Hopfield Model on Incomplete Graphs

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Abstract

We consider the Hopfield model on graphs. Specifically we compare five different incomplete graphs on 4 or 5 vertices's including a cycle, a path and a star. Provided is a proof of the Hamiltonian being monotonically decreasing under asynchronous network dynamics. This result is applied to the treated incomplete graphs to derive exact values for the incremental drop in energy on pattern sizes 2, 4, and an arbitrary m under restriction. Special cases provided includes evaluating the network on a graph as a union of two independent components, and additionally one example using a deterministic dilute variable. Furthermore we study the stability of patterns considering a Hopfield model with synchronous network dynamics for two different incomplete graphs using simulations.

Keywords Neural Network · Hopfield Model · Incomplete Graph

Contents

1 Introduction

The Hopfield model is a model of an artificial neural network with great importance in a multitude of scientific fields. Coupled with graphs this model provides a framework to study the stability of patterns and energy convergence of the network.

While there has been research on different angles of this topic in the past, for instance Jehoshua Bruck's *"On the Convergence Properties of the Hopfield Model"* (1990), Anton Bovier and Véronique Gayrard's paper *"Rigorous Results on the Thermodynamics of the Dilute Hopfield Model"* (1993), and Néstor Parga and Edmund Rolls *Transform-Invariant Recognition by Association in a Recurrent Network* (1998), surprisingly sparse material has been produced on the topic since the early nineties.

We aim to study the Hopfield model on an incomplete or deluted graph G. The main appeal of a Hopfield model on a deluted graph is the idea that delution might enable memorization of a large number of patterns provided less information, the Hopfield model still working as usual.

Delution can be implemented randomly and it is worth mentioning the Erdős-Rényi model for generating random graphs. Here we either delete an edge with probability p in the case of model $G(n, p)$, or we randomly choose a graph with a fixed M number of edges in model $G(n, M)$. Note, that it is also possible to choose the patterns of the network randomly. In particular, choosing the state of a neuron to be updated at random implies stochastic network dynamics.

The storage capacity of a Hopfield network with N number of neurons is usually measured by $\frac{P}{N}$ where P denotes the number of patterns. The capacity is bounded from above by $\frac{1}{4 \ln n}$. However, Hopfield networks can correct some of the errors upon memory retrieval. It was shown by Amit et al. in 1990 that when using Hebbian learning to update the weights the network storage capacity is limited to $0.138N$ number of patterns, if neurons are updated at random. In this thesis it is assumed that we know the incomplete graph G so that the network dynamics are deterministic in nature. The weights are updated using Hebbian learning, and expressed in terms of the patterns.

The main idea is to construct the patterns so that we do not use the edges in the complete graph K_n on n vertices's which are not present in G if such system exist, if not we prove the contrary. We want to find out if the given approach is meaningful and if the Hopfield model works as assumed on graph G. The purpose of this thesis is to analyze the convergence properties of the energy function and the stability of the patterns of a Hopfield model on an incomplete graph G. A final decision can then be made if the Hopfield model on graph G has a rich enough profile.

The thesis is organized as follows. Sect. 2 is devoted to the Hopfield model and the Hopfield model on various incomplete graphs will be treated in Sect. 3. Simulation results for two examples of incomplete graphs is treated in Sect. 4. In Sect. 5 we present a summary of findings and in Sect. 6 conclusions.

2 Hopfield model

We start this section aiming to introduce the Hopfield model, by making a short note on Artificial Neural Networks, which can be viewed as computing systems originally developed to mimic basic biological neural systems [Zhang et al., 1998]. The neuron can be defined as the information processing unit [Haykin, 2009], where the input signals are weighted by connectivity weights and summed up in the input potential, sometimes including bias, and then mapped to an output.

The Hopfield model was introduced by John J. Hopfield in 1982 with the aim to store and retrieve memory similar to how a brain works. The standard procedure is for the network to learn a number of binary patterns and then to return the one most similar to a given initialization pattern. The Hopfield model is a feed-back neural network since its architecture can be described as an undirected graph. Note, that the neurones in a Hopfield model are two-state neurones. A connection or synapse between two neurons is expressed in terms of a synaptic weight or connectivity weight. The network is said to be fully connected if the output of each neuron is connected to all the other neurons, this is equivalent to a Hopfield model on a complete graph. A Hopfield model constitutes a recurrent neural network and a dynamical system in time, to get a clearer view we introduce the following notation presented in next subsection.

2.1 Hopfield Model

2.1.1 State space

For a discrete dynamical system a *state space* can be defined as the set of all possible configurations of such system, and in the Hopfield model the time-dependent states of the neurons constitutes the configurations being used.

Let n be the number of neurons in a Hopfield network and let $\vec{\sigma}$ be an element of the *state space*, defined in the following way

$$\Omega_n = \{\vec{\sigma} = (\sigma_1, ..., \sigma_n) : \sigma_i \in \{-1, +1\}\}, \tag{1}$$

and observe that $|\Omega_n| = 2^n$.

2.1.2 Dynamics in time

The Hopfield model utilizes updates, i.e., changes the states of the neurons in time. The network can be updated synchronously or asynchronously, i.e. the states of all neurons being updated once or each at the time in the latter case.

To update the state of a neuron we will use the input potential, which rely on the network weights which are updated according to the Hebbian learning rule and expressed in terms of the *patterns*, as we explain below.

Given a set $\{\xi^1, ..., \xi^m\} \in \Omega_n$ which we call patterns, define for all $i \neq j$, the *connectivity weights* in the following way

$$w_{ij} = \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_i^{\mu} \xi_j^{\mu}, \quad 1 \leqslant i, j \leqslant n. \tag{2}$$

Then the *deterministic update rule* governing the network dynamics we define as follows. Let $\vec{\sigma}(0) = \vec{\eta}$. The *initial state* $\vec{\eta}$ is often chosen randomly, however in this thesis we will study different choices of $\vec{\eta}$. Then

$$\sigma_i(t+1) = \operatorname{sgn}\left(h_i(t)\right), \quad i = 1, .., n, \tag{3}$$

where

$$h_i(t) = h_i(\sigma(t)) = \sum_j w_{ij} \sigma_j(t), \quad i = 1, ..., n, \tag{4}$$

meaning that the state of neuron i is being updated; we call $h_i(t)$ the *input potential* for neuron i.

Asynchronous network dynamics. In asynchronous network dynamics only one state is being updated at each moment of time, the rest being unchanged. Note that if the state of a neuron k is changed then $\sigma_k(t+1) = -\sigma_k(t)$ since $\sigma_i \in \{+1, -1\}$ for $i = 1, ..., n$. Let $k(t)$ be the index of a neuron updated at time t. Observe, that

$$\sigma_{k(t)}(t+1) \in \{\sigma_{k(t)}(t), -\sigma_{k(t)}(t)\} \tag{5}$$

but for all $j \neq k(t)$

$$\sigma_j(t+1) = \sigma_j(t). \tag{6}$$

We will denote such network dynamics as *asynchronous network dynamics*.

2.1.3 Hamiltonian

A Hopfield network defines an *energy function* for each configuration $\vec{\sigma} \in \Omega_n$. We call this energy function a *Hamiltonian* and we denote the Hamiltonian in the following way

$$H(\vec{\sigma}) := H(\vec{w}, \vec{\sigma}) = -\sum_i \sum_j w_{ij}\sigma_i\sigma_j, \quad 0 \leqslant i, j \leqslant n. \tag{7}$$

The major property of a Hopfield network is that following repeated updating of different neurons the network converges to a local minimum in the energy function. This means that if a state is a local minimum of the energy function it is a *stable state* for the network. We will study the difference in energy

$$\Delta H(t) := H(\vec{\sigma}(t+1)) - H(\vec{\sigma}(t)) \tag{8}$$

following an update of the state of a neuron.

2.1.4 Storing and recalling patterns

A Hopfield network is constructed to be able to store and recall patterns. Intuitively when provided with a new pattern, the network will return one of the stored patterns that agrees most with the new pattern. Denote the m

patterns ξ^μ indexed by $\mu = 1, ..., m$, which we aim for the network to store and recall, in the following way

$$\{\xi^1, ..., \xi^m\} \subset \Omega_n. \tag{9}$$

The network is said to correctly represent a pattern ξ^μ if condition

$$\sigma_i(t) = \sigma_i(t+1) = \xi_i^\mu \tag{10}$$

holds for all neurons $1 \leqslant i \leqslant n$.

