consider now a different asset with zero mean pay-off $\mathbb{E}[Z] = 0$. Let the investor choose between two strategies after having asset Y and the strategy is decided beforehand. The final wealth, after the outcome of Y, is $X_a = X_0$ if $Y = 0$ or $X_a = X_0 - k$ if $Y = -k$.*

1. ***A**: The investor can decide to add Z if $Y = -k$ but not to add Z if $Y = 0$. This gives the total pay-off $X_b = X_0 - k + Z$.*

2. ***B***: *The investor can decide to add* Z *if* $Y = 0$ *but not to add* Z *if* $Y = -k$. *This gives the final pay-off* $X_b = X_0 + Z$.

An investor with aversion to down-side risk (prudence) will prefer B *over* A.

In some literature, as Eeckhoust and Schlesinger (2006), is Example 1 used to define prudence. It is proven in Eeckhoust and Schlesinger (2006) that these two definitions are equivalent. Some important proprieties for an investor is.

Definition 3. *Absolute and relative prudence is defined as:*

- *Absolute prudence:* $AP(x) = -u'''(x)/u''(x)$.
- *Relative prudence:* $RP(x) = x \cdot AP(x)$.

Absolute and relative prudence is used in a similar way as absolute and relative risk aversion. A precautionary premium can be found in the same way as risk premium in the previous part, see Danthine and Donaldson (2015) page $125 - 130$.

The investors in this thesis will also be considered to have so called temperance behavior. With temperance behavior will an investor avoid new independent risk in a situation with unavoidable risk, that is the same as Definition 4.

Definition 4. *Consider two i.i.d. random payoffs* Z_1 *and* Z_2 *with zero expected value. The investor, with temperance preferences, will prefer lottery* $A = [Z_1; Z_2]$ *over* $B = [0; Z_1 + Z_2]$.

It is shown in Eeckhoust and Schlesinger (2006) that Definition 4 is equivalent to the following definition:

Definition 5. *Let an investor be represented with a utility function* u. *Let* $u''(x)$ *be strictly concave then the investor is said to have temperance behavior. When an investor has temperance behavior than the following equivalent:*

1. *The second derivative of the utility function is strictly concave.*
2. $u''''(x) < 0$.

The derivatives in Definition 5 comes directly from the properties of a strictly concave function.

2.2.3 HARA and CRRA utility functions

Consider an investor who satisfies the four VNM-axioms and exhibits prudence and temperance preference. A result of VNM utility theorem is knowledge about existence of a utility function. This part will focus on deriving the important functions that will be used.

A logical assumption is that the investor has decreasing absolute relative risk aversion (ARA), see Hult et al. (2012). This means that the investor is willing to risk more if the investor has more wealth. One possible construction to have this behavior is to define

$$ARA(x) = -\frac{u''(x)}{u'(x)} = \frac{1}{\tau + \gamma x} \text{ where } \tau + \gamma x > 0. \tag{2.7}$$

This is called hyperbolic-ARA (HARA). The unit of ARA should be $\$^{-1}$ so the units of the parameters are $[\tau] = [\$]$ and $[\gamma] = [\ ]$. Equation (2.7) is a ordinary differential equation (ODE) on the form

$$u''(x) + \frac{1}{(\tau + \gamma x)} u'(x) = 0. \tag{2.8}$$

Equation (2.8) can be solved, for $u'(x)$. The solution to the ODE is

$$u'(x) = C \begin{cases} (\tau + \gamma x)^{-1/\gamma} \text{ if } \gamma \neq 0 \\ \exp(-x/\tau) \text{ if } \gamma = 0, \end{cases}$$

for some constant C. This gives the final solution for a HARA-utility function to be

$$u(x) = \begin{cases} \frac{1}{\gamma-1}(\tau+\gamma x)^{1-1/\gamma} & \text{if } \gamma \neq 1, 0 \\ \log(\tau + x) & \text{if } \gamma = 1 \\ -\tau \exp(-x/\tau) & \text{if } \gamma = 0, \tau \neq 0. \end{cases} \quad (2.9)$$

The integration constants C has been normalized in the solution to the ODE in Equation (2.9) have. This is the family of HARA-utility functions. A utility function with $\tau = 0$ has constant relative risk aversion (CRRA).

2.2.4 Properties of HARA and CRRA utility function

In Equation (2.9) are three different function depending on γ. The RRA for the tree different types are equal

$$RRA(x) = \frac{x}{\tau + \gamma x} = \frac{1}{\frac{\tau}{x} + \gamma}. \quad (2.10)$$

As x increases than does the behavior of an investor with HARA-utility go to a behavior of an investor with CRRA-utility

$$\lim_{x\to\infty} RRA(x) = \lim_{x\to\infty} \frac{1}{\frac{\tau}{x} + \gamma} = \frac{1}{\gamma}.$$

This shows that CRRA can be used to approximate a behavior of a HARA-investor when the investor that has a big wealth. To simplify the parameters let's use the following notation for CRRA, also called power utility

$$u(x) = \frac{x^{1-\lambda}}{1-\lambda},$$

where parameter λ is the RRA $\forall x > 0$. All investors that will be studied in this report will be those who exhibit prudence and temperance behavior and that is related to the third and fourth derivative. The first four derivatives of a HARA utility (with $\gamma \neq 0$ or 1) is:

$$\begin{aligned} \tfrac{\partial u(x)}{\partial x} &= (\tau+\gamma x)^{-1/\gamma} > 0 \\ \tfrac{\partial^2 u(x)}{\partial x^2} &= -(\tau+\gamma x)^{-(1+1/\gamma)} < 0 \\ \tfrac{\partial^3 u(x)}{\partial x^3} &= (1+\gamma)(\tau+\gamma x)^{-(2+1/\gamma)} > 0 \\ \tfrac{\partial^{IV} u(x)}{\partial x^{IV}} &= -(1+\gamma)(1+2\gamma)(\tau+\gamma x)^{-(3+1/\gamma)} < 0. \end{aligned} \quad (2.11)$$

From the definition of HARA $(\tau + \gamma x) > 0$, where $\lambda = 1/\gamma$ for CRRA. To guarantee prudence and temperance, the sign of the third and fourth derivative must be as in Equation (2.11). The third inequality shows that $\gamma > -1$ and $\tau > \gamma x$. From the fourth derivative it is known that $\gamma \in]-\infty, -1[$ or $]-1/2, \infty[$ and the intersection for the third and fourth derivative gives the set $\gamma \in]-1/2, \infty[$ so long as $(\tau + \gamma x) > 0$. This result gives limitations for the parameters. To ensure prudence and temperance for a CRRA-investor should $\lambda > 0$.

The relative prudence for CRRA-utility function is $RP(x) = \lambda + 1$, so a CRRA-investor has infact also constant relative prudence. The higher the value is the more prudent is the investor. For HARA-utility is relative prudence

$$RP(x) = x(1+\gamma)\frac{1}{\tau+\gamma x} = \frac{1+\gamma}{\frac{\tau}{x}+\gamma}.$$

In similar way is the limit

$$\lim_{x\to\infty} RP(x) = \frac{1+\gamma}{\gamma} = \frac{1}{\gamma} + 1 = \lambda + 1,$$

a CRRA-utility.

2.2.5 Maximum Expected Utility

Utility theory is often used to calculate the optimal strategy for an investor in a market. In general, is an investor interested in to maximizing the expected utility. Consider a portfolio with m different assets and the dynamics of the assets are known and the portfolio is held for one time period. The value of the portfolio (x) dependents of the weights $\mathbf{w} \in \mathbb{R}^m$ in the m assets, $x(\mathbf{w})$. The goal is to find optimal weights $\mathbf{w}$ that maximize expected utility where

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \mathbb{E}[u(x(\mathbf{w}))] \text{ s.t. } \sum_{i=1}^{m} w_i = 1.$$

In practical applications, can this problem become really difficult to solve, especially if m is large and the assets have complicated dynamics. Instead of weights can the amount be used instead as in Example 2.

Example 2. *There are only two assets, one risk-free rate and a risky asset. The initial wealth is X. The risk-free rate is considered to be constant and known r_f, the return of the risky asset $\tilde{r}$ ($\mathbb{E}[\tilde{r}] > r_f$) and let a be the amount invested in the risky asset. In this case is $\chi \in [0, X]$ if no leverage is allowed. This gives the following problem*

$$\Omega(X) = \max_{a} \mathbb{E}[u(X(1 + r_f) + a(\tilde{r} - r_f))] \quad a \in \chi,$$

where X_1 is the wealth after one time period and Ω is defined as the maximum expected utility. The solution to this problem depends both on the distribution of the assets and the utility function. The solution can be found by solving:

$$\mathbb{E}[u'(X(1 + r_f) + a(\tilde{r} - r_f))] = 0.$$

The problem can be generalized over several periods where the investor can consume at every moment. Remember that the utility function measures utility from consumption and not wealth.

The total expected utility $U(c(t), X_0)$ in the continuous case is

$$U(c(t), X_0) = \mathbb{E}\Big[\int_0^T D(t)u(c(t))dt + D(T)u(X_T)\Big], \tag{2.12}$$

where $c(t)$ is a continuous consumption function, $D(t)$ is the discount function, $u(\cdot)$ the utility function and X_t is the wealth at t. It is assumes that the investor's portfolio is consumed at the final time T. Notice that U has two inputs for the continuous consumption since the total expected utility depends in the function $c(t)$. In fact can the function $c(t)$ depend on several variables but this fact is not important here since this thesis will focus on the discrete case. Maximization of Equation (2.12) is called stochastic optimal control and is widely used in control theory.

The solution to optimize Equation (2.12) has only a few analytically solutions, see Björk (2009) chapter 19. Many research articles for financial application, has been published that either show that a maximization of Equation (2.12) exists or solve the actual problem for some set of dynamics of the assets and given functions ($u(c)$ and $D(t)$) e.g., Bouchard and Pham (2004), Duffie et al. (1997) and Pirvu and Haussmann (2009).

It is not possible in practical and numerical applications to have continuous consumption and trading so Equation (2.12) has to be discretized. The investor can consume and trade after time step Δt and after the discretization is the expected utility:

$$U(c_t, X_0) = \mathbb{E}\Big[\sum_{i=1}^{N} D(t_i)u(c_i) + D(T)u(X_T)\Big], \tag{2.13}$$

where N is the number of time steps. $T = t_N$ can either be fixed or it can be the first time when the portfolio X_t becomes zero i.e., the investor has gone bankrupted. Also for the discrete case can c_{t_i} depend on several variables, this will be handled later.

When solving a maximization problem like those in (2.12) and (2.13) one has to take budget constraints into account.

Example 3. *The assets are one risk-free asset with constant intrest rate r_f and one risky asset that follows the dynamics $dS = \alpha S dt + \sigma S dW$. Let $w^1_{t_i}$ be the weight invested in the risk free asset and $w^2_{t_i}$ be the weight invested in the risky asset at time t_i. The investors preference can be described by the utility function $u(c)$ and discount function $D(t)$. The investor can consume and trade at times t_i and $t_i - t_{i-1} = \Delta t \ \forall \ i$. The investor starts with wealth x_0. The optimal control problem with constraint and portfolio dynamics is:*

$$
\begin{gathered}
\max_{w^1, w^2, c} \mathbb{E}\Big[\sum_{i=1}^{N} D(t_i)u(c_i)dt + D(T)u(X_T)\Big] \\
\Delta X_t = X_{t-1}[w^1_{t_i} r_f + w^2_{t_i}\alpha]\Delta t - c_{t_i}\Delta t + w^2_{t_i}\sigma \Delta W_t \\
X_0 = x_0 \\
c_{t_i} \geq 0 \quad \forall t_i \geq 0 \\
w^1_{t_i} + w^2_{t_i} = 1 \forall t_i \geq 0.
\end{gathered}
$$

This is a discrete example of Björk (2009), page $282 - 283$.

How to solve problems like Example 3 and other stochastic optimal control problems will be handled later.

2.2.6 Discount function

The discount function $D(t)$ in Equation (2.12) and (2.13) is the investors discount function and not the markets. The function is a measure of time preference for the investor.

Example 4. *Consider an investor that is indifferent to consume 1 today or 1.2 tomorrow, $u(1) = D(1)u(1.2)$. Consider now that the investor can make the same decisions today about consumption after time 30 and 31 days, i.e., he can consume 1 after 30 days or 1.2 after 31 days. Then it is rational that $D(30)u(1) \leq D(31)u(1.2)$ since this shifts to later reward.*

Definition 6. *An investor has a rational discount factor if and only if where $y > x$ and $t > 0$*

$$
\begin{gathered}
u(x) = D(t)u(y) \\
D(s)u(x) \leq D(s+t)u(y).
\end{gathered}
$$

This can be summarized in a theorem:

Theorem 2. *If an investor has a rational discount function, then is $D(t)$*

$$
D(t) = (1 + \alpha t)^{-\beta/\alpha}.
$$

The limit of this function is $\lim_{\alpha \to 0} D(t) = \exp(-\beta t)$. For all x and y there exist s such that $u(x) = D(s)u(y)$ furthermore there exists $k = k(x, y)$ and t such that $D(t)u(x) = D(kt + s)u(y)$ where $k = 1$ if and only if $\alpha = 0$.

Proof. (Barbera et al. (1998)) Consider t' and $\gamma t + (1-\gamma)t'$:

$$
\begin{gathered}
D(t')u(x) = D(kt' + s)u(y) \\
D(\gamma t + (1-\gamma)t')u(x) = D(k(\gamma t + (1-\gamma)t') + s)u(y).
\end{gathered}
\tag{2.14}
$$

Define $r = u(x)/u(y)$, $w = D(t)$, $z = D(t')$ and $v = D^{-1}$. Insert this into (2.14) after some calculation

$$
rv^{-1}(\gamma v(w) + (1-\gamma)v(z)) = v^{-1}(\gamma v(rw) + (1-\gamma)v(rz)). \tag{2.15}
$$

From Aczél (1966), page 152, the only solution to Equation (2.15) is

$$
\begin{gathered}
v(t) = c \cdot \log(t) + d \\
v(t) = ct^{\lambda} + d.
\end{gathered}
$$

The inverse of $v(t)$ is the discount factor and is either hyperbolic or exponential. □

In the case where an investor has an exponential discount factor is when the investor has a stationary discount utility and implied that $u(x) = D(t)u(y) \iff D(s)u(x) = D(s+t)u(y) \forall s \geq 0$, Koppmans (1960). If the investors preference changes when s is added, always to the later choice, than the investor has 'common discount effects'. The name is derived from the fact that the two options have a common time difference t, Barbera et al. (1998). In this case the investor's discount function a hyperbolic function with $\alpha > 0$.

2.2.7 Myopic solution and the Merton problem

It has been argued that here are not many analytically solution to Equation (2.12). There is a case where a solution exist, this is called the Merton problem. This is summarized in a theorem:

Theorem 3. *Merton (1969) Consider only two assets, one risky asset with B-S dynamics and one constant risk-free asset and CRRA-utility function with RRA λ. Furthermore is the investors discount function exponential $D(t) = \exp(-\beta t)$ and $T < \infty$. The solution of maximizing Equation (2.12) is*

$$w^*(t,x) = \frac{\mu - r}{\sigma^2 \lambda}.$$

Notice that the optimal weight is independent in both time and wealth, $w^* = w^*(t,x), \ \forall t, x \ \in [0,T], \mathbb{R}_+$.

The optimal consumption is

$$\begin{aligned} c^*(t,x) &= \tfrac{x}{(T-t+\epsilon)} && \nu = 0 \\ c^*(t,x) &= \tfrac{\nu}{1+(\nu\epsilon-1)\exp(\nu(t-T))} x && \nu \neq 0 \\ \nu &= \tfrac{\beta}{\lambda} - (1-\lambda)\left(\tfrac{(\mu-r)w^*}{2\lambda} + \tfrac{r}{\lambda}\right). \end{aligned}$$

The proportional optimal consumption is independent of the wealth.

The parameter $0 \leq \epsilon \ll 1$ is related to bequest preference of the investor that the CRRA utility function does not take into account. If that is not taken into account should $\epsilon = 0$.

The proof of Theorem 3 is omitted but it can be found in Merton (1969). The solution for optimal weight in the risky asset can be shown to be equal for the discrete case and also without consumption. This is called the Myopic solution. Therefore, this solution can be used as a validation.

2.2.8 Certainty Equivalent

Certainty Equivalent (CE) is an important property when handling with utility function and expected utility. The value generated by a utility function is unit-less and does not mean that much on its own, it only works as a comparison tool. The value that one gets by calculating Equation (2.13) is heavily dependent on the utility function u and discount function $D(t)$. To be able to compare trading and consumption strategies and different investors is CE used. The definition of CE is

$$CE = u^{-1}(U),$$

where u^{-1} is the inverse of the investors utility function and U is the is the total expected utility defined in Equation (2.12). The unit of CE is money so it is comparable between investors with different utility functions. One can think of CE as the amount of money an investor needs to consume today to be indifferent to invest and consume in the future. For utility function that $u(0) = C > -\infty$ and $\lim_{x\to\infty} u(x) = \infty$ should the CE-value be much higher than the actual amount spent during the time period as Example 5 shows.

Example 5. *Consider an investor that has* \$900 *and he has to spend* \$300 *each year for the next three years and all the consumption occurs at one date every year. Assume that he has them in cash so he does not get any returns. The investor has a power utility function (CRRA) with $\lambda = 1/2$ and an exponential discount factor with $\beta = 0.02$. The total discounted 'expected' utility for the investor today is:*

$$U = \sum_{t=1}^{3} \exp(-0.02t) \frac{300^{1/2}}{1/2} = 99.86.$$

The CE of this plan is how much does the investor need to spend today to experience 99.8615 *'utils'. CE is calculated as:*

$$CE = u^{-1}(U) = (U(1-\lambda))^{1/(1-\lambda)} = \$2431.$$

Now one will notice that the CE amount is much higher than the actual amount spending (\$900*). There is no reason that the investor's CE is feasible at time* $t = 0$*, it is just a hypothetical amount that one needs to consume to receive as much utility. This is in fact in line with the economic theory about investors that are prudent. A result is that a prudent investor prefers regular and certain cash flows rather than big ones.*

Consider that the investor is considering to consume \$150 *a years for six years instead. The utility for that plan is* 137.11*. Now one can see that the investor should perform the second plan, but what does* $\Delta U = 137.11 - 99.86 = 37.25$ *mean? The actual number does not tell one that much. The CE for the second strategy is* \$4700*. That number gives an intuition on how the investor relates to the choices in the unit of* \$*.*

It seems unreasonable in a pure economic sense that the CE is so much higher than the actual amount spent (\$900 compared to \$2431 and \$4700). This is a consequence of the property of decreasing marginal utility and prudence. Remember that utility measures 'happiness' or 'the investors value of consumption' (not money!). To make this a bit clearer is Example 5 used.

Assume that the \$900 is the investor's travel budget for the next coming years. The 'value' of a trip does not depend on the amount of money in the 'travel account' but the actual trip(s). The result in Example 5 shows that the investor clearly prefers to travel once a year for the next three or even six years compared to just one trip today. The CE value gives us a number on how big, in terms of money, a trip today needs to be so that the investor is indifferent to take a trip now (and not for the next few years) or once a year. The investor is indifferent to making three smaller trips costing \$300 once a year for the next three years and a luxurious trip now for \$2431 and then spending the next three year at home. If the investor finds the value \$2431 too big (or too small) is simply the parameter λ not right for the investor, assuming that the investor can be represented with a CRRA-utility function.

