The classification theorem for closed surfaces

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Populärvetenskaplig sammanfattning

De flesta människorna har nog en idé vad en yta är för något. Det kan vara skalet på en apelsin eller ovansidan av ett bord. En sluten yta är inom matematiken en sådan yta som inte har några kanter. I detta arbete är målet att kunna klassificera alla möjliga slutna ytor. Denna klassifikation är med avseende på ett ämne inom matematiken som kallas topologi. Där klassas två ytor som samma sak om man kan deformera den ena till den andra. Exempelvis kan man ta en kub och genom att runda av hörnen få en sfär. En sfär och en badring är dock inte densamma eftersom badringen har ett hål medan sfären inte har det och just hål är en sådan sak som inte får tillföras i deformationen. I fallet av att klassificera slutna ytor är det antalet hål som är den avgörande faktorn.

Det är dock svårt att direkt se på en allmän sluten yta hur många hål den har, därför kommer verktyg från algebra och geometri behövas till vår hjälp. Först undersöks antalet hål på ytan genom att introducera en så kallad grupp som ska kolla på hur många sätt en ögla kan träs runt ytan utan att öglan försvinner när den dras åt. Denna grupp kallas ytans fundamentalgrupp och det är en av de huvudsakliga objekten vi undersöker.

Abstract

In this thesis we will study some basic concepts in algebraic topology such as the fundamental group, simplicial complexes and simplicial homology. These are then used together with a method called surgery to prove a complete topological classification of closed surfaces.

Acknowledgements

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Contents

Introduction

One of the aims of this thesis is to give a small introduction to some of the key concepts used in algebraic topology. It starts by defining the fundamental group of a space in the first chapter and proving some elementary properties of it. In this thesis, all spaces are assumed to be topological spaces with the most natural topology if not stated otherwise. In the second chapter, we introduce the more geometric tool of simplicies and simplicial complexes that we use mainly to simplify determining the fundamental group of certain spaces. In the third chapter, we take a slight detour looking at the concept homology and how it relates to the fundamental group. Finally, in the last chapter, we come to the main part of this thesis which is a proof of the classification theorem for closed surfaces. The method here is not the most common. It uses a technique called surgery from Armstrong's book on basic topology.

The reader is assumed to have basic knowledge in point-set topology and group theory. This includes notions such as compactness, connectedness, homeomorphisms, groups, presentations of groups, isomorphisms and theorems such as the first isomorphism theorem.

1 The fundamental group

1.1 The fundamentals

Definition 1.1. A *loop* in a space X is a continuous function $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1)$. We say that $\alpha(0)$ is the *base point* for the loop or that α is *based* at $\alpha(0)$.

Definition 1.2. Given two loops α and β on X with equal base point, their product is defined as

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & s \in [0, 1/2] \\ \beta(2s-1) & s \in [1/2, 1]. \end{cases}$$

This does however not define an associative operation on the space of all loops with fixed base point. To try and get around that problem, we decide to not look at all possible loops separately, but instead group them according to the following definition.

Definition 1.3. Let $f, g : X \to Y$ be continuous functions. Then f is *homotopic* to g, written $f \simeq g$, if there exists a continuous function $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. F is called a *homotopy* from f to g. If in addition F satisfies that $F(a,t) = f(a) = g(a)$ for all $a \in A \subseteq X$ and all $t \in [0,1]$, f is said to be homotopic to g relative to A, written $f \simeq g$ rel A.

The notation for homotopic functions indicates that it might be an equivalence relation and this is indeed the case, but before this can be shown, a lemma is needed.

Lemma 1.4 (Gluing lemma)**.** *Let $X, Y \subseteq T$ and $f : X \to Z$, $g : Y \to Z$ be continuous functions such that $f(x) = g(y)$ for $x, y \in X \cap Y$. If X and Y are both closed in $X \cup Y$ then $h : X \cup Y \to Z$ with $h(x) = f(x)$ for $x \in X$ and $h(y) = g(y)$ for $y \in Y$ is continuous.*

Proof. Let $F \subseteq Z$ be closed. Then $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$ is closed in $X \cup Y$. This follows from that $f^{-1}(F)$ is closed in X by continuity of f and thus closed in $X \cup Y$ since X is closed. Similarly for $g^{-1}(F)$. Hence $h^{-1}(F)$ is closed in $X \cup Y$ and therefore h is continuous. □

Lemma 1.5. *The notion of homotopic functions is an equivalence relation on the set of continuous function from X to Y.*

Proof. Let $f, g, h : X \to Y$ be continuous functions. Firstly, $f \simeq f$ via the homotopy $F(x,t) = f(x)$. Secondly, if $f \simeq g$ via the homotopy $F(x,t)$, then $g \simeq f$ via $F(x, 1-t)$. Finally, if $f \simeq g$ via $F(x,t)$ and $g \simeq h$ via $G(x,t)$, then $f \simeq h$ via

$$H(x,t) = \begin{cases} F(x,2t) & t \in [0,1/2] \\ G(x,2t-1) & t \in [1/2,1] \end{cases}$$

where the continuity of H follows from the gluing lemma. □

We note that if all the above homotopies were relative some set A, then it would not affect the argument and hence homotopic functions relative some set is also an equivalence relation.

One or two examples of homotopic functions might be useful here.

Example 1.6. Let $i : \mathbb{S}^1 \to \mathbb{S}^1$ be the identity function on $\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$ and $f : \mathbb{S}^1 \to \mathbb{S}^1$ the antipodal function, $f(x) = -x$. Then i is homotopic to f via the homotopy $F(x,t) = e^{\pi i t} x$.

Example 1.7. Let $f, g : X \to Y$ be any continuous functions and Y a convex space. Then f and g are homotopic via the straight line homotopy

$$F(x,t) = tg(x) + (1-t)\, f(x).$$

Consider the space of loops on a space X with a fixed base point p. From Lemma 1.5, it follows that homotopic loops rel $\{0,1\}$ is an equivalence relation on the space. Call the resulting equivalence classes homotopy classes and let $[\alpha]$ denote the homotopy class of the loop α. The multiplication of such homotopy classes is then defined in the natural way

$$[\alpha][\beta] = [\alpha \cdot \beta].$$

Lemma 1.8. *Multiplication of homotopy classes as defined above is a well defined operation.*

Proof. Assume that $\alpha \simeq \alpha'$ rel $\{0,1\}$ via the homotopy F and that $\beta \simeq \beta'$ rel $\{0,1\}$ via the homotopy G. Then $\alpha \cdot \beta \simeq \alpha' \cdot \beta'$ rel $\{0,1\}$ via the homotopy H defined by

$$H(x,t) = \begin{cases} F(2x,t) & x \in [0,1/2] \\ G(2x-1,t) & x \in [1/2,1]. \end{cases}$$

□

Note that the continuity of H follows from the gluing lemma. The use of this lemma will be implicit for the rest of this thesis. The transition from single loops to homotopy classes does now take away our problem with associativity encountered in Definition 1.2. It can even be said more than that.

Theorem 1.9. *The set of homotopy classes of loops in a space X with a fixed base point p forms a group under multiplication of homotopy classes.*

Before this can be proven we need a small lemma.

Lemma 1.10. *Composition with continuous functions preserve homotopy equivalence.*

Proof. If $f, g : X \to Y$ and $h : Y \to Z$ are continuous functions where $f \simeq g$ rel A via the homotopy F, then $hf \simeq hg$ rel A via the homotopy hF. Similarly, if $f : X \to Y$ and $g, h : Y \to Z$ are continuous functions where $g \simeq h$ rel B via the homotopy G, then $gf \simeq hf$ rel $f^{-1}(B)$ via the homotopy $H(x,t) = G(f(x),t)$. $\square$

Proof of Theorem 1.9. We start by showing that the multiplication is associative. Let α, β and γ be three loops in X based at p. By definition we have that

$$((\alpha \cdot \beta) \cdot \gamma)(s) = \begin{cases} \alpha(4s) & s \in [0, 1/4] \\ \beta(4s-1) & s \in [1/4, 1/2] \\ \gamma(2s-1) & s \in [1/2, 1] \end{cases}$$

and

$$(\alpha \cdot (\beta \cdot \gamma))(s) = \begin{cases} \alpha(2s) & s \in [0, 1/2] \\ \beta(4s-2) & s \in [1/2, 3/4] \\ \gamma(4s-3) & s \in [1/2, 1]. \end{cases}$$

Now define $f : [0,1] \to [0,1]$ as

$$f(s) = \begin{cases} 2s & s \in [0, 1/4] \\ s + \frac{1}{4} & s \in [1/4, 1/2] \\ \frac{s+1}{2} & s \in [1/2, 1]. \end{cases}$$

Note that f satisfies $f(0) = 0, f(1) = 1$ and since $[0,1]$ is convex, we can use the straight line homotopy from Example 1.7 to see that f and the identity function i are homotopic rel $\{0,1\}$. We also have that $(\alpha\cdot\beta)\cdot\gamma = (\alpha\cdot(\beta\cdot\gamma))\circ f$. Using Lemma 1.10 we get

$$\begin{aligned}(\alpha\cdot\beta)\cdot\gamma &= (\alpha\cdot(\beta\cdot\gamma))\circ f\\ &\simeq (\alpha\cdot(\beta\cdot\gamma))\circ i \text{ rel } \{0,1\}\\ &= \alpha\cdot(\beta\cdot\gamma).\end{aligned}$$

Hence $[\alpha\cdot\beta][\gamma] = [\alpha][\beta\cdot\gamma]$ as desired. The unity in this group is the homotopy class of the constant path at $p, p(s) = p$ for $s \in [0,1]$. Define

$$f(s) = \begin{cases} 2s & s \in [0,1/2] \\ 1 & s \in [1/2,1]. \end{cases}$$

Then f is again homotopic to the identity function and we have that

$$\alpha\cdot p = \alpha\circ f \simeq \alpha\circ i \text{ rel } \{0,1\} = \alpha.$$

Hence $[\alpha][p] = [\alpha]$ and similarly one gets $[p][\alpha] = [\alpha]$. Finally, the inverse of the homotopy class containing the loop α is obtained by taking the class of the reversed loop $\alpha^{-1}(s) = \alpha(1-s)$. This time we define

$$f(s) = \begin{cases} 1-2s & s \in [0,1/2] \\ 2s-1 & s \in [1/2,1] \end{cases}$$

and note that f is homotopic to the constant path at 1 rel $\{0,1\}$. Therefore

$$\alpha^{-1}\cdot\alpha = \alpha\circ f \simeq \alpha\circ 1 \text{ rel } \{0,1\} = \alpha(1) = p$$

and similarly $[\alpha][\alpha^{-1}] = [p]$. □

Definition 1.11. The group of loops in a space X based at p is called the *fundamental group of X based at p* and is denoted by $\pi_1(X,p)$. If it happens that $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$ for all $p,q \in X$, then we will disregard the base point and write $\pi_1(X)$.

This omission of base point for the fundamental group can be done in a lot of spaces as seen from the following theorem.

Theorem 1.12. *If X is path-connected then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$ for all $p,q \in X$.*

Proof. Since X is path-connected, we can choose a path γ in X such that $\gamma(0) = p$ and $\gamma(1) = q$. Now define $\gamma_* : \pi_1(X, p) \to \pi_1(X, q)$ by letting γ act via conjugation, i.e.

$$\gamma_*([\alpha]) = [\gamma^{-1} \cdot \alpha \cdot \gamma].$$

Then $\gamma_*([\alpha])(0) = \gamma_*([\alpha])(1) = q$ and

$$\begin{aligned}\gamma_*([\alpha][\beta]) = \gamma_*([\alpha \cdot \beta]) = [\gamma^{-1} \cdot \alpha \cdot \beta \cdot \gamma] &= [\gamma^{-1} \cdot \alpha \cdot \gamma \cdot \gamma^{-1} \cdot \beta \cdot \gamma] \\ = [\gamma^{-1} \cdot \alpha \cdot \gamma][\gamma^{-1} \cdot \beta \cdot \gamma] &= \gamma_*([\alpha]) \cdot \gamma_*([\beta]),\end{aligned}$$

hence γ_* is a homomorphism. Since γ_* has an inverse $\gamma_*^{-1} : \pi_1(X, q) \to \pi_1(X, p)$ defined as conjugation by γ^{-1}, γ_* is an isomorphism. □

Definition 1.13. A space X is called *simply connected* if it is path-connected and $\pi_1(X)$ is the trivial group consisting of a single element.

Example 1.14. Any convex space X is simply connected. Here the fundamental group is trivial since any two loops α and β are homotopic via the straight line homotopy from Example 1.7. In particular, $\mathbb{R}^n$ is simply connected for any n.

In view of Lemma 1.10, the following construction is well defined.

Definition 1.15. Let $f : X \to Y$ be a continuous function where $f(p) = q$. Then we define the *induced homomorphism* $f_* : \pi_1(X, p) \to \pi_1(Y, q)$ by $f_*([\alpha]) = [f \circ \alpha]$.

Note that f_* is indeed a homomorphism since $f \circ (\alpha \cdot \beta) = (f \circ \alpha) \cdot (f \circ \beta)$.

Lemma 1.16. *If $f : X \to Y$ and $g : Y \to Z$ are continuous functions such that $f(p) = q$ and $g(q) = r$, then $(g \circ f)_* = g_* \circ f_* : \pi_1(X, p) \to \pi_1(Z, r)$.*

Proof. Let α be a loop in X. Then $(g_* \circ f_*)([\alpha]) = g_*([f \circ \alpha]) = [(g \circ f) \circ \alpha] = (g \circ f)_*([\alpha])$. □

The above lemma does not seem like much, but with it we can prove the following important result, essentially saying that the fundamental group is a topological invariant.

Theorem 1.17. *If X and Y are homeomorphic, path-connected spaces, then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$.*

Proof. Let $h : X \to Y$ be a homeomorphism with $h(p) = q$ and consider $h_* : \pi_1(X, p) \to \pi_1(Y, q)$ and $h_*^{-1} : \pi_1(Y, q) \to \pi_1(X, p)$. By Lemma 1.16, we have $h_*^{-1} \circ h_* = (i_X)_* : \pi_1(X, p) \to \pi_1(X, p)$ and $h_* \circ h_*^{-1} = (i_Y)_* : \pi_1(Y, q) \to \pi_1(Y, q)$ where i_X and i_Y are the identity function on X and Y respectively. Since the identity function induces the identity homomorphism, we get that h_* is an isomorphism. □

In our final classification of closed surfaces, this result will be used to show that all surfaces on our list are unique.

1.2 Homotopy equivalence

Definition 1.18. Two spaces X, Y are called *homotopy equivalent* and are said to have the same *homotopy type*, if there exists continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq i_X$ and $f \circ g \simeq i_Y$. We then write $X \simeq Y$ and call f a *homotopy inverse* for g.

Lemma 1.19. *Homotopy equivalence is an equivalence relation on topological spaces.*

Proof. The relation is reflexive since one can take $f = g = i_X$. Symmetry follow directly from the definition. For transitivity, we let $f : X \to Y$ have homotopy inverse g and $u : Y \to Z$ have homotopy inverse v. Then using Lemma 1.10, we get

$$(g \circ v) \circ (u \circ f) = g \circ (v \circ u) \circ f \simeq g \circ i_Y \circ f = g \circ f \simeq i_X$$

and

$$(u \circ f) \circ (g \circ v) = u \circ (f \circ g) \circ v \simeq u \circ i_Y \circ v = u \circ v \simeq i_Z.$$

Hence $X \simeq Z$ via the homotopy inverses $g \circ v$ and $u \circ f$. □

We wish to combine our earlier study of the fundamental group with the notion of homotopy type. To do this, we first need to see how much the induced homomorphism of homotopic maps differ.