The network is initialized in a state $\vec{\sigma}(0) = \vec{\eta}$. Following repeated updating of the states the goal is to converge to a fixed point corresponding the pattern μ most similar to the initial state $\vec{\eta}$. We say that a pattern is retrieved if

$$\sigma_i(t) = \xi_i^\mu \tag{11}$$

for all $i = 1, ..., n$.

2.2 Hopfield Model on Graphs

2.2.1 Undirected Graph

A Hopfield net can be defined in terms of an *undirected graph*, which is a graph in which edges have no orientation, i.e. the edges can be described as unordered pairs so that the edge (i, j), is identical to the edge (j, i), where $i \neq j$.

2.2.2 Delute variable.

Let G be an undirected graph on vertices $\{1, ..., n\}$ and denote the delute variable by

$$\epsilon_{ji} = \epsilon_{ij} = \begin{cases} 1, & \text{if } (i,j) \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}, \tag{12}$$

where $\epsilon_{ii} = 0$ is assumed to hold for all i. Note, that $(\epsilon_{ij})_{1 \leqslant i,j \leqslant n}$ is called the adjacency matrix for graph G.

2.3 Proof that the Hamiltonian is monotonically decreasing in time

We define for any undirected graph G and any $\vec{\sigma} \in \Omega_n$ the energy function similar to (7);

$$H^G(\vec{\sigma}) := H(\vec{w}^G, \vec{\sigma}) = -\sum_i \sum_j w_{ij}^G \sigma_i \sigma_j, \quad w_{ij}^G = w_{ij}\epsilon_{ij}, \quad 1 \leqslant i, j \leqslant n \tag{13}$$

Notice, that (13) is equivalent to (7) when graph G is complete.

In the following theorem we will look closer at the instance when $\sigma_k(t+1) = -\sigma_k(t)$, where asynchrnous network dynamics is considered.

Theorem 2.1. In a Hopfield model on any given graph G, the energy function is monotonically decreasing in time, under asynchronous network dynamics.

Proof. Assume that for some k,

$$\sigma_k(t+1) = -\sigma_k(t), \tag{14}$$

and

$$\sigma_j(t+1) = \sigma_j(t), \qquad j \neq k. \tag{15}$$

Consider the difference in energy $\Delta H^G(t)$ in the following.

$$H^G(\vec{\sigma}(t+1)) - H^G(\vec{\sigma}(t))\big|_{\vec{\sigma}(t)=\vec{\sigma}}$$

$$= \Delta H^G(t) = \sum_{i\neq j} \sum_{j \neq i} w_{ij}^G \left(\sigma_i \sigma_j - \operatorname{sgn}\left(\sum_k w_{ik}^G \sigma_k\right) \operatorname{sgn}\left(\sum_k w_{jk}^G \sigma_k\right)\right) \tag{16}$$

$$= \sum_{i\neq j} \sum_{j \neq i} \left(\tfrac{1}{n} \sum_\mu \epsilon_{ij} \xi_i^\mu \xi_j^\mu\right) \left(\sigma_i \sigma_j - \operatorname{sgn}\left(\sum_k w_{ik} \sigma_k\right) \operatorname{sgn}\left(\sum_k w_{jk} \sigma_k\right)\right)$$

Using the fact that $-\sigma_i(t+1)\sigma_j(t) + \sigma_i(t+1)\sigma_j(t) = 0$, one can factorise the expression (16) in the following (17) way.

$$\begin{aligned}\Delta H^G(t) &= \sum_{i,j} w_{ij}^G \Big(\sigma_i(t)\sigma_j(t) - \sigma_i(t+1)\sigma_j(t+1)\Big) \\ &= \sum_{i,j} w_{ij}^G \Big[(\sigma_i(t) - \sigma_i(t+1))\sigma_j(t) + \sigma_i(t+1)(\sigma_j(t) - \sigma_j(t+1))\Big]\end{aligned} \tag{17}$$

In the case that $\sigma_k(t+1) = -\sigma_k(t)$, and if $\sigma_j(t+1) = \sigma_j(t)$ holds for all $j \neq k$, the following is obtained.

$$\begin{aligned}\Delta H^G(t) &= \sum_j w_{kj}^G(\sigma_k(t) - \sigma_k(t+1))\sigma_j(t) + \sum_i w_{ik}^G \left(\sigma_k(t) - \sigma_k(t+1)\right) \sigma_i(t+1) \\ &\underbrace{=}_{\substack{\text{letting} \\ w_{ii}^G = 0}} \sum_{j \neq k} w_{kj}^G \left(\sigma_k(t) - \sigma_k(t+1)\right) \sigma_j(t) + \sum_{i \neq k} w_{ki}^G \left(\sigma_k(t) - \sigma_k(t+1)\right) \cdot \overbrace{\sigma_i(t)}^{\substack{\text{assuming} \\ \sigma_i(t+1)=\sigma_i(t) \\ \text{for all } i \neq k}} \\ &\underbrace{=}_{\substack{\text{assuming} \\ \sigma_k(t+1) \\ = -\sigma_k(t)}} 2 \sum_{j \neq k} w_{kj}^G \sigma_j(t) 2\sigma_k(t) = 4\sigma_k(t) \sum_j w_{kj}^G \sigma_j(t) \\ &\underbrace{=}_{\substack{\text{using that} \\ a = |a| \operatorname{sgn}(a)}} 4\sigma_k(t) |\sum_j w_{kj}^G \sigma_j(t)| \sigma_k(t+1) \\ &\underbrace{=}_{\substack{\text{assuming} \\ \sigma_k(t+1) \\ = -\sigma_k(t)}} 4\sigma_k(t)|h_k(t)|(-\sigma_k(t)) = -4|h_k(t)| \leqslant 0\end{aligned} \tag{18}$$

Thus

$$\Delta H(t) = \begin{cases} -4|h_k(t)|, & \text{for } \sigma_k(t+1) = -\sigma_k(t) \text{ and } \sigma_j(t+1) = \sigma_j(t) \\ 0, & \text{for } \sigma_k(t+1) = \sigma_k(t) \text{ and } \sigma_j(t+1) = \sigma_j(t) \end{cases} \leqslant 0. \tag{19}$$

□

Conclusion. We conclude that the energy function of a Hopfield model on a given graph under asynchronous network dynamics is monotonically decreasing.

3 Hopfield model on incomplete graphs

The fact that the energy H is non-increasing follows by results on the Hopfield model on a complete graph. We introduce a way to define a Hopfield model on an incomplete graph given two patterns that agrees with the weights corresponding to this graph. We study examples of the Hopfield model on various incomplete graphs. The aim is to derive the exact values for the incremental drop in energy.

3.1 Defining a graph given two patterns

Given an incomplete graph G on $\{1, ..., n\}$, our idea is to define $\xi^1, \xi^2 \in \Omega_n$ so that

$$w_{ij} = \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_i^\mu \xi_j^\mu = 0 \tag{20}$$

whenever $\epsilon_{ij} = 0$.

This is useful for providing values for the energy difference in time for a Hopfield model on an incomplete graph assuming asynchronous network dynamics. This is done by determining the patterns corresponding to a certain set of weights belonging to a Hopfield model on a graph. Then we can use the ordinary Hopfield model on a complete graph G since it does not use (i, j), so that $w_{ij}^G = w_{ij}$.

Definition 3.1. *A general two pattern dependency formula.* Let w_{ij} be the weights and denote by γ_{ij} a function

$$\gamma_{ij} := (-1)^{1\{w_{ij}=0\}}, \tag{21}$$

that takes the values 1, -1.