An issue with CE for CRRA-utility function occurs if the parameters $\lambda > 1$ i.e., that the utility function is strictly negative and $\lim_{x\to\infty} u(x) = 0$. The following problems are true for all utility functions of this form. Consider an investor with a normalized initial wealth $X_0 = 1$, CRRA utility function with $\lambda = 3$ and he that has the money 'under the mattress' and the discount factors are assumed to be irrelevant. The investor can choose to spend all money at one time T or split the money in half and spend them at $T/2$ and T. The two utilities will be

$$\begin{gathered} u_1 = \tfrac{1^{-2}}{-2} = -\tfrac{1}{2} \\ CE(u_1) = 1 \\ u_2 = 2 \cdot \tfrac{0.5^{-2}}{-2} = -\tfrac{1}{2}^{-2} = -4 \\ CE(u_2) \approx 0.3536. \end{gathered} \tag{2.16}$$

From Equation (2.16), it seems that the investor does prefer to spend all of the money at the final time point. But it is known that a CRRA is prudent and that means that the investor prefers to split up all consumption as much as possible, this is clearly a contradiction. The error in Equation (2.16) is that the utility for the first strategy does not calculate the utility at $T/2$. The investor does not consume anything at that time point and $u(0) = -\infty$. So the correct calculations are:

$$\begin{gathered} u_1 = \tfrac{1^{-2}}{-2} + \lim_{x\to 0} \tfrac{x^{-2}}{-2} = -\tfrac{1}{2} + -\infty \\ CE(u_1) = 0 \\ u_2 = 2 \cdot \tfrac{0.5^{-2}}{-2} = -4 \\ CE(u_2) \approx 0.3536. \end{gathered}$$

(2.2.8) It is obvious in this case that the second strategy is better since $0.3536 > 0$. One can notice that the CE is usually (a lot) less than the initial wealth than an investor has a utility function of this type. An economic interpretation of this behavior can be that if the investor has payments that must be paid at every time-period and if this is not done will the investor default on a payment. It is not as straight forward to analyze CE for these kinds of utility functions. The CE is (approximately) 0.3536 in this example but the investor can consume 1 today, so that it seems to be a better strategy. But this is an inadequate conclusion since the investor can only spend zero on the two following time points. The CE of spending $1, 0, 0$ at time points $0, T/2, T$ is actually 0 just as if he consumes everything at the final time point. Therefore is the strategy that splits up the spending still better.

2.2.9 Portfolio value dependent utility function

The investor's utility function has so far in this thesis been assumed to be constant and independent from wealth. This is reasonable when the entire wealth is consumed in the final time T because then the utility function actually measures the expected wealth, $\mathbb{E}[X_T]$ only for one time point. It can be argued that if it is only a small part of the wealth that is consumed (optimally) over a long time-span then the investor should have different utility functions for different times and wealth. The total expected utility defined as

$$U(c(t), X_0) = E\Big[\int_0^T D(t)u(c_t)dt + D(T)\Phi(X_T)\Big]$$

in Björk (2009). Let the utility function $u = u(x, c)$, i.e., the investor's utility function is a function of wealth and consumption at time t. The function $\Phi(X_T)$ is a 'legacy' function and measures the utility of having capital at the final time point. In this thesis is $u(X_T) = \Phi(X_T)$.

In the ODE from Equation (2.8) is the wealth the variable x at time t and not the consumption. This is a problem since the wealth is not constant and therefore should not the utility function be equal for different wealth. This is solved directly with CRRA. One other approach to this will be presented:

Wealth dependent parameter τ: One solution to this can be that an investor with HARA-utility ($\gamma \neq 0, 1$), for every $t \in (0, T)$ have constant relative risk aversion (not CRRA-utility function!) for any given wealth X_t and assume that γ is known and well defined. By using this approach is $\tau = \tau(X_t)$. Let C be the value for the RRA ($RRA(X_t) = C, \ \forall X_t > 0$) but an investor has a HARA-utility function. The solution is

$$\frac{1}{\frac{\tau(X_t)}{X_t} + \gamma} = C \iff \tau(X_t) = X_t\Big(\frac{1}{C} - \gamma\Big). \tag{2.17}$$

This gives the following utility function

$$u(X_t, c_t) = \frac{1}{\gamma - 1}\Big(X_t\Big(\frac{1}{C} - \gamma\Big) + \gamma c_t\Big)^{1-1/\gamma}.$$

The constraints $\tau(X_t) + \gamma X_t > 0, \ \forall X_t > 0$ must still hold. Let's define this utility function as 'special-HARA' Notice that this is done to calculate the value of τ for any given wealth X_t. The values of the parameters can be calibrated so that the marginal utility is correct for a given investor. This makes the utility function 'path-dependent'. The process for the parameter is, if the dynamics for X_t is known,

$$d\tau = \Big(\frac{1}{C} - \gamma\Big)dX_t.$$

It can be argued that the utility function should, or at least can, be time-dependent, Musiela and Zariphopoulou (2006), but that will not be taken into account in this thesis.

2.2.10 Draw-down control and utility functions

The concept of draw-down control is important for investors that dislike big fluctuations in the portfolio value X_t. Draw-down (D_t) is defined as

$$M_t = \max_{s \leq t} X_s$$
$$D_t = 1 - \frac{X_t}{M_t}.$$

This is a measure of how much the investor has lost from the maximum value of the portfolio. If $D_t = 0$ then the portfolio value is at record height. A multi-period trading strategy built on draw-down control is presented in Nystrup et al. (2018). The idea by combining draw-down control and utility theory is to make the investor more risk averse if the portfolio value decreases. It is clear in the previous section that the current wealth X_t is a factor in the utility function but it is memory-less. One possibility is to add draw-down to the utility function to define it as

$$u(c_t, D_t) = (1 - \alpha D_t)u(c_t),$$

and α is a unit-less parameter that controls how much the investor takes draw-down into account. The reason for adding draw-down is to decrease (or increase) the utility from consumption for the investor if losses has been made. If one considers a path-dependent utility function with both draw-down and a wealth dependent $\tau(x)$ will the HARA-utility function be

$$u(c_t, X_t, D_t) = (1 - \alpha D_t)\frac{1}{\gamma - 1}\Big(X_t\Big(\frac{1}{C} - \gamma\Big) + \gamma c_t\Big)^{1-1/\gamma}.$$

Notice that the VNM-axioms still hold because a constant in front of the utility function does not change the preference at a certain time t. But it can, in fact, move preferences in time and if D_t is large, the investor will consume less at time t and move consumption to later time points.

Another way to model this is to change the RRA λ so $\lambda = \lambda(D_t)$ (where $\lambda = 1/\gamma$). Define $D_{max} \in (0, 1)$ as the maximal draw-down that an investor is willing to except i.e., the maximal draw-down in the period $0, t$. Define RRA now as

$$\lambda(D_t) = \lambda_0 \frac{D_{max}}{D_{max} - D_t},$$

where λ_0 is the RRA for the investor if the $D_t = 0$. This definition is used in Nystrup et al. (2018). The investor's relative risk aversion will go to infinity when the draw-down is near D_{max}. This means that the investor will not be willing to take on any risk. However will this type of utility function not be considered in the result.

2.3 Trading-consumption strategies

One of the goals of this thesis is to find a good method to compare two or more trading-consumption strategies (TCS's)/control rule for a given investor. It is assumed that an investor has a utility function $u(c)$ and a discount function $D(t)$ and an initial wealth X_0. Furthermore, it is assumed that there are N assets and the dynamics of those are considered to be known. Let an TCS's operator be denoted by $\mathscr{T} : (w_{t_-}, X_{t_-}; \mathcal{F}_t) \to (w_t, c_t)$ where $w_{t_-} \in \mathbb{R}^N$ is the weights in N different assets before trading at time t and X_{t_-} is the portfolio value before consumption c_t and $\mathcal{F}_t$ is market information, i.e., filtration. The weights after consumption c_t are traded so the weights are w_t according to TCS $\mathscr{T}$. In this thesis, trading happens with a frequency 1 or 12 i.e., once a year or month. The operator $\mathscr{T}$ can be divided into two functions, a consumption rule $c_t = C(w_{t_-}, X_{t_-}; \mathcal{F}_t)$ and a trading rule $w_t = W(w_{t_-}, X_t; \mathcal{F}_t)$. Notice that the new weights are calculated after consumption. The wealth at time t after consumption is $X_t = X_{t_-} - c_t$. The total expected utility for using $\mathscr{T}$ is

$$\begin{aligned} U(X_0; \mathscr{T}) = \mathbb{E}\Big[\sum_{i=1}^{N_{STEPS}} & D(t_i)u(c_{t_i}) + D(T)u(X_T)\Big] \\ c_{t_i} &= C(w_{t_{i-}}, X_{t_{i-}}; \mathcal{F}_t) \\ w_{t_i} &= W(w_{t_{i-}}, X_{t_i}; \mathcal{F}_t). \end{aligned} \tag{2.18}$$

An investor has two different TCS's ($\mathscr{T}_1, \mathscr{T}_2$) to choose from. The difference can be in W, C or both. It is important to notice that the utility and discount function is the same even for different TCS's. The investor wants to use the TCS that has larger expected utility according to Equation (2.18) but it is not self-evident how to calculate this. The technique that will be used to compare two strategies is based on Monte-Carlo methods and will be presented in chapter 3. There will not be presented any specific consumption strategies. These will be found solving the Bellman equation.

Trading strategies

There will be three relatively simple trading strategies used in this thesis and those will be compared for different investors with different utilities and discount functions. It will not add any additional value to this thesis to have more advanced strategies since the goal of this section is not to find an optimal strategy but rather a methodology to compare them.

Buy and Hold (B&H): This strategy is as simple as is sounds. The investor chooses some initial weights w_0 for the N assets and holds the assets to the final date T. Notice that the weight for this strategy are not constant. The returns for the different assets will differ and therefore are the weights not constant. The trading function for this strategy is $w_t = w_{t_-} = W(w_{t_-}, X_t)$

Constant Weights (CW): The trading function for this strategy is $w_t = w_0 = W(w_{t_-}, X_t)$ there $t = 0, \Delta t, 2\Delta t, ..., T - \Delta t, T$.

Constant proportion portfolio insurance (CPPI) This strategy is a bit more complicated than the first two. There are some differences in how this strategy is constructed in different literature. The CPPI-strategy that will be used here is inspired from the definitions in Black and Perold (1992).

The general idea of CPPI is to have a protection for future big losses. These types of strategies are sometimes used in e.g., pensions plans and life insurance. Big losses in those cases can have a huge impact on the client if there pension savings take on a big loss right before retirement. Big losses do not change the way the investor allocates for neither B&H or CW. This can be a down-side if the wealth is needed in relative short-term consumption.

Let the value of the risk-free asset and the risky asset be R_t (reserve asset) and A_t (active asset). Define a floor F_t, cushion C_t and exposure E_t. The floor is a fixed amount of money for time t. The cushion is how much more the wealth is from the floor $C_t = X_t - F_t$ and the exposure is the amount invested in the risky asset. Let m be a parameter that decides on how much the investor invests in the risky asset over the cushion i.e., $E_t = m \cdot C_t$, where $m > 1$, so long the maximum leverage is not reached. Let b be the investor's maximum leverage ratio, where $b = 1$ means no leverage allowed. The maximum leverage is $b \cdot X_t$. According to the CPPI-strategy, the exposure is

$$E_t = \min(m \cdot C_t, b \cdot X_t), \tag{2.19}$$

where $1 \leq b < m$.

Example 6. *Assume that an investor is using a CPPI for a portfolio. Let the current portfolio value be* $X_t = \$1000$*, the floor* $F_t = \$800$*,* $b = 1.5$ *and* $m = 2$*. By using the CPPI-rule in equitation* (2.19)*, the exposure will be* $E_t = \min(2 \cdot (1000 - 800), 1.5 \cdot 1000) = \$400$*, i.e., the investor will invest* \$400 *in the risky asset and* \$600 *in the risk-free asset.*

There are clear down-sides with the CPPI model if one does not need capital in the near future and the floor is a psychological floor rather than a 'real floor'. If one hits the floor (or perhaps goes slightly through it before rebalancing), the strategy is to invest all of the remaining capital in the risk-free asset. This is exactly what this strategy should do but an effect of this is that the capital is locked in low-return investments, let's call this 'lock-effect'. So CPPI should only be used if one really has a floor, a good example of that is if one plans to buy a house and needs an certain amount for a down payment. The down-side to this strategy is grater when the risk-free rate is low. This strategy would have a great down-side in these days where there is negative interest-rate in part of the world.

If one makes a special case of CRRA, it can be shown that a certain CPPI-strategy with the right parameters is the optimal investing strategy for B-S and constant interest rate for optimizing Equation (2.12) when you let $T \to \infty$, see details in Black and Perold (1992). The corresponding optimal consumption is derived as well.

The parameters of the optimal CPPI-process is dependent on every parameter in the model i.e., μ, σ, r, λ and X_0. Consider that one uses that strategy to optimize the utility and holds for one period. The next time step have X_0 change and therefor the parameters of the CPPI-strategy if one would 'update the strategy'. The special construction of the CRRA-utility function is that under the floor F of the CPPI-strategy the investor is risk-neutral and over the floor is the utility function a standard CRRA. This means that the utility function is dependent on X_t. I came to the conclusion that I did not want to investigate this strategy for a couple of reasons. The main reason is that the strategy is optimal for consumption for an infinite time in theory and I am looking at a 10 year period.

Another reason is that it does not seem reasonable to calculate a strategy at time 0 and simply hold that even if it was the optimal strategy considering that the information at that time would be different. Instead is the problem of optimal consumption and trading be solved with dynamic programming. Even if CPPI is not evaluated it was decided to present it since it was studied in detail and I came to the conclusion to leave it out.

Volatility control (VC) The goal of this strategy is to allocates the assets so that the portfolio has the same volatility, σ^*, for all time periods and it is the volatility for the entire portfolio. In this case is the portfolio only one stock/index and a risk free bank account. Assume that we have a rule that gives an optimal w at every time point. The volatility model that will be used is the one in S&P Dow Jones Indices (2018), in this paper is this method called risk control instead of volatility control. The model is based on a Risk Control Index (RC_t). The return for investing in RC is denoted r_{RC}. RC is defined

$$RC_t = RC_{t-\Delta t} \cdot (1 + r_{RC,t}).$$

The portfolio is rebalanced every $\Delta\hat{t}$. The volatility must be estimated, for this is an exponential-weighted moving average (EWMA) model used. The variance of the underlying asset according to a EWMA-model is, where λ is the decay factor and S_t is the price of the risky asset

$$V_t = \lambda V_{t-\Delta t} + (1 - \lambda) \log(S_t/S_{t-\Delta t})^2.$$

Notice the difference between $\Delta\hat{t}$ and Δt. For example, the volatility can be estimated every day and the portfolio is rebalanced every month. The volatility at the starting point can be considered known if one is simulating data or one can estimate the volatility after historical realizations and calculate the volatility of the returns time-series and use that as an initial 'guess'. The new weight at time t is defined

$$w_t = \min(w_{MAX}, \sigma_*/\sigma_t),$$

where w_{MAX} is the maximal weight that the investor is willing to invest in the risky asset. In this thesis is $w_{MAX} = 1$. This model is independent of the risk-free interest rate and the expected return of the risky asset. This can be down-side for the model. There are extended models that use excess returns (return minus risk-free return) instead of absolute return, these models can also be found in S&P Dow Jones Indices (2018). Only the simpler model is used here. An example of this strategy is:

Example 7. *An investor wants to allocate an index fund according to a volatility control measure. The estimated volatility is* $\sigma_t = 0.24$*, the investor has a target* $\sigma_* = 0.16$ *and* $w_{MAX} = 1$ *as some rebalance time point* t*. The weight in the index fund will be:*

$$w_t = \min(1, \sigma_*/\sigma_t) = 2/3$$

Note that if the simulates trajectories from a model with constant volatility, it is unreasonable to use volatility control. For Merton model that has constant volatility but if no jump has occurred is the volatility underestimated. On the other side is an EWMA framework overestimate the volatility after a jump has occurred. The Heston and Bates model has stochastic volatility so for those models a volatility strategy can be considered. A program calculating a volatility control model was provided from Kidbrooke Advisory.

2.4 Dynamic Programming for Utility Optimization

Dynamic programming and recursive methods have been used for many applications in economics and finance. Several strategies have been discussed in the previous section but they are most likely not the optimal. It is possible to use dynamic programming and use to numerical method to find an optimal strategy. The mathematical technique used comes from stochastic optimal control and is a part of classical control theory.

The first step is to define a **value function** $\Omega(t, X_t)$. The value function is the value of Equation (2.12) or (2.13) when an optimal strategy is used, for both allocation and consumption. The optimal TCS, or 'optimal control rule' is denoted $\hat{\mathscr{T}}$ for a given problem. Notice that both $\Omega(t,x)$ and $\hat{\mathscr{T}}$ are both unknown from the start and that's why regular methods for optimization do not work. It is assumed in this thesis that there exist a value function and an optimal control rule for the problem ahead but this is not self-evident for a general problem of this type.

2.4.1 The Hamilton-Jacobi-Bellman Equation

In this section will the main consulting of HJB-equation be mentioned. There is a lot of mathematical theory behind these results and this is omitted here. Perhaps will a reader, familiar to the topic, react to 'abuse of notations' and that the functions and spaces are not properly defined. If you are interested in these details is Björk (2009) recommend for an introduction of HJB-equation for finical applications and if you want to go into really technical details is Fabbri et al. (2017) recommended.

Assume that the investor wants to optimize the value of Equation (2.12) at time $t \in [0,T)$ and uses the optimal control rule $\hat{\mathscr{T}}$. Assume another rule $\mathscr{T}$ is used for a short time interval $(t, t+h]$, $(t+h<T]$ and than is the optimal rule used, then it the following is obviously true

$$\Omega(t, X(t)) \geq \mathbb{E}\Big[\int_t^{t+h} D(s)u(t,x,\mathscr{T}))ds + D(t+h)\Omega(t+h, X^{\mathscr{T}}(t+h))\Big|\mathcal{F}_t\Big], \tag{2.20}$$

since $\Omega(t,X_t)$ is the expected total utility if acted optimally and equality in Equation (2.20) only if $\mathscr{T} = \hat{\mathscr{T}}$. Therefore is $\Omega(t,X(t)) = \sup_{\mathscr{T}\in\mathcal{T}}(U(t,X(t);\mathscr{T}))$ where $U(\cdot)$ is from Equation (2.12), $\mathcal{T}$ are all feasible strategies $\mathscr{T}$. $u(t,x,\mathscr{T})$ is the utility generated at time t when having capital x and control rule $\mathscr{T}$ so there exist some c s.t. $u(c) = u(t,x,\mathscr{T})$ where $u(c)$ is the 'original version' of the utility function. In $\Omega(t+h, X^{\mathscr{T}}(t+h))$ has $\mathscr{T}$ been added since the value at $X(t+h)$ depends on the control rule.

The value of $\Omega(t+h, X(t+h)^{\mathscr{T}})$ can be expressed with $\Omega(t,X(t))$ by using Itô's formula, see Equation 19.21 in Björk (2009). If the time increment h goes to zero can the following PDE be derived:

$$\begin{aligned} &\tfrac{\Omega(t,x)}{dt} + \sup_{\mathscr{T}\in\mathcal{T}}\left\{u(t,x,\mathscr{T}) + \mathcal{A}^{\mathscr{T}}\Omega(t,x)\right\} = 0 \quad \forall t\in(0,T) \text{ and } x>0 \\ &\Omega(T,x) = u(x) \qquad\qquad \forall x>0 \end{aligned} \tag{2.21}$$

This PDE is called the Hamilton-Jacobi-Bellman Equation (HJB-Equation), and where $\mathcal{A}^{\mathscr{T}}$ is an Itô operator, see Appendix A.2. $\mathscr{T}$ denotes that the dynamics are generated by the control rule $\mathscr{T}$. The solution to the PDE is the optimal value function and the 'maximum argument' of the PDE is the optimal control rule. There is a clear problem that both Ω and $\hat{\mathscr{T}}$ are unknown. One can use the fact that the value function is known for the final time point, i.e., the utility of consuming all the money. There are still several problems and details for the continuous case that will not be addressed here, see chapter 19 in Björk (2009). If Equation (2.21) is satisfied, than it is the value funtion and the optimal control rule. This can be shown with a verification theorem, Theorem 19.6 Björk (2009) i.e., the HJB-equation is a necessary and sufficient condition. In this thesis will the focus on solving a discrete version of Equation (2.21).

2.4.2 Discrete form - Bellman Equation

It can be verified that the following equation must be true for the discrete form of the HJB-equation (the Bellman equation), with time increments Δt

$$\begin{aligned}\Omega_t(X_t) &= \max_{(c_s,w_s)_{s=t}^T} \mathbb{E}_t\Big[\sum_{s=t}^{T} D(s-t)u(c_sX_s)\Big] \\ &= \max_{c_t,w_t}\big[u(c_tX_t)+D(\Delta t)\mathbb{E}_t[\Omega_{t+\Delta t}(X_{t+\Delta t})]\big], \quad t=0,\Delta t,2\Delta t,..,T-\Delta t,T \\ &\qquad \Omega_T(X_T)=u(X_T) \quad \forall X_T>0.\end{aligned} \tag{2.22}$$

For simplicity put $\Delta t = 1$. Equation (2.22) is however only a necessary condition.