Lemma 1.20. *Let $f, g : X \to Y$ be continuous functions where $f \simeq g$ via the homotopy F. Then $g_* = \gamma_* f_*$ where $g_* : \pi_1(X, p) \to \pi_1(Y, g(p))$, $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ and $\gamma(s) = F(p, s)$ with γ_* acting via conjugation.*

Proof. Let α be a loop in X with base point p. Since $g_*([\alpha]) = [g \circ \alpha]$ and $\gamma_* f_*([\alpha]) = [\gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma]$, we need to show that $g \circ \alpha$ is homotopic to $\gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma$ rel $\{0, 1\}$. Define $G : [0,1] \times [0,1] \to Y$ by $G(s,t) = F(\alpha(s), t)$. The sought homotopy is then given by

$$H(s,t) = \begin{cases} \gamma(1-4s) & s \in \left[0, \dfrac{1-t}{4}\right] \\ G\left(\dfrac{4s+t-1}{3t+1}, t\right) & s \in \left[\dfrac{1-t}{4}, \dfrac{1+t}{2}\right] \\ \gamma(2s-1) & s \in \left[\dfrac{1+t}{2}, 1\right]. \end{cases}$$

□

The main result of this section is the following.

Theorem 1.21. *Homotopy equivalent spaces have isomorphic fundamental groups.*

Proof. Let $X \simeq Y$ via the homotopy inverses f and g. Say $g \circ f \simeq i_X$ via the homotopy F and $f \circ g \simeq i_Y$ via G. Pick $p = g(q) \in X$, we show $f_* : \pi_1(X,p) \to \pi_1(Y, f(p))$ is an isomorphism. Define γ by $\gamma(s) = F(p,s)$ and σ by $\sigma(s) = G(q,s)$. By Lemma 1.20, we get the following commutative diagram.

$$\begin{array}{ccc} \pi_1(Y,q) & \xrightarrow{\sigma_*} & \pi_1(Y, f(p)) \\ \Big\downarrow g_* & \overset{f_*}{\nearrow} & \Big\downarrow g_* \\ \pi_1(X,p) & \xrightarrow{\gamma_*} & \pi_1(X, gf(p)) \end{array}$$

By the argument from Theorem 1.12, σ_* and γ_* are isomorphisms. The lower triangle in the diagram forces f_* to be injective and the upper triangle forces f_* to be surjective. Hence f_* is an isomorphism and thus $\pi_1(X)$ is isomorphic to $\pi_1(Y)$. □

We note that homeomorphic spaces have the same homotopy type. A more non-trivial example is the following.

Example 1.22. For $n \geq 1$, $\mathbb{R}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$. Define $f(x) = x$ as the inclusion and $g(x) = \frac{x}{\|x\|}$. Then $g \circ f = i_{\mathbb{S}^{n-1}}$ and $f \circ g \simeq i_{\mathbb{R}\setminus\{0\}}$ via $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$.

The above example is a special case of the following.

Definition 1.23. Let $Y \subseteq X$. A *deformation retraction* of X onto Y is a homotopy $F : X \times [0,1] \to X$ rel Y where $F(x,0) = x$ and $F(x,1) \in Y$.

If Y is a deformation retraction of X, then X and Y can be seen to have the same homotopy type by taking $f(x) = x$ and $g(x) = F(x,1)$.

2 Simplicial complexes

2.1 Triangulation

Determining the fundamental group of an arbitrary space X can in general be very difficult. We therefore turn our attention to a more strict class of spaces that can be thought of as being built up of finitely many simple parts. Before this can be made precise, a few definitions are needed.

Definition 2.1. Points $v_0, v_1, \ldots, v_m \in \mathbb{R}^n$ are called *affinely independent* if the vectors $v_i - v_0$, $i = 1, 2, \ldots, m$, are linearly independent.

Definition 2.2. Let $v_0, v_1, \ldots, v_k$ be $k+1$ points that are affinely independent. The smallest convex set containing all the points is called a *k-simplex*. The points are then called the *vertices* of the simplex.

Example 2.3. For the lowest dimensions we see that a 0-simplex is a point, a 1-simplex a closed line segment, a 2-simplex a triangle including its interior and a 3-simplex a solid tetrahedron.

We can also generalize the concept of faces for an arbitrary simplex.

Definition 2.4. Let σ and τ be simplexes and assume the vertices of τ forms a subset to those of σ. Then τ is said to be a *face* of σ, written $\tau < \sigma$. A face of dimension 1 is often called an *edge*.

This allows us to construct more complicated structures as follows.

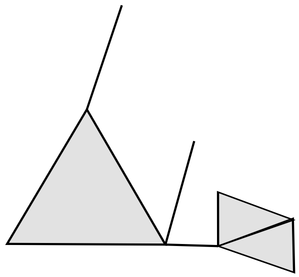

Figure 1: A simplicial complex.

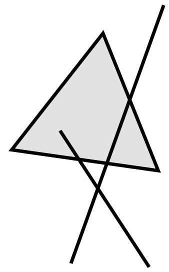

Figure 2: Not a simplicial complex.

Definition 2.5. A finite collection K of simplexes in $\mathbb{R}^n$ is called a *simplicial complex*, or more often just a complex, if whenever the simplex σ is in K, then all faces of σ is also in K. Also if two simplexes in K intersect, they do so in a common face.

To be able to use this concept for our future study of closed surfaces, we need the following definition.

Definition 2.6. Let K be a simplicial complex in $\mathbb{R}^n$. The *polyhedron* of K, denoted $|K|$, is the topological space obtained by forming the union of all simplexes in K and giving them the subspace topology in $\mathbb{R}^n$. A *triangulation* of a space X is then a simplicial complex K and a homeomorphism $h : X \to |K|$.

Note that K can be seen as a purely combinatorial object containing a collection of vertices and information about those vertices, telling which subcollections forms a simplex. On the other hand, $|K|$ is the geometric realization of that structure, embedded into some $\mathbb{R}^n$.

Example 2.7. Let $X = \mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. An example of a triangulation of X is then given by the simplicial complex K consisting of a hollow tetrahedron with $|K|$ embedded in $\mathbb{R}^3$ such that $|K| \subset \{x \in \mathbb{R}^3; \|x\| < 1\}$ together with the homeomorphism h that is radial projection from $|K|$ to X.

Example 2.8. The non orientable surface called the Klein Bottle can be triangulated as in Figure 3. Note that the arrows indicates identifications to be made for the edges.

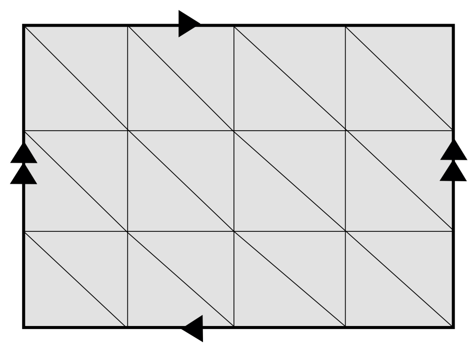

Figure 3: Triangulation of a Klein bottle.

Sometimes, a given triangulation might not be suitable, instead one would like some kind of refinement of it. This can be achieved in the following manner. Let σ be a simplex having vertices $v_0, v_1, \ldots, v_m$. Then $x \in \sigma$ if and only if $x = \lambda_0 v_0 + \cdots + \lambda_m v_m$ with $\sum_{i=0}^{m} \lambda_i = 1$ and $\lambda_i \geq 0$. Define the barycentre of σ as

$$\hat{\sigma} = \frac{1}{m+1}\left(v_0 + v_1 + \cdots + v_m\right).$$

Using the barycentre, we can from a given complex K create a new complex K^1 as follows. Let the vertices of K^1 be the barycentres of all the simplexes in K. Also, $\hat{\sigma_0}, \cdots, \hat{\sigma_k}$ form the vertices of some k-simplex of K^1 if and only if $\sigma_{i_0} < \cdots < \sigma_{i_k}$ where $\{i_0, \cdots, i_k\} = \{0, \cdots, k\}$.

Definition 2.9. Given a simplicial complex K, the *barycentric subdivision* of K is the complex K^1, constructed as above.

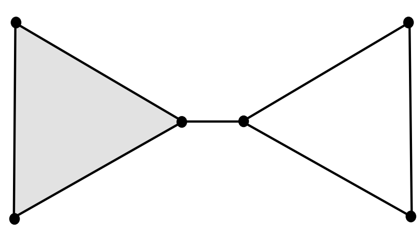

Figure 4: A simplicial complex.

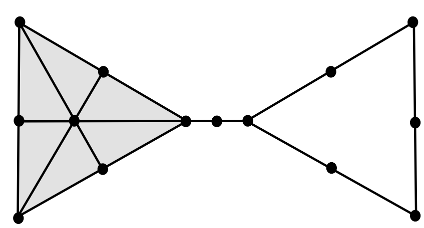

Figure 5: Its barycentric subdivision.

Some basic properties of simplicial complexes and its first barycentric subdivision is presented in the subsequent lemma. A full prof of each statement can be found in [1].

Lemma 2.10. *Let K be a simplicial complex in $\mathbb{R}^n$. Then*

a) $|K|$ is a compact space.

b) If $|K|$ is connected, then it is also path-connected.

c) If we consider the polyhedron of every simplex in K separately and take the identification topology on their union, we get $|K|$.

d) Each simplex of K^1 is contained in some simplex of K.

e) K^1 is a well defined simplicial complex and $|K^1| = |K|$.

Since we always consider a simplex in some $\mathbb{R}^n$, we can define the diameter of a simplex σ as $\operatorname{diam}(\sigma) = \sup_{x,y\in\sigma} |x - y|$. The diameter of a simplicial complex K can from this be defined as $\operatorname{diam}(K) = \max_{\sigma\in K} \operatorname{diam}(\sigma)$. One of the main properties of the barycentric subdivision is that it makes the diameter of the simplicial complex smaller. To prove that, two lemmas are needed.

Lemma 2.11. *The diameter of a simplex is the length of its longest edge.*

Proof. Let σ be a simplex having the vertices $v_0, \dots, v_m$ and set $d = \max\{|v_i - v_j|; 0 \le i \le j \le m\}$. Pick $x \in \sigma$, $x = \sum_{i=0}^m t_i v_i$, $\sum_{i=0}^m t_i = 1$. Then

$$|x - v_j| = \left|x - \left(\sum t_i\right) v_j\right| = \left|\sum t_i \left(v_i - v_j\right)\right| \le \left(\sum t_i\right) d = d.$$

Now given $y \in \sigma$, $y = \sum_{i=0}^m s_i v_i$, $\sum_{i=0}^m s_i = 1$, we have

$$|x - y| = \left|\left(\sum s_i\right) x - y\right| = \left|\sum s_i \left(x - v_i\right)\right| \le \left(\sum s_i\right) d = d.$$

□

Lemma 2.12. *Let σ be an arbitrary n-simplex. Then $\sup_{x\in\sigma} |x - \hat{\sigma}| \le \frac{nd}{n+1}$ where $d = \operatorname{diam}(\sigma)$.*

Proof. Let $v_0, \dots, v_n$ be the vertices of σ. Then

$$|\hat{\sigma} - v_j| = \left|\left(\frac{1}{n+1}\sum_{i=0}^{n} v_i\right) - v_j\right| \le \frac{1}{n+1}\sum_{i=0}^{n} |v_i - v_j| \le \frac{nd}{n+1}$$

since $|v_i - v_j| = 0$ if $i = j$. Let $x \in \sigma$, $x = \sum_{i=0}^{m} t_i v_i$, $\sum_{i=0}^{m} t_i = 1$. Then

$$|\hat{\sigma} - x| = \left|\sum t_i \left(\hat{\sigma} - v_i\right)\right| \leq \sum t_i \left(\frac{nd}{n+1}\right) = \frac{nd}{n+1}.$$

□

Defining the dimension of a simplicial complex as the maximum dimension of its simplexes, we arrive at the following theorem.

Theorem 2.13. *Let K be a simplicial complex of dimension n, then*

$$\operatorname{diam}(K^1) \leq \frac{n}{n+1} \operatorname{diam}(K).$$

Proof. Since the diameter of a simplicial complex is the maximum of the diameters of its simplexes, it suffices to prove it for a single n-simplex. The statement is trivial for a 0-simplex. Now assume it holds for all simplexes of dimension up to $n-1$. Let $\sigma_0 < \sigma_1 < \cdots < \sigma_m$ be simplexes where σ_m has dimension n. Then the simplex $\tilde{\sigma_k}$ in σ_m^1 with vertices $\hat{\sigma_{i_0}}, \ldots, \hat{\sigma_{i_k}}$, $i_j \neq m$, satisfies $\operatorname{diam}(\tilde{\sigma_k}) \leq \frac{m'}{m'+1} \operatorname{diam}(\sigma_m)$ for some $m' < n$ being the dimension of $\tilde{\sigma_k}$ and since $\tilde{\sigma_k}$ lies in some face of σ_m. Since $\frac{m'}{m'+1}$ is increasing in m', we get $\operatorname{diam}(\tilde{\sigma_k}) \leq \frac{n}{n+1} \operatorname{diam}(\sigma_m)$. By Lemma 2.11 and the definition of diameter for a complex, we need only consider the edges of σ_m^1 to determine its diameter. Any edge not having $\hat{\sigma}_m$ as a vertex is in some $\tilde{\sigma_k}$ and hence satisfies the bound. The bound for an edge having $\hat{\sigma}_m$ as a vertex is then shown in Lemma 2.12. □

Thus if we define the m:th barycentric subdivision of K inductively as $(K^{m-1})^1$, we can make the diameter of our simplicial complex arbitrarily small.

2.2 Simplicial approximation

Definition 2.14. Let K and L be simplicial complexes. A function $s : |K| \to |L|$ is called *simplicial* if it takes simplex linearly onto simplexes. That is if $v_1, \ldots, v_n$ are vertices of a simplex in K and $x = \lambda_1 v_1 + \cdots + \lambda_n v_n$, then $s(v_1), \ldots, s(v_n)$ span a simplex of L and $s(x) = \lambda_1 s(v_1) + \cdots + \lambda_n s(v_n)$.

Definition 2.15. Let K be a complex and v a vertex of K. Let $\sigma \in K$ have the vertices $v_0, \dots, v_m$ and $x \in \sigma$. Then x is said to be in the *interior* of the simplex σ if $x = \lambda_0 v_0 + \dots + \lambda_m v_m$ with $\sum_{i=0}^{m} \lambda_i = 1$ and $\lambda_i > 0$. The *open star* of v in K, denoted $\text{star}(v, K)$, is the union of the interiors of all simplexes of K having v as a vertex.

Definition 2.16. Let K and L be simplicial complexes and $f : |K| \to |L|$ a continuous function. The *carrier* of $f(x)$ is the unique simplex in L that has x in its interior. A simplicial function $s : |K| \to |L|$ such that $s(x)$ is in the carrier $f(x)$ for all $x \in |K|$ is called a *simplicial approximation* of f.

Simplicial approximations are useful in the sense that they are homotopic to the original function via the straight line homotopy. Since they are simplicial, they are also easier to work with. This will be used for the rest of this section. Before the main result of this section can be stated and proven, we are in need of two preliminary lemmas.