Define $\mathscr{j}(i) := \mathscr{j} \in \{j : 1 \leqslant j \leqslant n\}$ to be the index of the node connected to node i in G so that $\gamma_{i\mathscr{j}} = 1$. Observe that here we have

$$w_{i\mathscr{j}} := \pm \tfrac{2}{n} \tag{22}$$

which follows from the definition (2) of weights and patterns. Using property (22) along with the constraints (26) of the weights posed previously, one can define a *general pattern dependency formula* ξ_i^μ, for $i \neq j$ in the following way.

$$\xi_i^\mu := \xi_i^\mu(\vec{\xi}) = \gamma_{ij}\xi_j^\mu \xi_i^{3-\mu}\xi_j^{3-\mu}, \quad \mu = 1, 2 \tag{23}$$

This follows by division in

$$\begin{cases} \xi_i^1\xi_j^1 + \xi_i^2\xi_j^2 = 0, & \text{iff} \quad \xi_i^1\xi_j^1 = -\xi_i^2\xi_j^2 \\ \xi_i^1\xi_j^1 + \xi_i^2\xi_j^2 \neq 0, & \text{iff} \quad \xi_i^1\xi_j^1 = \xi_i^2\xi_j^2 \end{cases} \tag{24}$$

by ξ_j^1, so that

$$\begin{cases} \xi_i^1\xi_j^1 + \xi_i^2\xi_j^2 = 0, & \text{iff} \quad \xi_i^1 = -\xi_j^1\xi_i^2\xi_j^2 \\ \xi_i^1\xi_j^1 + \xi_i^2\xi_j^2 \neq 0, & \text{iff} \quad \xi_i^1 = \xi_j^1\xi_i^2\xi_j^2. \end{cases} \tag{25}$$

3.2 Hopfield model on a two-component disconnected graph with four nodes

Example. Define a Hopfield model on a graph G_1, with $n = 4$ nodes and consisting of two disconnected components. This graph contains two edges corresponding to

$$w_{12} = w_{13} = w_{24} = w_{34} = 0 \tag{26}$$

$$w_{14}, w_{23} \neq 0, \tag{27}$$

with the graph depicted in Figure 1. First observe that for such graph an even number of patterns is required to be able to construct the zero weights, this fact appears from the definition of the weights (2).

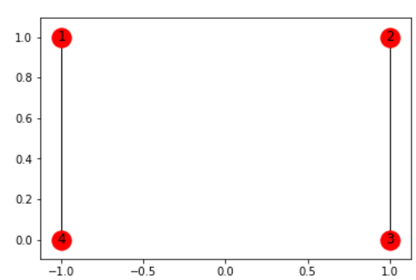

Figure 1: Two-component disconnected graph G_1 with four nodes

In order to define a graph specific pattern dependency formula, we start by defining the indexes of the weights. It will be convenient for us to use the following notation. For all $1 \leqslant i \leqslant n$

$$\begin{aligned} j_1 &:= j_1(i,n) = |n+1-2i| \\ j_2 &:= j_2(j_1,n) = n+1-j_1 \\ \ell &:= \ell(i,n) = n+1-i, \end{aligned} \tag{28}$$

this implies that the index node set $\{i, j_1, j_2, \ell\} = \{1,2,3,4\}$. We proceed to define a graph specific pattern dependency formula as follows. Note that in graph G_1, any node i is connected to node ℓ (see (28)). Then we define

$$\gamma_{ij}^{G_1}(w_{ij}) = \begin{cases} 1, & \text{for } j = \ell \\ -1, & \text{for } j = j_1, j_2. \end{cases} \tag{29}$$

The *connected weights* of graph G_1 take values in the following set

$$w_{i\ell}^{G_1} \in \{w_{14}, w_{23}\}, \quad w_{14}, w_{23} \in \{+\tfrac{1}{2}, -\tfrac{1}{2}\}. \tag{30}$$

Consider ξ^μ, $\mu = 1,2$ which satisfy (22)-(25). From the system of equations (23), we derive

$$\xi_i^\mu = -\xi_{j_1}^\mu \xi_i^{2-\mu} \xi_{j_1}^{3-\mu} = -\xi_{j_2}^\mu \xi_i^{3-\mu} \xi_{j_2}^{3-\mu} = \xi_\ell^\mu \xi_i^{3-\mu} \xi_\ell^{3-\mu}. \tag{31}$$

Using Definition 3.1, we shall express the component ξ_i^μ from pattern μ in terms of another component ξ_j^μ belonging to the same pattern where $i \neq j$. Recall that to define a zero weight with two patterns the sign of $\xi_i \xi_j$ must differ for the two patterns, whereas for non-zero weights it has to be equal for both patterns. It follows from equation (23) that

$$\xi_i^\mu = \xi_j^\mu \tag{32}$$

if condition

$$\operatorname{sgn}(\gamma_{ij} \xi_i^{3-\mu} \xi_j^{3-\mu}) = 1 \tag{33}$$

is satisfied. Keeping equation (31) in mind, (33) is the case if either of the following conditions are satisfied:

$$\xi_i^\mu = \xi_\ell^\mu, \quad \mu = 1,2 \tag{34}$$

or

$$\xi_i^{3-\mu} = -\xi_j^{3-\mu}. \tag{35}$$

Indeed a counter example using (31) confirms this equality by taking $\xi_1^\mu \neq \xi_4^\mu$ for at least one μ. Assume for instance $\xi_1^1 \neq \xi_4^1$ with $\xi_1^2 = \xi_4^2$ to hold, then the equality

$$\xi_1^1\xi_2^1 + \xi_1^2\xi_2^2 = \xi_2^1\xi_4^1 + \xi_2^2\xi_4^2 \tag{36}$$

now becomes equality

$$-\xi_4^1\xi_2^1 + \xi_4^2\xi_2^2 = \xi_2^1\xi_4^1 + \xi_2^2\xi_4^2 \tag{37}$$

which is a contradiction, this is true also if $\xi_1^\mu \neq \xi_4^\mu$ for all μ holds.

Finally, one but not the only way to express pattern ξ_i^μ in terms of ξ_j^μ, $\gamma_{ij}^{\mu-1}$ is in the following way stated below

$$\xi_i^\mu = \gamma_{ij}^{\mu-1}\xi_j^\mu. \tag{38}$$

3.2.1 Difference in energy updating neuron k in a network with 4 neurons and 2 patterns

Let

$$\sigma_j(0) = \eta_j, \qquad j \neq i, \tag{39}$$

and assume the following condition for the patterns

$$\begin{cases} \xi_i^\mu\xi_{j\neq\ell}^\mu = -\xi_i^{3-\mu}\xi_{j\neq\ell}^{3-\mu} \\ \xi_i^\mu\xi_\ell^\mu = \xi_i^{3-\mu}\xi_\ell^{3-\mu} \end{cases}, \qquad \mu = 1, 2, \tag{40}$$

which corresponds to the weights related to G_1 as defined in (26) and (27). Then when

$$\begin{gathered} \sigma_k(t+1) = -\sigma_k(t), \\ \sigma_j(t+1) = \sigma_j(t), \quad j \neq k, \end{gathered} \tag{41}$$

we derive using result (18)

$$
\begin{aligned}
\Delta H^{G_1}(0) &= -4|h_k(0)| \\
&\overset{\text{eq.(2)}}{=} -4|\sum_{j\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \eta_j| \\
&\overset{\text{eq.(28)}}{=} -4|\sum_{\substack{j\neq \ell\\ j\neq k}} \tfrac{1}{n} \left(\sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu\right) \eta_j + \sum_{j=\ell} \tfrac{1}{n} \left(\sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu\right) \eta_\ell| \\
&\overset{(m,n)}{=} -4|\sum_{\substack{j\neq \ell\\ j\neq k}} \tfrac{1}{4} \left(\sum_{\mu=1}^{2} \xi_k^\mu \xi_j^\mu\right) \eta_j + \sum_{j=\ell} \tfrac{1}{4} \left(\sum_{\mu=1}^{2} \xi_k^\mu \xi_j^\mu\right) \eta_\ell| \\
&= -4|\sum_{\substack{j\neq \ell\\ j\neq k}} \tfrac{1}{4} \left(\xi_k^1 \xi_j^1 + \xi_k^2 \xi_j^2\right) \eta_j + \tfrac{1}{4} \left(\xi_k^1 \xi_\ell^1 + \xi_k^2 \xi_\ell^2\right) \eta_\ell| \\
&\overset{\text{eq.(40)}}{=} -4|\sum_{\substack{j\neq \ell\\ j\neq k}} \tfrac{1}{4} \left(\xi_k^1 \xi_j^1 - \xi_k^1 \xi_j^1\right) \eta_j + \tfrac{1}{4} \left(\xi_k^1 \xi_\ell^1 + \xi_k^1 \xi_\ell^1\right) \eta_\ell| \\
&= -4|\tfrac{1}{4} \left(\xi_k^1 \xi_\ell^1 + \xi_k^1 \xi_\ell^1\right) \eta_\ell| \\
&= -2|\xi_k^1 \xi_\ell^1 \eta_\ell| \\
&\overset{\text{eq.(1)}}{=} -2|\eta_\ell| \\
&\overset{\text{eq.(1)}}{=} -2.
\end{aligned}
\tag{42}
$$

