Lund University/LTH

Bachelor thesis

FMAL01

# Approximation of $\pi$

**Abstract**

The inverse tangent function can be used to approximate $\pi$. When approximating $\pi$ the convergence of infinite series or products are of great importance for the accuracy of the digits. This thesis will present historical formulas used to approximate $\pi$, especially Leibniz–Gregory and Machin's. Also, the unique Machin two-term formulas will be presented and discussed to show that Machin's way of calculate $\pi$ have had a great impact on calculating this number.

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June 13, 2019

## Contents

# 1 Introduction

The number $\pi$ is defined as the relationship between the circle's circumference and its diameter and it has been of great interest amongst many mathematicians throughout time. The number $\pi$ is an irrational number, which means that $\pi$ can not be written as a ratio of integers, and there are different proofs showing that this indeed is the case. However, the proofs are rather complicated and this is the reason why the first proof that $\pi$ is an irrational number was not established until 1770 by Johann Heinrich Lambert.

Many mathematicians proved different formulas to obtain an approximation of $\pi$, however, an important difference between the formulas is how much effort it requires to compute $\pi$ to a certain accuracy. One way of approximating $\pi$ is to use infinite series, that is, the sum of the terms of an infinite sequence. The infinite series did not really start to play a role within mathematics until the second part of the $17^{\text{th}}$ century [16]. Some renowned mathematicians that have calculated $\pi$ with this mathematical tool are James Gregory and Gottfried Wilhelm Leibniz, as well as John Machin. A Japanese Google employee, Emma Haruka Iwao, has recently calculated $\pi$ to 31.4 trillion digits using Google's cloud-based compute engine. This approximation is the one with most digits ever calculated.

This thesis will introduce some history of $\pi$ as well as different methods to calculate its digits. Also, in particular, study the Gregory–Leibniz formula, Machin's formula and Machin-like formulas, and the convergence rate of their respective approximations of $\pi$.

# 2 History

The history of $\pi$ goes back a long way in history, starting already in the ancient Egypt around 1650 before Christ (B.C.). Mathematicians in Greece, China and Medieval Arabo–European countries have over the years also contributed to the development of this fascinating number.

## 2.1 – 500 Anno Domini

The number $\pi$ can be traced back all the way to the ancient Egypt. The main source of our knowledge of historic calculations of $\pi$ comes from ancient Egypt and the so called Rhind Mathematical Papyrus from the Middle Kingdom of Egypt which was found in 1858 but has been dated back to 1650 B.C. The papyrus contains a collection of 85 mathematical problems. Solutions to these problems have also been found and the solutions were written down by a man called Ahmes, who also is known as "the earliest known contributor to the field of mathematics" [2, p.9]. One of these problems, number 50, includes calculating the area of a circle, shown in the two figures below.

Figure 1: A part of the original Rhind Mathematical Papyrus [13].

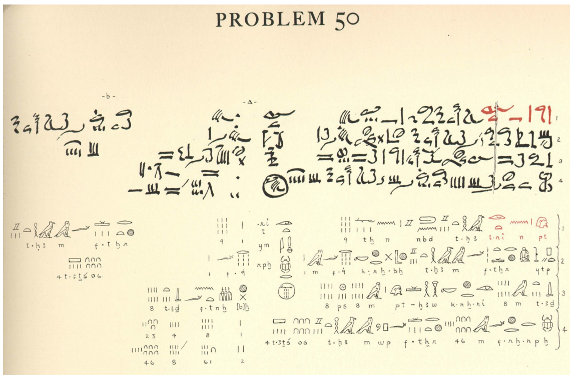


Figure 2: Problem 50, written out fair, of the Rhind Papyrus, which gives rise to an approximation of $\pi$ [14].

Today the translation of problem 50, figure (2), is: Example of a round field of diameter 9 khet. What is its area? According to Ahmes, the answer was 64 setat [3, p.1]. Khet is a length measurement and a setat is a measurement of area, $(\text{khet})^2$ . The units are very old and not used today.

The method for calculating this problem was according to the rule: shorten the diameter of the circle by one ninth to get the side of a square [3, p.3]. This can be written in formulas as

$$\pi r^2 = \pi(\frac{d}{2})^2 \approx (\frac{8d}{9})^2.$$

The approximation of $\pi$ using the modern formula for the area of a circle would then be as follows

$$\pi(\frac{d}{2})^2 = A \Rightarrow \pi \approx (\frac{2}{9})^2 \cdot 64 \approx 3.1605$$

where $d$ is the diameter and $A$ the area of the circle. The error is small, since $\pi \approx 3.1416$ for the first four decimals, so the Egyptians calculated areas of circles with a rather good accuracy for being back in 1650 B.C. It is however worth mentioning that this small error in $\pi$ might have a large impact on calculations [3, p.3]. For example calculating a circle of radius 50 meters gives, with nine correct decimals, an area of approximately 7854 $m^2$, meanwhile, with $\pi$ as Ahmes calculated the same calculations for this circle gives 7901 $m^2$. The difference is almost 50 $m^2$ and this is a rather large error when building for instance a tower.

An even better approximation of $\pi$ was made by the mathematician Archimedes around 200 B.C. His approach was to draw a regular hexagon inscribed in a circle and another circumscribed, then, simultaneously doubling the number of sides of the polygons and count the sides of each of the two at each step of doubling. The goal was to eventually have the polygons sides being so short that they almost coincide with the circle. This method is known as the principle of exhaustion [3, p.7–14].

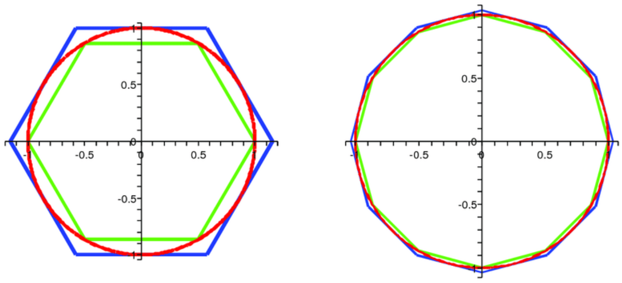


Figure 3: In– and circumscribed circle. From the left; hexagon (inscribed in green and circumscribed in blue) and a doubled hexagon, dodecagon. This shows the idea of the principle of exhaustion.

As the number of sides increase, the approximation of the circle becomes more accurate. After doubling the regular hexagons four times each there were 96 sides in each polygon and Archimedes could determine an interval containing $\pi$ by calculating the area of the inscribed and circumscribed polygon [3, p.14]. Since the decimal number system was not established yet the interval was written as

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

and is read as $3.1408 < \pi < 3.1429$. By using geometry, Archimedes approximated $\pi$ to be somewhere between 3.1408 and 3.1416, which is a rather good approximation compared to the real value of $\pi \approx 3.1416$.

In ancient Asia, there were also mathematicians working to achieve an approximation of $\pi$. Amongst these there are two men that stand out prominently: Liu Hui in the 3rd century and Zu Chongzhi in the 5th century. Liu Hui first discussed why the ratio of the circumference of the diameter was generally taken as 3 and later derived a more precise value of $\pi$. The work is similar to that of Archimedes, but instead of using both an inscribed and a circumscribed hexagon, Liu Hui only used one hexagon inscribed in the circle. Also, Hui adopted the decimal number system to his calculations to make the answer look more elegant. The decimal number system was developed in China but only worked for five decimals (starting from $10^{-1}$ they were called; fen, li, hao, miao and hu) and from the sixth decimal the digits were represented by fractions. Liu Hui increased the number of sides of the polygons more than Archimedes and in contrast to Archimedes, Hui did not encapsulate $\pi$ in an interval but gave one precise number [3, p.22]. By comparing the circle's diameter with its circumference Liu Hui approximated $\pi$ to $\frac{3927}{1250} = 3.1416$ [3, p.31].