The notation is change to $\Omega_t(\cdot)$ from $\Omega(t,\cdot)$ since the value function is only defined in the discrete time point, $t=0,1,2,...T-1,T$. This equation is called the Bellman equation and can be solved with dynamic programming. There will be two cases presented. The first one is where no money is consumed until the end, but the investor can still trade and the second where the investor can both invest and consume.

Without consumption:

In this framework is no consumption until the final time point. The general problem is

$$\Omega_t(X_t) = \max_{\{w_s\}_{s=t}^{T-1}} \mathbb{E}_t[u(X_T)] = \max_{w_t} \mathbb{E}_t[\Omega_{t+1}(X_{t+\Delta t})].$$

This can easily be verified from Equation (2.22). Remember that the discount function can be ignored if the investor only consumes at one time point. Since the value function is known for all values X_T at time $t=T$ can this be solved backwards. The wealth X_{t+1} can be calculated as $X_{t+1} = X_t(w_t' R_{t+1}^e + r_t)$, where R_t^e is the excess return of the risky asset (return minus the risk free return) and r_t is the risk free return over $t \to t+1$. Since there will only be two assets in this thesis will $X_{t+1} = X_t(w_t R_t^e + r_f)$. The idea now is to use the fact that the final value of Ω is known for all wealth's X_T and use that and go backward.

A problem when solving a dynamic asset allocation problem is the problem of branching with MC. Assume that you only wants to solve the problem for a two-period case with one underlying dynamics. If one does a simulation of 10000 trajectories to the first time point and at that time point do 10^4 new simulations from every trajectory to solve $w^*(x_t)$. Then only after two time point are you up at $(10^4)^2 = 10^8$ trajectories. This technique is obviously not feasible, even for a few periods. It is feasible to use the method LSMC with so called inner and outer scenarios for a two-period problem. This method was presented to price American option in Longstaff and Schwartz (2001).

Unfortunately, even this method is not a good fit for the problem of dynamic utility optimization. This is because the problem with branching still occurs. The only way to solve this is simply by accepting that many trajectories are needed and then to derive a method that efficiently solves the problem as accurately as possible. There are several possible ways, in Brandt et al. (2005) is regressions done for functions related to the value function (e.g., its derivative), in Cong (2016) is a Stochastic Grid Bundling Method (SGBM) presented. In both this method are the results for all trajectories used to solve the value (of either the value function and/or the optimal weight) for a given value of the underlying process.

It means, with an underlying process, a process that the current state change the result of the optimal control, so $\Omega = \Omega(X_t, Z_t)$ where Z_t is the process value at time t. A simple example is volatility in the Heston model, if the volatility is high then should the optimal weight be lower.

In Brandt et al. (2005) is Taylor expansion of the value function used around the value if the investor would only invest in the risk free asset. Since the program solves the problem backwards is the optimal weights for $s=t+1,..,T-1$ known at t. So the wealth that the expansion is done 'around' is $X_{t+1} = X_t r_f$. If one considers when one uses an expansion of order two for a CRRA- utility function will the first order of condition be

$$\mathbb{E}_t\Big[\frac{\partial \Omega_{t+1}(X_t(w_tR_{t+1}^e + r_f), Z_t)}{\partial(X_t(w_tR_{t+1}^e + r_f))} R_{t+1}^e\Big] \approx \mathbb{E}_t\Big[\frac{\partial u(X_t(w_tR_{t+1}^e + r_f))}{\partial(X_t(w_tR_{t+1}^e + r_f))}\phi_{t+1}(Z_{t+1})R_{t+1}^e\Big] = 0,$$

where ϕ is a function of the underlying variable Z, see Brandt et al. (2005) for details. The value function depends, sometimes indirectly, on the underlying process Z since the return and/or the portfolio value depends on the state variable. An approximated solutions for CRRA-utility function is:

$$
\begin{aligned}
w_t &\approx -[\mathbb{E}(B_{t+1})X_t]^{-1} \cdot \mathbb{E}[A_{t+1}] \\
A_{t+1} &= \tfrac{\partial \Omega_{t+1}(X_t r_t, Z_{t+1})}{\partial (X_t r_t)} R^e_{t+1} \\
B_{t+1} &= \tfrac{\partial^2 \Omega_{t+1}(X_t r_t, Z_{t+1})}{\partial (X_t r_t)^2} (R^e_{t+1} R^e_{t+1}{}').
\end{aligned}
$$

Using simulation techniques as discussed previously can these values be approximated numerically. For a given simulated value X_t^m is the weights at time t for simulation m:

$$\hat{w}_t^m = -(\hat{B}^m_{t+1} X_t^m)\hat{A}^m_{t+1}$$

where the notion with 'hats' mean numerical approximations.

2.4.3 Dynamic programming and CRRA-utility function

How a CRRA-investor would consume assuming no investment is summarized in a Theorem:

Theorem 4. *An investor has a CRRA-utility function with risk parameter $\lambda > 0$. The investor has a wealth X and has his money is 'under the mattress' and the discount factor is assumed to be 1 i.e., no discounting. The investor can consume at every time point $t_0, t_1 \dots, t_N = T$ and the number of time points are $N + 1$ consuming the remaining wealth at t_N. The is equivalent to say that the investor wants to spread out the consumption as much as possible. The Bellman equation for the optimal consumption for the investor is*

$$c^* = \frac{X}{N+1} \quad \forall t_i, i \in [0, N].$$

Proof. The theorem will be proven by induction. It is trivial that for the one period case is $c = 1$.

Two period case: Assume that the investor can only consume at two time points t_0, t_1. According two theorem should the optimal consumption be $c = 1/(1 + 1) = 0.5$. The Bellman equation for this case is:

$$U = \frac{(cX)^{1-\lambda}}{1-\lambda} + \frac{((1-c)X)^{1-\lambda}}{1-\lambda}$$

After taking the derivative with respect to c and cancelling out some terms, the optimization problem will be

$$
\begin{aligned}
X^{1-\lambda}(c^{-\lambda} - (1-c)^{-\lambda}) &= 0 \\
\iff \quad c^{-\lambda} &= (1-c)^{-\lambda} \\
\iff c &= \tfrac{1}{2}
\end{aligned}
$$

and the theorem has been proved for the two period case.

General case: Let $N > 1$ and assume that it is true the theorem is true for all $1 \leq \hat{N} < N$. The optimization problem is:

$$U = \frac{(cX)^{1-\lambda})}{1-\lambda} + \sum_{n=1}^{N} \frac{\Big((1-c)\prod_{s=1}^{n-1}(1 - \frac{1}{N-s+1})\frac{1}{N-n+1}X\Big)^{1-\lambda}}{1-\lambda}. \tag{2.23}$$

The terms in the product in Equation (2.23) is the amount that the investor has consumed between the time point t_n and $1/(N - n + 1)$ is the amount consumed at time point t_n assuming that the theorem is true. The product is a telescope product and all except two terms can be cancelled out. The product can be simplified to:

$$\prod_{s=1}^{n-1}(1 - \frac{1}{N-s+1}) = \prod_{s=1}^{n-1}(\frac{N-s}{N-s+1}) = \frac{N-n}{N}.$$

After taking the derivative of Equation (2.23) and doing some algebra:

$$
\begin{aligned}
c^{-\lambda} + \sum_{n=1}^{N} \big((1-c)\frac{1}{N}\big)^{-\lambda}\frac{1}{N} = 0 \\
\iff c = (1-c)\tfrac{1}{N} \\
\iff c = \tfrac{1}{N+1}.
\end{aligned}
$$

Thus, the theorem has been proven for the general case.

□

Theorem 4 is a simplification of a real world problem since there are no investments, no discounting and no income of capital but the theorem gives us a good indication of how the investor should approximately act. If it can be assumed that the investor can invest, and the exempted return of the investment strategy is greater than the discount factor should the investor consume more than in the case in theorem. The reason for this is that the prudent behavior of the investor still wants to have equal cash flows. Let's show this is an easy example:

Example 8. *Consider an investor with CRRA-utility function and risk parameter λ. Assume that the investor has wealth X today and can consume today and in ten years. This seems quite a long time-horizon but it this illustrates better the effect of the discount function and returns. The investor's optimal portfolio has an expected return 6% per year and the risk-free rate and the discount factor is 4%. Assume that the expected value of the return has realized, so the return turn out to be $(1+0.06)^{10} = (1+R) = 1.79$ and the discount factor is $\delta = (1+0.04)^{-10} \approx 0.676$. Notice that this is theoretically incorrect since it is assumed that the investor will get the expected return. The goal of this example is not to calculate optimal portfolio but instead to show how returns affect the consumption in an intuitive way rather than in a theoretical way. The investor consumed the remaining capital at $t = 10$. The optimization problem becomes*

$$
\begin{aligned}
\max_c(U(c)) = \tfrac{(cX)^{1-\lambda}}{1-\lambda} + \delta\tfrac{((1-c)X(1+R))^{1-\lambda}}{1-\lambda} \\
\iff c = \Big(1 + \tfrac{1}{\delta^{-(1/\lambda)}(1+R)^{1-1/\lambda}}\Big)^{-1}.
\end{aligned}
\tag{2.24}
$$

In Theorem 4 is $\delta^{-(1/\lambda)}(1+R)^{1-1/\lambda} = 1$ and therefore is $c = (1+1)^{-1} = 1/2$. If we consider a relatively risky investor with $\lambda_A = 0.5$ and one relatively risk averse $\lambda_B = 5$. The optimal c for these investor are $c_A = 0.55$ and $c_B = 0.63$. This shows that both investor want to consume more today than in Theorem 4. This is not true if the investor has a very low λ but those investors are not considered. The limit for increasing lambda $\lim_{\lambda\to\infty} c \approx 0.642$ since $(1-c)\cdot(1+R) = 0.642$ i.e., the more prudent the investor is the more does the investor want equal out consumption. Notice that the discount factor $\delta^{-1/\lambda} \to 1$ and $(1+R)^{1-1/\lambda} \to (1+R)$ when λ is large in Equation (2.24).

This can therefore be expected for the result from a numerical examples with proper theoretical framework for investors with high relative prudence. If the number of days that the investor consumes, for example, every year, the effect from the return would become even smaller. If the discount factor is change to $(1+0.02)^{-10}$ is the optimal weight instead 0.62 for λ_B, so the effect of a decrease on the discount factor change the optimal consumption surprisingly little. However for λ_A is the optimal $c = 0.45$. In this case is λ so low that the investor has such a low relative prudence that the investor is willing to wait with consumption since the return is that big compared to his discount function and relative prudence. Investors like that are not considered in the Result chapter.

2.4.4 Proxy-function method

This is only done if there is a underlying process. Instead of doing a regression on the value function or its derivatives have it been tried to do regression directly on the maximum argument

$$
w_t^*(Z_t) =' \arg_{w_t} \Omega_t(Z_t)' = \varphi(Z_t)'\theta_t, \tag{2.25}
$$

where θ_t is a vector of regression parameters, $\varphi(Z_t)$ is a proxy function and Z_t the regression parameter at time t. There is abuse of notation in Equation (2.25), but the term $\arg_{w_t} \Omega_t(Z_t)$ means the optimal weight at time t given the regression variable Z_t. That is equivalent to the maximum argument of the weight between $t \to t+1$ of the total expected utility U given that the investor acts optimally over $t+1 \to T$.

The proxy function that have been used are polynomial up to second order, so $\varphi(Z_t)' = [1, Z_t, Z_t^2]$ and $\theta' = [\theta_0^t, \theta_1^t, \theta_2^t,]$. To get a more accurate approximation from the proxy function one can have higher order polynomials but the risk of over-fitting the noise from the MC-simulations increases. If one wants to use higher order polynomials are more trajectories therefore needed.

A mathematical argument is that the optimal weight should be able to be approximated a second order Taylor Expansion around $\mathbb{E}_0[Z_t]$

$$\begin{gathered} w_t^*(Z_t) =' \arg_{w_t} \Omega_t(Z_t)' \approx \\ w^*(\mathbb{E}_0[Z_t]) + w^{*\prime}(\mathbb{E}_0[Z_t])(\mathbb{E}_0[Z_t] - Z_t) + \tfrac{w^{*\prime\prime}(\mathbb{E}_0[Z_t])}{2}(\mathbb{E}_0[Z_t] - Z_t])^2 = \\ \theta_0^t + \theta_1^t Z_t + \theta_2^t Z_t^2. \end{gathered} \tag{2.26}$$

However $w_t^*([Z_t])$ and its derivatives are unknown. It is assumed that the function is continuous and $w^*(Z_t) \in C^2$. Therefore is simply a proxy function used to estimated $w^*(Z_t)$. One can instead use the notation of U and define the optimal weight $w^*(Z_t) = \arg\max_{w_t} U(Z_t; \hat{\mathscr{T}}^{t+1:T})$ where $\hat{\mathscr{T}}^{t+1:T}$ is the optimal control rule over time $t+1 \to T$.

The regression is done with scipy's package `minimize`. Notice that a ordinary regression can't be done since the values of $w_t(Z_t)$ are unknown so there are no data points to make a regression on from the start. This will be gone through more in detail in the next chapter. One can use different optimization algorithms to solve a problem. In this case is Powell, SLSQP and Nelder-Mead used. The algorithms behind these methods are omitted in this text. The reason behind the choice was performance in validation tests i.e., examples where there are theoretical answers so that the results can be compared, speed and possible extra features such as possibility to have boundary conditions as input. An example with regression is presented here. The details of this method are presented in the next chapter.

Regression on volatility:

The Heston and Bates models have stochastic volatility and is the underlying asset for one case. The choice in optimal weight should be lower for higher volatility. Given a set of regression parameters can the approximated value function can be calculated. Remember that in this case is the value function a function of three variables, both time, portfolio value and volatility.

The MC-trajectories of the underlying asset is an approximation of the distribution function, number of trajectories is N. At every time t is an approximated value function and the future optimal weight known at $t+1$ and given the underlying asset. The following optimization is solved when finding the optimal weights at time t

$$\begin{aligned} \theta_t^* &= \arg\max_{\theta_t} \frac{1}{N} \sum_{i=1}^{N} u(X_T(w(\theta, V_t^i), \hat{\mathscr{T}}^{t+1:T})) = \\ &= \arg\max_{\theta_t} \frac{1}{N} \sum_{i=1}^{N} u\big(X_t(w(\theta, V_t^i) R_{t+1}^{e,i} + r_f) \cdot \prod_{s=t+1}^{T-1} (w_s^*(V_s^i) R_{s+1}^{e,i} + r_f)\big) \end{aligned} \tag{2.27}$$

where θ is the parameters of the proxy function, V_t is the variance and $R_s^{e,i}$ is excess return from the simulated trajectories. The arguments $w(\theta_t, V_t) = \varphi(V_t)'\theta_t = \theta_t^0 + \theta_t^1 V_t + \theta_t^2 V_t^2$, when a second order proxy function is used, and $\hat{\mathscr{T}}^{t+1:T}$ in $X_T(\cdot,\cdot)$ means simply the portfolio value given that weight $w(\theta_t, V_t)$ is used over $t \to t+1$ and that control rule $\hat{\mathscr{T}}$ is used over $t+1 \to T$.

When CRRA is used and there is no deposits doesn't the the optimal weights depend in X_t and it is set to $X_t = 1$ for all t and all trajectories. It is therefore not needed as as input to Ω for this problem. Notice that the arg max operator is taken over θ in Equation (2.27) but w in Equation (2.25), however are the weights and the parameters θ connected with the proxy function, see Equation (2.26).

I have not been able to prove how well this method works theoretically and that can be done in future research. However, practical numerical examples show convergence.

2.5 Numerical issues

There are several numerical issues that can occur when calculating utilities and solving the optimization problem. To demonstrate the problem for utility calculations CRRA-function will be used but the same happens for HARA. If one have an investor with a CRRA with capital of one million and parameter $\lambda = 10$ will the utility be $u(x) = (1000000)^{-9}/(-9) = 1.11^{-55}$. It it not feasible to optimize the utility since the accuracy of a computer is around 10^{-15}. To solve this problem one can simply problem for utility calculations the wealth normalize to 1. If one have a money dependent utility functions, than the parameters change as well so the utility function is equal.

For HARA is the RRA in Equation (2.10) and from this equation is an important property concluded. With regard to risk aversion, is τ/x the dependant part. So if one want to normalize a HARA utility function should $\tau_{OLD}/x = \tau_{NEW}/1$, so the normalized τ is simply the quota of the old τ and the wealth.

Chapter 3

Method

In this chapter, I present the methodology used in this thesis. The chapter starts with moment-matching and MC-algorithm for the financial assets. The general problem to solve when having trajectories is optimization of different types. For all problems is the time-horizon 10 years. A Golden Section optimizer was implemented for the static problem i.e., when one does not solve the dynamic Bellman equation. When solving the Bellman equation a numerical package is used. There are several technical difficulties left to solve even if a numerical package is used. An algorithm is presented to solve a general problem with one underlying process. A final iterative algorithm is presented for the cases where the value of the underlying process is unknown. A regression can be made for the estimated values of the underlying process but the regression will be biased since the values are only an estimation (and even sometimes more or less a guess).

3.1 Scenario generation and moment-matching

In further analyses are the trajectories used to study different strategies and optimize them. In the result B-S, Merton and Heston and the Bates model will be used and the effect of adding the dynamics in the Heston and Merton model is analyzed. The Bates model has both the added components in Merton and Heston, compared to B-S and it is therefor difficult to analyze which dynamics affect the results and how. The initial stock (or index) price is $S_0 = \$100$. The goal is to make the two first moments equal for both models or equivalently the expected value and the variance. The focus will not be on parameter estimation but rather on analyzing the utility. For the B-S model are there only two parameters, the drift (μ) and the volatility (σ). The drift is set to the approximated average drift of the $S\&P-500$ for the last 40 years and is put to $\mu = 0.0676$. This is not an 'exact answer' since there are difficulties calculating the exact return due to inflation etc, but these difficulties are not important for this thesis. The volatility will be set to $\sigma = 0.160$. The volatility is approximately the average volatility of S&P-500. It is important to be clear that these parameters are not found through any statistical parameter estimation but instead put to 'reasonable' numbers.

Given some parameters, MC-simulations can generate scenarios. For the Vasicek model, the parameters are inspired from Brandel (2018) and when the risk-free interest rate is considered to be constant, it is put to 2.0% which is approximately the interest rate in the US today. However, it has gone a bit higher in the last few months e.g., was the overnight LIBOR-rate in the US 2.394% at 4-th of January 2019.

3.1.1 Moment-matching

This is more easy done with the log-scale for the stock price. The real-value dynamics for the B-S model are given in Equation (2.4). By using Itô's-formula (see Theorem 5 in Appendix A.2), the dynamics of $X(t) = log(S_t)$:

$$dX(t) = \big(\mu - \frac{\sigma^2}{2}\big)dt + \sigma dW^{\mathbb{P}}(t).$$

When adding a jump process to B-S, i.e. the Merton model, diffusion and volatility are added from the jumps. The log-price process for a Merton model is (the parameters μ and σ do not need to be equal to the B-S parameters)

$$dX(t) = \big(\mu - \frac{\sigma^2}{2}\big)dt + \sigma dW^{\mathbb{P}}(t) + ln(1 + J(Q))dP,$$

where P is the Poisson process and $Q = ln(1+J)$ is the log-return jump amplitude. This notation is inspired from Hanson and Westman (2008). The expected value $\mathbb{E}[J(Q)] = \exp(\mu_J + \sigma_J^2) - 1$ and the variance is $Var[J(Q)] = E^2[\exp(Q)] \cdot (\exp(\sigma_J^2) - 1)$. The expected value and the variance of the log-return is:

$$\begin{cases} \mathbb{E}[dX(t)] = \big(\mu - \frac{\sigma^2}{2}\big)dt + \lambda\mu_J dt = \big(\mu_{B-S} - \frac{\sigma_{B-S}^2}{2}\big)dt \\ Var[dX(t)] = (\sigma^2 + \lambda(\sigma_J^2 + \mu_J^2))dt = \sigma_{B-S}^2 dt. \end{cases} \tag{3.1}$$

Unfortunately, there are three more parameters in the Merton model compared to B-S and therefor there does is not exist a unique solution to match the moments. To solve this problem is the drift term μ given a premium compared to the B-S model and that value is fixed. The jumps are considered to have a negative mean. The first moment does only depend on σ, λ and μ_J. These values are found with the Pythons package Scipy's function `mininize`. Reasonable numbers are used as an initial guess and the optimizer finds a closed by minimum. The function that is minimized is the squares error, i.e:

$$\texttt{ERR} = \Big(\big(\mu - \frac{\sigma^2}{2}\big) + \lambda\mu_J - \big(\mu_{B-S} - \frac{\sigma_{B-S}^2}{2}\big)\Big)^2.$$

I will want to consider rare events with big jumps since it is reasonable to consider that to be the case in the real world, especially for index jumps. These are considered to be 'big news shocks' like Brexit, large terror attacks, big defaults etc. The initial guess is therefore that the jump intensity is $\lambda = 0.2$, so once every five years. The initial guess for μ_J is -0.08. Finally is the initial guess for the volatility slightly lower than for B-S model. It is a requirement that $\sigma < \sigma_{B-S}$ from Equation(3.1). Given that we have found reasonable values for σ, λ and μ_J there is an analytic solution to σ_J given by:

$$\sigma_J^2 = \frac{\sigma_{B-S}^2 - \sigma^2}{\lambda} - \mu_J^2.$$

The next step is to do the same for the Heston model. It is the same type of problem, there are more parameters than equations. Therefore, some parameters are fixed and some calculated after that. A big problem is that the estimated value and the variance of the log-return is not linear. In this case when the investor only consumes in the final date, it is considered to be after 10 years. The moments are matched for 10 years. The parameters that need to be found are μ, κ, θ, σ_V and ρ. The first step was to fix some of the parameters, the correlation parameter is put to $\rho = -0.5$ and $\kappa = 0.1$. These parameters were found in Brandel (2018). No good direct method was found to calculate the other parameters. Because of this fact is the parameters change manually until the expected value and the variance was reasonably close to the other models. This is clearly a big down-side of this method. More study would be needed to find a better way.