Note that if $\sigma_k(t+1) = \sigma_k(t)$ for all k, then $\Delta H^{G_1}(t) = 0$ holds. Since the input potential of neuron k in G_1 is defined according to

$$h_k^{G_1}(0) = \tfrac{1}{4} \sum_{\mu=1}^{m} \xi_k^\mu \xi_\ell^\mu \eta_\ell, \tag{43}$$

we know that if equality

$$\operatorname{sgn}\left(\tfrac{1}{4} \sum_{\mu=1}^{m} \xi_k^\mu \xi_\ell^\mu \eta_\ell\right) = \eta_k \tag{44}$$

is satisfied, then the difference in energy will equal zero. This corresponds to choice of $\vec{\eta}$ according to

$$\tfrac{1}{2}(\xi_k^1 \xi_\ell^1 + \xi_k^2 \xi_\ell^2)\eta_\ell = \eta_k \tag{45}$$

or equivalently

$$\eta_k \eta_\ell = \xi_k^\mu \xi_\ell^\mu, \quad \mu = 1, 2. \tag{46}$$

The computations of result (42) points at a value of -2 for the delta energy function following an update of a neuron k where $\sigma(t+1) = -\sigma(t)$ is assumed. This can be summarized in the following way

$$\Delta H^{G_1}(0) = \begin{cases} -2, & \sigma_k(1) = -\sigma_k(0) \\ 0, & \sigma_k(1) = \sigma_k(0). \end{cases} \tag{47}$$

Conclusion. Given a graph G_1 and two patterns chosen as above, we conclude that the energy function is monotonically decreasing under asynchronous network dynamics, and that $\Delta H^{G_1}(0) \in \{0, -2\}$, where $\Delta H^{G_1}(0) = 0$ only in the trivial case that $\sigma_k(1) = \sigma_k(0)$.

3.2.2 G as a union of two independent components

Definition 3.2. Split the graph G_1 in two subgroups each on two vertices's as depicted in Figure 2 and Figure 3. Denote the two subgraphs by $G_1^{P_\kappa}$, $\kappa = 1, 2$.

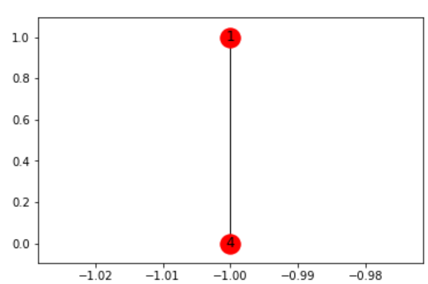

Figure 2: Subgraph $G_1^{P_1}$

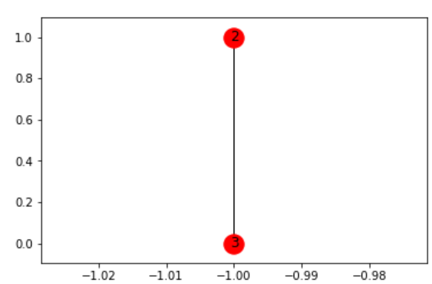

Figure 3: Subgraph $G_1^{P_2}$

To obtain the delta energy of any of the two subgraphs updating neuron

k, let $k = \ell$ hold and consider

$$
\begin{aligned}
\Delta H^{G_1^{P_\kappa}}(0) &= -4|h_k(0)| \\
&= -4|\sum_{j\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \eta_j| \\
&= -4|\sum_{j\neq \ell} \tfrac{1}{2} \sum_{\mu=1}^{2} \xi_k^\mu \xi_j^\mu \eta_j| \\
&= -4|\sum_{j=5-\ell} \tfrac{1}{2} \sum_{\mu=1}^{2} \xi_k^\mu \xi_j^\mu \eta_j| \\
&= -2|\sum_{\mu=1}^{2} \xi_\ell^\mu \xi_{5-\ell}^\mu \eta_{5-\ell}| \\
&= -2|(\xi_\ell^1 \xi_{5-\ell}^1 + \xi_\ell^2 \xi_{5-\ell}^2)\eta_{5-\ell}| \\
&= -2|(\xi_\ell^1 \xi_{5-\ell}^1 + \xi_\ell^1 \xi_{5-\ell}^1)\eta_{5-\ell}| \\
&= -2|2\xi_\ell^1 \xi_{5-\ell}^1 \eta_{5-\ell}| \\
&= -4|\xi_\ell^1 \xi_{5-\ell}^1 \eta_{5-\ell}| \\
&= -4.
\end{aligned}
\tag{48}
$$

From result (48) one can see that the two subgraphs give identical results in terms of the energy delta function.

Conclusion. Given a subgraph $G_1^{P_\kappa}$ and two patterns chosen in accordance to the set of weights agreeing with this graph, we conclude that the energy function is monotonically decreasing under asynchronous network dynamics, and that $\Delta H^{G_1^{P_\kappa}}(0) \in \{0, -4\}$. Note, that $\Delta H^{G_1^{P_\kappa}}(0) = 0$ only if $\sigma_k(1) = \sigma_k(0)$.

3.2.3 Difference in energy updating neuron k in a network with 4 neurons and with m arbitrary patterns under restriction

In the following subsection m_G denotes the number of patterns for a network on a graph G. We omit the subscript if it is clear from context. Consider a Hopfield network on an incomplete graph on n vertices's. To enable zero weights we assume $m = 2\iota$. For a network with n number of neurons on a complete graph the total number of possible configurations is 2^n. Thus for any incomplete graph the number of patterns is $m = 2\iota \leqslant 2^n$ and consequently $m_{G_1} = 2\iota_{G_1} \leqslant 16$ for $n = 4$.

Recall from (28) that in graph G_1 for any neuron indexed by $i = 1, ..., n$ there exists an edge to neuron $\ell(i, n)$ such that $w_{i\ell} \neq 0$. Consider then

$$\begin{aligned}\Delta H^{G_1}(0) &= -4|h_k(0)| \\ &= -4|\sum_{j\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \eta_j| \\ &= -|\sum_{j\neq k} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \eta_j| \\ &= -|\sum_{j\neq k} (\xi_k^1 \xi_j^1 + \xi_k^2 \xi_j^2 + ... + \xi_k^{m-1} \xi_j^{m-1} + \xi_k^m \xi_j^m)\eta_j| \\ &= -|(\xi_k^1 \xi_\ell^1 + \xi_k^2 \xi_\ell^2 + ... + \xi_k^{m-1} \xi_\ell^{m-1} + \xi_k^m \xi_\ell^m)\eta_\ell|,\end{aligned} \tag{49}$$

where we used the properties of patterns, pertaining to G_1, as defined previously. Since any term $\xi_k^\mu \xi_j^\mu \in \{-1, 1\}$ we derive from here that

$$\Delta H^{G_1}(0) = -2|\tfrac{m}{2} - \alpha|, \tag{50}$$

where $0 \leqslant \alpha < \frac{m}{2}$ denotes the number of terms $\xi_k^\mu \xi_\ell^\mu$ for which there exists a negative in the sum corresponding to weight $w_{k\ell}$. Relaxing the strictness of this inequality we get that the term α is restricted so that

$$0 \leqslant \alpha \leqslant \tfrac{m}{2} - 1. \tag{51}$$

The term α can never be larger than $\frac{m}{2}$ since if this is true it is impossible to find a negative for each of the α number of terms. It can never be $\frac{m}{2}$ since this implies an equal number of terms $\xi_i^\mu \xi_j^\mu$ with the same sign resulting in

$w_{k\ell} = 0$ which is false. It follows that

$$\Delta H^{G_1}(0) \in \{-m, -(m-2), ..., -(m-(m-4)), -2, 0\} = D_1, \quad m = 2\iota \leqslant 16, \tag{52}$$

and that

$$\Delta H^{G_1}(0) \in \{-16, -14, ..., -2, 0\}, \tag{53}$$

since $D_1 \subset \{-16, -14, ..., -2, 0\}$. The smallest value $\min \Delta H^{G_1}(0) = -16$ is obtained with $\alpha = 0$ and $m = 16$.

Conclusion. Given a graph G_1 and with m patterns chosen as above, we conclude that the energy function is monotonically decreasing under asynchronous network dynamics and that $H^{G_1}(0) \in \{-16, -14, ..., -2, 0\}$ and $\Delta H^{G_1}(0) \in \{-m, -(m-2), ..., -(m-(m-4)), -2, 0\}$. Note, that $\Delta H^{G_1}(0) = 0$ only if $\sigma_k(1) = \sigma_k(0)$.