Lemma 2.17. *The vertices $v_1, \dots, v_n$ span a simplex in K if and only if the intersection of their open stars is non-empty.*

Proof. If $v_1, \dots, v_n$ span a simplex of σ of K, then the open star $\text{star}(v_i, K)$ contains the interior of σ for all i, hence their intersection is non empty. Conversely, pick $x \in \bigcap_{i=1}^{n} \text{star}(v_i, K)$ and let σ be the carrier of x. From the definition of open star we get that each v_i must be a vertex of σ. Since any collection of vertices of a simplex span a face of that simplex, $v_1, \dots, v_n$ span a face of σ. □

Lemma 2.18 (Lebesgue's lemma)**.** *Let X be a compact metric space and $\mathscr{F}$ an open cover of X. Then there exists $\delta > 0$ such that for all $V \subseteq X$ of diameter less than δ, there exists $U \in \mathscr{F}$ such that $V \subseteq U$. The number δ is called the Lebesgue number of the covering $\mathscr{F}$.*

Proof. Assume that there exists sets A_i, $i = 1, 2, \dots$, each of diameter less than $1/i$, such that A_i is not included in any $U \in \mathscr{F}$. Pick $x_i \in A_i$ and form the sequence $(x_i)_{i=1}^{\infty}$. Since X is compact, the sequence has a converging subsequence, say convergent to y. Choose $U \in \mathscr{F}$ such that $y \in U$ and pick $\epsilon > 0$ such that $B(y, \epsilon) \subset U$. Now fix N large enough such that $\text{diam}(A_N) < \epsilon/2$ and $x_N \in B(y, \epsilon/2)$. Then $d(x, y) \leq d(x, x_N) + d(x_N, y) < \epsilon$ for all $x \in A_N$. Hence $A_N \subseteq B(y, \epsilon) \subset U$ contradicting the construction of the A_i's. □

Theorem 2.19 (Simplicial approximation theorem)**.** *Let K and L be simplicial complexes and $f : |K| \to |L|$ a continuous function. Then there exists $n \in \mathbb{N}$ and $s : |K^n| \to |L|$ where s is a simplicial approximation of $f : |K^n| \to |L|$.*

Proof. Begin by assuming that for each vertex u of K, there exists a vertex v of L such that

$$f(\text{star}(u, K)) \subseteq \text{star}(v, L). \tag{1}$$

Define s on the vertices of K by letting $s(u) = v$ with u and v as in (1). If $u_1, \ldots, u_m$ span a simplex of K, then by Lemma 2.17, $\bigcap_{i=1}^m \text{star}(u_i, K)$ is non empty which by (1) implies that $\bigcap_{i=1}^m \text{star}(s(u_i), K)$ is non-empty, which again by Lemma 2.17 implies that $s(u_1), \ldots, s(u_m)$ span a simplex. Then extend s linearly over all simplexes of K to get $s : |K| \to |L|$ as a simplicial map. Let $x \in |K|$ and $u_1, \ldots, u_k$ be the vertices of the carrier of x. Again, by repeated use of Lemma 2.17 and equation (1), we get $x \in \bigcap_{i=1}^k \text{star}(u_i, K) \implies f(x) \in \bigcap_{i=1}^k \text{star}(s(u_i), K)$. Hence the carrier of $f(x)$ has the simplex spanned by $s(u_1), \ldots, s(u_k)$ as a face and thus $s(x)$ is in the carrier of $f(x)$. Therefore s is a simplicial approximation of f. Now we show that (1) can be achieved if not by K, then at least by K^n for some $n \in \mathbb{N}$. Since the open stars in L form an open cover of L and $f : |K| \to |L|$ is continuous, we get that the preimage of those opens stars under f is an open cover of $|K|$. Because $|K|$ is compact we can use Lebesgue's lemma and pick a Lebesgue number δ of the covering. If we choose n large enough such that $\text{diam}(K^n) < \delta/2$, then given a vertex $u \in K^n$, the open star at u satisfies $\text{diam}(\text{star}(u, K^n)) < \delta \implies \text{star}(u, K^n) \subseteq f^{-1}(\text{star}(v, L))$ for some vertex $v \in L$ as desired. □

The simplicial approximation theorem is one of the key parts in our path to proving the classification theorem.

Corollary 2.20. *The set of homotopy classes from one triangulable space to another is at most countable.*

Proof. By the simplicial approximation theorem, any continuous function $f : |K| \to |L|$ is homotopic to a simplicial map from $s : |K^n| \to |L|$ for some $n \in \mathbb{N}$. Since simplicial maps are completely determined by their effect on the vertices and that vertices map to vertices, there are only finitely many possible such s for at a given number of barycentric subdivisions. Because there are countably many possible barycentric subdivisions, there exists countably many simplicial maps from some $|K^n|$ and hence at most countably many homotopy classes. □

Theorem 2.21. *$\mathbb{S}^n$ is simply connected for $n \geq 2$.*

Proof. Triangulate $\mathbb{S}^n$ by the boundary of an $(n+1)$ simplex σ, denoted Σ, and let $I = [0,1]$. Let $\alpha : I \to \mathbb{S}^n$ be a loop based at p and since $\mathbb{S}^n$ is path-connected we can from Theorem 1.12 w.l.o.g. assume that p is a vertex of Σ. By the simplicial approximation theorem, α is homotopic to some simplicial function $s : |I^m| \to \mathbb{S}^n$ for some m. This is a homotopy rel $\{0,1\}$ since $s(0)$ and $s(1)$ must lie in the carrier of $\alpha(0) = \alpha(1)$ which is p since p is a vertex. Since s is a simplicial map, it is linear between all vertices of Σ and thus only attains values on the edges of Σ. In particular the antipodal point $-p$ of p, corresponding to the barycentre of the simplex spanned by all other vertices of Σ, does not lie on an edge because for $n \geq 2$, Σ has more than 3 vertices. Hence

$$F(x,t) = \frac{(1-t)s(x) + tp}{||(1-t)s(x) + tp||}$$

is a homotopy from s to the constant path at p rel $\{0,1\}$ and thus $[\alpha] = [s] = [p]$ in $\pi_1(\mathbb{S}^n, p)$ so $\mathbb{S}^n$ is simply connected. □

2.3 The edge group

Given a space X that is triangulated by some simplicial complex K, we can determine $\pi_1(X)$ by calculating $\pi_1(|K|)$ since the groups are isomorphic. This will in turn be determined by constructing two other isomorphic groups. The first of them is constructed as follows.

A sequence of vertices $v_0 v_1 \cdots v_n$ in K where each pair $v_i v_{i+1}$ span a simplex of K is called an edge path in K. If $v_0 = v_n = v$, then the sequence is called an edge loop based at v in K. To mimic that two loops in $\pi_1(|K|)$ are equivalent if there is a homotopy between them, we say that two edge loops are equivalent if they differ by finitely many relations of the following type. A double vertex uu is equivalent to u. If uvw span a simplex then uvw is equivalent to uw. Intuitively, this allows us to replace two sides of a triangle by the opposite third side and to disregard a path that goes back and forth along an edge. Denote the equivalence class of the edge loop $vv_1 \cdots v$ by $\{vv_1 \cdots v\}$. We define the multiplication of two equivalence classes by

$\{vv_1 \cdots v_n v\} \cdot \{vw_1 \cdots w_m v\} = \{vv_1 \cdots v_n vw_1 \cdots w_m v\}$. The identity element is $\{v\}$ and the inverse of $\{vv_1 \cdots v_n v\}$ is $\{vv_n \cdots v_1 v\}$.

Definition 2.22. The group described above is called the *edge group* of K based at v and is denoted by $E(K, v)$.

Theorem 2.23. *$E(K, v)$ is isomorphic to $\pi_1(|K|, v)$.*

Proof. Consider an edge loop $vv_1 \dots v_{m-1}v$ in K. From this, we can consider a corresponding loop α in $|K|$ by letting $\alpha(0) = \alpha(1) = v$, $\alpha(i/m) = v_i$ for $i = 1, 2, \dots m-1$ and extending it linearly between those points. Since equivalent edge paths correspond to homotopic loops, we can define $\phi : E(K, v) \to \pi_1(|K|, v)$ by $\phi(\{vv_1 \dots v_{m-1}v\}) = [\alpha]$ which clearly is a homomorphism. We want to show that ϕ is even an isomorphism. To show surjectivity, we pick a loop $\alpha : [0, 1] \to |K|$ based at v and consider a complex I triangulating $[0, 1]$ with only the end points as vertices. By the simplicial approximation theorem, there exists $n \in \mathbb{N}$ and a simplicial map $s : |I^n| \to |K|$ that is homotopic to α. Denoting $v_i = s(i/2^n)$, we get $\phi(\{vv_1 \dots v_{2^n-1}v\}) = [s] = [\alpha]$. For injectivity, we consider an edge loop $vv_1 \dots v_m v$ that corresponds to an edge loop α that is homotopic to the constant loop at v via a homotopy F. Must show $vv_1 \dots v_m v$ is equivalent to v. We note that $F : [0, 1] \times [0, 1] \to |K|$ satisfies

$$F(x, 0) = \alpha(x), F(x, 1) = F(0, t) = F(1, t) = v.$$

This can be expressed as F sending the lower edge of the complex $I \times I$ in Figure 6 linearly to $|K|$ with $F(a_i) = v_i$ for $a_i = (i/(m+1), 0)$ and the outer edges constantly mapped to the vertex v. We argue that the edge paths $aa_1a_2 \dots a_m d$ and $abcd$ are equivalent in $I \times I$. Letting $\sim$ denote the relation of equivalent edge loops, we get

$$\begin{aligned} a(bcd)a_m \dots a_2a_1a &\sim a(bd)a_m \dots a_1a \sim ab(a_m d a_m)a_{m-1} \dots a_1a \\ &\sim a(ba_m)a_{m-1} \dots a_1a \sim ab(a_{m-1}a_m a_{m-1})a_{m-2} \dots a_1a \\ &\sim a(ba_{m-1})a_{m-2} \dots a_1a \end{aligned}$$

where the brackets indicate on what part of the edge loop a equivalence relation will be used. Continuing the same procedure of eliminating parts of the lower edge, we get after finitely many steps to

$$a(ba_1)a \sim ab(aa_1a) \sim aba \sim a$$

and hence $abcd \sim aa_1 \dots a_m d$.

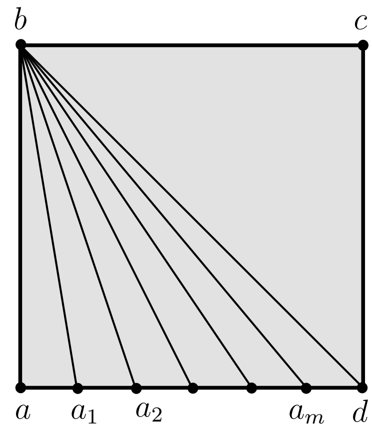

Figure 6: The complex $I \times I$.

Now consider the complex $(I \times I)^k$, we show that the edge path traversing the lower edge, denoted E_1, is still equivalent to the edge path traversing the other three edges, denoted E_2. Inductively, we may assume they are equivalent in $(I \times I)^{k-1}$. Since all edges of $(I \times I)^{k-1}$ are still present in $(I \times I)^k$, just divided in half, the result follows if we can show that the relations used in $(I \times I)^{k-1}$ still can be used in $(I \times I)^k$. Considering the complex in Figure 7, we must show $(adc)(cda) \sim a$ and $(aeb)(bfc) \sim adc$. The first relation follows via

$$ad(cc)da \sim a(dcd)a \sim ada \sim a$$

and the second via

$$\begin{aligned} ae(bb)fc &\sim a(eb)fc \sim aeg(bf)c \sim ae(gbg)fc \\ &\sim (aeg)fc \sim a(gfc) \sim (ag)c \\ &\sim a(dgc) \sim adc. \end{aligned}$$

Hence E_1 and E_2 are equivalent in $(I \times I)^k$. Now we may use the simplicial approximation theorem to get a simplicial map $S : |(I \times I)^n| \to |K|$ that simplicially approximates F. Since the relations defining edge equivalence are in terms of spanning a simplex, a simplicial map will map equivalent edge paths to equivalent edge paths. Since F is constantly equal to v on E_2, S maps E_2 to the constant loop at v. On E_1, since $F(a_i) = v_i$, this forces S to map every extra vertex created between a_i and a_{i+1} by the barycentric subdivision to either v_i or v_{i+1}. Hence S maps E_1 to an edge loop equivalent to $vv_1 \dots v_m v$. Thus $vv_1 \dots v_m v \sim v$ and ϕ is injective. $\square$

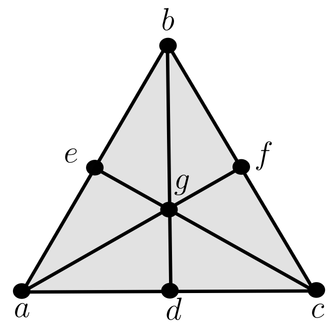

Figure 7: A barycentrically subdivided triangle.

From $E(K, v)$, we can construct another group that is easier to determine by using even more of the graph structure of the 1-simplex of K. Recall that a tree is a graph where any two vertices is connected by exactly one path. If we define a partial order on trees in a complex K by inclusion, we get the following lemma.

Lemma 2.24. *A maximal tree contains all the vertices of a connected complex K.*

Proof. Let T be a maximal tree in K and assume that T does not contain the vertex v. Now pick a vertex u in T. Since $|K|$ is path-connected, there exists an edge path $uv_1 \cdots v_n v$ in K from u to v. Let v_i be the last vertex not in T, possibly v. Form T' by adjoining v_i and the edge spanned by v_{i-1} and v_i to T. Then T' is a tree that strictly includes T, a contradiction to the maximality of T. Hence T contains all the vertices of K. □

Let L denote the simplex for a such a maximal tree in K. Since there are no loops in a tree, any edge loop in L will traverse each edge equally many times in either direction and hence be equivalent to the constant loop. Thus any edge loop in L will not contribute to $E(K, v)$. Let $v_0 = v$ and list all the edges in K as $v_0, \cdots, v_m$ and let $G(K, L)$ be the group determined by the following generators and relations. For each ordered pair of vertices v_i, v_j that span a simplex of K, there is a generator g_{ij}. Now $g_{ij} = 1$ if v_i, v_j span a simplex of L and $g_{ij}g_{jk} = g_{ik}$ if v_i, v_j, v_k span a simplex of K.

Theorem 2.25. *$G(K, L)$ is isomorphic to $E(K, v)$.*

Proof. Let E_i be the edge path in L from v to v_i for each vertex of K and define $\phi : G(K, L) \to E(K, v)$ by $\phi(g_{ij}) = \{E_i E_j^{-1}\}$. If v_i, v_j span a simplex of L, then $\{E_i E_j^{-1}\}$ is an edge loop in L and hence equal to the identity since L is a tree. If v_i, v_j, v_k span a simplex of K, then

$$\begin{aligned}\phi(g_{ij})\phi(g_{jk}) &= \{E_i E_j^{-1}\}\{E_j E_k^{-1}\} = \{E_i E_j^{-1} E_j E_k^{-1}\} \\ &= \{E_i v_j E_k^{-1}\} = \{E_i E_k^{-1}\} = \phi(g_{ik})\end{aligned}$$

where we used that $\{v_i v_j v_k\} = \{v_i v_k\}$. Hence ϕ is a well defined homomorphism. Now define $\theta : E(K, v) \to G(K, L)$ by $\theta(\{v v_{i_1} v_{i_2} \dots v_{i_n} v\}) = g_{0i_1} g_{i_1 i_2} \cdots g_{i_n 0}$. By an analogue argument as for ϕ we see that θ is also a homomorphism. We now argue that in fact ϕ is invertible with inverse θ and hence an isomorphism. $\theta(\phi(g_{ij})) = \theta(\{E_i E_j^{-1}\}) = g_{ij}$ since any consecutive vertices in E_i and E_j span a simplex of L. Since $\theta\phi$ is the identity on the generators of $G(K, L)$, it is the identity on the whole group. Furthermore, writing $\{v v_{i_1} v_{i_2} \dots v_{i_n} v\} = \{E_0 E_{i_1}^{-1}\}\{E_{i_1} E_{i_2}^{-1}\} \dots \{E_{i_n} E_0^{-1}\}$, we see that $\phi\theta$ is the identity on each factor of the product and since it is a homeomorphism, it is also the identity on the entire edge loop. □

Before turning to our final main tool for calculating the fundamental group, we wish to see what we can do so far.