For the Bates model is the parameters from the Merton and Heston model used. This obviously gives too low expected value and too high variance. The expected value is lower than the target since the Heston models expected value is right and then is negative jumps added. From the jumps are extra variance added so the variance is larger. The parameters are changed (relatively small changes are needed) to fit the target moments as good as possible.

3.1.2 Scenario generation

The models of the financial assets are in continuous time but simulations have to be done in discrete time. For models that have closed form solutions, such as a stock following B-S model, are there no need for a discretization schemes.

When the model does not have a closed form solution is a discretization scheme is needed. For advanced model may there not exist exact methods and approximations are needed. The algorithms that are used in this thesis is heavily inspired from Brandel (2018).

Vasicek model: Some properties for the Vasicek model is for $0 \leq s \leq t$ are:

$$
\begin{aligned}
r(t) &= r(s)e^{-a(t-s)} + \frac{a}{b}(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dW^{\mathbb{Q}}(u), \\
\mathbb{E}^{\mathbb{Q}}[r(t)|r(s)] &= r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)}), \\
\mathbb{V}^{\mathbb{Q}}[r(t)|r(s)] &= \frac{\sigma^2}{2a}(1 - e^{-2a(t-s)}).
\end{aligned}
$$

See Equation (2.3) for the dynamics. Trajectories can be simulated from the algorithm:

Algorithm 1 Vasicek

Require: $r_0, a, b, \sigma, \Delta, T, nbrSims, nbrSteps$

$Z \sim N(0,1)$ size $[nbrSims, nbrSteps]$
$s^2 = \frac{\sigma^2}{2a}(1 - e^{-2a\Delta})$
$r(0) = r_0$
$\alpha = e^{-a\Delta}$
$\beta = (b/a)(1 - e^{-a\Delta})$
for $i = 1 \to nbrSteps$ **do**
    $e(t_i) = Z(i)s$
    $r(t_i) = \alpha r(t_{i-1}) + \beta + e(t_i)$
end for

Algorithm 1 is exact. When calculating the interest rates for the portfolio values are they considered to be constant on Δ. When using the stochastic rates are they approximated to be constant on the interval Δ. Assume that the investor has $X_b(t)$ on the bank account and the simulated rate is r_t. At time $t + \Delta$ will the value in the bank account be $X_b(t + \Delta) = X_b(t)e^{r_t \Delta}$. When using the constant is the return to $(1 + 0.02)$, the corresponding log-scale rate is $r_0 = log(1.02) \approx 0.0198$. This value is uses as r_0 in Algorithm 1. The parameters used in the results are $a = 0.2, \sigma = 0.01$ and clearly $b = r_0 \cdot a$.

CIR-process: The variance model for the stock price in Heston (and Bates) are the same and it is a CIR-process. It is possible to simulate this process exact since the closed form solution is distributed as a non-central chi-square distribution with $d = 4\kappa\theta/\sigma^2$ degrees of freedom and non-centric parameter $\lambda(t) = (4\kappa V_0 e^{-\kappa t})/(\sigma^2(1 - e^{\kappa t}))$, Brigo and Mercurio (2006). The problem with this approach is that drawing a non-central chi-squared distributed random variable is 'expensive' for the computer and takes a long time compared to Gaussian and uniform random variables. Therefore, a discretization schemes is used instead.

The discretization is done with a quadratic exponential scheme. Let's consider a general CIR-model, $dx(t) = \kappa(\theta - x(t))dt + \sigma\sqrt{x(t)}dW$ and $x(0) = x_0$. Simulating this method is derived in Andersen et al. (2001). The distribution is known for $x(t + \Delta)$ given $x(t)$. Since the distribution is given by a non-central chi-squared distribution one can use the fact that $\chi_d'^2(\lambda) \stackrel{d}{=} (Z + \sqrt{\lambda})^2 + \chi_{d-1}^2$ where $\stackrel{d}{=}$ means equality in distribution and $Z \sim N(0,1)$ and χ_{d-1}^2 is a central chi-squared RV with $d - 1$ degrees of freedom. By condition on $x(t)$ can x be approximated by

$$
x(t + \Delta)|x(t) \stackrel{d}{\approx} a(b + Z)^2. \tag{3.2}
$$

where a and b are moment-matching constants and Δ the step size. The approximation in distribution in Equation(3.2) is sufficiently good for practical applications. The values are defined:

$$
\begin{gathered}
m = \mathbb{E}[x(t + \Delta)|x(t)] = \theta + (x(t) - \theta)e^{\kappa\Delta}, \\
s = \mathbb{V}[x(t + \Delta)|x(t)] = \tfrac{x(t)\sigma^2 e^{\kappa\Delta}}{\kappa}(1 - e^{\kappa\Delta}) + \tfrac{\sigma^2}{2\kappa}(1 - e^{\kappa\Delta})^2, \\
\text{Define: } \psi_s = \tfrac{s^2}{m^2}, \\
b = 2\psi_s^{-1} - 1\sqrt{2\psi_s^{-1}(2\psi_s^{-1} - 1)}, \quad a = \tfrac{m}{1+b^2}, \\
p = \tfrac{\psi_s - 1}{\psi_s + 1}, \quad \beta = \tfrac{2}{m(\psi_s + 1)}.
\end{gathered}
$$

Now one simply can simulate the next value by $x(t+\Delta) = a(b+Z)^2$. However this moment-matching technique does not work if $x(t)$ becomes too small and one will need another approach. If the value of $x(t)$ is 'too small' is the process simulated as

$$x(t+\Delta) = \begin{cases} 0, & U \leq p \\ \beta^{-1} \log\left(\frac{1-p}{1-U}\right), & U > p. \end{cases} \tag{3.3}$$

where $U \sim \mathcal{U}(0,1)$ is a uniform distribution. It is a sufficient rule to define any $\psi_c \in (1,2)$ according to Andersen et al. (2001). If $\psi_s \leq \psi_c$ than is Equation(3.2) be used and otherwise is Equation(3.3) is used. In this thesis, ψ_c is set to 1.5.

Bates: The Bates model consists of three related dynamics, see Equation(2.5). The volatility model can be simulated as the CIR-process. The stock process for the Bates model is the same as for Heston plus a jump diffusion process. The two processes are independent and can be separated in the algorithm. An algorithm for simulating a stock price with Heston model was presented in van Haastrecht and Pelsser (2010). The stock process for a the Heston model is $dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dW_s^P$. The stock price model will be simulated in the real-world measure $\mathbb{P}$ since we are only interested in the movements and not a derivative price (where you need $\mathbb{Q}$-dynamics). By integrating the process from $t \to t+\Delta$ one gets:

$$S(t+\Delta) = S(t)\exp\Big(\int_t^{t+\Delta}(\mu + \tfrac{1}{2}V(\tau))d\tau + \int_t^{t+\Delta}\sqrt{V(\tau)}dW_s^{\mathbb{P}}(\tau)\Big),$$
$$V(t+\Delta) = V(t) + \int_t^{t+\Delta}\kappa(\theta - V(\tau))d\tau + \sigma\int_t^{t+\Delta}\sqrt{V(\tau)}dW_v^{\mathbb{P}}.$$

The integral of the V-process is complicated to simulate. The BM's are correlated (with ρ) and from definition is

$$\begin{pmatrix} W_s(t+\Delta) - W_s(t) \\ W_v(t+\Delta) - W_v(t) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Delta\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) = N(\mathbf{0}, \Sigma).$$

To handle the correlation is Cholesky decomposition used, see the Appendix, and the BM for the stock process is $W_s(t) = \rho W_s(t) + \sqrt{1-\rho^2}W(t)$ where W is independent. By using this decomposition and using Itô's formula, see the Appendix, the log-price can be calculated as:

$$\log(S(t+\Delta)) = \log S(t) + \tfrac{1}{2}\int_t^{t+\Delta}V(\tau)d\tau +$$
$$\rho\int_t^{t+\Delta}\sqrt{V(\tau)}dW_v(\tau) + \sqrt{1-\rho^2}\int_t^{t+\Delta}\sqrt{V(\tau)}dW^P(\tau).$$

By using the integrated solution for the variance CIR-process by doing the approximation:

$$\int_t^{t+\Delta} V(\tau)d\tau \approx (\gamma V(t+\Delta) + (1-\gamma)V(t)))\Delta \quad 0 \geq \gamma \geq 1.$$

In this thesis, γ is set to $1/2$. By using all the results one gets

$$\log(S(t+\Delta)) = \log S(t) + K_0 + K_1 V(t) + K_2 V(t+\Delta) + \sqrt{K_3 V(t) + K_4 V(t+\Delta)}Z,$$

where the K-values and Z are

$$K_0 = \Delta\Big(\mu + \frac{\rho\kappa\theta}{\sigma}\Big), \qquad K_1 = \gamma\Delta\Big(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\Big) - \frac{\rho}{\sigma},$$
$$K_2 = (1-\gamma)\Delta\Big(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\Big) + \frac{\rho}{\sigma}, \qquad K_3 = \gamma\Delta(1-\rho^2),$$
$$K_4 = (1-\gamma)\Delta(1-\rho^2), \qquad Z \sim N(0,1).$$

An algorithm is now in place for the variance dynamics and for the stock process, i.e. the algorithm for the Heston model is completed. The last thing is to add the jumps and that is relatively easy. One draws a random variable $Y \sim Poi(\lambda(t)\Delta)$ where $\lambda(t)$ is the intensity. If N is greater than 0, than $J = \mu_J Y + \sigma_J\sqrt{Y}Z$ where $Z \sim N(0,1)$. Let $logS_{-}(t)$ denote the price before a jump. When added the jump to the logarithmic stock price:

$$\log S(t+\Delta) = \log S_{-}(t+\Delta) + J.$$

3.2 Optimization

3.2.1 Optimal weights for static strategy

Given that the investor has a trading strategy $w_{t_i} = W(w_{t_i-}, X_{t_i}, \mathcal{F}_t)$ where $t_i = 0, \Delta t, ...T$ and w_{t_i-} is the weight before trading, X_t is the current wealth and $\mathcal{F}_t$ is the filtration. Notice there is no consumption until T. In this section covers only optimization problems that depend on the initial weights w_0 solved. The optimization problem is

$$w_0^* = \arg\max_{w_0} \mathbb{E}[u(X_T)|X_0],$$
$$w_{t_i} = W(w_{t_i-}, X_{t_i}, \mathcal{F}_{t_i}).$$

where $W(\cdot)$ is defined in Equation(2.18). This problem can be hard to solve even with good MC-methods, especially if the portfolio consist of many assets, complex dynamics and correlations between the assets. This problem will be solved for several cases:

CRRA-utility and B&H strategy: The CRRA utility function look like $u(c;\lambda) = c^{1-\lambda}/(1-\lambda)$. The investor can invest in a risk-free investment for 10 years that yields 2% and invest in the risky asset. The portfolio value at time 10 will be

$$X_T = w_B X_0 (1 + 0.02)^{10} + w_R X_0 r_{RISKY}(10). \tag{3.4}$$

Where w_B and w_R are the weighs in the risk free asset (bank account) and the risky asset (such that $w_B + w_R = 1$), X_0 is the initial wealth and $r_{RISKY}(10)$ is the total return at 10 years for the risky asset.

The simulations are used to find $r_{RISKY}(10)$. Due to the relation between the two weights, w_B is replaced by $w_B = (1 - w_R)$.

The optimization problem of interest is

$$w_R^*(\lambda) = \arg\max_{w_R \in (0,1)} \mathbb{E}[u(X_T(w_R); \lambda].$$

where $w_R^*(\lambda)$ is the optimal weight in the risky asset and $X_T(w_R)$ is a function of w_R, see Equation (3.4). The expected utility is simply calculated:

$$\mathbb{E}[u(X_T(w_R); \lambda, w_R)] = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \frac{X_T^i(w_R)^{1-\lambda}}{1-\lambda}.$$

where $X_T^i(w_R)$ is the portfolio value at time T of trajectories $i \in (1, N_{sims})$ with weight w_R.

CRRA-utility and CW: In this case, the investor rebalance the portfolio at every time period Δt to its original weights, w^*. This can be done without transaction cost.

The reason for solving this problem is to find good starting values for the weighs for the different strategies that will be studied. It also gives reasonable values for λ. The discount factor does not affect the solution in the one period case and therefore it is ignored.

Golden Section

When calculating optimal weight for B&H and CW strategy invested in the risky asset was a Golden Section algorithm used. To be consistent with the optimization literature do I consider the minimum of the negative utility, i.e., $w^* = \arg\min_w[-\mathbb{E}[u(X_T(w))] = \arg\min_w[F(w)], w \in (0,1)$.

Consider the function $F(w)$ that is defined on the interval $w \in \mathscr{L}_0 = [a_0, b_0]$ and is relatively heavy to compute, the derivative is unknown or numerically difficult to approximate and the function is convex or at least unimodal. Define the (unknown) optimal weight w^*. This method is an iterative algorithm and k is the current number of iteration. Define a_k and b_k the current lower and upper and the current interval $L_k = b_k - a_k$. Consider the starting points at $k = 0$ and the interval $L_0 = b_0 - a_0$. Two point, θ_0 and μ_0 are chosen in $\mathcal{L}_0$. The goal is that the algorithm converges optimally. Consider the improvement $L_k = \alpha L_{k-1}, \quad 0 < \alpha < 1$. The two points are defined:

$$\begin{cases} \theta_k = a_k + (1-\alpha)(b_k - a_k), \\ \mu_k = a_k + \alpha(b_k - a_k). \end{cases}$$

Given a α, can some conclusions be made from the result so far:

Case 1: If $F[\mu_k] < F[\theta_k]$. It is known that the minimum argument is on the interval $[\theta_k, b_k]$. Therefor can the points be updated:

$$\begin{cases} a_{k+1} = \theta_k, \\ b_{k+1} = b_k, \\ \theta_{k+1} = \mu_k, \\ \mu_{k+1} = a_{k+1} + \alpha(b_{k+1} - a_{k+1}). \end{cases}$$

For the updated θ_{k+1} is this equality only true for one α. The value of alpha is given by $\alpha = (\sqrt(5)-1)/2 \approx 0.618$ and this number is called 'the golden section number'. It is quite self-evident to find this number if one defines $\theta_{k+1} = \mu_k$.

Case 2: If $F[\mu_k] > F[\theta_k]$. The section will be on the other side. The new points are:

$$\begin{cases} a_{k+1} = a_k, \\ b_{k+1} = \mu_k, \\ \theta_{k+1} = a_{k+1} + (1-\alpha)(b_{k+1} - a_{k+1}), \\ \mu_{k+1} = \theta_k. \end{cases}$$

When the interval $L_{k-1} - L_k <$ `TOL` is the algorithm terminated and an approximation of optimal weight is found from the minimum arguments of θ and μ.

The method will converge if the function is unimodal. The iteration is stopped when the difference in the optimal point from one iteration to the next is smaller than `TOL`. There is a problem if the initial limit point is the optimal weight. That is the case if the investor has a low risk aversion and $w^* = 1$. But the algorithm still converge to $w^* = 1$. Remember that the optimal weight is the minimum argument and not the minimum of the function itself.

3.2.2 Analysis of different $\mathscr{T}$ with non-parametric bootstrap

Consider that an investor has two different trading and consumption strategies $\mathscr{T}$ i.e., $\mathscr{T}_1$ and $\mathscr{T}_2$ where $\mathscr{T}_1 \neq \mathscr{T}_2$. The investor has a defined utility function $u(\cdot)$, U is the total expected discounted utility and a discount function $D(t)$ and an initial wealth X_0. The expected utility of the different TCS's is as in Equation(2.18). The goal is the find the expected value and the distribution of $Pr(U(\mathscr{T}_1, X_0) \leq U(\mathscr{T}_2, X_0))$. The two strategies are used for the N simulated trajectories and a vector (u_1 and u_2) of N utilities found and an approximated expected value of the utility can be calculated. These utilities are uniformly drawn from the utility vector and a new expected value is calculated. This is repeated for B-times. The probability that $\mathscr{T}_1$ is better or worse than $\mathscr{T}_2$ can be calculated with this technique. Algorithm 2 describes the steps.

Algorithm 2 Bootstrap

Require: u_1, u_2, N, B
$k = 0$
for $b = [1, B]$ **do**
  $idx = RANDINT([1, N], N)$ Index: N uniform random integers between (1,N)
  $U_1^{MEAN} = MEAN(u_1[idx])$
  $U_2^{MEAN} = MEAN(u_2[idx])$
  if $U_1^{MEAN} > U_2^{MEAN}$ **then**
    $k++$
  end if
end for
$p = k/B$ Probability that strategy 1 is better

In Algorithm 2 is idx the indices of the vectors u_1, u_2. Notice that the probability calculated is strictly better/positive. The reason behind this is that for some riskier investors, the optimal weight is one or higher. In this thesis, leverage is not allowed so all of these investor will have weight one. In this case is B&H and CW identical and therefore if the algorithm calculates 'equal or positive' would the answer be 100% and that would be misleading.

Bootstrap will also be used in the same way to calculate confidence intervals for the optimal weights. In that case are trajectories bootstrapped and the optimization is done B time.

3.2.3 Probability plots and box plots

To visualize the results from Bootstrap will probability plots and box-plots be used. The probability plot should not be confused with P-P or Q-Q plots. An example of both the probability plot and box plot is in Figure 3.1 The result is from randomly sample of $N = 1000$ i.i.d $z \sim N(50, 10)$ where 10 is standard deviation, denote this vector $\mathbf{z}$, z_i is the i-th element in the vector.

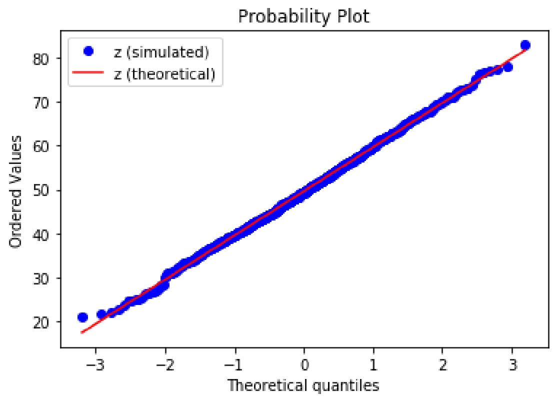
The definition of the probability plot is the following. The data is fit to a normal distribution ($N(\mu^*, \sigma^*)$). The x-axis in Figure 3.1a is the theoretical quantiles of a standard random normal distributed random variable. This corresponds to $y = (z - \mu^*)/(\sigma^*)$ where $y \sim N(0, 1)$. Define the order statistics of $\mathbf{z}$ as $\hat{\mathbf{z}}$ with elements $z_{(1)} \leq z_{(2)} \leq ... z_{(N-1)} \leq z_{(N)}$. Approximation of the theoretical distribution of the order statistics using Filliben's estimate, defined:

$$m(i) = \begin{cases} 1 - 0.5^{1/N} & \text{if } i = 1, \\ 0.5^{1/N} & \text{if }, i = N \\ \frac{i-0.3175}{N+0.365} & \text{else.} \end{cases}$$

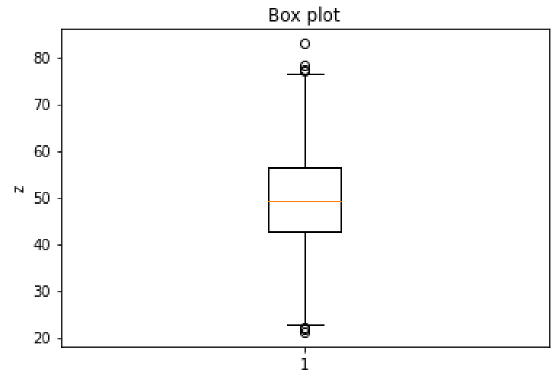
The theoretical quantiles for the vector m given a standard normal, $x = \Phi^{-1}(m)$ where Φ is the CDF for a standard normal distribution. The red line in Figure 3.1a is $(x, \mu^* + x \cdot \sigma^*)$, notice that $(\mu^* + x \cdot \sigma^*) \sim N(\mu^*, \sigma^*)$. The blue dots are $(x, \hat{\mathbf{z}})$.