3.3 Hopfield model on a two-component disconnected graph with five nodes

Example. Define a Hopfield model on a two-component disconnected graph G_2 so that it contain $n = 5$ nodes, symmetry in weights (2) still assumed. In addition to the set of weights in the four node example two more weights are zero so that

$$\begin{gathered} w_{12} = w_{13} = w_{24} = w_{34} = w_{25} = w_{35} = 0 \\ w_{14}, w_{15}, w_{23}, w_{45} \neq 0, \end{gathered} \tag{54}$$

where an even number of patterns is still required. The visualization of graph G_2 is presented in Figure 4 below.

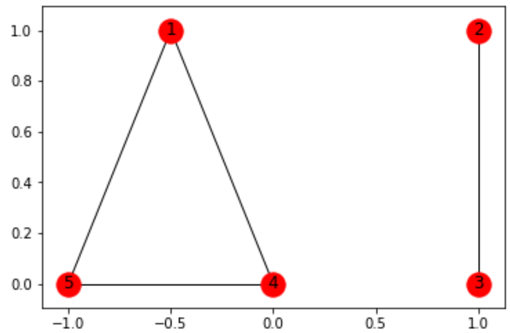

Figure 4: Two-component disconnected graph G_2 with five nodes

3.3.1 Defining the graph with two patterns.

To express the weights in terms of patterns condition (54) need to be satisfied. A quick inspection should convince oneself that the definition of the weights (26) and the pattern dependency formula (31) of G_2 is identical to that of G_1 for all $i, j \neq 5$.

Observe that in contrast to the graph G_1 examined previously that contained one connection for each node, in G_2 there also exist nodes that connects to two nodes.

Exchanging $n+1$ for n in the definition belonging to (31), then for $i, j \neq 5$ the first three leftmost products in pattern dependency formula for G_2 will be identical.

3.3.2 Difference in energy updating neuron k in a network with 5 neurons and 2 patterns

Define the following indexis as below.

$$\begin{aligned} j_1 &\in \{1, 4, 5\} \\ j_2 &\in \{2, 3\} \end{aligned} \tag{55}$$

We have that

$$
\begin{aligned}
\Delta H^{G_2}(t) &= -4|h_k(\sigma_k(t))| \\
&= -4|\sum_{j\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \sigma_j(t)| \\
&= -4|\sum_{j\in j_1\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \sigma_j(t) + \sum_{j\in j_2\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \sigma_j(t)|.
\end{aligned} \tag{56}
$$

If $k \in j_2$ the first sum in the last line below becomes zero, and there is just one possible connecting weight in j_2 so that

$$
\begin{aligned}
\Delta H^{G_2}(0) &= -4|h_k(0)| \\
&= -4|\sum_{j\in j_2\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \eta_j| \\
&= -\tfrac{4}{n}|\xi_k^1 \xi_{5-k}^1 \eta_{5-k} + \xi_k^2 \xi_{5-k}^2 \eta_{5-k}| \\
&= -\tfrac{4}{n}|\xi_k^1 \xi_{5-k}^1 \eta_{5-k} + \xi_k^1 \xi_{5-k}^1 \eta_{5-k}| \\
&= -\tfrac{8}{n}|\xi_k^1 \xi_{5-k}^1 \eta_{5-k}| \\
&= -\tfrac{8}{5}|\xi_k^1 \xi_{5-k}^1 \eta_{5-k}| \\
&= -\tfrac{8}{5}.
\end{aligned} \tag{57}
$$

Define $j_1 \in \{k, \tilde{j}_1, \tilde{j}_2\}$ and $\tilde{j} \in \{\tilde{j}_1, \tilde{j}_2\}$. If $k \in j_1$, the following result appears

$$
\begin{aligned}
\Delta H^{G_2}(0) &= -4|h_k(0)| \\
&= -4|\sum_{j\in j_1\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \eta_j| \\
&= -\tfrac{4}{5}|\sum_{j\in j_1\neq k} (\xi_k^1 \xi_j^1 + \xi_k^2 \xi_j^2)\eta_j| \\
&= -\tfrac{8}{5}|\sum_{j\in j_1\neq k} \xi_k^1 \xi_j^1 \eta_j| \\
&= -\tfrac{8}{5}|\xi_k^1 \xi_{\tilde{j}_1}^1 \eta_{\tilde{j}_1} + \xi_k^1 \xi_{\tilde{j}_2}^1 \eta_{\tilde{j}_2}| \\
&= -\tfrac{8}{5}|\xi_k^1 (\xi_{\tilde{j}_1}^1 \eta_{\tilde{j}_1} + \xi_{\tilde{j}_2}^1 \eta_{\tilde{j}_2})| \\
&= -\tfrac{8}{5}|\xi_{\tilde{j}_1}^1 \eta_{\tilde{j}_1} + \xi_{\tilde{j}_2}^1 \eta_{\tilde{j}_2}|
\end{aligned} \tag{58}
$$

which is maximized if

$$
\xi_{\tilde{j}_1}^1 \eta_{\tilde{j}_1} = \xi_{\tilde{j}_2}^1 \eta_{\tilde{j}_2} \tag{59}
$$

or equivalently

$$\eta_{\tilde{j}_1}\eta_{\tilde{j}_2} = \xi^1_{\tilde{j}_1}\xi^1_{\tilde{j}_2} \tag{60}$$

and

$$\eta_{\tilde{j}_1}\eta_{\tilde{j}_2} = \xi^2_{\tilde{j}_1}\xi^2_{\tilde{j}_2}, \tag{61}$$

resulting in the value

$$\Delta H^{G_2}(0)\big|_{k\in j_1} = -\tfrac{16}{5}. \tag{62}$$

Choosing an initialization pattern $\vec{\eta}$ according to (60) and (61), will always ensure the smallest possible delta energy function.

We get that

$$\Delta H^{G_2}(0) = \begin{cases} 0, & \text{if } \ k \in j_1 \ \text{ and } \ \xi^1_{\tilde{j}_1}\eta_{\tilde{j}_1} = -\xi^1_{\tilde{j}_2}\eta_{\tilde{j}_2} \\ -\tfrac{8}{5}, & \text{if } \ k \in j_2 \\ -\tfrac{16}{5}, & \text{if } \ k \in j_1 \ \text{ and } \ \xi^1_{\tilde{j}_1}\eta_{\tilde{j}_1} = \xi^1_{\tilde{j}_2}\eta_{\tilde{j}_2}. \end{cases} \tag{63}$$

Conclusion. Given a graph G_2 and with two patterns chosen in accordance to the set of weights agreeing with this graph, we conclude that the energy function is monotonically decreasing under asynchronous network dynamics, and that $\Delta H^{G_2}(0) \in \{0, -\frac{8}{5}, -\frac{16}{5}\}$.

3.3.3 Difference in energy updating neuron k in a network with 5 neurons and 2 patterns deleting one edge

If $k \in j_2$, then deleting the edge corresponding to w_{23} brings a result of zero in (57), while deleting any other edge do not change the result. Denote by $\varepsilon_0 = \{\epsilon : \epsilon_{23} = 0 \text{ and } \epsilon_{ij} = 1 \text{ for } i,j \neq 2,3\}$, the set of delute variables corresponding to deleting the edge between neuron 2 and 3.

Denote by i_0, j_0 a particular value of i, j, then denote $\varepsilon_1 = \{\epsilon : \epsilon_{ij} = 0 \text{ and } \epsilon_{i_0 j_0} = 1 \text{ for } i_0, j_0 \neq i, j \neq 2, 3\}$.

If $k \in j_1$ it holds by the result of (58) that deleting the edge corresponding to w_{23} does not affect the delta energy function and neither does deleting the edge $w_{\tilde{j}_1\tilde{j}_2}$ where $k \neq \tilde{j}_1, \tilde{j}_2$.