Later, Zu Chongzhi adapted Liu Hui's method and improved the calculations for $\pi$ even more. The achieved ratio was $\frac{355}{113} \approx 3.1415929$ where only the last decimal is wrong. This result or approximation was not improved in almost 1,000 years until the Indian mathematician Madhava of Sangamagrama (ca.1350–1425) calculated $\pi$ with 11 correct decimals. Eventually, the Persian mathematician and astronomer Ghiyath al–Kashi (1380–1429) calculated $\pi$ to 16 correct decimals [3, p.24].

### 2.2 1500 – 1800 Anno Domini

Not until the end of the $16^{\text{th}}$ century the method of calculating $\pi$ changed. It was the French solicitor and amateur mathematician, Françoise Viète, who first introduced the infinite product in 1579. The same basic method as Archimedes had presented was used, namely, the principle of exhaustion. However, the polygons had 393216 sides and the calculations were described as an infinite product. In his book, "Variorum de rebus mathematicis responsorum, liber $VIII$" [18] the polygons were represented by triangles. By dividing the polygon into triangles, Viète could observe a relationship between the circumference of a polygon of $n$ sides and another polygon with $2n$ sides and this relationship was $\cos\theta$ [2, p.32]. His infinite product was obtained by using this relationship and in Section 2.4.1, the proof for this formula is performed using trigonometric identities. The formula Viète achived was

$$\frac{2}{\pi} = \lim_{n\to\infty} \prod_{n=1}^{N} \frac{a_n}{2} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}}} \cdots$$

where $a_n = \sqrt{2 + a_{n-1}}$ and $a_1 = \sqrt{2}$.

For Viète, the formula was not very convenient since it demands many iterations over $n$ to obtain just a few correct decimals. However, this discovery of the infinite product was an important step for future mathematics [2, p.33].

In 1649, John Wallis presented a different infinite product obtained by using analytic calculus, equation (1). This mathematical formula became very important for the later so-called integral calculus [2, p.40]. In contrast to Viète's formula this one did not contain any square roots and therefore it was easier to handle than Viète's, however, it was still not feasible to calculate $\pi$ this way since many terms had to be used in order to obtain just a few correct digits of $\pi$ [3, p.68–77]. The formula that Wallis presented was

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \tag{1}$$

and its proof follows in Section 4.2.2. Something that is rather interesting with this product is its pattern: the first term is larger than $\pi$, the second is smaller and then the third is larger again and then this continues.

Later, a series for $\frac{\pi}{4}$ was obtained independently by Gottfried Wilhelm Leibniz (1646–1716), James Gregory (1638–1675) and an Indian mathematician whose identity is unknown but usually ascribed to Nikalantha. The formula obtained is the arctan series and is named after Leibniz and Gregory, Gregory–Leibniz formula, equation (2). The arctan series was obtained in different ways by each of these mathematicians but none of them used the principle of exhaustion since $\pi$ was not the actual number that they tried to calculate.

The Gregory–Leibniz series has had a great impact on future mathematics since many mathematicians worked out other formulas due to the arctan series [3, p.92].

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} + \dots \tag{2}$$

In 1706 the mathematician John Machin improved the approximation of $\pi$ by expressing the difference between two arctangents as

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}. \tag{3}$$

In 1706, a mathematician named William Jones published his work and used the Greek symbol $\pi$ for the first time to represent the number. This symbol was chosen because it is the first letter in the Greek word for circumference, $\pi\varepsilon\rho\iota\mu\varepsilon\tau\rho o\sigma$ (read as perimetros). The mathematician Leonard Euler used the symbol $\pi$ in his work and it was after this that the symbol $\pi$ became more frequently used to describe the number [2, p.77].

Later, in 1750 the number $\pi$ had been expressed by infinite series, infinite products as well as infinite continued fractions. In 1766–1767 Johann Heinrich Lambert proved the irrationality of $\pi$, which means that the number can not be expressed as a fraction and the decimal expansion is not periodic and nor does it terminate [3, p.141]. The idea was first to prove that the continued fraction expression

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{\vdots}}}}$$

holds. Then he showed, using the argument of infinite descent, that if $x \neq 0$ is rational then the right hand side in his formula is irrational. Since $\tan \frac{\pi}{4} = 1$ is rational this implies that $\pi$ is irrational [10]. This proof is rather complicated but later in 1974 a different proof was presented by the number theorist Ivan Niven which is referred to as a “simple” proof, however, it is still very technical. Worth mentioning is that, more proofs showing that $\pi$ is irrational exists but is not discussed in this paper.

## 2.3 1800 – Anno Domini

An English man called William Shanks lived between 1812 and 1882 in a small village close to the Scottish border. He was particularly interested in mathematical constants, and his most ambitious project was a record-setting computation of $\pi$ to 707 decimal places. It took Shanks almost 20 years to compute, by hand, the digits to $\pi$ and he published his results in 1873. The computation was made using Machin’s formula, equation (3). Unfortunately, in 1944 Shanks approximation was proved wrong by F. D. Ferguson, who discovered that only the first 527 decimal places were right. Ferguson found this error by using the formula

$$\frac{\pi}{4} = 3\arctan\frac{1}{4} + \arctan\frac{1}{20} + \arctan\frac{1}{1935}$$

and making a computation using a calculator, then comparing his result with the one from Shanks [9].

Moreover, in 1882, the original proof of the transcendency of $\pi$ was published by Ferdinand von Lindemann. That is, $\pi$ is not a solution to any polynomial equation with rational coefficients. This proof has been simplified by other mathematicians such as Weierstrass, Hilbert, Gordon and many more [11, p.169].

Around the 1950's and onwards computers have been used to calculate $\pi$ with iterative algorithms. A Japanese Google employee, Emma Haruka Iwao, has recently calculated $\pi$ to 31.4 trillion digits using Google's cloud-based compute engine [15]. This approximation is the one with most digits ever calculated and mathematicians are still working on algorithms to achieve even better approximations.

Worth mentioning is that this many digits are not used for any calculations today. For example, the radius of the visible universe is about 46 billion light years. To calculate the circumference of a circle with this radius with an accuracy equal to the diameter of a hydrogen atom, which is approximately $1.06 \cdot 10^{-10}$ m, one will need 39 or at most 40 decimal places for $\pi$ [8].

## 2.4 Proofs

### 2.4.1 Viète's formula

$$\frac{2}{\pi} = \lim_{n\to\infty}\prod_{n=1}^{N}\frac{a_n}{2} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}} \cdots$$

Viète's formula can be proved using the trigonometric identities $\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{\cos(\theta+1)}{2}}$. First, these two trigonometric formulas will be proved and after that the proof of Viète's formula will follow.

**Theorem 1.** *Double angle formula for sine*

$$\sin 2\theta = 2\cos\theta\sin\theta$$

for all $\theta$.

*Proof.* Using the sum-formula $\sin(\theta+\beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$. Now, set $\beta = \theta$ then

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\cos\theta\sin\theta.$$

□

**Theorem 2.** *Half-angle formula for cosine*

$$\cos\frac{\theta}{2} = \sqrt{\frac{\cos\theta + 1}{2}}$$

*when* $-\pi \leq \theta \leq \pi$.