Box plot is a way to evaluate data with no assumption of the distribution i.e., non-parametric. Define the empirical quartiles $Q1$ (25-percentile), $Q2$ (50-percentile) and $Q3$ (75-percentile). In figure Figure 3.1b, the orange line is $Q2$ (the median) and the big box is the interval $[Q1, Q3]$. Define inter-quartile range as

$IQR = Q3 - Q1$. The line out of the box range is $[Q1 - 1.5IQR, Q3 + 1.5IQR]$. For a normal random variable would this be a 99.3-confidence interval. The data outside this interval is presented as circles in the plot. The theoretical value of 99.3-confiance interval for z is $[23.0, 77.0]$, similar to Figure 3.1b.



(a) Example of a probability plot



(b) Example of a box plot

Figure 3.1: Example of probability plot and box plot for 1000 simulated i.i.d $N(50, 10)$

3.2.4 Bellman equations

When solving the Bellman equation numerically is Scipy's package `minimize` used. The problem was solved for four cases. All cases were solved for a ten-year period and the time step was one year, so there where 11 time points and 10 time periods. The first plan was to solve it for monthly strategy but due to computational time was it deduced to yearly consumption/investments. Here is a summary of the problems and after will follow a detailed description of the algorithms used to solve the problems. In all cases a CRRA-utilty function will be used if nothing else is said. A CRRA-function is used because the solution is independent of current wealth. HARA-utility function will be solved in the last problem.

First the algorithm is tested if it gives the right answers for a model with no consumption (beside at the final time point), B-S and constant rate. The optimal weights are known (the Myopic solution) in that case and therefore the numerical answers are compared. The second validation test is only for consumption and no investment. The solution to this problem is presented in Theorem 4.

The second case is with no consumption but instead of B-S is the Bates model used. At every time point, the optimal weight is approximated with a proxy function of the simulated variance. A proxy function is set to $w(\theta, V_t) = \varphi(V_t)'\theta_t = \theta_t^0 + \theta_t^1 V_t + \theta_t^2 V_t^2$. An identical set-up is also done for Vasicek rate and B-S, where $w(\theta, r_t) = \varphi(r_t)'\theta_t = \theta_t^0 + \theta_t^1 r_t + \theta_t^2 r_t^2$.

The third case is with consumption, B-S and constant risk-free rate. Theorem 4 and Example 8 provided realistic solutions. For all cases is consumption calculated as proportion of wealth, so $c \in (0, 1)$.

The final case is the most complicated. Deposits of capital are considered in this case. The future deposits are considered to be deterministic. From a practical viewpoint, it is considered to be labor income and/or how much the investor can save every month. For this case is B-S and constant risk-free rate be used. In this case was a linear proxy funtion used.

Algorithm

The most important thing to know about the algorithm is that the problem is solved 'backwards'. information is required as to how an investor will act in the future to be able to know how an investor should act at time t. This means that the algorithm start at point $T-1$ and optimize the decision for that point. In the next step is the optimal solution for $T-2$ solved given that the investor makes an optimal choice at time $T-1$, which is found in the first step. There are several relatively easy recursive algorithm methods to solve this. For the four problems presented above there are some differences in the algorithm but a general pseudo code is presented.

Algorithm 3 Bellman-algorithm

Require: $(ret, r_f, X_0, \lambda, T, \Delta t, c_{BOOL}, reg_{DATA}, IN\_DATA, v_{RULE})$
$[nbrSims, nbrSteps]$ = shape of return matrix (ret)
$v_{FUNCTION} = zeros(nbrSims, nbrSteps)$
if $reg_{DATA}! =$ `Null` **then**
  $f = f_{REG}$
else
  $f =$ `Null`
end if
for $t = (T-1) \rightarrow 0$ **do**
  $x(t) = \arg\max_x$ `maximize`(v_func, IN_DATA)
  $w(t), c(t) = f(reg_{DATA}, x(t))$ or $x_t(if f ==$ `Null`$)$
  $IN\_DATA = x(t)$
  Save what is needed (e.g. the value function)
end for
  Value function$(v_func(IN_DATA))$
return $v_{TEMP} = v_{RULE}(t, IN\_DATA, v_{FUNCTION}(t+1), f)$

In Algorithm 3 are the inputs ret the simulated returns, r_f the risk-free rate (can be constant or simulated Vasicek rates), X_0 initial portfolio value, λ the utility parameter(s), T is the final time point, Δt time step,

c_{BOOL} is `True` if the investor consumes during $t = [0, T]$ and False otherwise, reg_{DATA} is a boolean if any regression is done or not and it also contains a proxy function object (f), IN_DATA is the initial guess for the optimizer at the first time step $T-1$ and v_{RULE} is a function that calculates the expected utility (the letter v comes from the general term 'value function' in dynamic programming).

Algorithm 3 is very general and in reality is a bit more complicated. To simplify the notation excess return can be used, that means the return over the risk-free return $R^e = (r_{RISKY ASSET} - r_f)$. The notation and some math is inspired from Brandt et al. (2005).

Problem 1: This is the most simple case and algorithm can be more or less used directly. In this case there is no regression on any state variables so c_{BOOL} = `False` and f = `Null`. The initial guess, $IN\_DATA = w_0$ where $w_0 = (\mu - r_{log})/(\lambda\sigma^2)$ (r_{log} is the corresponding exponential rate for $r = 0.02$ i.e., $r_{log} = ln(1.02)$). So the algorithm starts at $t = T-1$, optimize the weight for the final step (w^*_{T-1}). After that is the weight optimized for the period $T-2 \to T-1$ given that that the w^*_{T-1} is used for period $T-1 \to T$. This continues until $t = 0$.

Problem 2: Now is the risky asset simulated from the Bates model. The simulated variance process is used as the regression variable (or state variable) at every time point. The optimal weight will be considered to be found from a second order polynomial i.e., $f(t) = w(V_t) = \theta^t_0 + \theta^t_1 V_t + \theta^t_2 V_t^2$ where V_t is the variance from the simulated trajectories. In this case is $IN\_DATA = [\theta^t_0, \theta^t_1, \theta^t_2]$. The optimal weight is $w^*(V_t) = \varphi(V_t)'\theta^{t*}$ where θ^{t*} is the optimal parameter vector found with `minimize`.

In both problem 1 and 2 is the relationship (v_{RULE}) almost the same. The only different is that in the second one it depends on the state variable V_t. The Bellman equation is (for problem 1 is V_t constant so it does not affect the equation),

$$
\begin{aligned}
\Omega_t(X_t, V_t) &= \max_{w_t} E_t\Big[\max_{\{w_s\}_{=t+1}^{T-1}} E\Big[\frac{X_T^{1-\lambda}}{1-\lambda}\Big]\Big|V_t\Big] \\
&= \max_{w_t} E_t\Big[\max_{\{w_s\}_{=t+1}^{T-1}} E_{t+1}\Big\{\frac{\big[X_t \prod_{s=t}^{T-1}(w_s(V_s)R^e_{s+1} + r_f)\big]^{1-\lambda}}{1-\lambda}\Big\}\Big|V_t\Big] \\
&= \max_{w_t} E_t\Big[\frac{[X_t(w_t(V_t)R^e_{t+1} + r_f)]^{1-\lambda}}{(1-\lambda)} E_{t+1}\Big[\big[\prod_{s=t+1}^{T-1}(w_s(V_s)R^e_{s+1} + r_f)\big]^{1-\lambda}\Big]\Big|V_t\Big] \\
&= \max_{w_t} E_t\Big[u(X_{t+1}) \max_{\{w_s\}_{=t+1}^{T-1}} E_{t+1}\Big[\big[\prod_{s=t+1}^{T-1}(w_s(V_s)R^e_{s+1} + r_f)\big]^{1-\lambda}\Big]\Big|V_t\Big] \\
&= \max_{w_t} E_t\big[u(X_{t+1})\phi_{t+1}(V_{t+1})\big|V_t\big].
\end{aligned}
\tag{3.5}
$$

From Equation (3.5), it can be concluded that the function $\phi_{t+1}(V_{t+1})$ does not depend on wealth but only variance and time and one can set $X_t = 1$. This function is saved when the optimal weights are found for time t. In the next step $(t-1)$, the optimization problem is relatively simple since $\phi_t(V_t)$ is known. For problem 1, this is simply a constant and for problem 2 this is a vector with size number of simulations (100000 in my test case). One would expect that the optimal weight would decrease for increased volatility, since the expected value is equal (for real value returns). An equal set up can be done for Vasicek rate instead of stochastic volatility. In that case is B-S model used for the risky asset.

Problem 3:

The risky asset is simulated from B-S, constant rate is 2% and the discount factor is set to 2% over the risk free rate i.e., $\delta = (1 + 0.04)$. Notice that the utility is discounted and not the wealth i.e., the net present value of utility is $U(X_t, t=0) = \delta^{-T}\mathbb{E}[u(X_T)]$ and not $U(X_t, t=0) \neq \mathbb{E}[u(\delta^{-T}X_t)]$).

There are no state variable and therefore is no regression needed. For every time step does the optimization depend on two variables (w_t, c_t) and therefore is the problem a bit more complicated, especially numerically. The value function is

$$\Omega_t(X_t) = \max_{\{w_s,c_s\}_{s=t}^T} E\Big[\sum_{s=t}^{T} \delta^{s-t} u(c_s X_s)\Big]$$
$$= \max_{\{w_t,c_t\}_{s=t}^T} u(X_t c_t) + \delta^{-1} E_t[\Omega_{t+1}(X_{t+1}(w_t, c_t))].$$

The value function Ω depends on both the weight and consumption and the initial wealth is set to $X_t = 1 \forall t$. One can calculate the value function in time $t+1$ efficiently since the one knows the optimal choices in the future steps. Notice that for every time step follows the wealth after consumption $X_s^- = (1 - c_s)X_s^+$ where $^+$ means before consumption and $^-$ after consumption. When calculating the value function is the wealth $X_{s+1}^- = (1 - c_s)X_s(w_s R_{s+1}^e + r_f) \forall s \in (t, T)$.

To validate the quality if the algorithm can two theoretical results be used. The first is that the optimal weight should be the same as the myopic solution. The second is if all returns are 0 (the money 'is under the mattress') the optimal consumption should follow Theorem 4.

Problem 4:

This problem is clearly the most complicated one since it depends on an investor's current wealth but that is unknown. The reason that the current wealth does matter ever for CRRA-utility is because of the relationship between the current wealth and the future deposits effect the optimal choice. The problem can be summarized as follows: you want to make a regression of something that you do not know, the portfolio value. An easy way to solve this is simply to 'guess' a distribution for the wealth. The problem with this approach is that the most important thing to know is how one should act today. To get that information one has to use the right portfolio values in the regressions and in the backward algorithm. Algorithm 4 is pseudo code to solve this problem.

Algorithm 4 Problem 4

Require: $ret, r_f, X_0, \lambda, T, \Delta t, reg_{DATA}, IN\_DATA, v_{RULE}$, TOL, DEP

1: $w_0 = (\mu - r_f)/(\lambda\sigma^2)$
2: $w_{ALL} = w_0$ The weights for all time point and all trajectories
3: ERR $= 1$
4: $iter = 0$ #-of iteration
5: **while** ERR $>$ TOL **do**
6: $w_{OLD} = w_{ALL}$
7: $X_{ALL} = wealth(w_{ALL}, DEP)$ Calculate the wealth for all time points if one invest as w_{ALL} for all trajectories and with deposit DEP
8: w_{ALL} solve Algorithm 3
9: $\Omega(iter) = \Omega(w_{ALL})$ value function given w_{ALL}
10: ERR $= ||w_{OLD} - w_{ALL}||_2$
11: $\Omega_0(iter) = mean(\Omega(iter)(0))$ expected value of the value function at $t = 0$
12: **if** $(\Omega_0(iter) < \Omega_0(iter - 1)$ and $iter > 0$ **then**
13: QUIT
14: **end if**
15: $iter + +$
16: **end while**
17: $w_{ALL}^* = w_{ALL}$
18: $X_{ALL} = wealth(w_{ALL}^*, DEP)$

The required parameters and variables that have an equal name as in Algorithm 3 are the same. The only extra input, TOL tolerance of the matrix-norm of the difference in optimal weights and DEP is deposit.

Line $10-12$ stops the algorithm if the new value function (utility) is worst the previous. This can happen when the proxy function is bad, when the algorithm is already close to the optimum or if the 'noise' from the MC simulations affect the answer. Remember that the proxy function is only an estimate of the true function and if the form of the function is far from right e.g., if a second order polynomial is not sufficient, the utility can decrease. The fact that MC is used can there be some problems due to lack of trajectories.

The problem with using a HARA-utility function is that the result depends on the current wealth. Algorithm 4 calculated the current wealth and this problem can be solved even for HARA-utility.

Chapter 4

Result

100 different risk parameters (λ) for CRRA between 0.1 and 10 was studied from one simulation. Closed study of risk parameter on index 25 and 60 and that corresponds to $\lambda_1 \approx 2.53$ and $\lambda_2 \approx 6.06$ is done. For those are confidence intervals of optimal weight analyzed for all four equity models. For CW-strategy was monthly rebalancing used.

In the Appendix are the parameters used for the four equity moments, the four first standardized moments moments and quantities from a simulation of 100000 trajectories.

Some numerical examples is presented from the dynamic programming algorithm in the last part of this chapter.

4.1 Static strategy optimization

4.1.1 Confidence interval for static strategies

To get a more accurate confidence intervals of the optimal weight is bootstrap used. In the Golden Section Algorithm was a tolerance on w^* set to $\texttt{TOL} = 10^{-5}$. An optimal weight is calculated for every bootstrap and from the result can a confidence interval be found. This is done for the two CRRA-functions with λ_1 and λ_2 for both B&H and CW.

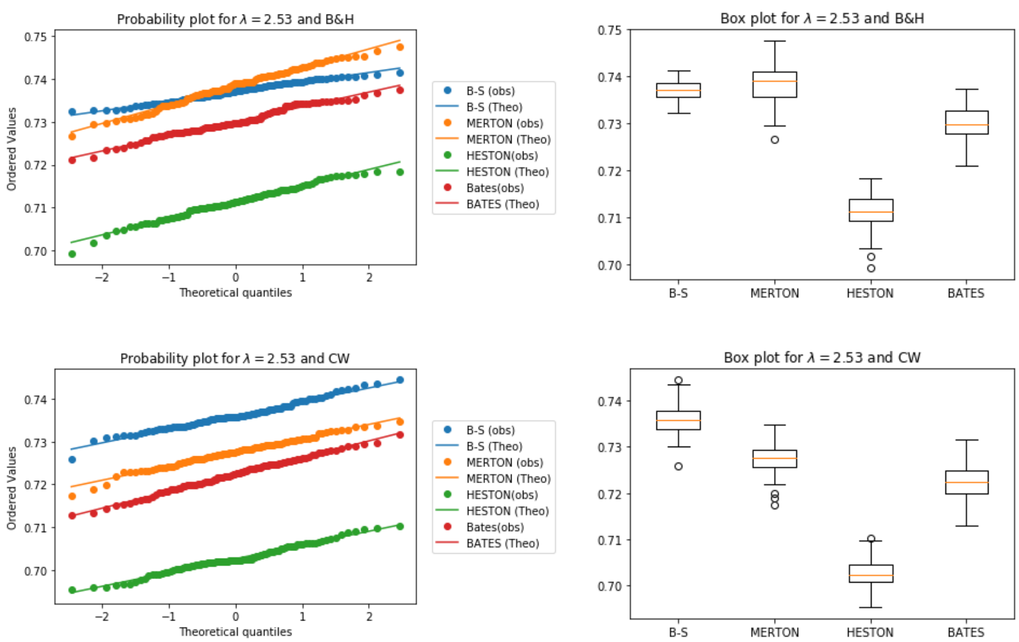


Figure 4.1: Probability plot and box plots for optimal weights for $\lambda \approx 2.53$ for both B&H and CW. Number of simulations 100000 and number of bootstrap was 100.

In Figure 4.1 show that the optimal weights are a bit lower for CW than BH. Look at e.g, the median in the box plots. The theoretical value for both B&H and CW for B-S model is $w^* = (\mu - r)/(\lambda_1 \sigma^2) = 0.7372$ and that is inside both confidence interval and close to the mean of the bootstrapped interval.

The jumps in the Merton model does not seem to have a big effect for B&H and relatively small for CW. The probability that no jump occurs during 10 years is only $Pr(N = 0) = \exp(-10 \cdot \lambda_J) = 0.154$ so most of the trajectories experience at least one jumps. This statistic difference makes the confidence intervals wider. For B&H does it not seem that the jumps effect the mean of the optimization.

In Figure 4.2 are the probability plots and box plots for λ_2.

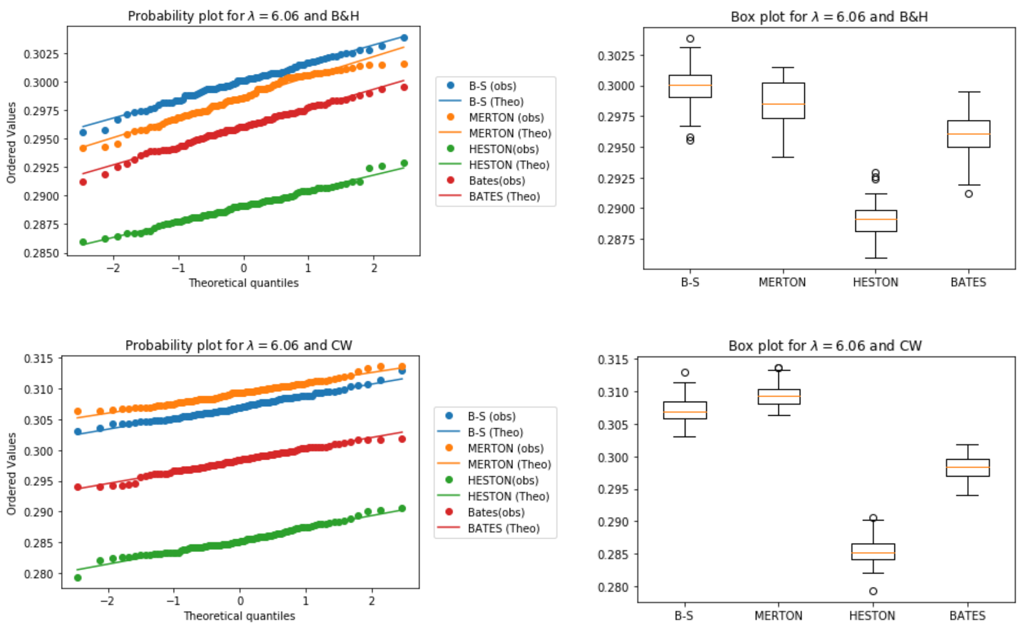
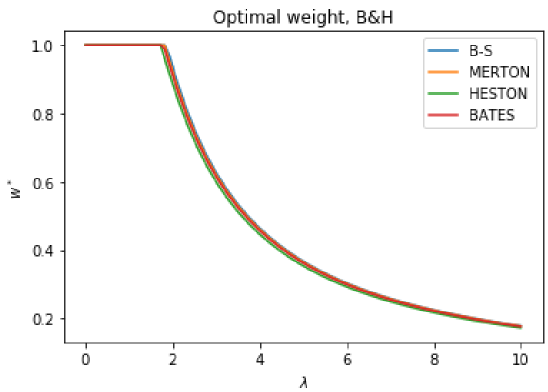


Figure 4.2: Probability plot and box plots for optimal weights for $\lambda_2 \approx 6.06$ for both B&H and CW. Number of simulations 100000 and number of bootstrap was 100.