However, deleting the edge corresponding to $w_{j_1\tilde{j}_1}$, so that $\epsilon_{j_1\tilde{j}_1} = 0$, affects the delta energy function. This can be seen from result (64), building on the computations of result (58), but with the delute variable not always non-zero. Denote by $\varepsilon_2 = \{\epsilon : \epsilon_{j_1\tilde{j}_1} = 0 \text{ and } \epsilon_{ij} = 1 \text{ for } i, j \neq j_1, \tilde{j}_1\}$ the set of delute variables corresponding to deleting the edge between neuron j_1 and $\tilde{j}_1$, then the following must hold.

$$
\begin{aligned}
\Delta H^{G_2^{\varepsilon_2}}(0)\Big|_{k\in j_1} &= -\tfrac{8}{5}|\epsilon_{j_1\tilde{j}_1}\xi^1_{j_1}\xi^1_{\tilde{j}_1}\eta_{\tilde{j}_1} + \epsilon_{j_1\tilde{j}_2}\xi^1_{j_1}\xi^1_{\tilde{j}_2}\eta_{\tilde{j}_2}| \\
&= -\tfrac{8}{5}|1\cdot\xi^1_{j_1}\xi^1_{\tilde{j}_1}\eta_{\tilde{j}_1} + 0\cdot\xi^1_{j_1}\xi^1_{\tilde{j}_2}\eta_{\tilde{j}_2}| \\
&= -\tfrac{8}{5}|\xi^1_{j_1}\xi^1_{\tilde{j}_1}\eta_{\tilde{j}_1}| \\
&= -\tfrac{8}{5}
\end{aligned}
\tag{64}
$$

From the previous result (62) it becomes clear, together with computations (64) and as stated below

$$
H^{G_2^{\varepsilon_2}}(0)\Big|_{k\in j_1} = -\tfrac{8}{5} \quad > \quad -\tfrac{16}{5} = H^{G_2}(0)\big|_{k\in j_1}, \tag{65}
$$

that the delta energy function has increased with the deletion of this $w_{j_1\tilde{j}_2}$ weight. Furthermore we denote by ε_3 the set $\varepsilon_3 = \{\epsilon : \epsilon_{i_0j_0} = 0 \text{ and } \epsilon_{ij} = 1 \text{ for } i, j \neq i_0, j_0 \neq j_1, \tilde{j}_1\}$.

Conclusion. Given a graph G_2 and with two patterns chosen in accordance to the set of weights agreeing with this graph, we conclude that deleting one edge, the energy function is still monotonically decreasing under asynchronous network dynamics, and that

$$
\Delta H^{G_2^{\varepsilon 0}}(0)\Big|_{k=j_2} \in \{0\}, \tag{66}
$$

$$
\Delta H^{G_2^{\varepsilon 1}}(0)\Big|_{k=j_2} \in \{0, -\tfrac{8}{5}, -\tfrac{16}{5}\}, \tag{67}
$$

$$
\Delta H^{G_2^{\varepsilon 2}}(0)\Big|_{k=j_1} \in \{0, -\tfrac{8}{5}\}, \tag{68}
$$

and

$$
\Delta H^{G_2^{\varepsilon 3}}(0)\Big|_{k=j_1} \in \{0, -\tfrac{8}{5}, -\tfrac{16}{5}\}. \tag{69}
$$

3.4 Hopfield model on a star graph with five nodes

Example. Define a Hopfield model on a star graph G_3 so that it contain $n = 5$ nodes, symmetry in weights (2) assumed. The weights are defined according to

$$\begin{gathered} w_{23} = w_{24} = w_{25} = w_{34} = w_{35} = w_{45} = 0 \\ w_{12}, w_{13}, w_{14}, w_{15} \neq 0, \end{gathered} \tag{70}$$

with an even number of patterns required. The graph G_3 can be seen in Figure 5 below.

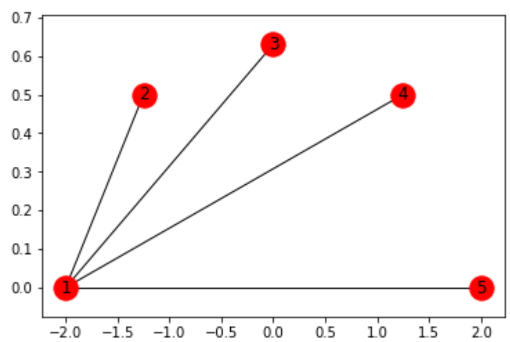

Figure 5: Star graph G_3 with five nodes

3.4.1 Difference in energy updating neuron k in a network with 5 neurons and 2 patterns

We continue in a similar fashion to previous sections on delta energy. First consider the case when $k \neq 1$, the following must then hold.

Case 1. $k \neq 1$

Consider

$$\begin{aligned}
\Delta H^{G_3}(0) &= -4|h_k(0)| \\
&= -4|\sum_{j\neq k}\tfrac{1}{n}\sum_{\mu=1}^{m}\xi_k^{\mu}\xi_j^{\mu}\eta_j| \\
&= -4|\sum_{j=1}\tfrac{1}{n}\sum_{\mu=1}^{m}\xi_k^{\mu}\xi_j^{\mu}\eta_j| \\
&= -\tfrac{4}{5}|\sum_{\mu=1}^{m}\xi_k^{\mu}\xi_1^{\mu}\eta_1| \\
&= -\tfrac{4}{5}|\sum_{\mu=1}^{m}\xi_k^{\mu}\xi_1^{\mu}\eta_1| \\
&= -\tfrac{4}{5}|2\xi_k^{1}\xi_1^{1}\eta_1| \\
&= -\tfrac{8}{5}|\xi_k^{1}\xi_1^{1}\eta_1| \\
&= -\tfrac{8}{5}.
\end{aligned} \tag{71}$$

Case 2. $k = 1$

Let j_1, j_2, j_3, j_4 be arbitrary indexes for j. Consider then

$$\begin{aligned}
\Delta H^{G_3}(0) &= -4|h_k(0)| \\
&= -4|\sum_{j\neq k}\tfrac{1}{n}\sum_{\mu=1}^{m}\xi_k^{\mu}\xi_j^{\mu}\eta_j| \\
&= -\tfrac{4}{5}|\sum_{j\neq k}(\xi_k^{1}\xi_j^{1}+\xi_k^{2}\xi_j^{2})\eta_j| \\
&= -\tfrac{4}{5}|\sum_{j\neq k}(\xi_k^{1}\xi_j^{1}+\xi_k^{1}\xi_j^{1})\eta_j| \\
&= -\tfrac{8}{5}|\sum_{j\neq k}\xi_k^{1}\xi_j^{1}\eta_j| \\
&= -\tfrac{8}{5}|\xi_k^{1}\sum_{j\neq k}\xi_j^{1}\eta_j| \\
&= -\tfrac{8}{5}|\sum_{j\neq k}\xi_j^{1}\eta_j| \\
&= -\tfrac{8}{5}|\xi_{j_1}^{1}\eta_{j_1}+\xi_{j_2}^{1}\eta_{j_2}+\xi_{j_3}^{1}\eta_{j_3}+\xi_{j_4}^{1}\eta_{j_4}|.
\end{aligned} \tag{72}$$

There are three possible conditions for $\xi_j^1 \eta_j$ as defined below, with the following results for the delta energy function.

$$\Delta H^{G_3}(0) = \begin{cases} 0, & \text{if} \quad \xi_{j_1}^1 \eta_{j_1} = \xi_{j_2}^1 \eta_{j_2} = -\xi_{j_3}^1 \eta_{j_3} = -\xi_{j_4}^1 \eta_{j_4} \\ -\frac{16}{5}, & \text{if} \quad \xi_{j_1}^1 \eta_{j_1} = \xi_{j_2}^1 \eta_{j_2} = \xi_{j_3}^1 \eta_{j_3} = -\xi_{j_4}^1 \eta_{j_4} \\ -\frac{32}{5}, & \text{if} \quad \xi_{j_1}^1 \eta_{j_1} = \xi_{j_2}^1 \eta_{j_2} = \xi_{j_3}^1 \eta_{j_3} = \xi_{j_4}^1 \eta_{j_4} \quad (*) \end{cases} \tag{73}$$

Here $(*)$ in (73) constitutes the minimum for the delta energy function of graph G_3 updating neuron k.

Conclusion. Given a graph G_3 and with two patterns chosen in accordance to the set of weights agreeing with this graph, we conclude that the energy function is monotonically decreasing under asynchronous network dynamics, and that $\Delta H^{G_3}(0) \in \{0, -\frac{16}{5}, -\frac{32}{5}\}$.

3.5 Hopfield model on a path graph with five nodes

Example. Define a Hopfield model on a path graph G_4 so that it contain $n = 5$ nodes, with symmetry in weights (2) assumed. The weights are defined according to

$$\begin{gathered} w_{13} = w_{14} = w_{15} = w_{24} = w_{25} = w_{35} = 0 \\ w_{12}, w_{23}, w_{34}, w_{45} \neq 0, \end{gathered} \tag{74}$$

with an even number of patterns required. Visualization of G_4 is provided in Figure 6 below.