Example 2.26. Since a complex K consists of finitely many simplexes, there are only finitely many generators and relations for $G(K, L)$. Hence if X is a triangulable space, then $\pi_1(X)$ is finitely presented.

Example 2.27. Consider a bouquet of n circles, that is, n circles glued together at a common point. This can be triangulated by a complex K consisting of the boundary of n triangles all meeting at a common vertex v. Take a maximal tree L consisting of all the vertices as well as the edges from v to all those vertices. Then each triangle has two edges in L and one in $K \setminus L$. The non-trivial generators of $G(K, L)$ is then in bijective correspondence with those n edges not in L. Since K does not have any simplexes of higher dimension, there are no relations between them. Hence $G(K, L)$ is a free group on n generators. In particular, for $n = 1$, we get the following result.

Theorem 2.28. $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

Our final tool for calculating fundamental groups will allow us to determine the fundamental group of a more complicated structure by considering the fundamental group of smaller subsets of the original space.

Theorem 2.29 (The Seifert-van Kampen theorem)**.** *Let J, K be simplicial complexes in $\mathbb{R}^n$ where $|J|$,$|K|$ and $|J \cap K|$ are path-connected. Let also $j : |J \cap K| \to |J|$ and $k : |J \cap K| \to |K|$ be the inclusion mappings and v a vertex of $|J \cap K|$. Then $\pi_1(|J \cup K|, v) = \pi_1(|J|, v) * \pi_1(|K|, v)$ with the additional relation that $j_*(\alpha) = k_*(\alpha)$ $\forall \alpha \in \pi_1(|J \cap K|, v)$. Here $*$ denotes the free product of groups.*

Proof. Let T_0 be a maximal tree in $|J \cap K|$ and extend it to maximal trees T_1 in $|J|$ and T_2 in $|K|$. By Theorem 2.23 and 2.25, $\pi_1(|J \cup K|, v)$ is generated by generators g_{ij} corresponding to edges of $J \cup K \setminus T_1 \cup T_2$ since $T_1 \cup T_2$ is a maximal tree in $|J \cup K|$. These generators have relations $g_{ij} g_{jk} = g_{ik}$ corresponding to each 2-simplex of $|J \cup K|$. But this group can also be described by defining generators h_{ij} for each edge of $J \setminus T_1$ and f_{ij} for each edge of $K \setminus T_2$, subject to $h_{ij} h_{jk} = h_{ik}$ and $f_{ij} f_{jk} = f_{ik}$ if they span a 2-simplex of J or K respectively and $h_{ij} = f_{ij}$ if they correspond to the same edge of $J \cap K$. Since the edges of $J \cap K \setminus T_0$ generate $G(J \cap K, T_0)$, j_* of an edge in $J \cap K$ is some h_{ij} and k_* of an edge in $J \cap K$ is some f_{ij}, the last relation can be rephrased as $j_*(\alpha) = k_*(\alpha)$ $\forall \alpha \in \pi_1(|J \cap K|, v)$. Hence the statement is proven. □

Note that this is not the most common nor general version of the Seifert-van Kampen theorem. For this more general approach, see [8].

Example 2.30. Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ denote the torus. Following the notation from the Seifert-van Kampen theorem, we let K be a triangle and J a triangulation of the torus with the interior of that triangle removed, see Figure 8. Then $J \cap K$ is the edges of a triangle and $J \cup K$ is a triangulation of the torus. Now $|K|$ is convex so $\pi_1(|K|, v) = 1$. For $|J|$, the space deformation retracts onto its boundary which becomes a bouquet of two circles, hence $\pi_1(|J|, v)$ is a free group on two generators by Example 2.27. For the extra relations, we need to consider a single loop in $|J \cap K|$. In $|K|$ this is homotopic to the identity since $|K|$ is simply connected. In $|J|$, this is homotopic to $\beta\gamma\beta^{-1}\gamma^{-1}$. Hence by Seifert-van Kampen, the fundamental group of the torus is given by $\langle \beta, \gamma | \beta\gamma\beta^{-1}\gamma^{-1} = v \rangle = \mathbb{Z}^2$.

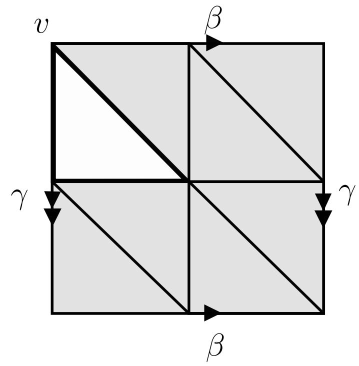

Figure 8: Complex triangulating a punctured torus.

3 Simplicial homology

3.1 Construction of homology groups

Even though the Seifert-van Kampen theorem gives us a strong tool for determining the fundamental group of a space in terms of generators and relations, it is in principle hard to determine if two different presentations of a group determine the same group or not. It has even been proven that an algorithm to determine whenever two words in a presentation determine the same group element does not exist, see [3]. When simplifying these presentations, another invariant of topological spaces called homology groups will come into play and that is what we will study in this section.

At first, we need to define orientation for a simplex as well as induced orientations.

Definition 3.1. Let $v_0, \ldots, v_k$ be an ordering of the vertices for some k-simplex σ, written $\sigma = (v_0, \ldots, v_k)$. We say that two such orderings are equivalent if one can be obtained from the other by an even permutation. The equivalence classes obtained is then called the possible *orientations* of σ. A simplex with a specified orientation will be called an *oriented simplex.*

We note that for a given k-simplex, there are exactly two possible orientations. Only exception is if $k = 0$. In agreement with intuition, a vertex has only one orientation. An orientation of a simplex also induces an orientation of its faces in the following way.

Definition 3.2. Let $(v_0, \ldots, v_k) = \sigma$ and let τ be the simplex spanned by all the vertices of σ except v_i. For i even, we let τ be oriented by the natural ordering $v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$. For i odd, we take the opposite orientation.

Example 3.3. Consider a triangle with vertices v_0, v_1, v_2 determining its orientation. One can check that the intuitive orientations of the edges as (v_0, v_1), (v_1, v_2) and (v_2, v_0) is indeed obtained by the above definition.

From a simplicial complex, we can now define the following group.

Definition 3.4. Let K be a simplicial complex. Define the *qth chain group* of K, denoted $C_q(K)$, as the free abelian group generated by all the oriented q-simplexes of K, subject to the relation that $\sigma + \tau = 0$ if they represent the same simplex but with opposite orientations. An element in $C_q(K)$ is called a *q-dimensional chain*.

A q-chain can be written as a formal linear combination $\lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$ where σ_i is an oriented q-simplex and $\lambda_i \in \mathbb{Z}$. We also note that $(-\lambda)\sigma = \lambda(-\sigma)$ where $-\sigma$ stands for σ with opposite orientation. We can now define a homomorphism on $C_q(K)$ as follows.

Definition 3.5. Let K be a simplicial complex. The *boundary homomorphism* $\partial : C_q(K) \to C_{q-1}(K)$ is defined such that for a given oriented q-simplex, it gives the sum of its $(q-1)$-dimensional faces with their induced orientation, i.e.

$$\partial(v_0, \ldots, v_q) = \sum_{i=0}^{q} (-1)^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_q).$$

It is then extended linearly to an arbitrary element of $C_q(K)$.

We see that ∂ is a well defined homomorphism since if the orientation of σ is reversed, then so is the orientation of all its faces and hence $\partial\sigma + \partial(-\sigma) = 0$. If $q = 0$, then we define the boundary of a point to be zero. Thus $C_q(K) = 0$ if $q < 0$. An important property of the boundary homomorphism can be summarized as that a boundary has no boundary.

Lemma 3.6. *Let* $\partial^2 = \partial \circ \partial : C_{q+1}(K) \to C_{q-1}(K)$. *Then* $\ker(\partial^2) = C_{q+1}(K)$.

Proof. It suffices to prove $\partial^2(v_0, \ldots, v_{q+1}) = 0$ since elements of that form generate $C_{q+1}(K)$. Now

$$
\begin{aligned}
&\partial^2(v_0, \ldots, v_{q+1}) \\
&= \partial \sum_{i=0}^{q+1} (-1)^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{q+1}) \\
&= \sum_{i=0}^{q+1} (-1)^i \sum_{j=0}^{i-1} (-1)^j (v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{q+1}) \\
&+ \sum_{i=0}^{q+1} (-1)^i \sum_{j=i+1}^{q+1} (-1)^{j-1} (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{q+1}) \\
&= \sum_{i=0}^{q+1} (-1)^i \sum_{\substack{j=0 \\ j \neq i}}^{q+1} (-1)^{j-1} (1-1)(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{q+1}) \\
&= 0.
\end{aligned}
$$

□

Definition 3.7. For $\partial : C_q(K) \to C_{q-1}(K)$, we denote $\ker(\partial) = Z_q(K)$ and call it the group of *q-cycles* of K. If $\partial : C_{q+1}(K) \to C_q(K)$, we denote its image $\text{Im}(\partial) = B_q(K)$ and call it the group of *bounding q-cycles* of K.

We note that $Z_1(K)$ is generated by the elementary 1-cycles of the form $(v_1, v_2) + (v_2, v_3) + \ldots (v_k, v_1)$. By Lemma 3.6, $B_q(K)$ is a subgroup of $Z_q(K)$ and hence we can make the following definition.

Definition 3.8. Let K be a simplicial complex. The *qth homology group* of K is defined as the quotient group

$$H_q(K) = Z_q(K)/B_q(K).$$

Elements of $H_q(K)$ are called *homology classes*, denoted $[z]$ for $z \in Z_q(K)$, and two q-cycles in the same homology class are called *homologous* cycles.

For the reader with more knowledge in abstract algebra, the definitions of $Z_q(K)$ and $B_q(K)$ can be remembered via the short exact sequence

$$0 \longrightarrow Z_q(K) \longrightarrow C_q(K) \xrightarrow{\partial} B_{q-1}(K) \longrightarrow 0$$

where the unspecified morphisms are inclusion on the first two and the only possible on the last. Homology groups can be defined for topological spaces without the necessity of any underlying simplicial complex, for more on that, see e.g. [8]. By construction, $H_q(K)$ is a finitely generated abelian group. Such groups can by the fundamental theorem of finitely generated abelian groups be decomposed as $\mathbb{Z}^n \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}$ where $p_i | p_{i+1}$, see [2].

Definition 3.9. Let $H_q(K) = \mathbb{Z}^n \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}$. Then n is called the *qth Betti number* of K, denoted β_q.

3.2 Basic properties

Some homology groups can be expressed in terms of already known objects relating to the simplicial complex.

Theorem 3.10. *$H_0(K)$ is isomorphic to $\mathbb{Z}^n$ where n is the number of connected components of $|K|$.*

Proof. Since $C_{-1}(K) = 0, Z_0(K) = C_0(K)$. Hence we only need to determine $B_q(K)$. Let v, w be two vertices in the complex K that lie in the same component of $|K|$. Then there exists an edge path $vv_1 \dots v_m w$ from v to w in K. The 1-cycle $(v, v_1) + (v_1, v_2) + \cdots + (v_m, w)$ has then the boundary $v - w$, hence v is homologous to w. If the vertices lie in different components then they are not homologous since the boundary of any 1-cycle always contributes with an even number of vertices in each connected component. Finally, v is not homologous to λw for any $\lambda \neq 1$ since after taking the boundary of a 1-cycle, there are equally many vertices appearing positive as negative. □

For our purposes, the main result about homology groups is the following.

Theorem 3.11. *Let K be a connected simplicial complex. Then $H_1(K)$ is isomorphic to $\pi_1(|K|)/[\pi_1(|K|), \pi_1(|K|)]$ where $[\pi_1(|K|), \pi_1(|K|)]$ is the commutator subgroup.*

Proof. Pick a vertex v of K as base point. By Theorem 2.23, we can consider $E(K, v)$ instead of $\pi_1(|K|, v)$. Given an edge loop $\alpha = vv_1 \dots v_k v$, we denote the corresponding 1-cycle by $\zeta(\alpha) = (v, v_1) + (v_1, v_2) + \cdots + (v_k, v)$ where (v_i, v_{i+1}) is omitted if $v_i = v_{i+1}$. Define $\phi : E(K, v) \to H_1(K)$ by $\phi(\{\alpha\}) = [\zeta(\alpha)]$. Note that equivalent edge paths give homologous 1-cycles. For example, if (w_1, w_2, w_3) is an oriented 2-simplex, then $\partial((w_1, w_2, w_3)) =$

$(w_1, w_2) + (w_2, w_3) - (w_1, w_3)$ so $\phi(\{w_1, w_2, w_3\}) = \phi(\{w_1, w_3\})$. Hence ϕ is a well defined homomorphism. We first show that ϕ is surjective by proving that any elementary 1-cycle is in $\mathrm{Im}(\phi)$. Let $z_1 = (w_1, w_2) + (w_2, w_3) + \cdots + (w_n, w_1)$. Defining γ as an edge path in K from v to w_1 gives that $\phi(\{\gamma w_1 w_2 \dots w_n \gamma^{-1}\}) = z_1$ as desired. Since $H_1(K)$ is abelian, the commutator subgroup of $E(K, v)$ is included in $\ker(\phi)$. If we show that any element in $\ker(\phi)$ also is in $[E(K, v), E(K, v)]$, then the result follows from the first isomorphism theorem. Pick $\alpha \in \ker(\phi)$, then $\zeta(\alpha) \in B_1(K)$, say $\zeta(\alpha) = \partial(\lambda_1\sigma_1 + \cdots + \lambda_m\sigma_m)$ where σ_i are oriented 2-simplexes of K. Now assume $\sigma_i = (a_i, b_i, c_i)$ and define γ_i as an edge path in K from v to a_i. Then $\{\gamma_i a_i b_i c_i \gamma_i^{-1}\} = \{v\}$ and therefore also

$$\{\beta\} = \prod_{i=1}^{m} \{\gamma_i a_i b_i c_i \gamma_i^{-1}\}^{\lambda_i} = \{v\}.$$

Hence $\{\alpha\beta^{-1}\} = \{\alpha\}$. But since $\zeta(\gamma_i a_i b_i c_i \gamma_i^{-1}) = \partial(a_i b_i c_i)$ we get $\zeta(\alpha\beta^{-1}) = 0$. By construction of ζ, any such edge loop mapping to 0 must traverse the oriented edge (v_i, v_{i+1}) equally many times as (v_{i+1}, v_i). Hence, recalling $\theta : E(K, v) \to G(K, L)$ from Theorem 2.25, we get that $\theta(\{\alpha\beta^{-1}\})$ equals a product of elements where each element occur equally many times as its inverse. Thus $\theta(\{\alpha\beta^{-1}\}) \in [G(K, L), G(K, L)]$, which implies that $\{\alpha\} = \{\alpha\beta^{-1}\} \in [E(K, v), E(K, v)]$ since θ is an isomorphism. □

Since the number of components of a topological space as well as its fundamental group is a topological invariant of the space, so is the zeroth and first homology group by the above theorems. In general, the homology group of any order is an invariant, for a proof, see e.g. [1]. We will now take a look at two examples for how to determine the homology groups of a simplicial complex.