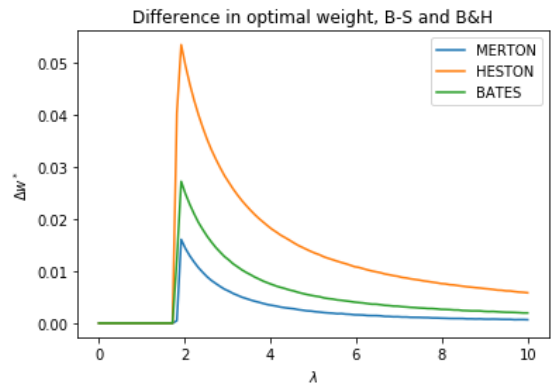
The theoretical value for the optimal weight for B-S is $w^* = 0.3078$. This value is not in the confidence intervals for B&H. In Appendix B are the functions that are optimized. It is clear in Figure B.1b and Figure B.2b that the utility function is much flatter and therefore more difficult to maximize. To improve the results could one decrease the tolerance `TOL` in the Golden Section algorithm and/or increase the number of trajectories.

4.1.2 Analysis without Bootstrap

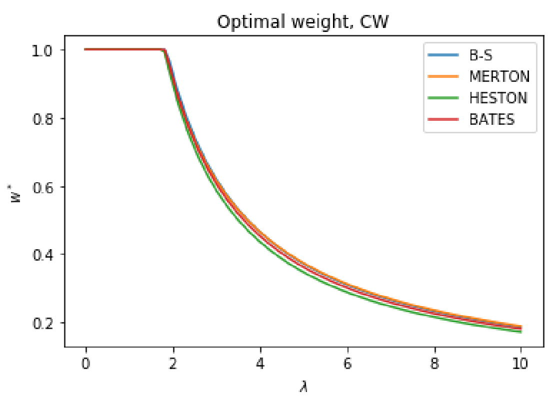
All results in this section is with constant interest rate of $r = 0.02$, this is the yearly rate and the monthly return from the bank account is calculated $(1+r)^{1/12}$. The optimal weight w^* in the risky asset for the four models and the difference, for the more complicated models, compared to B-S are in Figure 4.3.



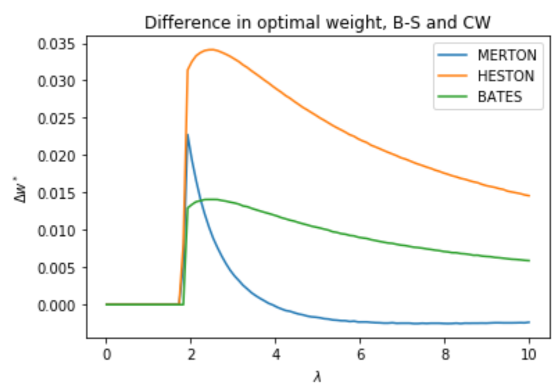
(a) Optimal weights in risky asset for the four model using B&H



(b) Difference in optimal weight compared to B-S for B&H, $(w^*_{B-S} - w^*_{\text{other model}})$



(c) Optimal weights in risky asset for the four model using CW



(d) Difference in optimal weight compared to B-S for CW, $(w^*_{B-S} - w^*_{\text{other model}})$

Figure 4.3: The optimal weights for the two strategies, B&H and CW. The difference compared to B-S are illustrated in Figure 4.3b and Figure 4.3d

It is clear from the result in Figure 4.3 that, when stochastic variance is added, does the optimal weights decreases $\forall \lambda \in (0.01, 10)$ compared to B-S if $W^* < 1$. The result is a bit more unclear for Merton. The difference in optimal weights for the four models using B&H and CW is in Figure 4.4. Theoretically should the difference be 0 for B-S.

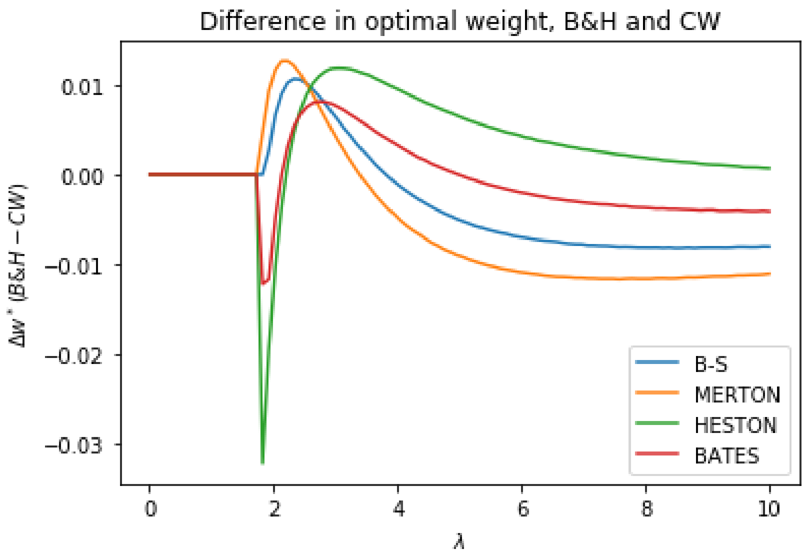
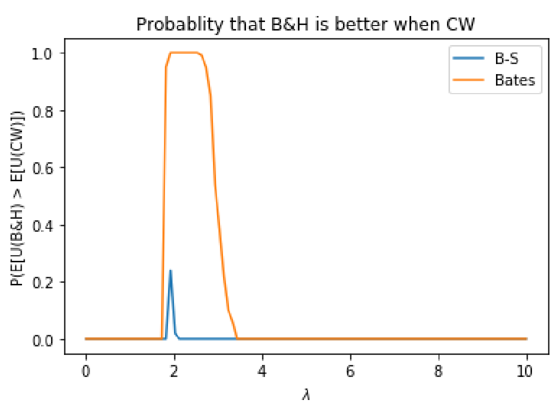


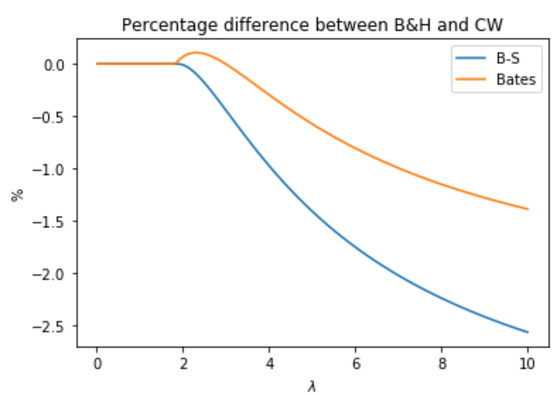
Figure 4.4: The difference in optimal weight between B&H and CW

The differences in optimal weights for the three strategies are quite small, see Figure 4.4. It is not sure that these differences are statistically significant.

The probability that a certain strategy is better than another is looked at for B-S and Bates. In Figure 4.5a, see the probability that B&H gives a higher expected utility compared with CW i.e., $Pr(E[U(B\&H)] > E[U(CW)])$. Notice that it is 'strictly grater than'. Since for low λ-value are the optimal weight $w^* = 1$ and for those λ-values are the strategies equivalent. Therefore, if one would study 'grater or equal as' would the result be misleading. For all strategies are the optimal weights used.



(a) Probability that B&H is better than CW for B-S and Bates model



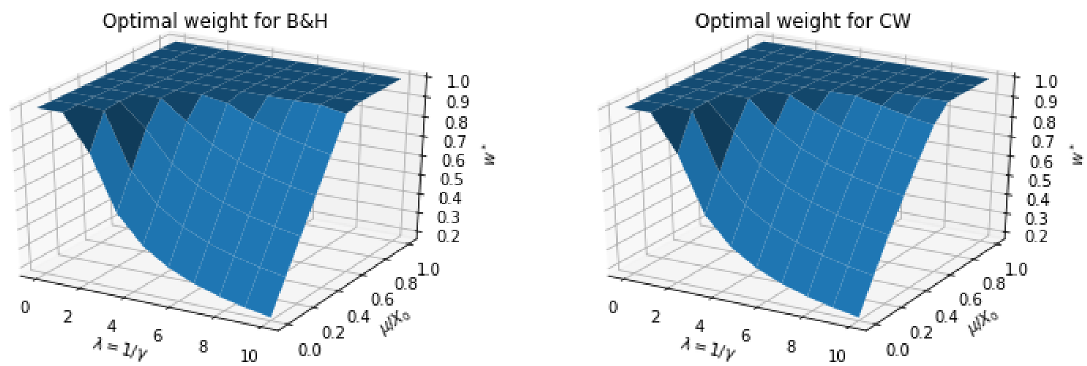
(b) Percentage difference in expected utility between B&H and CW

Figure 4.5: Main result for comparing B&H and CW for CRRA-utility function.

From Figure 4.5 does it seem that CW is better for practically all values for λ. For Bates model, it seems that B&H can be preferable for approximately $\lambda \in (2, 3.5)$. If one looks at Figure 4.5b, one sees that the improvement is not big. In this region are the optimal weights close to 1, see Figure 4.3a, and the strategies are not that different. See Appendix B.2 for the objective function for $\lambda_{1,2}$.

4.1.3 HARA-utility function

It is noticed from Equation (2.10) that for relative risk aversion is the value of τ/X_0 the important part when considering HARA-utility. By letting τ/X_0 go from $0 \to 1$ can the effect of hyperbolic risk aversion be studied. The optimization was done for 10 γ and τ-values, so the total number of combinations is 100 utility-functions. The results are for Bates model for both B&H and CW. The results for the Bates model are in Figure 4.6. When τ increases, decreases the risk aversion and therefore will the optimal weight increase.



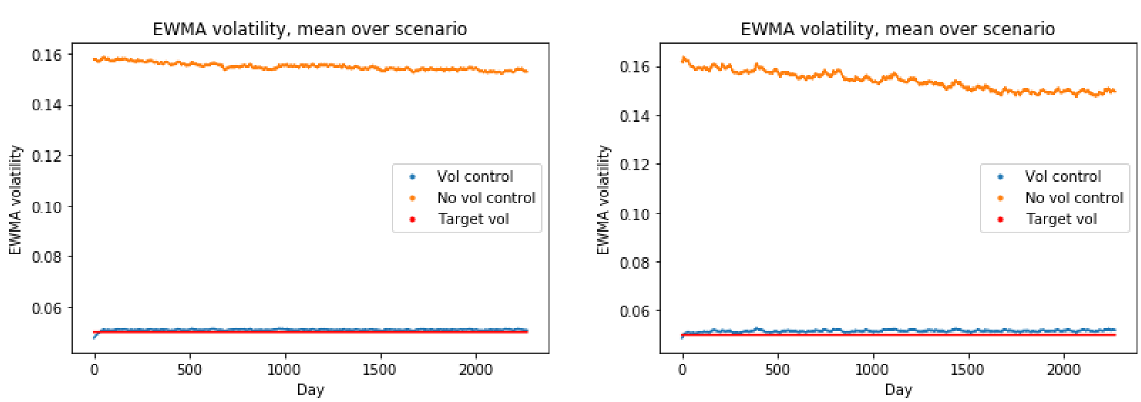
(a) Optimal weights for 100 different HARA-utility function for B&H.

(b) Optimal weights for 100 different HARA-utility function for CW

Figure 4.6: Main result for HARA-utility function for static optimization

4.1.4 Vol-control

For vol-control strategy are target volatilities $\sigma = [0.05, 0.10, 0.12, 0.15]$ used with monthly rebalancing, the trajectories from Bates model (daily simulated) was used, and the constant rate at 0.02. The probability that CW is better than vol-control is calculated with bootstrap. It was found that vol-control did not outperform the optimal CW-strategies for any of the 100 λ-values on the interval $(0.01, 10)$. Notice that the target volatility is not optimized since the goal was to conclude if a product already on the market was better and that was not found that it was better. To illustrate that the vol-control model underestimated the volatility a bit see Figure 4.7b. The reason for this is because of the big jumps. They occur rarely but effect the EWMA-volatility. The decay factor in EWMA was set to 0.955. In Figure 4.7a is Heston model used and for that model is the EWMA-volatility a much better estimator. The target volatility is 0.05 for Figure 4.7.



(a) Vol-control from Heston model with target vol $\sigma = 0.05$

(b) Vol-control from Bates model with target vol $\sigma = 0.05$

Figure 4.7: The average volatility for the index, volatility control index and target volatility

Draw-down control:

When adding draw-down dependent utility function did the optimal weights decrease. The result for λ_2 with B-S and Bates where the draw down-parameter is between $[0, 0.5]$ are in Figure 4.8.

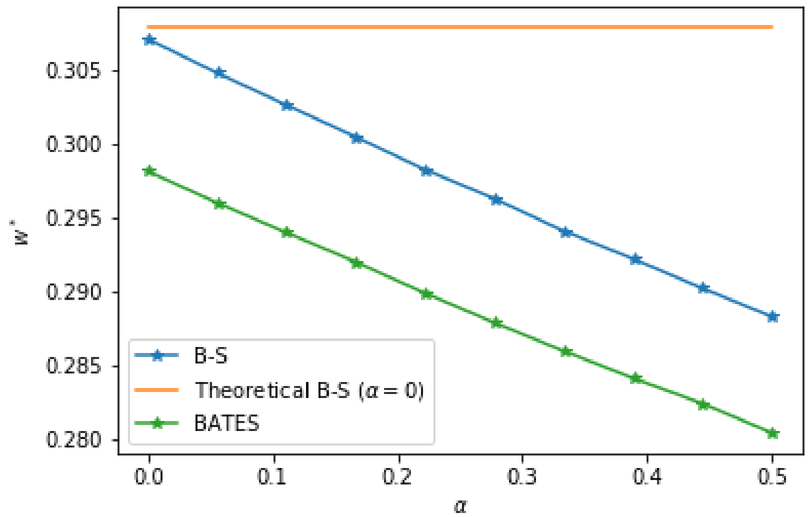
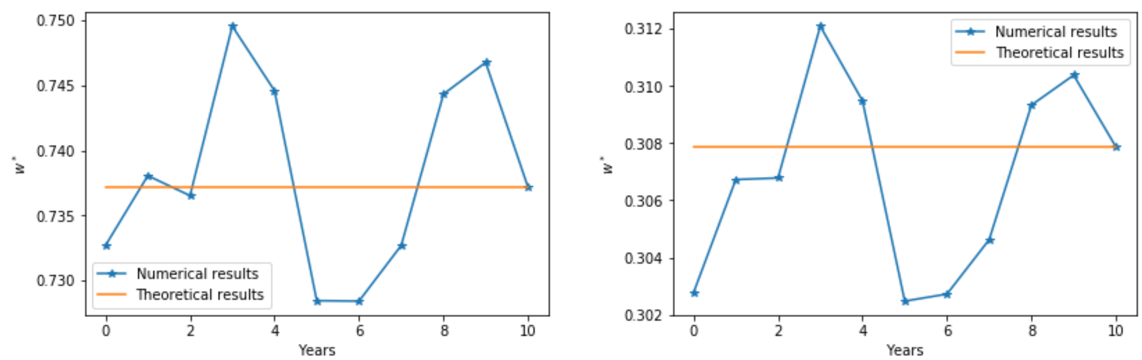


Figure 4.8: Optimal weight for different draw-down parameter α for B-S and Bates model

4.2 Dynamic programming and utility maximization

4.2.1 Validation

The first case considered is the simplest model, B-S model and constant interest rate for a CRRA-investor. The myopic theorem gives an analytic solution. The problem is solved with Algorithm 3 and in Figure 4.9 are the results.



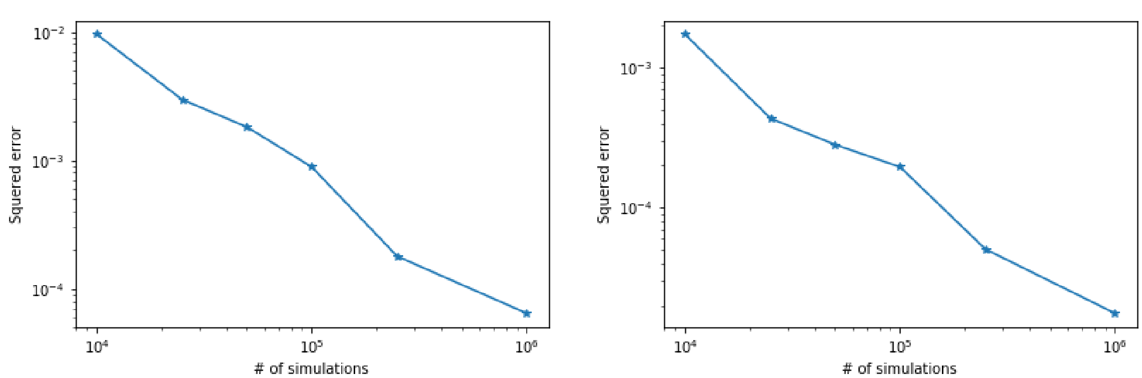
(a) Risk parameter $\lambda_1 \approx 2.53$. The theoretical value is $w^* \approx 0.737$

(b) Risk parameter $\lambda_2 \approx 6.06$. The theoretical value is $w^* \approx 0.308$

Figure 4.9: The optimal allocation from Algorithm 3 with 100000 simulations

The maximum error in Figure 4.9 is slightly more than 1%. It is known that the error does not increase the further back the algorithm is from T, see Brandt et al. (2005). The optimal weight found at $t-1$, w_{t-1}, seem to depend on w_t. The similarities in the figures are due to MC-noise.

There is a little bit more error in Figure 4.9a compared to Figure 4.9b. In figure Figure 4.10 is a converges of the squared error for different number of simulations.



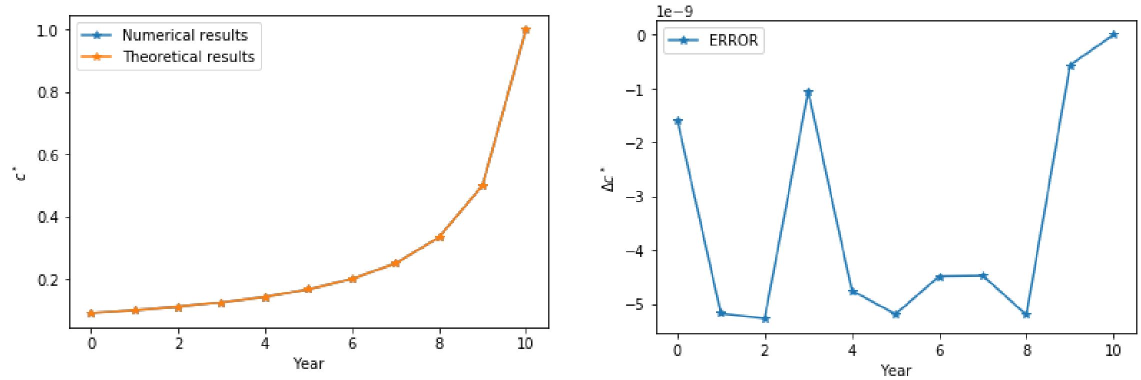
(a) Squared error for λ_1

(b) Squared error for λ_2

Figure 4.10: The squared error convergence until 100000 simulations.

Notice the scale difference in Figure 4.10a and Figure 4.10b. The error for λ_2 is much lower.

In the second validation test it is assumed that the investor does not invest and only consumes during the ten year period. The theoretical results are derived in Theorem 4. In Figure 4.11.