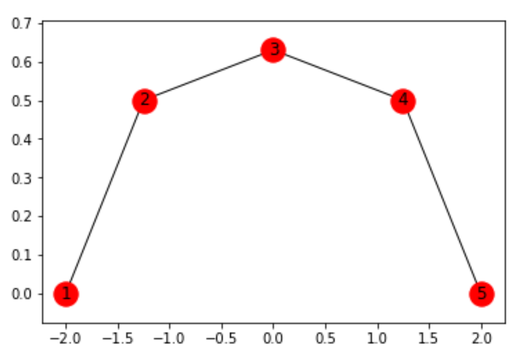

Figure 6: Path graph G_4 with five nodes

3.5.1 Defining the graph with two patterns

For a path defined as in G_4 two patterns are not enough to define its weights. This can be observed in the following way. Let the patterns ξ_1^1, ξ_1^2 contain an uneven number of minus signs, it follows that patterns ξ_3^1, ξ_3^2 has to contain an even number of negative terms. To satisfy the condition $w_{12} \neq 0$, then ξ_2^1, ξ_2^2 has to contain an uneven number of minus signs, but to get $w_{23} \neq 0$ an even number of minus signs are required. The realization of both requirements can for obvious reasons not occur at the same time.

3.5.2 Defining the graph with four patterns

What about four patterns? We claim that the graph G_4 is not defined on four patterns. Assume for some $i \neq j$, that each of the sets

$$\{\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4\} \neq \{\xi_j^1, \xi_j^2, \xi_j^3, \xi_j^4\}, \quad i \neq j, \tag{75}$$

contains an even number of negative elements. Then

$$\sum_{\mu=1}^{4} \xi_i^\mu \xi_j^\mu = 0, \quad i \neq j, \tag{76}$$

if and only if

$$\{\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4\} \neq \{-\xi_j^1, -\xi_j^2, -\xi_j^3, -\xi_j^4\}, \quad i \neq j, \tag{77}$$

and

$$\sum_{\mu=1}^{4} \xi_i^\mu \xi_j^\mu \neq 0, \quad i \neq j, \tag{78}$$

if and only if

$$\{\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4\} = \{-\xi_j^1, -\xi_j^2, -\xi_j^3, -\xi_j^4\}, \quad i \neq j. \tag{79}$$

Let there be a k for which $\{\xi_k^1, \xi_k^2, \xi_k^3, \xi_k^4\}$ contains an even number of negative elements. To satisfy (74), the following conditions need to be satisfied:

$$\{\xi_k^1, \xi_k^2, \xi_k^3, \xi_k^4\} \neq \{-\xi_{k+2}^1, -\xi_{k+2}^2, -\xi_{k+2}^3, -\xi_{k+2}^4\}, \quad k = 1, 2, 3, \tag{80}$$

$$\{\xi_k^1, \xi_k^2, \xi_k^3, \xi_k^4\} \neq \{-\xi_{k+3}^1, -\xi_{k+3}^2, -\xi_{k+3}^3, -\xi_{k+3}^4\}, \quad k = 1, 2, \tag{81}$$

and

$$\{\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4\} \neq \{-\xi_5^1, -\xi_5^2, -\xi_5^3, -\xi_5^4\}. \tag{82}$$

Note, that for any $i \neq j$ where set $\{\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4\}$ contains an even number of negative elements and set $\{\xi_j^1, \xi_j^2, \xi_j^3, \xi_j^4\}$ contains an odd number of negative elements, that (78) holds. Thus, given $\{\xi_k^1, \xi_k^2, \xi_k^3, \xi_k^4\}$ as above, all sets $\{\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4\}$, $i = 1, 2, 3, 4, 5$, contains even number of negative elements. Since

$$\sum_{\mu=1}^{m} \xi_{k-1}^{\mu} \xi_k^{\mu} \neq 0, \quad k = 2, 3, 4, 5, \tag{83}$$

we need

$$\{-\xi_{k-1}^1, -\xi_{k-1}^2, -\xi_{k-1}^3, -\xi_{k-1}^4\} = \{\xi_k^1, \xi_k^2, \xi_k^3, \xi_k^4\}, \quad k = 2, 3, 4, 5, \tag{84}$$

and

$$\{\xi_k^1, \xi_k^2, \xi_k^3, \xi_k^4\} = \{-\xi_{k+1}^1, -\xi_{k+1}^2, -\xi_{k+1}^3, -\xi_{k+1}^4\}, \quad k = 2, 3, 4, 5. \tag{85}$$

However, this holds if and only if

$$\{\xi_{k-1}^1, \xi_{k-1}^2, \xi_{k-1}^3, \xi_{k-1}^4\} = \{\xi_{k+1}^1, \xi_{k+1}^2, \xi_{k+1}^3, \xi_{k+1}^4\}, \quad k = 2, 3, 4, 5, \tag{86}$$

which violates (74) and (75). Similar results can be shown if the set of patterns indexed by k contains an odd number of negative elements.

Conclusion. We have shown that given a path graph G_4, its weights cannot be defined using two or four patterns.

3.6 Hopfield model on a cycle graph with five nodes

Example. Define a Hopfield model on cycle graph G_5 so that it contain $n = 5$ nodes, symmetry in weights (2) assumed. The weights are defined in the following way,

$$\begin{gathered} w_{13} = w_{14} = w_{24} = w_{25} = w_{35} = 0 \\ w_{12}, w_{23}, w_{34}, w_{45}, w_{51} \neq 0 \end{gathered} \tag{87}$$

with an even number of patterns required. The graph G_4 can be seen in Figure 7 below.

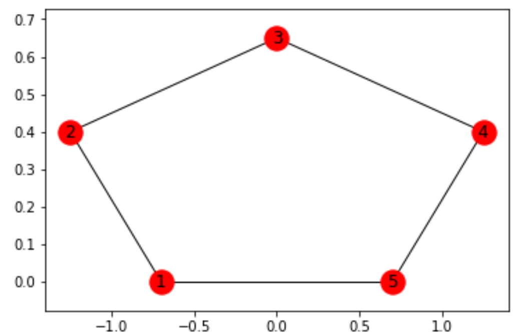

Figure 7: Cycle graph G_5 with five nodes

3.6.1 Difference in energy updating neuron k in a network with 5 neurons and 2 patterns

We shall prove that given two patterns, the energy delta function in a Hopfield network for a cycle graph is not defined. Continue in the same way as for the previous incomplete graphs by computing the difference in energy updating neuron k on graph G_5. Consider

$$\begin{aligned}
\Delta H^{G_5}(0) &= -4|h_k(0)| \\
&= -4|\sum_{j\neq k} \tfrac{1}{n} \sum_{\mu=1}^{m} \xi_k^\mu \xi_j^\mu \eta_j| \\
&= -\tfrac{4}{5}|\sum_{j\neq k} (\xi_k^1 \xi_j^1 \eta_j + \xi_k^2 \xi_j^2 \eta_j)| \\
&= -\tfrac{4}{5}|\xi_k^1 \xi_{k-1}^1 \eta_{k-1} + \xi_k^2 \xi_{k-1}^2 \eta_{k-1} + \xi_k^1 \xi_{k+1}^1 \eta_{k+1} + \xi_k^2 \xi_{k+1}^2 \eta_{k+1}| \qquad (88) \\
&= -\tfrac{4}{5}|2\xi_k^1 \xi_{k-1}^1 \eta_{k-1} + 2\xi_k^1 \xi_{k+1}^1 \eta_{k+1}| \\
&= -\tfrac{8}{5}|\xi_k^1 \xi_{k-1}^1 \eta_{k-1} + \xi_k^1 \xi_{k+1}^1 \eta_{k+1}| \\
&= -\tfrac{8}{5}|\xi_k^1 (\xi_{k-1}^1 \eta_{k-1} + \xi_{k+1}^1 \eta_{k+1})| \\
&= -\tfrac{8}{5}|\xi_{k-1}^1 \eta_{k-1} + \xi_{k+1}^1 \eta_{k+1}| \\
&= -\tfrac{8}{5}|\xi_{k-1}^2 \eta_{k-1} + \xi_{k+1}^2 \eta_{k+1}|.
\end{aligned}$$

The delta energy function can then be defined according to

$$\Delta H^{G_5}(0) = \begin{cases} 0, & \xi_{k-1}^\mu \xi_{k+1}^\mu = -\eta_{k-1}\eta_{k+1} \quad \text{for } \mu = 1 \text{ and } \mu = 2 \\ -\frac{16}{5}, & \xi_{k-1}^\mu \xi_{k+1}^\mu = \eta_{k-1}\eta_{k+1} \quad \text{for } \mu = 1 \text{ and } \mu = 2 \quad (*). \end{cases} \qquad (89)$$

Note however, that for every k we need that the following condition to be satisfied

$$\begin{cases} \xi_k^1\xi_{k+1}^1 = \xi_k^2\xi_{k+1}^2 \\ \xi_k^1\xi_{k-1}^1 = \xi_k^2\xi_{k-1}^2 \\ \xi_{k-1}^1\xi_{k+1}^1 = -\xi_{k-1}^2\xi_{k+1}^2, \end{cases} \tag{90}$$

in other words, for a node to be connected to a neighbour implies that $\xi_k^1\xi_{k+1}^1 = \xi_k^2\xi_{k+1}^2$, and that also $\xi_k^1\xi_{k-1}^1 = \xi_k^2\xi_{k-1}^2$ must hold. But for every two entries in (90) that holds, multiply them and you will see that the expression can never be true, for instance $\xi_{k-1}^1\xi_{k+1}^1 = \xi_{k-1}^2\xi_{k+1}^2$ would also have be true, which contradicts the definition of the weights in this graph.