Example 3.12. Consider a connected graph G with v vertices and e edges. Since $|G|$ is connected, $H_0(G) = \mathbb{Z}$ and $\beta_0 = 1$ by Theorem 3.10. To determine $H_1(G)$, we use Theorem 2.25 and 3.11 saying that $H_1(G)$ is isomorphic to $G(G, L)/[G(G, L), G(G, L)]$. Since G has v vertices, there are $v - 1$ edges in the spanning tree L and thus $e - (v - 1) = e - v + 1$ edges determining generators for $G(K, L)$. Since a graph does not have any 2-simplexes, there are no additional relations and $G(K, L)$ is a free group on $e - v + 1$ generators. Therefore $H_1(G) = \mathbb{Z}^{e-v+1}$ and $\beta_1 = e - v + 1$. Since there are no simplexes of higher order, $H_q(G) = 0$ for $q \geq 2$.

Another interesting structure that we can determine the homology groups of is a simplicial complex which can be obtained as the cone of another complex.

Definition 3.13. Let $v_0, \ldots, v_m$ be points in $\mathbb{R}^n$, seen as a subspace of $\mathbb{R}^{n+1}$. Let $v = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and call the simplex spanned by $v_0, \ldots, v_m$ for σ. The *cone* on σ is the simplex determined by the points $v_0, \ldots, v_m, v$ in $\mathbb{R}^{n+1}$. The cone on a complex K in $\mathbb{R}^n$ is a complex CK in $\mathbb{R}^{n+1}$ obtained by taking the cone on each simplex of K. We call v the *apex* of the cone CK.

Note that any other point of $\mathbb{R}^{n+1} \setminus \mathbb{R}^n$ could also have been used to define the cone since the relations between all the vertices would still be the same and thus their polyhedra would be homeomorphic.

Example 3.14. Let K be a simplicial complex which is also a cone, say $K = CL$ for some complex L with apex v. Since a cone is connected, $H_0(K) = \mathbb{Z}$ by Theorem 3.10. For $q > 0$, define $d : C_q(K) \to C_{q+1}(K)$ by $d(\sigma) = (v, v_0, \ldots, v_q)$ for an oriented q-simplex $\sigma = (v_0, \ldots, v_q) \in L$ and $d(\sigma) = 0$ for any other q-simplex in K. Since $d(\sigma) + d(-\sigma) = 0$, d is a homomorphism if we extend in linearly to any q-chain in $C_q(K)$. We now argue that $\partial d(\sigma) = \sigma - d\partial(\sigma)$ for any oriented q-simplex σ. If $\sigma \in L$, then

$$\begin{aligned} \partial d(\sigma) &= \partial(v, v_0, \ldots, v_q) \\ &= (v_0, \ldots, v_q) + \sum_{i=0}^{q} (-1)^{i+1} (v, v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_q) \\ &= \sigma - d\partial(\sigma). \end{aligned}$$

If σ is not in L, then one vertex of σ must be v. Therefore we may assume $\sigma = (v_0, \ldots, v, \ldots, v_q)$ with v in the jth position. Since $d(\sigma) = 0$, we need to show $d\partial(\sigma) = \sigma$. But since any oriented simplex containing the vertex v is in $\ker(d)$, we get

$$\begin{aligned} d\partial(\sigma) &= d((-1)^j (v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_q)) \\ &= (-1)^j (v, v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_q) = \sigma. \end{aligned}$$

Hence the relation $\partial d(\sigma) = \sigma - d\partial(\sigma)$ holds for any q-chain by linearity. Now let $z \in Z_q(K)$. Then $\partial d(z) = z - d\partial(z) = z$. Hence $z \in B_q(K)$, so $Z_q(K) = B_q(K)$, which in turn shows that $H_q(K) = 0$ for $q > 0$.

4 The classification theorem

4.1 Notes on surfaces

We have now built up enough machinery to turn our focus directly towards the classification theorem. Before we can state our first version of the theorem, we need some definitions.

Definition 4.1. An n-dimensional *topological manifold* M is a Hausdorff, second countable space where each point $x \in M$ has a neighbourhood homeomorphic to $\mathbb{D}^n = \{x \in \mathbb{R}^n ; \|x\| < 1\}$.

Definition 4.2. A *closed surface* is a compact and connected 2-dimensional manifold.

Hence objects such as the sphere $\mathbb{S}^2$, the torus $\mathbb{T}$ and even non-orientable surfaces such as the Klein bottle are closed surfaces. On the other hand, the Möbius strip is not since it has a boundary where points does not have a neighbourhood homeomorphic to $\mathbb{D}^2$. Note also the distinction between a closed surface and a closed topological space. One might think that a more suitable name for closed surfaces would be compact surfaces since that is the actual requirement in the definition. That is however a slightly larger class of surfaces.

Definition 4.3. A *compact surface* S is a compact and connected 2-dimensional manifold that can have boundary. The *boundary* of S, denoted ∂S, is defined as points with a neighbourhood homeomorphic to $\mathbb{D}^2_+ = \{(x, y) \in \mathbb{D}^2 | x \geq 0\}$.

Before stating our main theorem, we will mention two small facts about compact surfaces that will be useful later.

Lemma 4.4. *The boundary and interior of a compact surface are disjoint.*

Proof. Let S be a compact surface and assume that its boundary and interior are not disjoint. Then there exists $x \in S$ having neighbourhoods U and V such that $f : \mathbb{D}^2_+ \to U$ and $g : \mathbb{D}^2 \to V$ with $f(0) = g(0) = x$ are homeomorphisms. By replacing $\mathbb{D}^2_+$ with half discs of smaller radius if necessary, we may assume $f(\mathbb{D}^2_+) \subseteq V$. Define $\phi : g^{-1}f : \mathbb{D}^2_+ \to \mathbb{D}^2$. Then $\phi(\mathbb{D}^2_+)$ is a neighbourhood of 0 in $\mathbb{D}^2$. Pick another disc $D \subset \mathbb{D}^2$ centred at the origin with small enough radius R such that $D \subset \phi(\mathbb{D}^2_+)$. Let $r : \phi(\mathbb{D}^2_+) \setminus \{0\} \to \partial D$, $r(x) = R\,(x/\|x\|)$ be radial projection. Since r is the identity on ∂D, it induces a surjective homomorphism $r_* : \pi_1(\phi(\mathbb{D}^2_+) \setminus \{0\}) \to \pi_1(\partial D)$. Since $\phi(\mathbb{D}^2_+) \setminus \{0\}$ is homeomorphic to $\mathbb{D}^2_+ \setminus \{0\}$ via the homeomorphism ϕ^{-1} and $\mathbb{D}^2_+ \setminus \{0\}$ deformation retracts onto any point in $\{(x, y) \in \mathbb{D}^2_+ \setminus \{0\} | x > 0\}$, we get $\pi_1(\phi(\mathbb{D}^2_+) \setminus \{0\}) = 1$. But ∂D is homeomorphic to $\mathbb{S}^1$ and by Theorem 2.28 this gives $\pi_1(\partial D) = \mathbb{Z}$. Thus we get a contradiction from the surjectivity of r_*. □

Corollary 4.5. *Let $h : S_1 \to S_2$ be a homeomorphism between two compact surfaces. Then h is also a homeomorphism from ∂S_1 to ∂S_2. In particular, homeomorphic surfaces have homeomorphic boundary.*

Proof. Pick x in the interior of S_1 and let $f : \mathbb{D}^2 \to U$ be a homeomorphism onto a neighbourhood of x. Since h is a homeomorphism, $h(U)$ is a neighbourhood of $h(x)$ in S_2 and $fh : \mathbb{D}^2 \to h(U)$ is a homeomorphism. Thus $h(x)$ is in the interior of S_2 and h maps interior of S_1 into interior of S_2. By the same logic applied to h^{-1}, the interior of S_2 is mapped to the interior of S_1 and hence the interior of S_1 is mapped bijectively onto the interior of S_2. Since the interior and the boundary of a compact surface are disjoint, h must map ∂S_1 to ∂S_2 bijectively as well. □

We now turn to the statement of the main theorem of this thesis.

Theorem 4.6 (The classification theorem for closed surfaces)**.** *Any closed surface is homeomorphic to either the sphere $\mathbb{S}^2$, the sphere with a finite number of handles added or the sphere with a finite number of discs removed and replaced by Möbius strips. Furthermore, no pair of those surfaces are homeomorphic.*

The surfaces mentioned in the classification theorem will be called the standard surfaces. We do however need to clarify what we mean by adding handles. In short, by adding a handle, we mean removing two disjoint discs from a closed surface S and gluing opposite ends of a cylinder in their place. By gluing we mean to take the union of the spaces together with the equivalence relation that identifies the appropriate boundary circles and giving it the identification topology. The procedure can also be described in the following way. Consider a closed disc in S containing the two removed discs in its interior. Removing that disc and then gluing the cylinder to it, we get a surface as in Figure 9, clearly homeomorphic to a punctured torus. Hence, adding a handle can be described as removing an open disc from S and $\mathbb{T}$ respectively and forming the space obtained by gluing their boundary circles together. We therefore make the following definition.

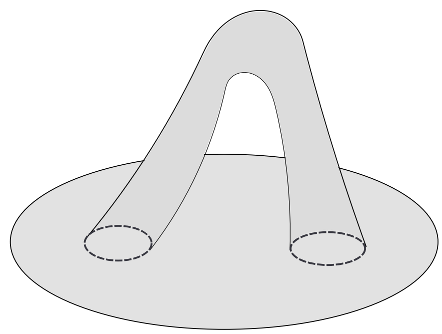

Figure 9: A punctured torus.

Definition 4.7. Let S_1 and S_2 be two closed surfaces. Their *connected sum*, denoted $S_1 \# S_2$, is defined as the surface obtained by removing an open disc from both S_1 and S_2 and then gluing their resulting boundary circles together.

We will see in the last section that this is a well defined operation, that no matter where we choose the discs on the surfaces, the connected sum gives homeomorphic surfaces. Replacing discs with Möbius strips can also be described in the language of connected sums, but for that, we need to define another closed surface.

Definition 4.8. The *projective plane*, denoted $\mathbb{P}$, is defined as the closed surface obtained by removing an open disc from $\mathbb{S}^2$ and gluing in a Möbius strip along the resulting boundary circle.

The definition of the projective plane is done to suit the applications we have in mind for it. A more common definition would be to take equivalence classes of points in $\mathbb{S}^2$ with the equivalence relation that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if there exists $\lambda \in \mathbb{R}$ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. The interested reader can check that this gives the projective plane. Going back, we see that removing an open disc from S and replacing it with a Möbius strip can be explained as forming the connected sum $S \# \mathbb{P}$ via the same logic as for adding a handle. Since $\mathbb{S}^2 \# \mathbb{T}$ is homeomorphic to $\mathbb{T}$ and $\mathbb{S}^2 \# \mathbb{P}$ is homeomorphic to $\mathbb{P}$, we get the following version of the classification theorem.

Theorem 4.9. *Any closed surface is homeomorphic to either* $\mathbb{S}^2$*, a connected sum of finitely many tori or a connected sum of finitely many projective planes. Furthermore, non of these surfaces are homeomorphic to another.*

A question that can come to mind when considering the classification theorem is what happens if we were to both form the connected sum with tori and projective planes. This is the content of the next lemma.

Lemma 4.10. *The space obtained by removing m disjoint discs from $\mathbb{S}^2$ and replacing them with Möbius strips as well as adding n handles is homeomorphic to $\mathbb{S}^2$ with $2n+m$ disjoint discs removed and replaced by Möbius strips. In other words, $\mathbb{P}\#\mathbb{P}\#\mathbb{P}$ is homeomorphic to $\mathbb{P}\#\mathbb{T}$.*

To prove this, we will use another lemma.

Lemma 4.11. *Let $\mathbb{K}$ denote the Klein bottle as defined by the identification made on the square in Figure 3. Then $\mathbb{P}\#\mathbb{P}$ is homeomorphic to $\mathbb{K}$.*

Proof. We first wish to acquire knowledge about $\mathbb{P}\#\mathbb{P}$. By the second characterisation of the projective plane, we se that it can be obtained by considering the northern hemisphere $\mathbb{S}^2_+ = \{(x,y,z) \in \mathbb{S}^2 | z \geq 0\}$ with the relation that diametrically opposite points on the equator should be identified. Since $\mathbb{S}^2_+$ is homeomorphic to $\overline{\mathbb{D}^2}$ which in turn is homeomorphic to a square, $\mathbb{P}$ can be obtained by doing identifications as in Figure 10. Also, considering $\mathbb{P}$ as $\overline{\mathbb{D}^2}$ with diametrically opposite points identified on its boundary, we can when removing an open disc from $\mathbb{P}$ choose the disc $D = \{(x,y) \in \overline{\mathbb{D}^2} |\, |x| > 1/2\}$ and we see that what remains of $\mathbb{P}$ is homeomorphic to the Möbius strip. Thus $\mathbb{P}\#\mathbb{P}$ is the space obtained by identifying the edges of two Möbius strips. But recalling the Klein bottle $\mathbb{K}$, we see that if we consider the square with identifications defining $\mathbb{K}$, we can divide the square into three rectangles. The middle rectangle becomes a Möbius strip by the identification of two of its edges and similarly the other two rectangles also form a single Möbius strip after the identifications, see Figure 11 for clarification. Hence $\mathbb{K}$ is obtained by identifying the boundary circles of two Möbius strips, thus $\mathbb{P}\#\mathbb{P}$ is homeomorphic to $\mathbb{K}$. □

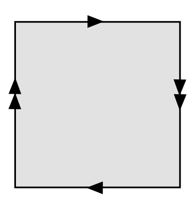

Figure 10: A projective plane.

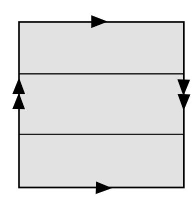

Figure 11: A divided Klein bottle.

proof of Lemma 4.10. By the preceding lemma, it suffices to show $\mathbb{P}\#\mathbb{T}$ is homeomorphic to $\mathbb{P}\#\mathbb{K}$. This will be done by showing that both spaces correspond to identical polygons in the plane with identifications on their edges. In Figure 12, we pick a disc with boundary d in a Klein bottle and a projective plane. Gluing the resulting spaces together in the direction indicated by the arrows on the bounding circles, we get a hexagon with edges to be identified in pairs. The same thing is then done for a torus and a projective plane in Figure 13. To prove that the two hexagons with identifications correspond to the same closed surface, we use a cut and paste technique. Consider the hexagon obtained from the connected sum of a torus and a projective plane. Insert a new edge denoted d as indicated in the first polygon in Figure 14. We can then do the identifications along the edges labelled c to get a new hexagon. In that hexagon, we make a new cut to form an edge labelled e so that we can glue the edges labelled a together. This creates a hexagon with identical identifications on its edges to that of $\mathbb{P}\#\mathbb{K}$, the only difference being that all arrows are pointing in the opposite direction. But since those identifications give the same closed surface (intuitively, one hexagon can be turned up side down to get the other), the statement is proven.