*Proof.* From the unit circle it is known that $1 = \sin^2\theta + \cos^2\theta$, also, the double angle formula for cosine is $\cos 2\theta = \cos^2\theta - \sin^2\theta$ (proven in a similar way as for sine above). Now, using these equations results in

$$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1 \Leftrightarrow 2\cos^2\theta = \cos 2\theta + 1$$

and replacing $\theta$ by $\frac{\theta}{2}$ gives

$$\cos^2\frac{\theta}{2} = \frac{1}{2}(\cos\theta + 1) \Rightarrow \cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta + 1}{2}}$$

for $\theta \neq \pi n$, where $n$ is an odd integer. □

*Proof.* Now, the proof of Viète's formula will follow using the double angle formula for sine with three different $\theta$, starting with $\theta$ and $\frac{\theta}{2}$,

$$\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} \tag{4}$$

$$\sin\frac{\theta}{2} = 2\cos\frac{\theta}{4}\sin\frac{\theta}{4}. \tag{5}$$

Now, substituting $\sin(\frac{\theta}{2})$ with equation (5) gives

$$\sin\theta = 2\cos\frac{\theta}{2}\Big(2\cos\frac{\theta}{4}\sin\frac{\theta}{4}\Big). \tag{6}$$

By substituting $\theta$ in equation (4) with $\frac{\theta}{4}$ gives

$$\sin\frac{\theta}{4} = 2\cos\frac{\theta}{8}\sin\frac{\theta}{8}$$

and finally, using this result in (6) gives

$$\sin\theta = 2\cos\frac{\theta}{2}\Big(2\cos\frac{\theta}{4}\Big(2\cos\frac{\theta}{8}\sin\frac{\theta}{8}\Big)\Big). \tag{7}$$

Referring to equation (4) as $n = 1$, (6) as $n = 2$ and (7) as $n = 3$ and expand for even more $n$ this can be written as a product

$$\sin\theta = 2^n \cdot \cos\frac{\theta}{2} \cdot \cos\frac{\theta}{4} \cdots \cos\frac{\theta}{2^n} \cdot \sin\frac{\theta}{2^n}$$

and dividing this expression by $\theta$ results in

$$\frac{\sin\theta}{\theta} = \frac{2^n}{\theta}\cdot\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^n}\cdot\sin\frac{\theta}{2^n}$$

also known as the normalized $sinc\theta$ function given by the Fourier transform from the rectangle function. Re-writing $\frac{2^n}{\theta}$ as $\frac{1}{\frac{2^n}{\theta}}$ we get

$$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^n}\cdot\frac{\sin\theta/2^n}{\theta/2^n}. \tag{8}$$

Taking the limit on $\frac{\sin(\theta/2^n)}{(\theta/2^n)}$ as $n\to\infty$ results in

$$\lim_{n\to\infty}\frac{\sin(\theta/2^n)}{(\theta/2^n)} = 1$$

where the known limit $\lim_{x\to\infty}\frac{\sin x}{x} = 1$ was used.

Now, for large $n$ it can be concluded that equation (8) can be approximated as

$$\lim_{n\to\infty}\frac{\sin\theta}{\theta} = \lim_{n\to\infty}\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^n}$$

To get Viète's formula $\theta$ is substituted to $\theta = \frac{\pi}{2}$.

$$\lim_{n\to\infty}\frac{\sin(\pi/2)}{(\pi/2)} = \frac{1}{(\pi/2)} = \lim_{n\to\infty}\cos\frac{\pi}{4}\cdot\cos\frac{\pi}{8}\cdots\cos\frac{\pi}{2\cdot 2^n}$$

Finally, using the formula for half angle of cosine in the equation above gives

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}}\cdots$$

and Viète's formula is proven. □

### 2.4.2 Wallis's formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\Big(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\Big) = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots$$

*Proof.* The integral

$$I_n = \int_0^{\pi}\sin^n x\,dx \tag{9}$$

can be re-written as

$$\int_0^{\pi}\sin^n x\,dx = \int_0^{\pi}\sin^{(n-1)} x\cdot\sin x\,dx$$

and applying integration by parts results in

$$\int_0^\pi \sin^{(n-1)} x \cdot \sin x \, dx = \Big[ -\sin^{(n-1)} x \cdot \cos x \Big]_0^\pi + \int_0^\pi (n-1) \cos^2 x \cdot \sin^{(n-2)} x \, dx.$$

Using the trigonometric identity $1 = \cos^2 x + \sin^2 x$ then $\cos^2 x$ can be expressed as $1 - \sin^2 x$. Now integral (9) can be written as

$$\int_0^\pi \sin^n x \, dx = 0 + \int_0^\pi (n-1)(1 - \sin^2 x) \sin^{(n-2)} x \, dx$$

and finally

$$\int_0^\pi \sin^n x \, dx = \frac{n-1}{n} \int_0^\pi \sin^{(n-2)} x \, dx. \tag{10}$$

This yields that $I_n$ can be written as

$$I_n = \frac{n-1}{n} I_{n-2} \tag{11}$$

where $I_0 = \pi$ and $I_1 = 2$. Now, the even and odd subscripts are separated as

$$\begin{aligned} I_{2n} = \frac{2n-1}{2n} I_{2n-2} &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} I_{2n-4} \\ &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots I_0 \\ &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \pi = \pi \prod_{n=1}^{k} \frac{2n-1}{2n} \end{aligned} \tag{12}$$

and

$$\begin{aligned} I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} I_{2n-3} \\ &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots I_1 \\ &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots 2 = 2 \prod_{n=1}^{k} \frac{2n}{2n+1} \end{aligned} \tag{13}$$

where step before the step in both equations are obtained from the definition of $I_0$ and $I_1$. For $x$ in the interval $[0, \pi]$ we have that $0 \leq \sin x \leq 1$ and

$$0 \leq \sin^{(2n+2)} x \leq \sin^{(2n+1)} x \leq \sin^{2n} x$$

which gives

$$0 < \int_0^\pi \sin^{(2n+2)} \, dx \leq \int_0^\pi \sin^{(2n+1)} x \, dx \leq \int_0^\pi \sin^{2n} x \, dx$$

and re-writing this in terms of $I_n$

$$0 < I_{2n+2} \leq I_{2n+1} \leq I_{2n}.$$

Using equation (11) we have

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2} \tag{14}$$

and the inequality (14) can now be re-written as

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1. \tag{15}$$

Moreover, taking the limit as $n \to \infty$ one gets

$$\frac{I_{2n+1}}{I_{2n}} \to 1$$

which is equivalent to

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} \to 1.$$

Finally, by using equation (11) and (12)

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = \prod_{n=1}^{\infty} \Big(\frac{2n-1}{2n} \cdot \frac{2n+1}{2n}\Big) = 1$$

and taking the recipricol

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \Big(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\Big)$$

which is Wallis product formula. □

### 2.4.3 Irrationality of $\pi$

This is a proof by contradiction based on the proof given by Ivan Niven in 1974 where the main idea is to state two properties, prove them both and finally conclude that they contradict each other [3, p.276]. In addition to Niven's proof there are some extra calculations, verifications and arguments added to give the reader a better understanding. Step 1 together with 2 proves Property 1, meanwhile, Step 3 together with 4 proves Property 2. The proof below will show that these two properties contradicts each other.