(a) Theoretical and numerical result for c^*

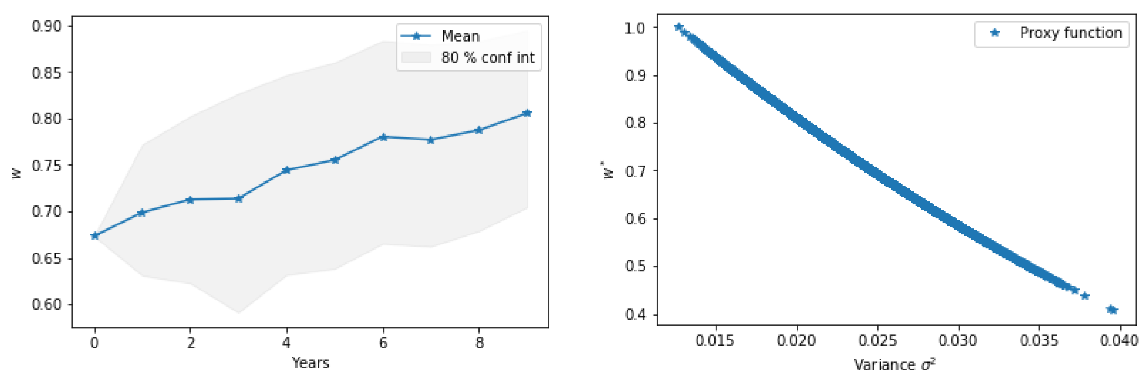
(b) Error, $c^*_{THEO} - c^*_{NUM}$

Figure 4.11: Optimal consumption without investments and the error.

It is clear from Figure 4.11b that the error is extremely small. The curves are practically on top of each other in Figure 4.11a.

4.2.2 Bates-model

At every time step is the optimal weights a function of the underlying variance process. It would be expected that the optimal weights at each time would decrease if the volatility is higher, since the drift term and the rate are constant. The proxy function shows this behavior. See Figure 4.12 for results.



(a) The mean and the 80% confidence interval for optimal weights.

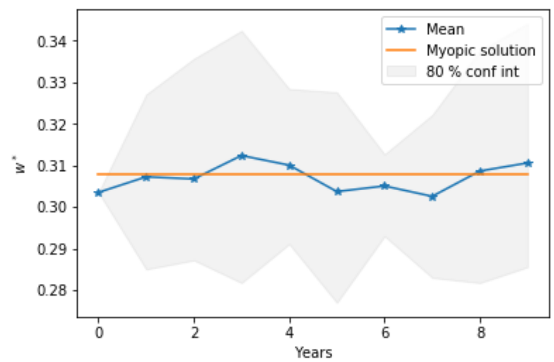
(b) An example of typical proxy function with Bates variance process as regression variable, for $t = 5$

Figure 4.12: The process for the optimal weight and a typical proxy function when doing regression on variance. The investors utility function is a CRRA-function with parameter λ_1

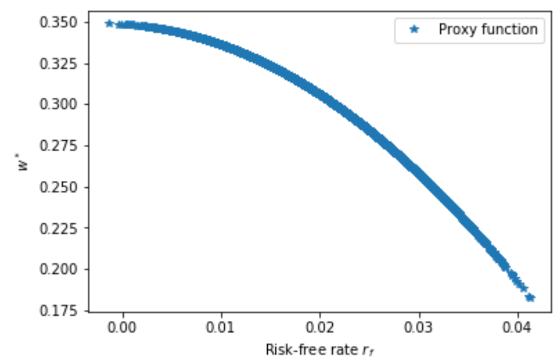
The proxy function in Figure 4.12b is strictly decreasing as expected. The average variance for the Bates model decreases since $\theta < V_0$ and it is therefore expected that the weights increase.

Vasicek rate

When using Vasicek rate as the underlying asset was the following results found. This result is for a CRRA-utility function with parameter λ_2.



(a) The mean and the 80% confidence interval for optimal weights.

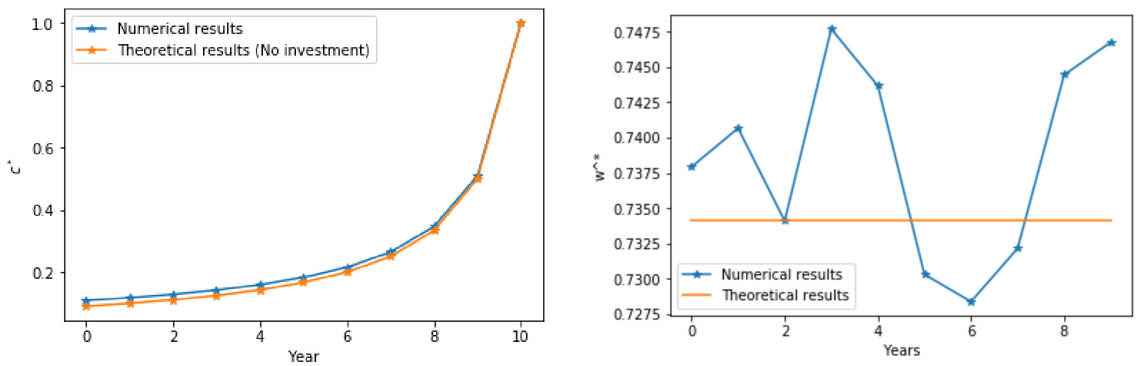


(b) An example of typical proxy function with Vasicek rate as regression variable, for $t = 5$

Figure 4.13: The process for the optimal weight and a typical proxy function when doing regression for Vasicek rate.

4.2.3 Consumption and investment

Consider an investor with CRRA-utility function with risk aversion parameter λ_1 and discount function $(1+0.04)^{-\Delta t}$ and 100000 simulations. In Figure 4.14 are the optimal c^* and w^*. Notice that the optimal w^* still is the myopic solution and it can be used as a benchmark. The optimal consumption is compared to the validation case but that is not the theoretical solution for this problem and it can not be found an error, only the difference. Remember that c is proportion of the wealth that is consumed.



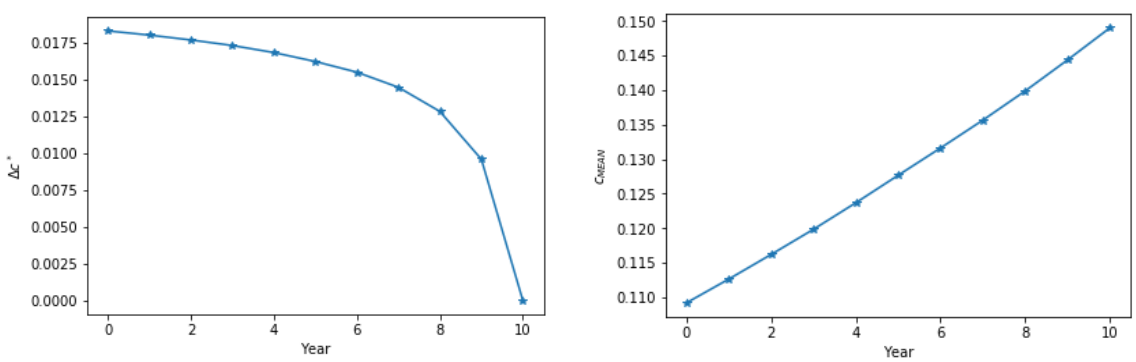
(a) Optimal c^* for λ_1

(b) Optimal w^* for λ_1

Figure 4.14: The optimal c^* and w^* with $D(t) = (1+0.04)^t$

In Figure 4.14b is the errors a little bigger in this case with equal number of trajectories compared to when only the weights are optimized, see Figure 4.9a.

The difference between the optimal consumption and the validation test and the expected value consumed at every time point if the investor starts with 1 unit of money are presented in Figure 4.15. Here is C_{MEAN} the value consumed i.e., $C = c_t X_t$ where X_t is the current wealth.



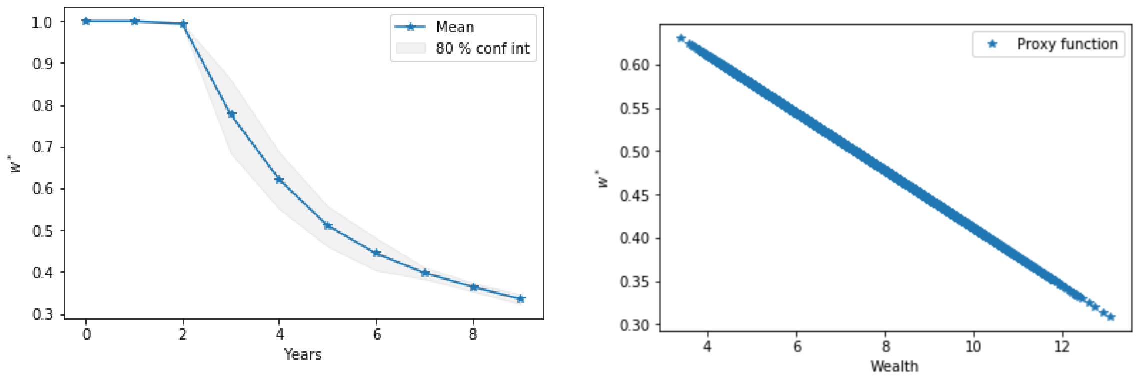
(a) Δc^* for λ_1

(b) C_{MEAN} for λ_1

Figure 4.15: The average optimal consumption plan for an investor with λ_1

4.2.4 With deposit

The first example is for an CRRA-investor with λ_2 as risk aversion parameter. The investor starts with no initial capital and has a yearly deposit of 1. The total inflow of capital is 11. Using Algorithm 4 is the following results found:



(a) The mean and the 80% confidence interval for optimal weights for λ_2.

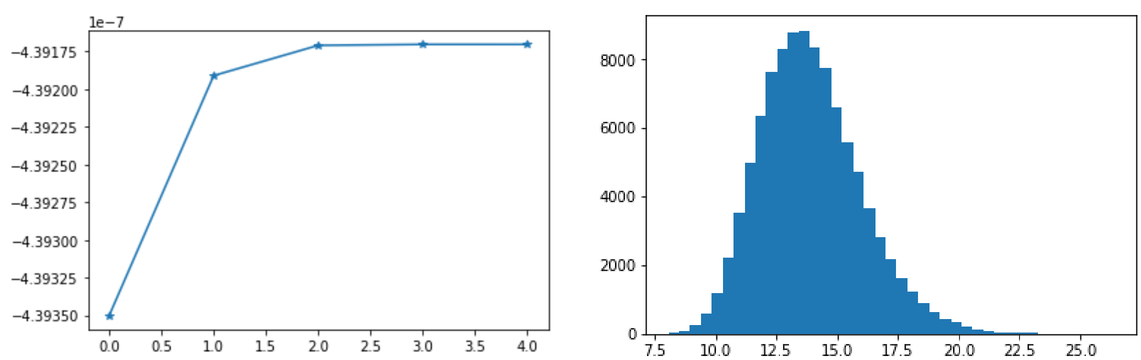
(b) Linear proxy function at $t = 5$

Figure 4.16: Optimal weights and a linear proxy function for CRRA-investor (λ_2)

To use a second order polynomial was not a good proxy function for this case. It is obvious that the optimal weight should decrease when the wealth is higher and sometime that behavior did not occur if a second order polynomial is used as a proxy function. It is therefore more suitable to simply use a linear proxy function or possibly a higher order polynomial if it does not act in the same way. The result with a linear proxy function is sufficient.

The myopic solution for this investor is ~ 0.30 and theoretically should not the optimal weight be lower than that. In Figure 4.16b are the lowest value around the myopic solution.

In Figure 4.17a are the expected utilities at time $t = 0$ for every iteration, where the first utility is simply the CW-strategy calculated from the myopic solution. It is a clear increase in the utility. It is quite close to the final strategy after just one iteration. In Figure 4.17b is the histogram of the portfolio value at the final time point $T = 10$ for the 100000 trajectories.



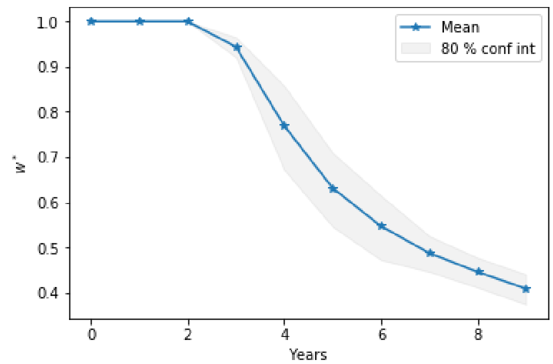
(a) Value function at $t = 0$ for the 5 iteration before termination

(b) Histogram of the wealth distribution at $T = 10$ where $DEP = 1$ and $X_0 = 0$.

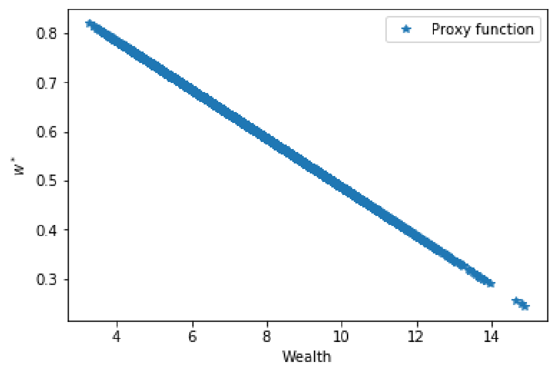
Figure 4.17: Value function at $t = 0$ and histogram of final wealth (λ_2)

HARA and special-HARA

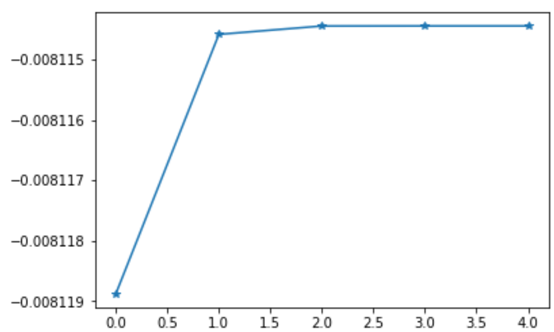
algorithm 4 can also solve the dynamic portfolio problem for a HARA-utility function since it takes the current portfolio value into account. In this case is the parameters $\gamma = 1/\lambda_2$ and $\tau = 0.5$. The investor is less risk averse compared to a CRRA-investor with parameter λ_2, for all X, therefore is the optimal weights be higher. The results are in Figure 4.18.



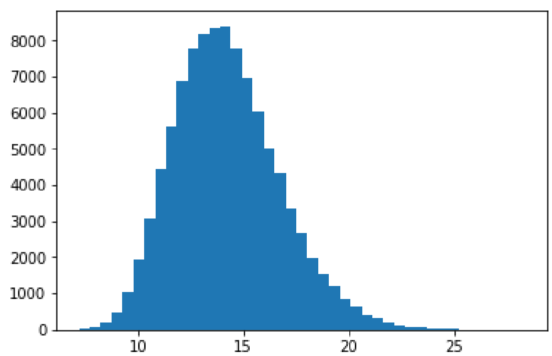
(a) The mean and the 80% confidence interval for optimal weights.



(b) Linear proxy function at $t = 5$



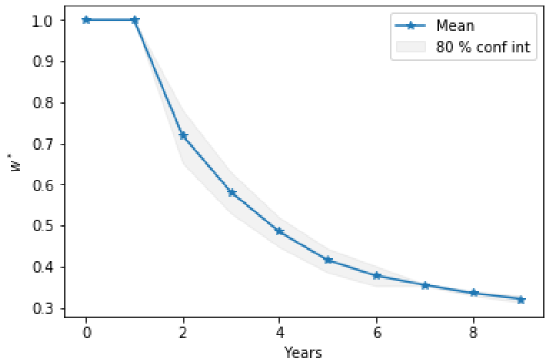
(c) Expected utility for the five iteration of Algorithm 4 until termination



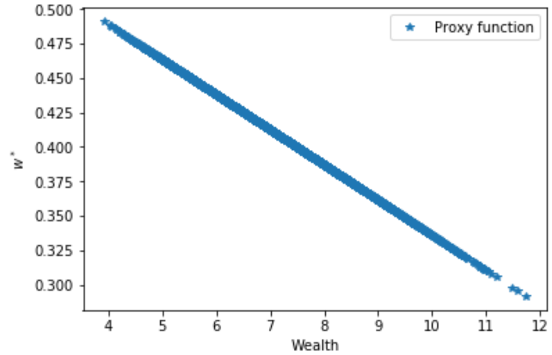
(d) Histogram of the portfolio wealth for the optimal strategy

Figure 4.18: Main result with HARA-utility function with $\gamma = 1/\lambda_2$ and $\tau = 0.5$

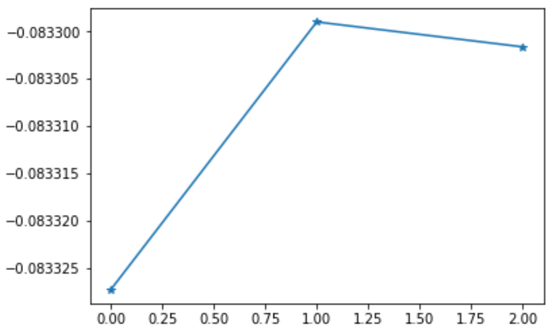
Consider an investor with a special HARA-utility function, see Equation (2.17). The investors CRRA-value is $C = \lambda_2$ and $\gamma = 0.1$. Remember the feature of this utility function, the relative risk aversion is constant but 'when looking forward' does the investor have a HARA-utility function. After the first iteration did the utility increase but after the second did it decrease a little bit. The reason for this can be that the linear proxy functions are not accurate approximations. The weights, proxy function and histogram are from the result of the optimal strategy found, see Figure 4.19.



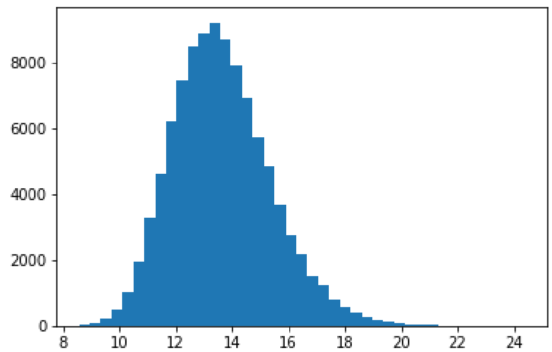
(a) The mean and the 80% confidence interval for optimal weights.



(b) Linear proxy function at $t = 5$



(c) Expected utility for the five iteration of Algorithm 4 until termination



(d) Histogram of the portfolio wealth for the optimal strategy found

Figure 4.19: Main result with special HARA-utility function with $C = \lambda_2$ and $\gamma = 0.1$

Chapter 5

Discussion

5.1 Scenario generation and moment-matching

It was decided at the beginning of the process that it would not to focus on the problem with parameter estimation since it would take up much time, especially for the Bates model but also jump models in general. The two first moments were match as good as possible for the four equity models. There are some technical problems with this approach. The variance in the Heston model is non-linear in time on log scale and there cannot the moment be match for all times i.e., the two first moments cannot be equal time-independently. The moments have to be equal for a certain time point and was decided to be at the final time point $T = 10$. Another problem with the moment-matching approach is that there are theoretically infinite many solutions for the parameters. The parameters have to be picked so that they seem reasonable.

The study of one model with the approach in this thesis would be straight forward. The problems lies when the models are compered. One alternative approach would to simulate the returns from the equity directly with e.g., a GARCH$(1, 1)$-model and have different distributions on the stochastic part. Notice that the problem only applies for comparison. Given the result for a certain equity model, an interest rate model and utility function can conclusions be made from the results.

The effect of the added dynamics in the Merton and Heston model did add negative skewness and positive excess kurtosis to the distribution of the trajectories, see Table B.2. The percentiles of the returns show that the consequence of the negative skewness and positive excess kurtosis compared to B-S makes all percentiles worse i.e., the biggest losses are bigger and the higher gains are lower, see Table B.3.

5.2 Static strategy optimization

The expected values and the variance is (almost) equal for the model trajectories. This means that the tails are heavier but it also means that the mode of the distribution are higher. It is clear from Figure 4.1, Figure 4.2 and Figure 4.3 that the optimal weight is higher for B-S compared to the other models except for Merton. The heavier tails, especially in the downside, affect the optimal weights to be a bit smaller. The modes for the models with heavier tails are a bit higher compared to B-S but this does not cancel out the effect from the tails. This is probably an effect from prudent preference i.e., aversion to down side risk.

The functions in Figure B.1 are strictly convex. For λ_2 is the second derivative much smaller at the optimal w^* compared to λ_1, for both strategies. This means that the expected utility does not change as much for an incremental change in w compared to λ_1. The distribution of trajectories of log portfolio value for CW seem to be (almost) normally distributed but the distribution of B&H is more skeweed, see Figure B.3. The minimum amount possible at $T = 10$ for B&H is $X_0(1 - w^*)(1 + r_f)^T$ but this is not the case for CW. In conclusion can CW take lower values than B&H but CW is still a better alternative, because of the rebalancing.