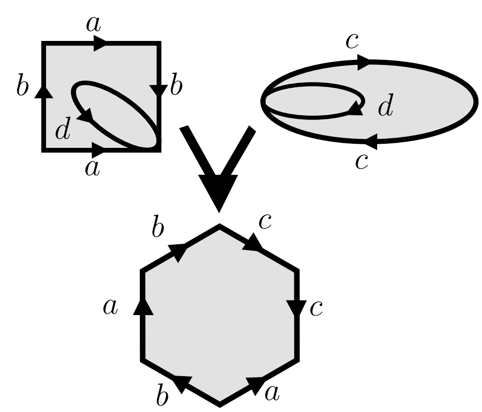

Figure 12: Connected sum of a Klein bottle and a projective plane.

□

A more detailed description on how to read of the identifications in the above figures is given at the start of the last section on polygonal representations. Recalling the definition of a monoid as a group where we do not

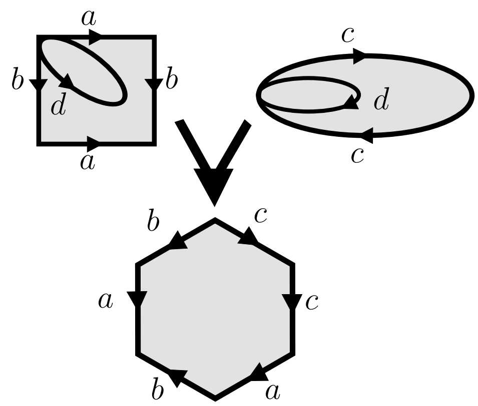

Figure 13: Connected sum of a torus and a projective plane.

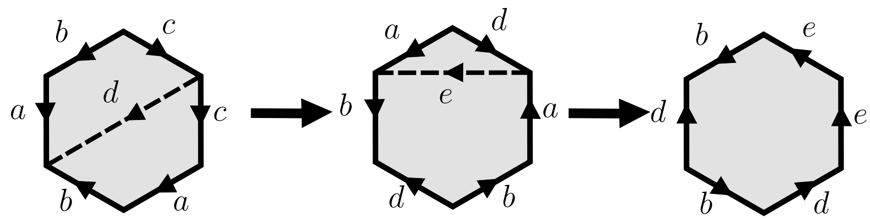

Figure 14: The cut and paste needed to go from $\mathbb{P}\#\mathbb{T}$ to $\mathbb{P}\#\mathbb{K}$.

require elements to have inverses, we can state another form of the classification theorem.

Theorem 4.12. *The set of closed surfaces forms a commutative monoid under taking connected sum with the sphere as identity element. Furthermore, it has presentation*

$$\langle \mathbb{T}, \mathbb{P} | \mathbb{P}\#\mathbb{P}\#\mathbb{P} = \mathbb{P}\#\mathbb{T} \rangle .$$

4.2 Combinatorial surfaces

A key fact for our proof of the classification theorem is the following theorem which allows our to utilize our knowledge about simplicial complexes.

Theorem 4.13. *Any closed surface can be triangulated.*

This will not be proven here but a proof can be found in [5]. Essential for that proof is a generalization of the classical Jordan-curve theorem.

Theorem 4.14 (The Jordan-Schönflies theorem)**.** *Any homeomorphism from a simple closed curve in* $\mathbb{R}^2$ *onto another can be extended to a homeomorphism of the whole plane.*

In particular, any simple closed curve in $\mathbb{R}^2$ bounds a region homeomorphic to $\overline{\mathbb{D}^2}$. A proof of this can in turn be found in [9]. For completeness, we also state the usual Jordan curve theorem.

Theorem 4.15 (The Jordan curve theorem)**.** *Any simple closed curve in* $\mathbb{R}^2$ *separates it.*

We note that the Jordan curve theorem works with $\mathbb{R}^2$ replaced with $\mathbb{S}^2$ as well. For a proof of both cases for the Jordan curve theorem, see [7].

Since any closed surface can be triangulated, we can turn our focus to the simplicial complex that triangulates it rather then the surface itself. We therefore wish to obtain knowledge about the structure of simplicial complexes that can triangulate a closed surface. A such example is what is called a combinatorial surface.

Definition 4.16. A *combinatorial surface* is a simplicial complex K with the properties that it has dimension 2, any two vertices can be connected by an edge path, each edge is a face of exactly two triangles and each vertex can be seen the apex of a simple polygonal curve in K.

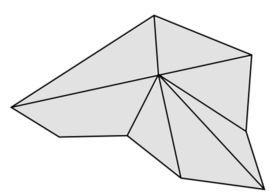

Figure 15: Structure around a vertex in a combinatorial surface.

The interesting part of combinatorial surfaces is that any complex triangulating a closed surface is indeed a combinatorial surface.

Theorem 4.17. *If* $h : S \to |K|$ *is a triangulation of the closed surface* S*, then the complex* K *is a combinatorial surface.*

Proof. **Connectedness:** The connectedness property follow from that any maximal tree in K must contain all the vertices of K given that $|K|$ is connected.

Number of triangles: We then turn to the number of triangles having a specific edge as its face. Let $h : |K| \to S$ be a homeomorphism and σ a 1-simplex in K. Assume first that σ is not the face of any triangle in K. Pick x in the interior of σ, then there exists a neighbourhood U of $h(x)$ and a homeomorphism $g : U \to \mathbb{D}^2$ with $gh(x) = 0$. Since $V = h^{-1}(U)$ is a neighbourhood of x, we get that $V \setminus \{x\}$ is homeomorphic to $\mathbb{D}^2 \setminus \{0\}$ where only the last set is connected, a contradiction. If σ is the face of exactly one triangle in K, then any point x in the interior of σ will clearly have a neighbourhood homeomorphic to $\mathbb{D}^2_+$ and thus also $h(x)$, contradicting that the interior and boundary of a compact surface are disjoint. Now assume that $n \geq 3$ triangles have σ as a face. Pick x in the interior of σ and consider the embedding of those triangles in $\mathbb{R}^3$ with x mapped to the origin. Call the space T_n. Pick a neighbourhood U of $h(x)$ and a homeomorphism $g : U \to \mathbb{D}^2$ with $gh(x) = 0$. By replacing $\mathbb{D}^2$ with discs of smaller radius if necessary, we may assume that $h^{-1}g^{-1}(\mathbb{D}^2)$ is a neighbourhood of 0 in T_n. We can then construct a space T'_n homeomorphic to T_n but with triangles of smaller side lengths such that $T'_n \subseteq h^{-1}g^{-1}(\mathbb{D}^2)$. The radial projection $r : h^{-1}g^{-1}(\mathbb{D}^2) \setminus \{0\} \to \partial T'_n$ will then induce a surjective homeomorphism $r_* : H_1(h^{-1}g^{-1}(\mathbb{D}^2) \setminus \{0\}) \to H_1(\partial T'_n)$. Now $h^{-1}g^{-1}(\mathbb{D}^2) \setminus \{0\}$ is homeomorphic to $\mathbb{D}^2 \setminus \{0\}$ which deformation retracts onto its boundary $\mathbb{S}^1$. Hence $H_1(h^{-1}g^{-1}(\mathbb{D}^2) \setminus \{0\}) = \mathbb{Z}$. But $\partial T'_n$ is a graph with $2n$ edges and $n + 2$ vertices and thus $H_1(\partial T'_n) = \mathbb{Z}^{n-1}$ by example 3.12, contradicting the surjectivity of r_*. Thus we have shown that any edge is a face of exactly two triangles.

Dimension: Next, to show K has dimension 2, we assume K contains a simplex σ of dimension $n \geq 3$. As usual, pick x in the interior of σ and a neighbourhood U of $h(x)$ such that $g : U \to \mathbb{D}^2$ with $hg(x) = 0$ is a homeomorphism. But we can also pick a neighbourhood V of x and a homeomorphism $f : V \to \mathbb{D}^n$ such that $f(x) = 0$. Let $\phi = ghf^{-1} : \mathbb{D}^n \to \mathbb{D}^2$ and pick a sphere $D^n_1 \subset \mathbb{D}^n$ such that $hf^{-1}(D^n_1) \subset U$. Then $\phi(D^n_1)$ is a neighbourhood of the origin in $\mathbb{D}^2$. Let D^2_2 be a disc such that $D^2_2 \subseteq \phi(D^n_1)$. The radial projection will then induce a surjective homeomorphism $r_* : \pi_1(\phi(D^n_1)\setminus\{0\}) \to \pi_1(\partial D^2_2)$. Now $\phi(D^n_1)\setminus\{0\}$ is homeomorphic to $D^n_1\setminus\{0\}$ which deformation retracts onto its boundary that is homeomorphic to $\mathbb{S}^{n-1}$. Since $n > 2$, $\pi_1(\mathbb{S}^{n-1}) = 0$. But $\pi_1(\partial D^2_2) = \pi_1(\mathbb{S}^1) = \mathbb{Z}$, contradicting the surjectivity of r_*.

Vertex structure: Lastly, for the fact that each vertex can be seen as the apex of a cone with polygonal base, we note that the other properties implies that any triangle having that vertex as a face must be a part of such a cone. The only thing that needs to be ruled out is the existence of several such layers of triangles having that vertex as a cone. But this follows from a connectedness argument almost verbatim as in the proof that any edge must be a face of a triangle. □

The first property of closed surfaces that we would like to translate onto a property of combinatorial surfaces is that of orientability. We remember that a surface is said to be orientable if there can be defined a continuously varying unit normal on the surface. For example, a sphere and a torus is orientable but a Möbius strip is not. Since the Möbius strip is non orientable, it follows that any surface containing a subset homeomorphic to it is also non orientable. Hence the projective plane and the Klein bottle are non orientable closed surfaces. For combinatorial surfaces, we note that the orientation of a single simplex was defined at the start of the chapter on homology.

Definition 4.18. A combinatorial surface K is said to be *orientable* if it is possible to define an orientation on all the triangles in K such that any two triangles intersecting at an edge will induce opposite orientation on that edge. Otherwise, the combinatorial surface is called *non orientable*.

The reader is encouraged to consider any combinatorial surface that can triangulate the sphere and see that it will be orientable. Similarly, it will not be possible to find such a triangulation for the Möbius strip. We will show that any complex triangulating an orientable surface must be orientable. But we need another definition at first that will be used several times later.

Definition 4.19. Let K be a combinatorial surface and L a 1-dimensional subcomplex. By *thickening* L, we mean taking the polyhedron of the subcomplex of K^2 consisting of all simplexes that meet L together with their faces.

Theorem 4.20. *Let $h : |K| \to S$ be a triangulation of the orientable surface S. Then the simplicial complex K is orientable.*

Proof. Assume that K is not orientable. Then there exists a sequence of distinct oriented 2-simplexes $\sigma_1, \ldots, \sigma_k$ such that σ_i shares an edge with σ_{i+1} for $1 \leq i \leq k-1$ and have compatible orientations but σ_k share an edge with σ_1, not having compatible orientations. Add simplexes to K in form of the barycetres of the σ_i's, the barycentres of their edges of intersection and the edges from $\hat{\sigma}_i$ to those intersection barycentres where the edge is a face of σ_i. Those edges forms an elementary 1-cycle L in K when viewed as a 1-chain. Since all pairs of 2-simplexes are pairwise compatible except for one pair, thickening L gives a strip homeomorphic to a Möbius strip in $|K|$, contradicting that S is orientable. □

We now show two other results on thickening 1-dimensional complexes. But first, we need some preliminaries.

Lemma 4.21. *Let A be homeomorphic to the unit disc $\overline{\mathbb{D}^2}$. Then any homeomorphism $g : \partial A \to \partial A$ can be extended to a homeomorphism on the whole of A.*

Proof. Let $h : A \to \overline{\mathbb{D}^2}$ be a homeomorphism. Then we get another homeomorphism $hgh^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$ that can be extended to a function $f : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ as follows. Let $f(0) = 0$ and $f(x) = \|x\| hgh^{-1}(x/\|x\|)$ for $x \in \overline{\mathbb{D}^2} \setminus \{0\}$. Then $h^{-1}fh : A \to A$ will be the sought extension of g. □

Lemma 4.22. *Let A and B both be homeomorphic to $\overline{\mathbb{D}^2}$ which intersect along their boundaries in an arc. Then $A \cup B$ is also homeomorphic to $\overline{\mathbb{D}^2}$.*

Proof. Let $\gamma = A \cap B$, $\alpha = \partial A \setminus A \cap B$ and $\beta = \partial B \setminus A \cap B$. View $\overline{\mathbb{D}^2}$ as the union of the two half-discs $D_- = \{(x, y) \in \overline{\mathbb{D}^2} | x \leq 0\}$ and $D_+ = \{(x, y) \in \overline{\mathbb{D}^2} | x \geq 0\}$ having boundaries $\gamma' = D_- \cap D_+$, $\alpha' = \partial D_- \setminus (D_- \cap D_+)$ and $\beta' = \partial D_+ \setminus (D_- \cap D_+)$. Since both γ and γ' are homeomorphic to $[0, 1]$, γ is homeomorphic to γ'. Similarly, $\overline{\alpha}$ is homeomorphic to $\overline{\alpha'}$. By doing identifications on the end points, we get that $\gamma \cup \overline{\alpha} = \gamma \cup \alpha$ is homeomorphic to $\gamma' \cup \overline{\alpha'} = \gamma' \cup \alpha'$. Call that homeomorphism $g : \partial A \to \partial D_-$. Let $h_1 : D_- \to \overline{\mathbb{D}^2}$ and $h_2 : A \to \overline{\mathbb{D}^2}$ be homeomorphisms. Then $f = g \circ h_1 \circ h_2^{-1} : \partial A \to \partial A$ can be extended to a homeomorphism $\tilde{f} : A \to A$ by Lemma 4.21. Thus $F = h_1^{-1} \circ h_2 \circ \tilde{f} : A \to D_-$ is a homeomorphism of A and D_- sending γ to γ'. Similarly we get a homeomorphism $G : B \to D_+$ sending γ to γ'. Using the gluing lemma we get a homeomorphism from $A \cup B$ to $D_- \cup D_+ = \overline{\mathbb{D}^2}$ as desired. □

Now, only as small graph theoretical lemma is needed before we can turn to the thickening complexes again.

Lemma 4.23. *Any tree contains a vertex with exactly one edge connected to it.*

Proof. Let T be a tree and assume every vertex is connected to at least two edges. Since a tree has a unique path between all its vertices and there are only finitely many vertices, there exits a path between two vertices containing a maximum number of edges. Call this path $P = v_0, v_1 \ldots, v_m$. Let $w \neq v_{m-1}$ be a vertex connected by an edge to v_m. If $w \in P$, this contradicts the uniqueness of path in a tree. If $w \notin P$, then $v_0, v_1, \ldots, v_m, w$ will be a path in T containing more edges than P, contradicting the maximality. Hence v_m can have only one edge connected to it. □

Definition 4.24. Let K be a simplicial complex and v a simplex in K. The *closed star* of v in K, denoted $\overline{\text{star}}(v, K)$ is the union of all simplexes in K having v as a vertex.