*Proof.* First, assume that $\pi$ is a rational number $\pi = \frac{a}{b}$ where $a$ and $b$ are integers and $b > 0$. For a positive integer $n$, which is specified later, we define the polynomial $f(x)$ based on $a$ and $b$ as

$$f(x) = \frac{x^n(a-bx)^n}{n!}. \tag{16}$$

Also, $F(x)$ is defined with even order derivatives of $f(x)$ and alternating sign as

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) + \cdots + (-1)^n f^{(2n)}(x). \tag{17}$$

*Property* 1. The integral

$$\int_0^\pi f(x) \sin x \, dx$$

is always an integer.

*Step* 1. It follows from the binomial theorem that $n!f(x)$ is a polynomial with integer coefficients and $n \leq \deg(x) \leq 2n$. Clearly, $f^{(k)}(0) = 0$ for $0 \leq k \leq n$ (as well as for $k > 2n$) and if $n \leq k \leq 2n$, by the binomial theorem we have

$$f^{(k)}(0) = k!\frac{p}{n!}$$

for any integer $p$, so $f^{(k)} = 0$ is an integer for each $k$. This means that $f(x)$ and $f^{(j)}(x)$ has integer values at $x = 0$, which is also the case when $x = \frac{a}{b} = \pi$, since

$$\begin{aligned} f(a/b - x) &= \frac{(a/b - x)^n(a - b(a/b - x))^n}{n!} \\ &= \frac{(a/b - x)^n(a - a + bx))^n}{n!} = \frac{x^n(a-bx)^n}{n!} = f(x) \end{aligned}$$

where $f(x)$ is the same as equation (16). It can be concluded that the corresponding function $f(x)$ and its derivative have integer values for both $x = 0$ and $x = \frac{a}{b}$.

*Step* 2. From elementary calculus we have

$$\begin{aligned} \frac{d}{dx}(F'(x)\sin x - F(x)\cos x) &= F''(x)\sin x + F'\cos x - F'(x)\cos x + F\sin x \\ &= F''(x)\sin x + F(x)\sin x \\ &= (F''(x) + F(x))\sin x = f(x)\sin x \end{aligned}$$

where the last step is achieved by taking the derivative of equation (17) two times and then add $F(x)$, eventually, what is left is $f(x)$. We have now computed one anti-derivative of $f(x)\sin x$ and the next step is to take the integral over $[0, \pi]$.

$$\int_0^\pi f(x)\sin x \, dx = \left[F'(x)\sin x - F(x)\cos x\right]_0^\pi = F(\pi) + F(0) = F(\frac{a}{b}) + F(0)$$

Now, we know that this integral always have to be an integer by Step 1 so Property 1 has been established.

*Property* 2.

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!}$$

where $\frac{\pi^n a^n}{n!} \to 0$ as $n \to \infty$.

*Step* 3. It can be seen that $0 < f(x)\sin x$ since in the interval $0 < x < \pi$ it follows that $0 < \sin x$. Moreover, studying equation (16) it can be seen that for $0 < x < \pi$ all values are greater than 0. Multiplying two positive functions gives a positive function and so the lower bound is proved.

Further, in the interval $0 < x < \pi$ the function $\sin x \leq 1$ and for $f(\pi)$ we obtain

$$f(x) = \frac{x^n(a-bx)^n}{n!} < \frac{\pi^n a^n}{n!}$$

since $x^n$ is growing and is at its largest $\pi^n$, also, $(a-bx)^n$ is positive and decreasing as $x$ grows, which gives $(a-bx)^n < a^n$. Multiplying these two positive function does not change the sign so the upper bound is proved. This means that Property 2 holds.

*Step* 4. In this step we will integrate

$$\int_0^\pi 0\,dx < \int_0^\pi f(x)\sin x\,dx < \int_0^\pi \frac{\pi^n a^n}{n!}\,dx \Leftrightarrow 0 < \int_0^\pi f(x)\sin x\,dx < \frac{\pi^{n+1}a^n}{n!} \tag{18}$$

and consider the Taylor series expansion

$$\pi e^{\pi a} = \pi + \frac{\pi^2 a}{1!} + \frac{\pi^3 a^2}{2!} + \frac{\pi^4 a^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{\pi^{n+1}a^n}{n!}.$$

Since $\pi e^{\pi a}$ is a convergent series, tends to a limit, the terms must tend to 0 as $n$ gets large. This implies that the right hand side in equation (18) tends to 0 as $n \to \infty$.

To summarize, Property 2 states that the integral will be between 0 and 1 for sufficiently large $n$, meanwhile, Property 1 states that the integral is always an integer. These two properties contradicts each other and therefore it can be concluded that our premise was wrong and $\pi$ is an irrational number. $\square$

# 3 Mathematical background

## 3.1 Precision and accuracy

When approximating a number, it is not certain that the approximated number is correct and there are different descriptive phrases commonly used depending on what is referred to. For instance, to describe the correctness of the digits in any number one is usually referring to the precision, amount of digits, and accuracy, correctness of digits. The error may occur for a variety of reasons, for example truncation, rounding or inherited errors [5, p.56].

*Example.* $\pi \approx 3.1416$.

$$\pi \approx 2.984327654 \tag{19}$$
$$\pi \approx 3 \tag{20}$$
$$\pi \approx 3.14160 \tag{21}$$
$$\pi \approx 3.1416 \tag{22}$$
$$\pi \approx 3.1415927 \tag{23}$$

Of the five statements above (19) is the most precise but least accurate, (20) is not very precise but it is accurate, (21) is more precise than (22) but the last digit in (21) is not accurate and finally (23) is accurate in all digits given.

### 3.2 Convergence of series

In mathematics, and especially when working with finite or infinite series, convergence is fundamental. In order to study convergence you will have to look at the first $N$ terms of the expansion, the partial sum, and observe any changes in the limit as $N \to \infty$ [7].

*Definition.* An infinite series, with a sequence of real numbers $a_1, a_2, a_3, \dots$ is defined as

$$\sum_{n\geq 1} a_n = \lim_{x\to\infty} \sum_{n=1}^{N} a_n.$$

If the limit exists in $\mathbb{R}$ then the infinite series is convergent. On the other hand, if the limit does not exist the series is called divergent.

An infinite series can be split up in its partial sum $S_N$, and the error $R_N$, as

$$S = \sum_{n=1}^{\infty} a_n = S_N + R_N$$

and it is the error that is related to the correctness of the approximated digits [6, p.3].

A series can converge slowly or more rapidly. Generally, a more rapid convergence is desired since the amount of computation needed to calculate the approximated number to any given accuracy is reduced. When reference is made to a more rapid convergence, such reference means that fewer terms are required to approximate the series with a good accuracy. In other words, the partial sum $S_N$ settles down [6, p.1].

For different $N$ (number of terms included in the partial sum) the approximation of the series can be determined to be in an interval

$$S_N \pm R_N^* = \begin{cases} S_N - R_N^* \\ S_N + R_N^*. \end{cases} \tag{24}$$

where $R_N^*$ is an upper bound for $|R_N|$, also called the absolute error.

However, there is nothing that says that the calculated partial sum obtained to calculate the interval above is the correct one and this problem may occur due to rounding, which will be discussed in the next section [11, p.59].