B&H seem to have a small advantage when the the investor is relatively risk tolerant and the optimal weight is close to 1 for the Bates model. That means that the strategies are similar, see Figure 4.5. To be able to guarantee this would more analysis needed to be done. The reason that CW is better for almost all λ's is because that it rebalances to the optimal weight in every time step. The optimization is done for CRRA and for those utility functions is the answer known, from myopic solution, this is however only known for B-S. But the results indicated that this is true also for Bates for almost all 's.

In this thesis was transaction cost omitted. The main reason for this decision was that the rebalanicing are almost always rather small so the transition cost of rebalancing are very small, for example rebalanicing $w = 0.32$ to 0.30 would only be $0.02 \cdot 0.001 = 0.00002$ of the portfolio value, if the transaction cost is 10 basis points.

Figure 4.8 shows that an investor with a high value draw-down parameter α wants to invest less in the risky asset. This is a result one would expect. The slopes of the curves are not as big as I would expect though. The difference in in the optimal weights for $\alpha = 0$ and $\alpha = 0.5$ are only approximately 0.02.

The vol-control strategy seems not to benefit an investor compared to CW or B&H. The target volatility are set to only direct feasible volatility control products on the market, see S&P Dow Jones Indices (2018). It may be the case that if the target volatility is optimized for a utility function then could it be better or as good as CW but that was not investigated.

The result for HARA-utility function, see Figure 4.6, was as expected. The optimal weight increase as τ increase since RRA decreases, for a constant γ.

5.3 Dynamic programming

The two validation test indicates that the algorithm converges to the right solution, see Figure 4.10 and Figure 4.11b. It was decided to use 10^5 trajectories for the following results even if the converges is better for 10^6 due to computational time.

The results for the Bates model and using the simulated variance as regression variables gave reasonable results if one studies the proxy function. The average volatility decrease for the model since $(V_0^{1/2} = 0.16) > (\theta^{1/2} \approx 0.135)$ and θ is the long-term average for the variance process, see Table B.1. This is an effect of the moment-matching discussed earlier. The confidence intervals were set to 80% for the optimal weights, see Figure 4.12a. If a higher confidence intervals would be presented would the upper limit almost always be almost 1 and therefore was it decided to present a bit smaller interval. This was used for the other examples as well to be consistent. Probably is the method less accurate for the cases if the underlying process are out in the tails of a process distribution (e.g. variance for a Bates/Heston model) since the proxy function is only a second order polynomial.

The results for the Vasicek model ware similar to the Bates model. A difference is that the confidence intervals for the optimal weights for the Vasicek rate are much smaller compared to stochastic volatility. The reason for this is that the rate dynamics change the expected returns much less than the variance dynamics.

There are some fluctuations in the confidence intervals in both Figure 4.12a and Figure 4.13a. These are results from MC-noise, the error in the proxy function and/or error in the optimization. The similarities in the fluctuations for different problems is probably from MC-noise since the same trajectories was used for the problems but it was different programs for different problems. The confidence interval should be smooth since the density function of the underlying processes are very similar for t and $t + \Delta t$. This problem is bigger for the Vacisek rate according to the results.

It is clear for Figure 4.14a that the investor wants to consume a little bit more at every time point compared to the validation case, this is argued in Example 8. The method seems to be more robust to solve problems for consumption compared to optimal weights. The reason for this is that the investors considered have high relative prudence. The higher the relative prudence, the higher is the relative difference in splitting up consumption. When solving the problem for optimal weights can the difference of $E[u(w^*)]$ and $E[u(w^* + \delta)]$ be very small, where w^* is the optimal weight and $1 \gg |\delta| > 0$. For this reason is the error for the weights bigger than consumption when using the same number of simulations.

Algorithm 4 seems to work when adding deposits. The algorithm was validated with no deposits and the results was the approximately the same as for Figure 4.9 and was therefore omitted. The result in Figure 4.16a can be the most surprising result in thie thesis. At the first three time points are the optimal weights practically 1 (the average for $w_3 \approx 0.98$). The myopic solution for the investor is ~ 0.3. The future depositors are considered to be guaranteed. To make a more clear economic argument can one see the deposits as the present value of the cash flows (no discount function is used) and the total present value of the cash flows is 11, including deposit at both $t = 0$ and $T = 10$. If the investor would have 11 to invest at $t = 0$ wound he put approximately 30% in the risky asset. Therefore is this the portfolio that the investor is striving for. This is why the investor wants to investor 100% of the deposits in the risky asset early in the investment horizon. The optimal weights do converge to the myopic solution if $T \to \infty$ (from above, meaning $w_t^* > w_{MYOPIC}^*$). When the portfolio value increases is the relative value less for the future deposits.

In Figure 4.17a are the utilities for the investor until the algorithm has converge to `TOL`. Already after the first step is a quite good strategy found. The most important thing is that the algorithm does never become worse after an iteration, this is also the case for HARA-utility function, see Figure 4.18. The quality (not the results) of the algorithm should not be to different for CRRA and HARA since in both problems is the value dependance similar. The results indicate this. For example are the numbers of iterations before convergence the same.

The algorithm did not behave as well for the special-HARA utility function. In the second iteration did the algorithm find a slightly worse strategy compare to the first. The utility will oscillate around the value after the first iteration when the algorithm is not stopped. It was not concluded what the problem is. The most likely problem is that there are so many different utility functions to optimize that one would need more trajectories and/or a higher order proxy function. Remember that the utility function is different in for every portfolio value. Therefore is the total number of utility function $10 \cdot 10^5 + 1$, the 1 is because at $t = 0$ is the utility function the same for all trajectories.

5.4 Further research

There are several topics from this thesis that needs more research. The confidence intervals in Figure 4.1 and Figure 4.2 can be made smaller. It was considered to use some kind of variance reduction techniques to reduce them but it was left out and decided to focus on dynamic programming instead. It could be done research on how one can more efficiently optimize a static strategy.

The study that compares B&H and CW can be done more general with for example trading cost and/or other strategies. A similar example was tried for vol-control but the target volatility was not optimized. Research about the optimal target volatility for different utility functions would be interesting. The target volatility optimization can in fact be compared to the dynamic portfolio when the volatility is used as regression variable. Maybe a vol-control strategy is preferred for an investor with a high draw-down parameter α since vol-control decreases the fluctuations in the portfolio and therefore the draw-down.

It can be argued that it is unnecessary to solve any static strategies since one can 'simply' solve the dynamic problem instead. A lot more research can be done for dynamic programming. It was not proven that the algorithms converge to the optimal solution but the validation tests indicates towards that.

A limit was put to have only one underlying process for the regression. This can be extended to a multivariate regression model. If this would be implemented could e.g, a problem with draw down utility function and stochastic volatility be solve where D_t and V_t would be the underlying processes.

The proxy functions are approximation functions to the true answer. There are several topics that can be studied more deeply. There was done no estimation of the uncertainty of the parameters in the proxy functions. Consider a linear proxy functions $w^*(t, Z_t) = \theta_0^t + \theta_1^t Z_t$ where Z_t is some underlying process. It would be interesting to evaluate the distribution of the parameters and w^*. The parameters at time t and $t \pm \Delta t$ probably dependent somehow. Therefore can one possibly find a density function $p(\theta^t | \theta^{t+1:T-1}, Z_t)$. The error could probably be reduced if this relation could be estimated. Perhaps this can be done with some filter method, however this is unclear at the moment. Finally could other proxy functions than polynomials be used or to use orthogonal polynomials to maybe improve the results.

The investment horizon for this thesis is 10 years for all examples. Many different research proposals can be made about how the investment horizon affect the result.

Chapter 6

Conclusions

Utility is optimized in this thesis for both static and dynamic portfolios and this is done with MC-methods. The main purpose of the first part (static optimization) was to evaluate the difference in optimal portfolio with CRRA and HARA utility functions for different equity models but with equal two first moments.

It was found, for both B&H and CW, that when adding stochastic volatility are the optimal weights in the risky asset lower and that is statistically significant for the parameters used. If only stochastic jumps are introduced, there are not statistically significant differences in the optimal weights with an investment period at $T = 10$. The results show also that almost all CRRA-investors prefer CW over B&H and vol-control products.

Validation tests indicate that the dynamic programming algorithm converge to the correct answer, however, this is not proven. The method of using regression directly on the maximum argument seems to work for one underlying process. The most interesting result from the dynamic programming is in Figure 4.16. It shows that even a relative risk averse investor should invest 100% of his capital at the beginning of an investment horizon if he starts with $X_0 = 0$ and has equal and guaranteed deposits.

The iterative algorithm worked for the cases evaluated, with some uncertainty for special-HARA function. The results show that already after one iteration, it comes relatively close to the maximum found.

There was error in the validation tests and research could be done to try to make them smaller. The practical use of this work is for robot advisory, for example for long-term pension plans. The error in estimating the client's utility function and the error in the models can be much bigger than the error from the dynamic programming. Therefore, the results can already be used to help people decide how they should save up for retirement. The increase of the precision of the algorithms is interesting mainly for big institutions and for pure academic purposes.

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Appendix A

Theory

A.1 Cholesky decomposition and correlated stochastic variables

Consider a positive-definite variance matrix Σ. Cholesky decomposition decomposes Σ to a lower triangle matrix (L) s.t. $\Sigma = LL^T$.

The Cholesky Decomposition of Σ for the Heston and Bates Model is

$$L = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}. \tag{A.1}$$

So $\Sigma = LL^T$ and define a new independent Brownian Motion W. The matrix L and W can be used to simplify the solution. The Brownian motions can be simulated as

$$\begin{pmatrix} W_v \\ W_s \end{pmatrix} = \begin{pmatrix} W_v \\ W \end{pmatrix} \cdot L = \begin{pmatrix} W_v \\ W_v\rho + W\sqrt{1-\rho^2} \end{pmatrix} \tag{A.2}$$

where W and W_v are independent.

A.2 Itô's formula

Here will only the one dimensional Itô's formula be presented, but it relatively straight forward to generalize the theorem to a multidimensional process, see Björk (2009) Chapter 4.7.

Theorem 5. *X is some stochastic process with known dynamics $dx(t) = \mu(t)dt + \sigma(t)sW(t)$, where μ and σ are both adapted processes and W is a standard Brownian motion. Define a new process, $Z(t) = f(t, X(t))$ where $f \in C^{1,2}$. The stochastic process for Z is*

$$\begin{aligned} df(t, X(t)) &= \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW(t) \\ &= \left\{ \frac{\partial f}{\partial t} + \mathcal{A}f \right\} dt + \sigma \frac{\partial f}{\partial x} dW(t). \end{aligned} \tag{A.3}$$

$\mathcal{A}$ is called the Itô operator.

The proof is omitted.

Appendix B

Result

B.1 Scenario generation and moment-matching

The initial price was $S_0 = 100$ and $\log S_0 \approx 4.61$ The parameters for the four equity model that was simulated from was:

Parameters	B-S	Merton	Heston	Bates
μ	0.0676	0.0776	0.0671	0.0760
σ	0.16	0.138		
κ			0.1	0.1
θ			0.0232	0.0182
σ_v			0.0204	0.0152
ρ			-0.5	-0.5
V_0			0.0256	0.0256
μ_J		-0.0727		-0.0527
σ_J		0.171		0.0854
λ_J		0.187		0.187

Table B.1: Parameters for all models, both for the low and high volatility.

A simulation of 100000 parameters had the following four first standardized moments in the log-price at the final $T = 10$.

Model	Mean return	Variance	Skewness	Excess Kurtosis
B-S	0.547	0.256	-0.002	0.014
Merton	0.546	0.252	-0.0857	0.096
Heston	0.547	0.256	-0.248	0.178
Bates	0.546	0.250	-0.190	0.069

Table B.2

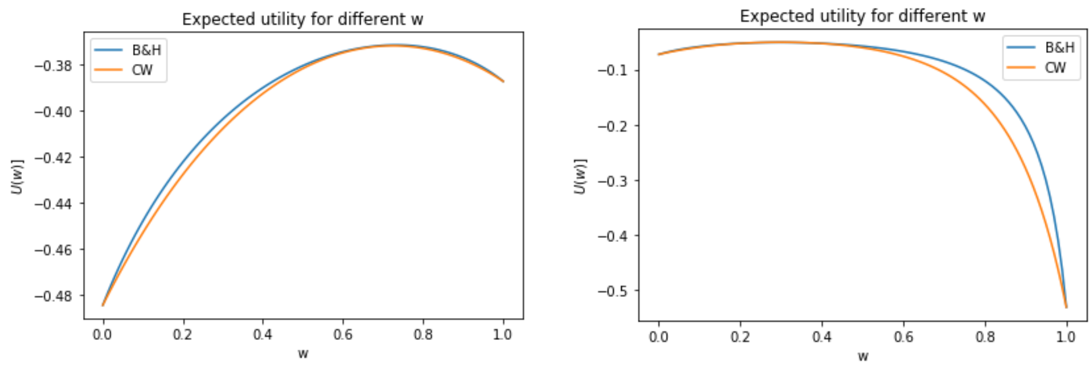
The $1, 5, 95, 99$-percentiles of the real-returns from the simulations at $T = 10$:

Model	1-percentile	5-percentile	95-percentile	99-percentile
B-S	-0.467	-0.247	2.98	4.62
Merton	-0.483	-0.255	2.89	4.44
Heston	-0.515	-0.273	2.85	4.17
Bates	-0.496	-0.263	2.81	4.19

Table B.3

B.2 Objective function

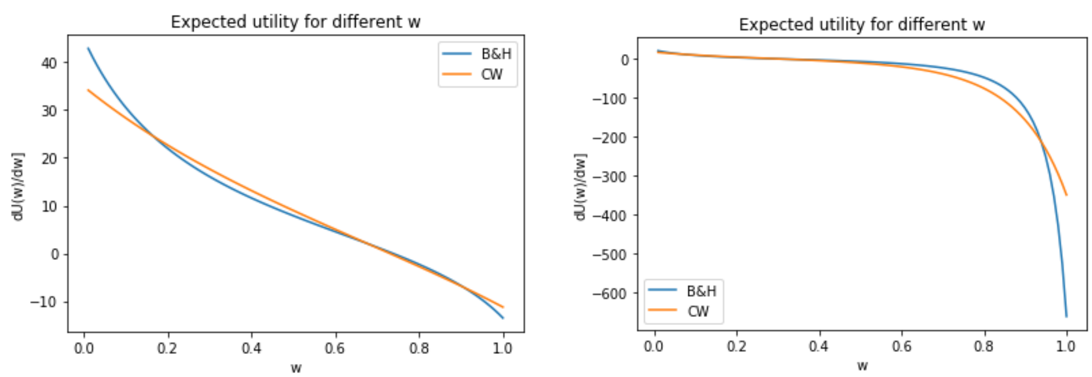
The function that are tried to optimized are in Figure B.1. All these figure are from the Bates model but they look similarly for all strategies.



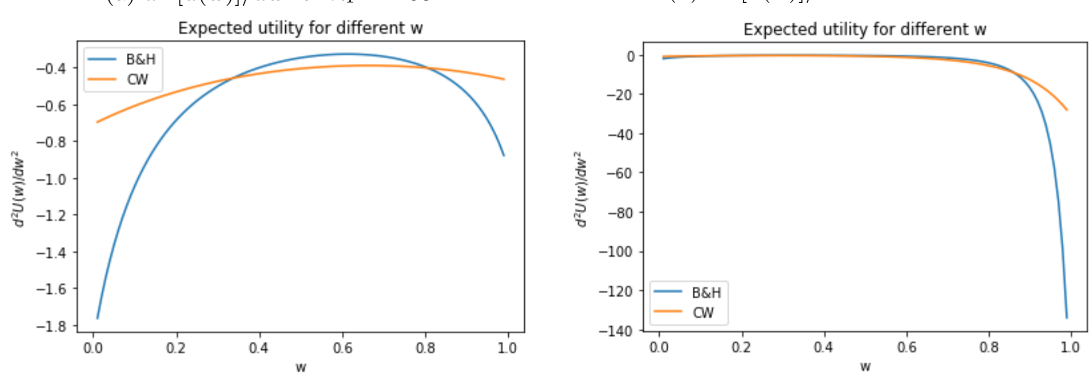
(a) $E[u(w)]$ for $\lambda_1 \approx 2.53$ (b) $E[u(w)]$ for $\lambda_1 \approx 6.06$

Figure B.1: The objective functions, $E[u(w)]$, for $\lambda_{1,2}$ and B&H and CW.

The numerical first and second derivative for these objective functions are in Figure B.2. To calculate the first derivative is symmetric difference quotient with Δw $(U'(w) \approx (U(w+\Delta w) - U(w-\Delta w))/2w$, where $U(w) = E[u(w)]$. The second derivative is calculated with the second order central formula, $U''(w) \approx (U(w+\Delta w) - 2U(w) + U(w-\Delta w))/\Delta w^2$.



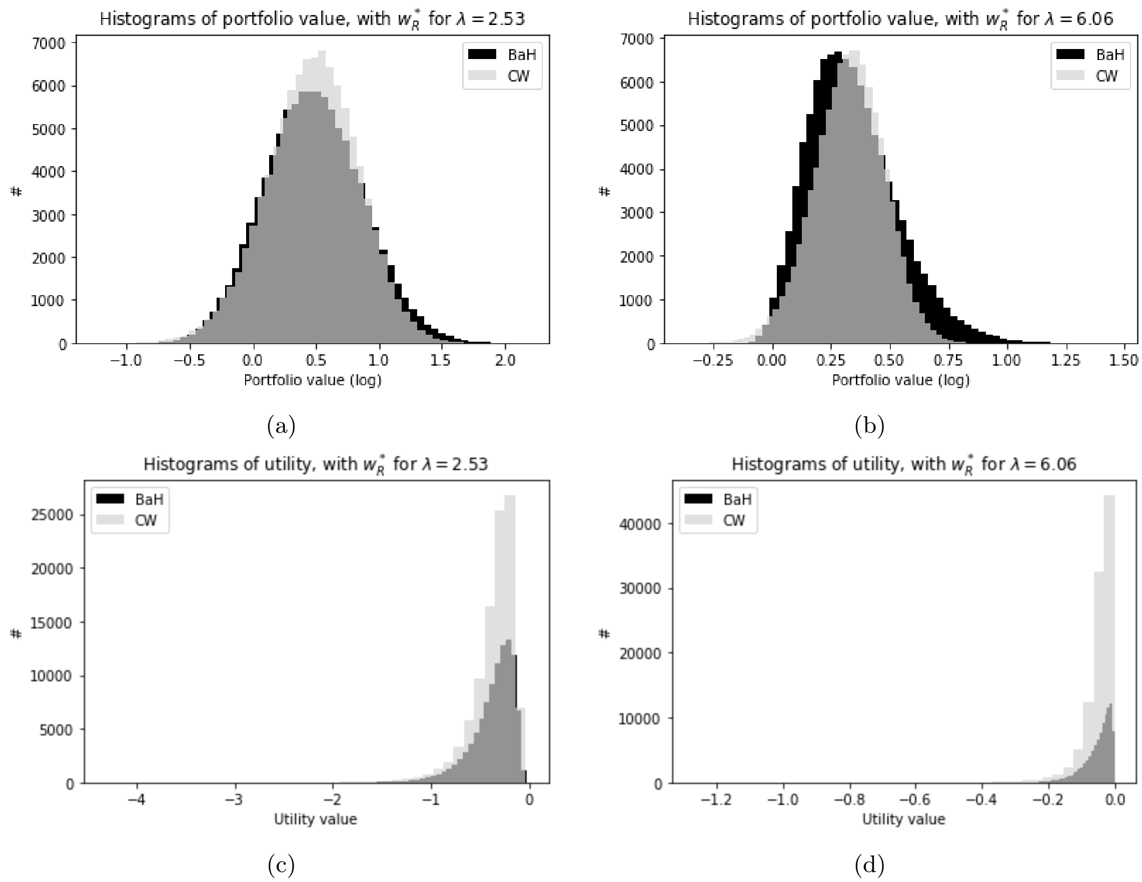
(a) $dE[u(w)]/dw$ for $\lambda_1 \approx 2.53$ (b) $dE[u(w)]/dw$ for $\lambda_1 \approx 6.06$



(c) $d^2E[u(w)]/dw^2$ for $\lambda_1 \approx 2.53$ (d) $d^2E[u(w)]/dw^2$ for $\lambda_1 \approx 6.06$

Figure B.2

The histograms of the wealth and utility for all 100000 trajectories with Bates model after 10 years are in Figure B.3. Two different λ and optimal weights are used for B&H and CW.



(a) (b) (c) (d)

Figure B.3: Histogram of portfolio value and utility both λ's and B&H and CW