Lemma 4.25. *Thickening a tree gives a disc.*

Proof. Let K be a simplicial complex and the subcomplex T a tree. If T consists of a single vertex v, thickening T gives $\overline{\text{star}}(v, K^2)$. This, being a region in $\mathbb{R}^2$ enclosed by a simple closed polygonal curve, is homeomorphic to $\mathbb{D}^2$. Now let T be a tree having n vertices and assume that thickening a tree of fewer vertices gives a disc. Choose a vertex v in T only belonging to one edge E of T. Then $T_1 = T \setminus \text{star}(v, T)$ is a tree with $n-1$ vertices, thus thickening T_1 gives a disc D. Thickening T gives D, union with $A = \overline{\text{star}}(\hat{E}, K^2)$ and $B = \overline{\text{star}}(v, K^2)$. Now A, B, D are all homeomorphic to $\mathbb{D}^2$. Since $A \cap D$ is an arc and $A \cap B$ is an arc, two applications of Lemma 4.22 shows $A \cup B \cup D$ is homeomorphic to $\mathbb{D}^2$. □

Theorem 4.26. *Thickening a simple closed polygonal curve gives either a cylinder or a Möbius strip.*

Proof. Let K be a simplicial complex. Let C be a simple closed polygonal curve in K and E an edge in C. Then $T = C \setminus \text{star}(\hat{E}, C)$ is a tree and therefore thickening T gives a disc D by Lemma 4.25. Thickening C gives the union of D and $\overline{\text{star}}(\hat{E}, K^2)$, intersecting in two disjoint arcs. Gluing them together in one of these arcs give a disc by Lemma 4.22. Thus we get a disc where we need to identify two disjoint arcs. This is in turn homeomorphic to a rectangle with the two arcs of the disc mapped to opposite sides. Gluing these sides together can be done in two different ways, either resulting in a cylinder or a Möbius strip. □

Thickening of complexes will be one of the tools in our main part of the proof of the classification theorem later.

4.3 Euler characteristic

One of our main tool for relating a general closed surface to our standard surfaces will be via their Euler characteristic.

Definition 4.27. Let K be an n-dimensional simplicial complex. The *Euler characteristic* of K is defined as the number

$$\chi(K) = \sum_{i=0}^{n} (-1)^i \alpha_i$$

where α_i is the number of i-simplexes in K.

Thus for a graph, its Euler characteristic equals its number of vertices minus its number of edges and for a combinatorial surface, its number of vertices, minus its number of edges, plus its number of faces. When explicitly calculating the Euler characteristic, we will make use of the following relations.

Lemma 4.28. *Let $K \cup L$ be a simplicial complex obtained as the union of the complexes K and L. Then $\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$.*

Proof. This is a direct consequence of the inclusion-exclusion principle applied to the number of simplexes of each dimension. □

The next relation is not as easily proven.

Theorem 4.29. *The Euler characteristic is invariant under barycentric subdivision. I.e. $\chi(K) = \chi(K^1)$ for any simplicial complex K.*

Before this can be shown, we need a better way of controlling the barycentric subdivision.

Definition 4.30. Let K be a simplicial complex and σ, τ simplexes in K. If $\sigma < \tau$, let L be the subcomplex of the boundary of τ that do not have σ as a face. The *stellar subdivision* of σ is then obtained by replacing τ with the cone with base L and apex $\hat{\sigma}$.

Lemma 4.31. *Applying stellar subdivision to a simplex and its faces finitely many times gives the barycentric subdivision of the simplex.*

Proof. The statement trivially holds for a 0-simplex. Let σ be an n-simplex and assume that the statement holds for simplexes of dimension $< n$. The first stellar subdivision of σ creates edges from all the vertices of σ to $\hat{\sigma}$. By the induction assumption, all faces of σ will after finitely many subdivisions be barycentrically subdivided. The only difference made by the fact that they are faces of σ is that an edge will be created from the barycentre of each face to $\hat{\sigma}$. Thus obtaining σ^1. □

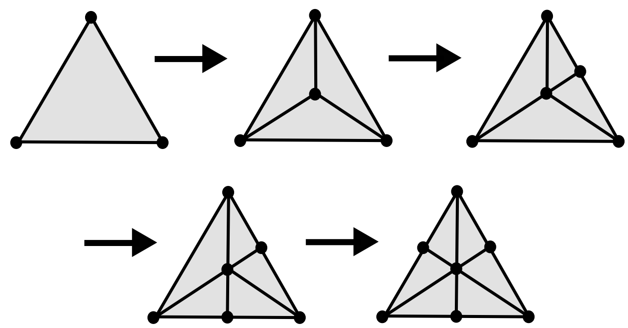

Figure 16: Stellar subdivision on a 2-simplex.

Lemma 4.32. *The Euler characteristic of a simplex is invariant under barycentric subdivision.*

Proof. Again, it holds trivially for a 0-simplex. Let σ be an $(n-1)$-simplex and assume it hold for all simplexes of dimension $< (n-1)$. Doing the first stellar subdivision of σ creates $\binom{n}{0}$ new vertices, $\binom{n}{1}$ new edges, $\binom{n}{2}$ new 2-simplexes, etc. up to $\binom{n}{n-1}$ new $(n-1)$-simplexes. All faces of σ are unaltered, its just the interior of σ that is changed. Therefore the Euler characteristic of σ is changed by

$$\begin{aligned}\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1}\left(\binom{n}{n-1} - 1\right) \\ &= (1-1)^n - (-1)^n + (-1)^n \\ &= (-1)^n(1-1) \\ &= 0,\end{aligned}$$

using the binomial theorem. Now let τ be a face of σ of dimension $(k-1)$. Doing the stellar subdivision on τ does not change its Euler characteristic by assumption. It does however alter an old k-simplex by creating $\binom{k}{0}$ new edges, $\binom{k}{1}$ new 2-simplexes etc. up to $\binom{k}{k-1}$ new k-simplexes. Hence the total Euler characteristic is changed by

$$
\begin{aligned}
&-\binom{k}{0}+\binom{k}{1}-\binom{n}{2}+\cdots+(-1)^k\left(\binom{k}{k-1}-1\right)\\
&=-\left(\binom{k}{0}-\binom{k}{1}+\cdots+(-1)^{k-1}\binom{k}{k-1}\right)+(-1)^{k+1}\\
&=(1-1)^k-(-1)^k+(-1)^{k+1}\\
&=(-1)^k(1-1)\\
&=0.
\end{aligned}
$$

Since any stellar subdivision does not change the Euler characteristic, the barycentric subdivision does not change it either by Lemma 4.31. □

Proof of Theorem 4.29. If the complex K consists of a single simplex, then the result follows from Lemma 4.32. Now assume K consists of n simplexes and that the statement holds for a collection of $< n$ simplexes. Pick a simplex L of maximum dimension in K and write $K = J \cup L$ for some simplicial complex J consisting of $n-1$ simplexes. Using the assumption and Lemma 4.28, we get

$$
\begin{aligned}
\chi(K) &= \chi(J)+\chi(L)-\chi(J\cap L)\\
&= \chi(J^1)+\chi(L^1)-\chi(J^1\cap L^1)\\
&= \chi(J^1\cup L^1)\\
&= \chi(K^1).
\end{aligned}
$$

□

We now turn to calculating the Euler characteristic for certain classes of complexes.

Lemma 4.33. *Let G be a connected graph. Then $\chi(G) \leq 1$ with equality if and only if G is a tree.*

Proof. If G is a tree, then G has one more vertex than edge, giving $\chi(G) = 1$. If G is a general connected graph, we let L be a spanning graph of G. By Lemma 2.24, L contains all the vertices of G. Hence L can be obtained from G by removing a finite number n of edges. Thus $\chi(G) = \chi(L) - n \leq 1$. □

We would like a similar theorem for combinatorial surfaces, but we need a little more work for that.

Definition 4.34. Let K be a combinatorial surface and T a spanning tree. The *dual graph* of T, denoted T^*, is a 1-dimensional complex with vertices at the barycentres of 2-simplexes in K. Two vertices in T^* determine an edge in T^* if the corresponding triangles in K intersect at an edge not in T.

Lemma 4.35. *Let K be a combinatorial surface, T a spanning tree and $N(T)$ the thickening of T. Then*

a) $N(T) \cup N(T^*) = |K|$,

b) $N(T) \cap N(T^*) = \partial N(T)$,

c) T^* *is a connected graph.*

Proof. To prove a), we note that since $|K| = |K^2|$ we have the inclusion $N(T) \cup N(T^*) \subseteq |K|$. For $x \in |K|$, let L be the carrier of x in K^2. If L is a vertex of K, then $x \in T$. If L is a part of an edge of K, then either its an edge of T so $x \in T$, or its a part of an edge of K not in T, giving that T^* goes through that edge. Hence in K^2, that edge has four parts where the outer parts are in $N(T)$ since they meet vertices in T, and the inner two parts are in $N(T^*)$ since an edge of T^* goes through there. In any case $x \in N(T) \cup N(T^*)$. If L is a part of a triangle of K, then either that triangle meets an edge of K or the barycentre of that triangle. In the first case, if the edge is in T, then $|L| \subseteq N(T) \cup N(T^*)$. If not, then either L meets T at a vertex of K or it must share an edge with T^*, implying $x \in N(T) \cup N(T^*)$. Finally, if L meets the barycentre of the triangle, then since the barycentre is in T^*, $x \in N(T) \cup N(T^*)$ and hence a) is proven.
For b), we know that K does not have any boundary, hence $\partial N(T)$ must be glued together with parts of $N(T^*)$. In other words, $\partial N(T) \subseteq N(T) \cap N(T^*)$. But from the construction of T and T^*, they cannot intersect in any larger set, hence $N(T) \cup N(T^*) = \partial N(T)$.
Lastly, for c), we remember from Lemma 4.25 that $N(T)$ is homeomorphic to $\overline{\mathbb{D}^2}$ and hence $\partial N(T)$ to $\mathbb{S}^1$. We first show $N(T^*)$ is connected. Pick $x, y \in N(T^*)$ and a path in $|K|$ from x to y. If that path is entirely in $N(T^*)$ we are done. Else, there exists a first point p and a last point q

in $N(T) \cap N(T^*)$ where that path goes between $N(T)$ and $N(T^*)$. Since $N(T) \cap N(T^*)$ is homeomorphic to $\mathbb{S}^1$, there exits a path in $N(T) \cap N(T^*)$ from p to q. Hence going from x to p, then via that path to q and then on to y, gives a path in $N(T^*)$ from x to y. Therefore $N(T^*)$ is connected, which gives that T^* is connected. $\square$

Lemma 4.36. *Let K be a combinatorial surface. Then $\chi(K) \leq 2$.*

Proof. Pick a spanning tree T of K and form its dual graph T^*. We have that $\chi(K) = \chi(T) + \chi(T^*)$ since $\chi(T)$ counts positively all the vertices of K, any edge in K is either in T and hence counted negatively in $\chi(T)$, or its not, giving an edge in $\chi(T^*)$ that is counted negatively. Lastly, for every 2-simplex of K, there is a vertex of T^* which is counted positively for $\chi(T^*)$. Hence by Lemma 4.33, $\chi(K) = \chi(T) + \chi(T^*) \leq 1 + 1 = 2$. $\square$

The next theorem is the main result of this section. It will allow us to go between just knowing the Euler characteristic of a combinatorial surface to knowing the topological structure of that surface.

Theorem 4.37. *Let K be a combinatorial surface. Then the following statements are equivalent.*

a) $\chi(K) = 2$,

b) $|K|$ *is homeomorphic to* $\mathbb{S}^2$.

c) Any simple closed curve in $|K|$ *consisting of edges in* K^1 *separates* $|K|$.

Proof. Assume that $\chi(K) = 2$ and write $\chi(K) = \chi(T) + \chi(T^*)$ where T is a spanning tree for K. Since $\chi(T) = 1$ by Lemma 4.33, this gives $\chi(T^*) = 1$ which again by the same lemma gives that T^* is also a tree. Then, by Lemma 4.25, $N(T)$ and $N(T^*)$ are both homeomorphic to $\overline{\mathbb{D}^2}$. Hence since $|K| = N(T) \cup N(T^*)$ and $N(T) \cap N(T^*) = \partial N(T)$, $|K|$ is obtained as the union of two discs, glued together along their boundary circles, which is homeomorphic to $\mathbb{S}^2$ by Lemma 4.22.
For *b)* $\implies$ *c)*, we note that its just a variant of the Jordan curve theorem as mentioned before.
Lastly, we assume that any simple closed curve in $|K|$ consisting of edges in K^1 separate $|K|$ into two connected components. Suppose that T^* is not a

tree. Then T^* contains a loop in K^1 which by assumption separates $|K|$. But each such component must contain vertices of T, contradicting that T is a tree. Hence T^* must be a tree and by Lemma 4.33, $\chi(K) = \chi(T) + \chi(T^*) = 2$. □

4.4 Surgery

To show that any surface is homeomorphic to one of the standard surfaces, we will use a method of doing so called surgery on the surface. The method will increase the Euler characteristic of the surface and after a finite number of surgeries, we can show that the space will homeomorphic to $\mathbb{S}^2$. Retracing our surgeries, we can then show that the original surface is homeomorphic to a standard surface. In detail, the method goes as follows.

Let K be a combinatorial surface in $\mathbb{R}^n$ and let L be a 1-dimensional subcomplex of K that do not separate K. If no such L exists, then by Theorem 4.37, $|K|$ is homeomorphic to $\mathbb{S}^2$ and we are done. Note that L might be a subcomplex of K^1 and not K, but to simplify notation, we consider K^1 as our new complex K. This will give us no problems since they have the same polyhedron and Euler characteristic. Now, thicken L to obtain the polyhedron $N(L)$ and call the underlying complex for N_L. Furthermore, denote the complement of the interior of N_L in K^2 for M_L. Then M_L can be seen as all simplexes in K^2 that do not meet L together with their faces. By Theorem 4.26, $N(L)$ is homeomorphic to either a cylinder or a Möbius strip. If $N(L)$ is homeomorphic to a cylinder, then $|M_L|$ will be a compact surface with boundary consisting of two disjoint circles, call the complexes that triangulate these circles by L_1 and L_2 respectively. Taking the cone on L_1 and L_2, we get a new closed surface

$$K_* = M_L \cup CL_1 \cup CL_2.$$

If $N(L)$ is homeomorphic to the Möbius strip, then $|M_L|$ would have only one circle as boundary whose triangulating complex we denote by L_1. Taking the cone on L_1, we get the closed surface

$$K_* = M_L \cup CL_1.$$

In any case, the combinatorial surface K_* is called the surface obtained from K by doing surgery along L. We will now see what we can say about the resulting complexes and surfaces obtained by this method.

Lemma 4.38. $\chi(N_L) = 0$

Proof. By the proofs of Lemma 4.25 and Theorem 4.26, we know that N_L consists of unions of closed stars of the form $\overline{\text{star}}(v, K^2)$ for $v \in L^1$. Since K is a combinatorial surface, $\overline{\text{star}}(v, K^2)$ has the structure of a cone with vertex v and a simple closed polygonal curve as base. Assume that the curve has n vertices. Then $\overline{\text{star}}(v, K^2)$ has $n+1$ vertices, $2n$ edges (one from each vertex of the curve to the next vertex of the curve and one from each vertex of the curve to v) and n triangles. Using Lemma 4.28, we get $\chi(\overline{\text{star}}(v, K^2)) = n + 1 - 2n + n = 1$. This closed star intersects another closed star in exactly three vertices and two edges, making their total Euler characteristic having value $1 + 1 - (3 - 2) = 1$. Joining another closed star to that complex, the intersection will still be three vertices and two edges, keeping the total Euler characteristic at 1. But when we will join the last closed star to the complex, it will intersect it at two different closed stars, at a total intersection of six vertices and four edges. Hence $\chi(N_L) = 1 + 1 - (6 - 4) = 0$. □

The main property of surgery is the following relation.