Instead, the estimated error $R_N^*$ is studied to determine the accuracy of the approximated number. This error can be estimated with different methods, for instance, with integrals. The estimated error gives information about the accuracy of the approximated digits in the partial sum $S_N$ by showing which digit that is affected. When referring to $t$ correct decimals one means that

$$|S - S_N| \leq 0.5 \cdot 10^{-t}$$

and all digits in position with units $\geq 10^{-t}$ is said to be significant digits.

*Example.*

$$\begin{aligned} a &= 1.414 \\ R^* &= 0.22 \cdot 10^{-3} \\ 1.41378 \leq\ & a \leq 1.41422 \end{aligned}$$

The error is $\leq 0.22 \cdot 10^{-3} \leq 0.5 \cdot 10^{-3}$ and $t = 3$ which means that there are three correct decimals.

*Example.*

$$\begin{aligned} a &= 1.414 \\ R^* &= 0.77 \cdot 10^{-3} \\ 1.41323 \leq\ & a \leq 1.41477 \end{aligned}$$

The error is $\leq 0.77 \cdot 10^{-3} \leq 0.5 \cdot 10^{-2}$ so there are two correct decimals.

### 3.3 Rounding

When working with a large number of decimals it is necessary to use rounding. That is, making the number more simple with less digits than available but still keeping it close to its actual value. Rounding is an efficient tool when working with numbers that have many digits but only needs a simple representation of it, in other words, fewer digits. However, rounding can also be a problem when one wants the correct representation and the rounding is done even though it is not wanted.

Rounding almost always occurs when calculating with computer programs, since by rounding the storage space in the computer can be used in a more efficient way. When an infinite series is approximated with a finite number of terms the rounding error has an impact on the approximation. This impact will be greater as the number of terms increases. Here, the rounding error is the difference between the actual result from a given algorithm and the result given from the same algorithm using a finite number of terms.

The round-off error appears because computers use float point numbers, finite and discrete, instead of real numbers, infinite and continuous. Floating point numbers are used in computers to store and represent real numbers. Depending if the computer has 32 or 64 bits, which are the most common, the amount of numbers which the computer can correctly represent differs. The number of bits are referred to as the computers word length and limits the computer in how many digits it can represent. The components to represent real numbers are the sign, exponents and the mantissa, see figure (4).

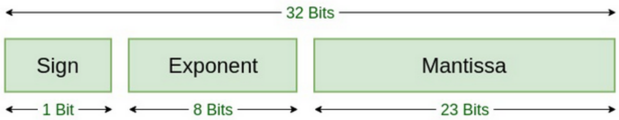


Figure 4: 32 bit number, IEEE Floating-Point Standard.

Moreover, when adding a very small number to a very large this would make the final sum to be shifted just a little bit on the number line for real numbers. However, this is not the case for the computer. Since the computer represents digits in $\mathbb{R}$ with floating point numbers this number line has gaps, see figure (5). Considering the first part of any partial sum, $\sum_{i=1}^{N-n} a_i$, and adding a small number from the remaining part, $\sum_{i=n}^{N} a_i$, instead of shifting just a little bit on the number line the computer rounds off and stays where it already is and this leads to a cancellation error. Computers are working with a restricted number of binary significant digits. It is constantly rounding and making trade-off between size and precision.

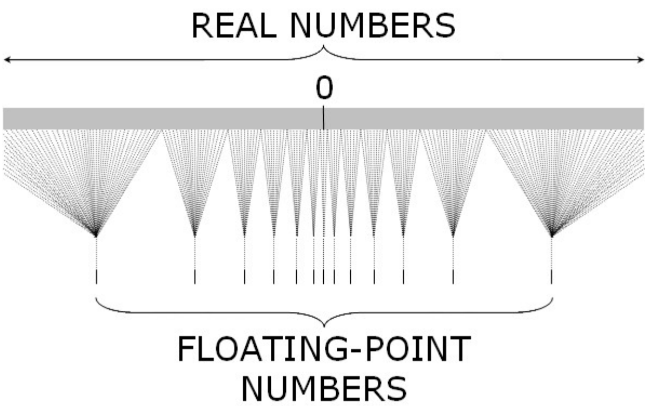


Figure 5: Real number line, infinite and continuous, versus floating point numbers, finite and discrete [4].

If the last values are very small in comparison to the first ones then the last ones might not have any impact on the approximation and therefore be left out, however, this last sequence might actually have an impact on the approximation. The propagation of the error will affect the approximation in such way that it is less accurate. One way to minimize the round-off error is, if $N$ is known, to estimate the series backwards [4, p.20–33]. The forward and backwards estimations can be written as

$$\begin{aligned} S_{\text{forward}} &= x_1 + x_2 + \cdots + x_{N-1} + x_N \\ S_{\text{backwards}} &= x_N + x_{N-1} + \cdots + x_2 + x_1 \end{aligned}$$

where the result of computing backwards is that the "tail" of decimals, which did not have any effect on the rounded number before, may have a greater impact now since they have been added up.

*Example.* Calculations of the harmonic series

$$S_N = \sum_{n}^{N} \frac{1}{n}$$

for $N = 10^7$ Matlab gives:

$$\begin{aligned} S_{\text{forward}} &= \sum_{1}^{N} \frac{1}{n} = 16.695311365857272 \\ S_{\text{backwards}} &= \sum_{N}^{1} \frac{1}{n} = 16.695311365859965 \end{aligned}$$

and it can be seen that the four last decimals are different which means that there is a rounding error. Also, worth mentioning is that when computing the harmonic series using fixed precision it looks like it is convergent but this is not the case. The wrong assumption is due to float number representation and word length in the computer. In this thesis the partial sum, $S_N$, may differ depending on the number of terms included and also what formula that has been used. Some formulas are more sensitive to rounding and as a result $S_N$ will differ.

One can conclude that to determine the correctness of an approximation it is important to have knowledge about different types of uncertainties such as precision, accuracy, error, sources and propagation of uncertainties. In this thesis, if nothing else is stated, approximations of series will mainly be calculated forward.

# 4 Gregory–Leibniz formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \tag{25}$$

## 4.1 History

As mentioned in Section 2.2, the arctan formula was obtained independently in around 1670 by Gottfried Wilhelm Leibniz (1646–1716), James Gregory (1638–1675) and an Indian mathematician (ascribed to Nikalantha) [3, p.92].

Leibniz earned a doctor's degree in law in February 1667 but studied mathematics as a side project. In 1672, Leibniz visited the prominent European physicist and mathematician, Christiaan Huygens. After discussions regarding physics and mathematics, Huygens provided Leibniz with the letters of Pascal, published under the name of Dettonville. Leibniz' studies of these letters played an important role in his development as a mathematician. Leibniz obtained the arctan formula by expanding $\frac{1}{1+x^2}$ as an infinite series and integrating term by term. Eventually, Leibniz was able to present the arctan formula and his results has been celebrated among many mathematicians [3, p.94–96].

Gregory was a Scottish mathematician who died from sudden illness and therefore the greater part of his work was never published. His main interest in mathematics was to find an infinite series representation of any given function. In his first book Gregory introduced important mathematical ideas, such as convergence and algebraic and transcendent functions. Gregory was a good friend of a man named John Collins and through him Gregory came in contact with English mathematicians, such as Isac Newton and John Pell. On December 24, 1670, Collins sent Gregory Newton's series for $\sin x$, $\cos x$, $\arcsin x$ and $x \cot x$ [3, p.97–101]. In a famous letter to Collins on February 15, 1671, Gregory gives expansions of different series, amongst others, $\arctan x$, which he thought was Newton's method [3, p.87–91]. This was not the case, instead he had found the relationship between the series and successive derivation of the given function, now known as the Taylor series. Gregory did not recognize his discovery and therefore his work was never published. Instead Brook Taylor presented his similar work almost forty years later [3, p.97–101].