Conclusion. We have showed that given a cycle graph G_5, its weights cannot be defined with two patterns.

4 Simulation results for stability of patterns in a Hopfield model with 5 neurons and 4 patterns for graph G2 and G3

We have implemented a computer algorithm in Python that finds all sets of patterns corresponding to graph G_2, G_3 and which gives information about the stability of the patterns.

Denote the network size by n, the pattern size by m, the number of possible sets of patterns in the graph by c_1, and the number of sets of patterns that contain a stable pattern for each eta by c_2.

Denote the initial states for the network by $\eta = (\eta_1, \eta_2, ..., \eta_n)$, where the initial state of neuron i is denoted by $\sigma_i(0) = \eta_i$ for $i = 1, ..., n$. Denote also the pattern μ of the network by $\xi^\mu = (\xi_1^\mu, ..., \xi_n^\mu)$, where the pattern μ of neuron i is denoted by ξ_i^μ for $\mu = 1, ..., m, i = 1, ..., n$.

Table 1: Comparison table

Graph	n	m	c_1	c_2	Example
G_2	5	4	192	24	$\eta = (1,1,1,1,1)$ $\xi^1 = (-1,-1,-1,-1,-1)$ $\xi^2 = (-1,1,1,-1,-1)$ $\xi^3 = (1,-1,-1,1,1)$ $\xi^4 = (1,1,1,1,1)$
G_3	5	4	6144	768	$\eta = (1,1,1,1,1)$ $\xi^1 = (-1,-1,-1,1,1)$ $\xi^2 = (-1,-1,1,-1,1)$ $\xi^3 = (-1,1,-1,-1,1)$ $\xi^4 = (1,1,1,1,1)$

5 Summary of findings

5.1 Definition of the weights given pattern size 2 or 4

Comparing the six different graphs, given either two or four number of patterns, we have proved that not all of the weights belonging to the particular graphs will be defined. The results are stated below where an x indicates that the graph is defined on the given pattern size provided in the top row.

Table 2: Comparison table

Graph	Def. on 2 patterns	Def. on 4 patterns
K_n	x	x
G_1	x	x
G_2	x	x
G_3		x
G_4		
G_5		x

We conclude that the weights of the graphs G,G_1, and G_2 are defined given two and four patterns, the weights of the graphs G_3,G_5 are defined given four patterns, and the weights of the graph G_4 are undefined given two patterns and given four patterns.

5.2 The delta energy function

We have studied the delta energy function for various incomplete graphs, applying asynchronous network dynamics. The result is provided in the table below.

Denote by $\beta = 0, ..., 10$, an index corresponding to each row in table (3) below and the set of all treated graphs by $\tilde{G} = \{G_1, G_1^{P_\kappa}, ..., G_5\}$. Denote the graph $\tilde{G}_\beta \in \tilde{G}$, the network size n_β, pattern size m_β and index k_β of the neuron which state is to be updated corresponding to row β. Furthermore denote by x_β the variable $x_\beta = (\tilde{G}_\beta, n_\beta, m_\beta, k_\beta)$, and given x_β denote the difference in energy at time $t = 0$ by $F(x_\beta) = \Delta H^{\tilde{G}_\beta}(0)\Big|_{k=k_\beta}$, where $h_k(\vec{\sigma}(0))|_{k=k_\beta} = \sum_{j=1}^{n_\beta} w_{k_\beta j}\eta_j = \sum_{j=1}^{n_\beta}(\frac{1}{n_\beta}\sum_{\mu=1}^{m_\beta}\xi_{k_\beta}\xi_j)\eta_j$. Denote by D_β the codomain of the map $f : x_\beta \to F(x_\beta)$ given x_β where asynchronous network dynamics is assumed.

Table 3: Codomain of the map $f : x_\beta \to F(x_\beta)$ given x_β, where asynchronous network dynamics is assumed.

β	Graph, G_β	Network size, n_β	Pattern size, m_β	k_β	Codomain D_β of f
0	G_1	4	2	k	$\{0, -\frac{10}{5}\}$
1	G_1	4	$m \leqslant 16$	k	$\{0, -2, -(m-(m-4)), ..., -(m-2), -m\}$
2	$G_1^{P_\kappa}$	4	2	k	$\{0, -\frac{20}{5}\}$
3	G_2	5	2	k	$\{0, -\frac{8}{5}, -\frac{16}{5}\}$
4	$G_2^{\varepsilon_0}$	5	2	j_2	$\{0\}$
5	$G_2^{\varepsilon_1}$	5	2	j_2	$\{0, -\frac{8}{5}, -\frac{16}{5}\}$
6	$G_2^{\varepsilon_2}$	5	2	j_1	$\{0, -\frac{8}{5}\}$
7	$G_2^{\varepsilon_3}$	5	2	j_1	$\{0, -\frac{8}{5}, -\frac{16}{5}\}$
8	G_3	5	2	k	$\{0, -\frac{16}{5}, -\frac{32}{5}\}$
9	G_4	5	2	k	not defined
10	G_5	5	2	k	not defined

Comparing the graphs we conclude that

$$\max F(x_\beta) = 0, \quad \beta = 0, ..., 10, \tag{91}$$

and that

$$\begin{aligned} \min F(x_8) < \min F(x_2) < \min F(x_3) = \min F(x_5) = \min F(x_7) \\ < \min F(x_0) < \min F(x_6) < \min F(x_4) = F(x_4), \end{aligned} \tag{92}$$

where additionally

$$\min F(x_1) < \min F(x_8), \text{ if } m_1 > \tfrac{32}{5}, \tag{93}$$

must hold.

Conclusion. In the 4 node graph G_1 there are fewer non-zero values for $\Delta^G H(0)$ compared to the 5 node graphs excluding the special cases. Thus we favor the 5 node graphs. The star graph G_3 contain the zero-value so that $\Delta^G H(0) = 0$ not just for the trivial case $\sigma_k(1) = \sigma_k(0)$, which is not favorable. The graphs G_4, G_5 are not favorable since we can not create the weights corresponding to these graphs with 2 number of patterns. Thus excluding special cases, we favor graph G_2 which we interpret to have the richest profile. However, including special cases, the graph G_1 with an arbitrary m (restricted) number of patterns, has the possibility to include the largest number of non-zero values for the incremental drop in energy depending on the value of m.

5.3 Stable patterns

Here we refer to Table 1 in Sect. 5.

6 Conclusion

The purpose of this thesis was to analyze the convergence properties of the energy function and the stability of the patterns of a Hopfield model on an incomplete graph, with the aim to study the Hopfield model on an incomplete graph.

We have provided examples of a Hopfield model on incomplete graphs such that the weights corresponding to the given graph can be expressed in terms of its patterns for $2, 4$ or m number of patterns as defined previously. Treated examples includes a cycle graph, path graph and a star graph. Results has been presented, containing exact values for the incremental drop in energy for the treated graphs under asynchronous network dynamics. In the case that the weights corresponding to the graph can not be expressed with the given number of patterns, a theoretical proof has been included. We have treated special cases such as an incomplete graph as a union of two independent components, and deluting one edge in an already incomplete graph using a deterministic delute variable. Results from computer simulations on the stability of the patterns and the number of configurations, was presented for two of the provided incomplete graphs as defined in preceding sections. Furthermore we have drawn conclusions on the richness of profile for the incremental drop in energy of the Hopfield model on different incomplete graphs, based on these findings.

Further research could examine the richness of profile for the Hopfield model on given or randomly generated incomplete graphs but for a larger number of vertices's and patterns.

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