Theorem 4.39. $\chi(K_*) > \chi(K)$

Proof. Let L_1 be a simplicial complex triangulating a circle. Then L_1 contains no 2-simplexes and has equally many vertices as edges, hence $\chi(L_1) = 0$. When taking the cone on L_1, we get a simplex of the same type as a closed star in a combinatorial surface. From the proof of Lemma 4.38, we know that $\chi(CL_1) = 1$. Turning to our combinatorial surface K, if $N(L)$ is a cylinder, we get by Lemma 4.28 that

$$\begin{aligned}\chi(K_*) &= \chi(M_L) + \chi(CL_1) + \chi(CL_2) - \chi(L_1) - \chi(L_2) \\ &= \chi(M_L) + 2.\end{aligned}$$

If $N(L)$ is a Möbius strip, then

$$\begin{aligned}\chi(K_*) &= \chi(M_L) + \chi(CL_1) - \chi(L_1) \\ &= \chi(M_L) + 1.\end{aligned}$$

In any case, $\chi(K_*) > \chi(M_L)$. Finally, by Theorem 4.29 and Lemma 4.38, we have

$$\chi(K) = \chi(K^2) = \chi(M_L) + \chi(N_L) - \chi(M_L \cap N_L) = \chi(M_L)$$

since $M_L \cap N_L$ is either one or two circles and thus having Euler characteristic zero. Hence $\chi(K_*) > \chi(M_L) = \chi(K)$. □

We would now like to apply surgery once again, this time on K_*. The problem is that we need the discs CL_i to be intact through the process. Hence if the polygonal curve Γ do not separate K_* and Γ intersects CL_i, we will need to carefully move those apart without moving Γ onto any other CL_i. For this, we need several lemmas.

Lemma 4.40. *Let $Z \subset Y \subset X$ be concentric discs in $\mathbb{R}^2$. Then there exists a homeomorphism $h : X \to X$ such that $h : \partial X \to X = i_{\partial X}$ and $h(Y) \subseteq Z$.*

Proof. We may assume w.l.o.g. that they are centred at the origin. We can also identify $\mathbb{R}^2$ with $\mathbb{C}$. Let Z have radius r_1, Y radius r_2 and X radius r_3. A homeomorphism h is then given explicitly by

$$h(\theta, r) = \begin{cases} e^{i\theta} \cdot \dfrac{rr_1}{r_2} & r \in [0, r_2] \\ e^{i\theta} \cdot \left(\dfrac{r(r_1 - r_3) - r_3(r_1 - r_2)}{r_2 - r_3} \right) & r \in [r_2, r_3]. \end{cases}$$

Since h is a continuous bijection from a Hausdorff space to a compact space, h is indeed a homeomorphism. □

Lemma 4.41. *Let C_1 and C_2 be two simple polygonal curves in $\mathbb{R}^2$ where C_2 is included in the interior of the region bounded by C_1. The region between C_1 and C_2 is then homeomorphic to an annulus.*

Proof. Pick a vertex of C_1 and connect it via a straight edge to a vertex of C_2. Then pick another vertex of C_1 and connect it via a simple polygonal curve to another vertex of C_2 such that it does not intersect the first edge created or intersect C_1 or C_2 except at the end points. This divides the region between C_1 and C_2 into two pieces, each with boundary a simple polygonal curve and hence homeomorphic to $\overline{\mathbb{D}^2}$. Keeping track of where the edges not in C_1 and C_2 are mapped to, we get two discs where on each disc, two disjoint arcs are corresponding to those edges. To get the sought region, we must identify the corresponding arcs on the two discs. Before this is done, we can map the discs to rectangles where the special arc are mapped to opposite edges. After that, gluing the rectangles together clearly gives a region homeomorphic to an annulus. See Figure 17. □

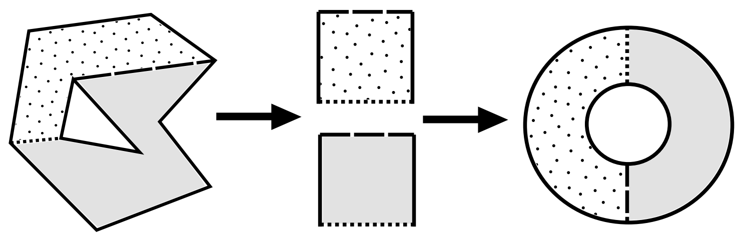

Figure 17: Idea for Lemma 4.41.

Lemma 4.42. *Let K be combinatorial surface, D a subcomplex homeomorphic to $\overline{\mathbb{D}^2}$ and σ a 2-simplex in D. Then there is a homeomorphism $h : |K| \to |K|$ such that $h(D) = \overline{\text{star}}(\hat{\sigma}, K^2)$ and $h = i_{|K|}$ on any simplex that do not meet D.*

Proof. We begin by thickening ∂D. The resulting complex will then have a boundary consisting of two disjoint simple polygonal curves in K^2. The curve that is not in D can be seen as the boundary ∂D_1 of a complex D_1 with D_1 containing D in its interior and homeomorphic to $\overline{\mathbb{D}^2}$. By two applications of Lemma 4.41, first on ∂D_1 and ∂D and then on ∂D and $\partial\left(\overline{\text{star}}(\hat{\sigma}, K^2)\right)$, we can see the situation as three concentric discs, $\overline{\text{star}}(\hat{\sigma}, K^2) \subset D \subset D_1$. By Lemma 4.40, there exists a homeomorphism $h : |D_1| \to |D_1|$ such that $h(D) = \overline{\text{star}}(\hat{\sigma}, K^2)$ and $h : \partial D_1 \to D_1 = i_{\partial D_1}$. By extending h to be the identity on $|K| \setminus |D_1|$, h will be a continuous bijection on $|K|$ and since $|K|$ is compact and Hausdorff, also an homeomorphism. □

By the above lemma, we know that if Γ happens to be a curve that do not separate K_* but goes through some disc CL_i, then we can refine K_* to K_*^2 and then replace CL_i with $\overline{\text{star}}(\hat{\sigma}, K_*^2)$. Since $\overline{\text{star}}(\hat{\sigma}, K_*^2)$ lies in the interior of the 2-simplex σ, we have $\Gamma \cap \overline{\text{star}}(\hat{\sigma}, K_*^2) = \emptyset$ and we can continue the surgery. We can now prove one part of the classification theorem.

Theorem 4.43. *Any closed surface is homeomorphic to a standard surface.*

Proof. Let S be a closed surface and K be a combinatorial surface triangulating S. If $\chi(K) = 2$, then $|K|$ is homeomorphic to $\mathbb{S}^2$ and we are done. Else, $\chi(K) < 2$ and there is a 1-dimensional subcomplex L of K^1 that do not separate K. Doing surgery along L, we get a new combinatorial surface with strictly higher Euler characteristic. If necessary, we do surgery on that surface as well and repeat the process until we arrive at a surface with Euler characteristic two and hence homeomorphic to $\mathbb{S}^2$. This sphere will have a number of disjoint discs marked on it from the surgeries. We now retrace the applied surgeries by sewing in a cylinder on a pair of discs if $N(L)$ was homeomorphic to a cylinder or sewing in a Möbius strip to a single disc if $N(L)$ was homeomorphic to a Möbius strip. If S is orientable, then we cannot get any Möbius strips and hence S is homeomorphic to a sphere with a finite number of handles added. If S is non-orientable, then we could get both Möbius strips and handles. But by Lemma 4.10, this is the same as sewing in only several Möbius strips instead. Hence S is homeomorphic to a sphere with a finite number of discs removed and replaced by Möbius strips. □

The first complete and published proof that any closed surface is homeomorphic to a standard surface was done by Brahana in his dissertation [4], published in 1921. This was done using more of the cut and paste technique used in the proof of 4.10. For more history about the classification theorem, see [6].

4.5 Polygonal representation

To complete the classification theorem, we need to show that none of our standard surfaces is homeomorphic to another. For this, we need to be able to describe any of our standard surfaces as a polygon in the plane with different sides identified. This will be called the polygonal representation of the surface. For example, in Figure 18, we see a polygonal representation of a torus where we read that the edges with the same letters are to be identified in the direction of the arrows. A way to express this is to write down all the letters as they occur in the polygon when traversing its edges clockwise. Hence for the torus we get $aba^{-1}b^{-1}$ where a^{-1} denotes that the arrow is pointing in the counter clockwise direction at that edge. Such an expression describing the identifications will be called a surface symbol. We note that the sphere has surface symbol aa^{-1} and the projective plane aa.

We would like to determine the surface symbol for the connected sum of n tori and also for n projective planes. To do this, we note that we can always cut up the surface such that the disc intersects the boundary of its polygonal representation in exactly one point, see Figure 19. We show that the surface symbol for the connected sum of n tori is $a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}$ by induction. Since the surface symbol of the torus is $aba^{-1}b^{-1}$, it holds for $n = 1$. Then we assume it holds for n and show it for $n + 1$. By removing a disc

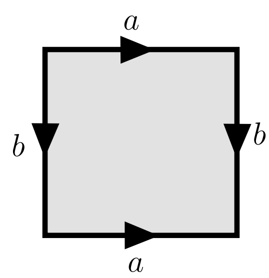

Figure 18: Polygonal representation of a torus.

in the connected sum of n tori and a single torus as indicated in Figure 19, gluing them together gives a polygonal representation as in Figure 20 which has the sought surface symbol.

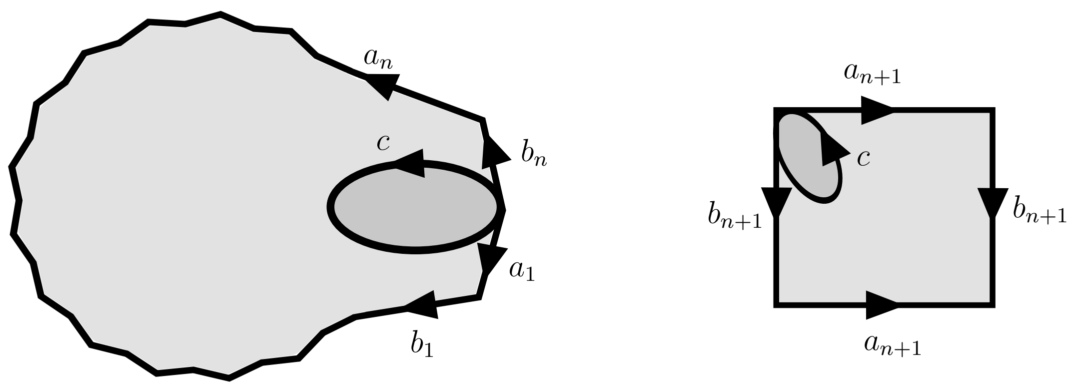

Figure 19: Discs in the connected sum of n tori and a single torus.

By an analogue argument, we find that the surface symbol for the connected sum of n projective planes is $a_1a_1a_2a_2\cdots a_na_n$. We summarize the result of our discussion in a lemma.

Lemma 4.44. *The surface symbol for the connected sum of* n *tori is*

$$\prod_{i=1}^{n} a_ib_ia_i^{-1}b_i^{-1}$$

and the surface symbol for the connected sum of n *projective planes is*

$$\prod_{i=1}^{n} a_i^2.$$

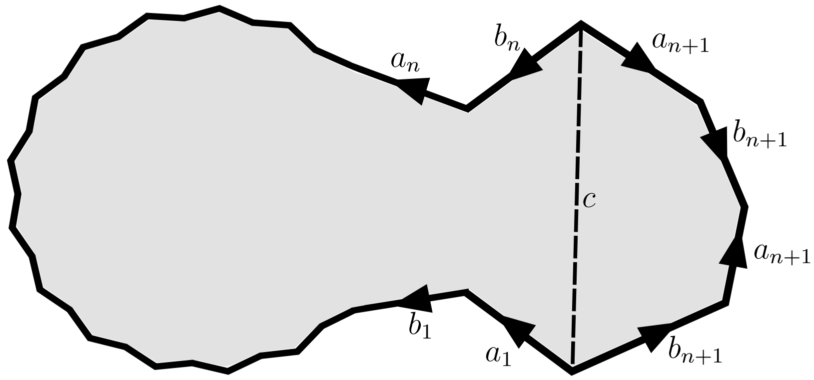

Figure 20: Polygonal representation of $n+1$ tori.

To show that all our standard surfaces are topologically distinct, we will show that they all have non isomorphic first homology group. Since it is a topological invariant, it will imply that the surfaces are not homeomorphic. By Theorem 3.11, this is the same as determining their abelianized fundamental group. For this, we will use the Seifert-van Kampen theorem. At first, we note that since $\mathbb{S}^2$ is simply connected, it has trivial first homology group. Now consider a polygonal representation for the connected sum of n tori and triangulate it. Let J be a 2-simplex and K the complex triangulating that polygon but with the interior of J removed. Then $\pi_1(|J|) = 1$ since $|J|$ is convex. For $|K|$, we see that the space deformation retract onto the boundary of the polygon which becomes a bouquet of $2n$ circles. Hence $\pi_1(|K|)$ is a free group on $2n$ generators, determined by generators a_i, b_i from the surface symbol. Then, we see first that a generator of $|J \cap K|$ in $|J|$ is homotopic to the identity while in $|K|$, it is clearly homotopic to the surface symbol $\prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1}$ after the deformation retraction. Letting $\mathbb{T}_n$ denote the connected sum of n tori, Seifert-van Kampen gives that

$$\pi_1(\mathbb{T}_n) = \left\langle a_1, b_1, \ldots, a_n, b_n \mid \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} = 1 \right\rangle .$$

Taking the quotient by its commutator subgroup, we get the free abelian group on $2n$ generators and no extra relations since $\prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1}$ is already included in the commutator subgroup. Thus $H_1(\mathbb{T}_n) = \mathbb{Z}^{2n}$. For the connected sum of n projective planes that we will denote $\mathbb{P}_n$, the exact same argument applies with triangulating its polygonal representation. The difference this time is that we will get n generators for $\pi_1(|K|)$ and that a generator of $|J \cap K|$ in $|K|$ will now be homotopic to the surface symbol $\prod_{i=1}^{n} a_i^2$. Hence the fundamental group for $\mathbb{P}_n$ has presentation

$$\pi_1(\mathbb{P}_n) = \left\langle a_1, a_2, \ldots, a_n \mid \prod_{i=1}^{n} a_i^2 = 1 \right\rangle.$$

This time, taking the quotient by its commutator subgroup, we get a free abelian group with generators $a_1, a_2, \ldots, a_n$ subject to the non trivial relation $\prod_{i=1}^{n} a_i^2 = 1$. Changing basis to $(a_1 a_2 \ldots a_n), a_2, \ldots a_n$, we see that $H_1(\mathbb{P}_n) = \mathbb{Z}_2 \times \mathbb{Z}^{n-1}$. Thus, since the connected sum of different number of tori or projective planes have different first betti numbers, they are not homeomorphic. Also, since the first homology group of $\mathbb{P}_n$ for any n has a non zero torsion part while $\mathbb{T}_n$ does not, they cannot be isomorphic and the spaces not homeomorphic. This establishes the last part of the classification theorem. Defining the genus of a orientable surface as the number of tori needed to get a connected sum of tori isomorphic to the surface and similarly for projective planes for the non orientable case, we can summarize as follows.

Theorem 4.45. *The following statements are equivalent for closed surfaces.*

a) The surfaces are homeomorphic.

b) The surfaces have isomorphic first homology group.

c) The surfaces have isomorphic fundamental group.

d) The surfaces have the same genus and are both either orientable or non orientable.

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