The mathematician from southern India, Nilakantha, had knowledge of the formula already in the middle of the $15^{\text{th}}$ century. This persons work, written in Sanskrit, was only discovered around 1835, in other words many years after Gregory and Leibniz had already presented the formula [16].

## 4.2 Proof

The proof of the arctan series varies between the three mathematicians, however, all of them play some part in today's most commonly used proof, which is made with Maclaurin expansion.

**Theorem 3.**

$$\begin{aligned}\arctan x = \int_0^x \frac{1}{1+t^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt\end{aligned} \tag{26}$$

*Proof.* Using the expansion of the finite geometric series

$$\frac{1-z^{n+1}}{1-z} = 1+z+z^2+z^3+\cdots+z^n$$

for $z \neq 1$ and adding $\frac{z^{n+1}}{1-z}$ on both sides

$$\frac{1}{1-z} = 1+z+z^2+z^3+\cdots+z^n+\frac{z^{n+1}}{1-z}$$

is achieved. Changing the variable $z=-t^2$ gives

$$\frac{1}{1+t^2} = 1-t^2+t^4-\cdots+(-1)^n t^{2n}+(-1)^{n+1}\frac{(-t^2)^{n+1}}{1+t^2}. \tag{27}$$

Now, $\frac{1}{1+t^2}$ is the derivative of $\arctan t$ and taking the antiderivative of (27) on the interval $[0,x]$ then

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+(-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}dt \tag{28}$$

is obtained. What remains to be proved is that the integral in (28) tends to zero as $n\to\infty$. For $|x|\leq 1$

$$\left|\int_0^x \frac{t^{2n+2}}{1+t^2}dt\right| \leq \int_0^{|x|} t^{2n+2}dt$$

where the right hand side can be re-written as

$$\int_0^{|x|} t^{2n+2}dt = \left[\frac{t^{2n+3}}{2n+3}\right]_0^{|x|} = \frac{|x|^{2n+3}}{2n+3}. \tag{29}$$

Since we have defined $|x|\leq 1$ this gives

$$\frac{|x|^{2n+3}}{2n+3} \leq \frac{1}{2n+3} \tag{30}$$

and as $n\to\infty$ the right hand side in equation (30) goes to zero, which proves that (26) holds. Further, for $x=1$ in equation (28) Gregory–Leibniz formula

$$\arctan 1 = \frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+(-1)^n\frac{1}{2n+1}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$$

is achieved. $\square$

## 4.3 Convergence

Gregory–Leibniz' formula converges very slowly, approximately 5000 terms must be calculated to obtain an accuracy of three digits for $\pi$ [11, p.54].

Equation (28) can also be written as $\arctan(x) = S_N(x) + R_N(x)$, recall from Section 3.2, where

$$S_N(x) = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and

$$R_N(x) = (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}.$$

In equation (30) it is shown that

$$R_N(x) \leq \frac{|x|^{2n+3}}{2n+3} \tag{31}$$

and in Gregory–Leibniz series for $x = 1$

$$\arctan 1 = \frac{\pi}{4} = S_N(1) + R_N(1)$$

$$\pi = 4S_N(1) + 4R_N(1) = 4\sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{2n+1} + 4R_N(1).$$

By studying the error term in equation (31) it can be seen that the error decreases with $2n$ which means that more terms will give a better accuracy for the approximation. To show that this series converges rather slowly values for $N = 1000, 5000$ will follow respectively and starting with $N = 1000$ gives

$$\pi \approx 4S_{1000}(1) = 4\sum_{n=0}^{1000} (-1)^n \frac{1}{2n+1}.$$

Computation in MatLab gives

$$\pi \approx 4S_{1000}(1) = 3.142591654339544$$

and the error estimation is

$$4R_{1000}(1) \leq \frac{4}{2003} < \frac{4}{2000} = 0.2 \cdot 10^{-2}.$$

Studying the error one can conclude that there will be an impact on the third decimal in $S_{1000}$, this gives that 3.14 are the only correct digits. The result of these calculations shows that with 1000 terms two correct decimals are obtained. Now, $N = 5000$ will be calculated which will give a better result than the previous calculation.

$$\pi \approx 4S_{5000}(1) = 4\sum_{n=0}^{5000} (-1)^n \frac{1}{2n+1}$$

Computation in MatLab gives

$$\pi \approx 4S_{5000}(1) = 3.141792613595791$$

and the error estimation is calculated to

$$4R_{5000}(1) \leq \frac{4}{10003} < \frac{4}{10000} = 0.4 \cdot 10^{-3}$$

Studying the error it shows that there will be an impact on the fourth decimal in $S_{5000}$ and one can conclude that 3.141 are the only correct digits.

After calculations one can see that a lot of terms are needed to get the third decimal correct. Since this formula was found before calculators and computers it was not feasible to do the calculations. However, the Gregory–Leibniz formula converge more rapidly if $x = 1$ is replaced by $x = \frac{1}{\sqrt{3}}$ as Abraham Sharp did and found 71 digits in 1699 [11, p.55].

### 4.4 $x = \frac{1}{\sqrt{3}}$

A rather small change to the Gregory–Leibniz formula gives a noticeable improvement to the convergence, which will be shown in this section. Changing the argument of the arctan series to $x = \frac{1}{\sqrt{3}}$ gives

$$\begin{aligned}
\arctan\Big(\frac{1}{\sqrt{3}}\Big) &= \frac{1}{\sqrt{3}} - \frac{\left(\frac{1}{\sqrt{3}}\right)^3}{3} + \frac{\left(\frac{1}{\sqrt{3}}\right)^5}{5} - \dots = \\
&= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\Big(\frac{1}{\sqrt{3}}\Big)^2 + \frac{1}{5}\Big(\frac{1}{\sqrt{3}}\Big)^4 - \dots\right) = \\
&= \frac{1}{\sqrt{3}}\Big(1 - \frac{1}{9} + \frac{1}{45} - \dots\Big) = \frac{\pi}{6}.
\end{aligned} \tag{32}$$

Now, to determine the speed of convergence the error is studied again by dividing the equation into

$$\begin{aligned}
\frac{\pi}{6} &= S_N\Big(\frac{1}{\sqrt{3}}\Big) + R_N\Big(\frac{1}{\sqrt{3}}\Big) \Leftrightarrow \\
\pi &= 6S_N\Big(\frac{1}{\sqrt{3}}\Big) + 6R_N\Big(\frac{1}{\sqrt{3}}\Big)
\end{aligned}$$

where

$$S_N\Big(\frac{1}{\sqrt{3}}\Big) = \sum_{n=0}^{N}(-1)^n \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1} = 2\sqrt{3}\sum_{n}^{N}(-1)^n\Big(\frac{1}{\sqrt{3}}\Big)^{2n}\frac{1}{2n+1} = 2\sqrt{3}\sum_{n}^{N}\frac{(-1)^n}{3^n(2n+1)}$$

and

$$6R_N\Big(\frac{1}{\sqrt{3}}\Big) \leq 6\frac{\left(\frac{1}{\sqrt{3}}\right)^{2N+1}}{2N+3} = \frac{2\sqrt{3}}{3^{N+1}(2N+3)}$$

where the error is approximated with equation (29). Comparing this error with equation (31) it can be seen that here it decreases much more rapid, as there is a dominant factor of $3^N$ in the denominator, that is, fewer terms will be needed for the series to converge. For $N = 4, 10$ the new approximation of $\pi$ will be calculated. Starting with $N = 4$ gives

$$\pi \approx 6S_4\Big(\frac{1}{\sqrt{3}}\Big) = 3.142192613595791 \tag{33}$$

$$6R_4\Big(\frac{1}{\sqrt{3}}\Big) \leq \frac{2\sqrt{3}}{3^5 \cdot 11} < 0.2 \cdot 10^{-2} \tag{34}$$

and here, the error says that there is an uncertainty in decimal three. So with this change of variable a correctness for two decimals is obtained by calculating the first four terms, that is, the convergence is much more rapid than before. Now, for $N = 10$

$$\pi \approx 6S_{10}\Big(\frac{1}{\sqrt{3}}\Big) = 3.141593304503082 \tag{35}$$

$$6R_{10}\Big(\frac{1}{\sqrt{3}}\Big) \leq \frac{2\sqrt{3}}{3^{11} \cdot 23} < 0.9 \cdot 10^{-6} \tag{36}$$

and this time the error will effect decimal seven and six. It can be concluded that with $N = 10$ terms there will be five correct decimals, 3.14159. These two examples show that just a small adjustment to $x$ can make the infinite series, used to approximate $\pi$, converge much more rapidly and as a result give more accurate digits for fewer terms in the approximation.

The so called Theodorus's constant $\sqrt{3}$ is an irrational number and to determine the fraction $\frac{1}{\sqrt{3}}$ it is necessary to first make calculations of $\sqrt{3}$ in order to calculate $\pi$. Just like there are different formulas for calculating $\pi$ this yields for $\frac{1}{\sqrt{3}}$ as well. For instance, this number can be approximated by infinite series. As a result of using $\sqrt{3}$, Abraham Sharp's formula, equation (32), was rather difficult to calculate, especially by hand.

# 5 Machin's formula

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \tag{37}$$

## 5.1 History

John Machin was a mathematician and astronomer who lived between 1680 and 1752. Not much is known about Machin, but it has been established that he had a great interest in human diseases and he was able to describe these in a fascinating way. Actually, he was the first one to describe "Distemperd Skin" which today is known as Ichthyosis Hystrix, a rare skin disorder. Not only did he work on diseases but he also took a big interest in mathematics and astronomy. In 1710 he was elected a Fellow of the Royal Society (a Fellowship for many of the British most eminent scientists), serving as their secretary [1].

Before Machin started as a secretary of the Royal Society he improved the convergence of the inverse tangent method $\frac{\pi}{4} = \arctan x$, using smaller arguments and created an alternative arctan series which converges more quickly. In 1706, Machin calculated $\pi$ to 100 digits and the formula played an important part in the calculation of decimals for $\pi$ from the beginning of the 18th century until the end of the 20th century [11, p.58].

## 5.2 Proof

The proof of Machin's formula will be made with the sum and difference identity for the tangent function. First, the tangents identities will be proved and secondly proof of Machin's formula will be provided.

**Theorem 4.** *Tangent of difference*

$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v} \tag{38}$$

*for all $u$ and $v$.*

*Proof.* Tangents of difference can be re-written with $\sin(u - v)$ and $\cos(u - v)$ as

$$\tan(u - v) = \frac{\sin(u - v)}{\cos(u - v)} = \frac{\sin u \cos v - \sin v \cos u}{\cos u \cos v + \sin u \sin v}$$

and dividing the numerator and denominator with $\cos u \cos v$ assuming $u, v \neq \frac{\pi}{2} + \pi n$, for $n \in \mathbb{N}$, gives

$$\tan(u - v) = \frac{\frac{\sin u \cos v - \sin v \cos u}{\cos u \cos v}}{\frac{\cos u \cos v + \sin u \sin v}{\cos u \cos v}} = \frac{\tan u - \tan v}{1 + \tan u \tan v}.$$

which is the tangents of difference. $\square$

Tangent of sum

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

is obtained by changing $v$ to $-v$ and using that the tangent function is odd.

*Proof.* We want to prove that the right hand side of equation (37) is equal to the left hand side and taking tangents on both sides of equation (37) gives

$$\tan \frac{\pi}{4} = \tan\left(4 \arctan \frac{1}{5} - \arctan \frac{1}{239}\right) \tag{39}$$

where $\tan \frac{\pi}{4} = 1$. Using tangents difference formula with $u = 4 \arctan \frac{1}{5}$ and $v = \arctan \frac{1}{239}$ the right hand side can be re-written as

$$\frac{\tan\left(4 \arctan \frac{1}{5}\right) - \tan\left(\arctan \frac{1}{239}\right)}{1 + \tan\left(4 \arctan \frac{1}{5}\right) \tan\left(\arctan \frac{1}{239}\right)} = \frac{\tan\left(4 \arctan \frac{1}{5}\right) - \frac{1}{239}}{1 + \tan\left(4 \arctan \frac{1}{5}\right)\left(\frac{1}{239}\right)}. \tag{40}$$

Now, $\tan\left(4\arctan\frac{1}{5}\right)$ can be expressed as

$$\tan\left(4\arctan\frac{1}{5}\right) = \tan\left(2\arctan\frac{1}{5} + 2\arctan\frac{1}{5}\right)$$

and using tangents sum formula then

$$\frac{\tan\left(2\arctan\frac{1}{5}\right) + \tan\left(2\arctan\frac{1}{5}\right)}{1 - \tan\left(2\arctan\frac{1}{5}\right)\tan\left(2\arctan\frac{1}{5}\right)}$$

is obtained. Re-writing the expression above one more time gives

$$\frac{\tan\left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right) + \tan\left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right)}{1 - \tan\left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right)\tan\left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right)}$$

and using tangents sum formula once more results in

$$\frac{\frac{\tan\left(\arctan\frac{1}{5}\right)+\tan\left(\arctan\frac{1}{5}\right)}{1-\tan\left(\arctan\frac{1}{5}\right)\tan\left(\arctan\frac{1}{5}\right)} + \frac{\tan\left(\arctan\frac{1}{5}\right)+\tan\left(\arctan\frac{1}{5}\right)}{1-\tan\left(\arctan\frac{1}{5}\right)\tan\left(\arctan\frac{1}{5}\right)}}{1 - \frac{\tan\left(\arctan\frac{1}{5}\right)+\tan\left(\arctan\frac{1}{5}\right)}{1-\tan\left(\arctan\frac{1}{5}\right)\tan\left(\arctan\frac{1}{5}\right)} \cdot \frac{\tan\left(\arctan\frac{1}{5}\right)+\tan\left(\arctan\frac{1}{5}\right)}{1-\tan\left(\arctan\frac{1}{5}\right)\tan\left(\arctan\frac{1}{5}\right)}}.$$

For $\tan(\arctan\frac{1}{5}) = \frac{1}{5}$ we get

$$\frac{\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}} + \frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}}}{1 - \frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}} \cdot \frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}}} = \frac{2\cdot\frac{50}{120}}{1-\left(\frac{50}{120}\right)^2} = \frac{\frac{5}{6}}{1-\frac{2500}{14400}} = \frac{72000}{71400} = \frac{120}{119}$$

and putting this result in to equation (40) gives

$$\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}} = \frac{\frac{28561}{28441}}{\frac{28561}{28441}} = 1$$

which means that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

and Machin's formula is proved. $\square$

## 5.3 Convergence

Due to the choice of smaller $x$-values, $x = \frac{1}{5}$ and $x = \frac{1}{239}$, in the arctan series Machin's formula converges more rapidly than those formulas presented earlier and this will be shown in the same way as in Section 4.2. By using the Maclaurin expansion of arctan, equation (28) Machin's formula

$$\begin{aligned} \frac{\pi}{4} &= 4\Big(\sum_{n=0}^{N}(-1)^n\frac{\left(\frac{1}{5}\right)^{2n+1}}{2n+1} + (-1)^{n+1}\int_0^{\frac{1}{5}}\frac{t^{2n+2}}{1+t^2}\Big) \\ &\quad - \Big(\sum_{n=0}^{N}(-1)^n\frac{\left(\frac{1}{239}\right)^{2n+1}}{2n+1} + (-1)^{n+1}\int_0^{\frac{1}{239}}\frac{t^{2n+2}}{1+t^2}\Big) \end{aligned} \tag{41}$$

is obtained. Now $S_N$ and $R_N$ is extracted.

$$\begin{aligned} S_N &= 4\sum_{n=0}^{N}(-1)^n\frac{\left(\frac{1}{5}\right)^{2n+1}}{2n+1} - \sum_{n=0}^{N}(-1)^n\frac{\left(\frac{1}{239}\right)^{2n+1}}{2n+1} \\ &= \frac{4}{5}\sum_{n=0}^{N}\frac{(-1)^n}{5^{2n}\cdot(2n+1)} - \frac{1}{239}\sum_{n=0}^{N}\frac{(-1)^n}{239^{2n}\cdot(2n+1)} \end{aligned} \tag{42}$$

$$R_N = 4\cdot(-1)^{n+1}\int_0^{\frac{1}{5}}\frac{t^{2n+2}}{1+t^2} - (-1)^{n+1}\int_0^{\frac{1}{239}}\frac{t^{2n+2}}{1+t^2}.$$

These expressions gives

$$\frac{\pi}{4} = S_N + R_N \Leftrightarrow \pi = 4S_N + 4R_N$$

$$\pi = \frac{16}{5}\sum_{n=0}^{N}\frac{(-1)^n}{5^{2n}\cdot(2n+1)} - \frac{4}{239}\sum_{n=0}^{N}\frac{(-1)^n}{239^{2n}\cdot(2n+1)} + 4R_N$$

and as before, an estimation of the error is made using equation (29) and (30) so that

$$|4R_N| \leq \frac{16\cdot\left(\frac{1}{5}\right)^{2n+3}}{2n+3} + \frac{4\cdot\left(\frac{1}{239}\right)^{2n+3}}{2n+3} = \frac{16}{5^{2n+3}\cdot(2n+3)} + \frac{4}{239^{2n+3}\cdot(2n+3)}. \tag{43}$$

This time, the error will decrease even faster than for the Gregory–Leibniz formula with argument $x = 1$ and $x = \frac{1}{\sqrt{3}}$. Here, the dominant terms in the denominators are $25^n$ and $239^{2n}$ which will grow fast as $n$ increases and as a result the error will decrease fast. Once again, different number of terms will be studied starting with $N = 4$.

$$\pi \approx 4S_4$$

MatLab gives

$$\pi \approx 3.1415926824044$$

and the approximated error is

$$|4R_4| \leq \frac{16}{5^{11}\cdot 11} + \frac{4}{239^{11}\cdot 11} < 0.3\cdot 10^{-7} \tag{44}$$

Here, the error will effect decimal eight so with $N = 4$ terms there will be seven correct decimals, 3.1415926. Further, calculations for $N = 8$ gives

$$\pi \approx 4S_8$$

MatLab gives

$$\pi \approx 3.141592653589837$$

and the approximated error is

$$|4R_8| \leq \frac{16}{5^{19}\cdot 19} + \frac{4}{239^{19}\cdot 19} < 0.5\cdot 10^{-13}$$

Here, the error will effect decimal 14 and the 13:th decimal might be effected by rounding, so with $N = 8$ terms there will be 12 correct decimals, 3.141592653589.

It can be concluded that Machin's formula converges more rapidly than the Gregory–Leibniz formula and as a result gives more accurate digits to $\pi$ for fewer terms. In addition, there are no irrational numbers included as in the modified Gregory–Leibniz formula with $x = \frac{1}{\sqrt{3}}$ which makes Machin's formula easier to calculate compared to the Gregory–Leibniz formula.

# 6 Machin-like formulas

As has been shown in this thesis, the arctan series is of importance for the calculation of $\pi$ and depending on the argument $x$ the series can converge slowly or more rapidly. Starting with $x = 1$, the Gregory–Leibniz formula, it was concluded that this series converged far too slow in order to obtain digits with good accuracy. Later, it was shown that other values of $x$ give a more accurate approximation of digits to $\pi$, still, there were some difficulties with the argument $x = \frac{1}{\sqrt{3}}$ since this is a fraction with the irrational number $\sqrt{3}$. However, in Machin's formula two arctan terms are added together, with arguments even smaller than the other formulas shown in this thesis, and the formula converges much more rapidly than such previous shown formulas.

There are several inverse tangent formulas that have been influenced by the one Machin discovered and these are called Machin-type or Machin-like formulas. These formulas differ from the original Machin formula since the arguments and scale are different and can, for example, be generated using complex numbers.

Around 1800, mathematicians started to work with, what is now called Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, which revolutionized the search for arctan identities. In 1894, the mathematician Dmitry A. Grave published a problem requesting all rational solutions to

$$\frac{\pi}{4} = m \arctan \frac{1}{p} + n \arctan \frac{1}{q}.$$

Shortly after, this problem was generalized by Carl Störmer as

$$k\frac{\pi}{4} = m \arctan \frac{1}{x} + n \arctan \frac{1}{y}$$

for $(m, x, n, y, k) \in \mathbb{Z}$. Störmer proved that there were only four unique solutions when the nominator in the equation above is equal to one [12]. Later, certain gaps in the proof that Störmer published were filled in by another mathematician, Ljunggren, in 1942 [17].

By assuming that $k, m, n \geq 0, x \neq \pm y, x \neq \pm 1, y \neq \pm 1$ and $\gcd(m, n)$, greatest common divisor, Störmer's formula only has four unique solutions, namely the ones published by Machin, Euler, Hermann and Hutton, which will be shown later. Störmer completed his proof in 1894 and he found that Gauss had already observed the connection between complex integers and arctan.

Further, for the one-term formula it can be proven that

$$k\pi = n \arctan \frac{b}{a}$$

only has the integer solutions $b = 0$ or $a = \pm 1$ and this proof was published for the first time by Störmer. He writes this equation as

$$\rho \arctan \frac{b}{a} = k \frac{\pi}{4}$$

and assumed that $\rho$ and $k$ are positive with $\gcd(\rho, k) = 1$ and $\gcd(a, b) = 1$. Today, this can be shown by using Gaussian integers and will be proven in Section 6.3. Two key points when working with arctan and Gaussian integers are: the complex numbers can provide insight into real problems, and unique factorization, when it exists, is a powerful tool [17].

## 6.1 Machin-like formula

$$c_0 \frac{\pi}{4} = \sum_{n=1}^{N} c_n \arctan \frac{a_n}{b_n} \tag{45}$$

where $a_n$ and $b_n$ are integers such that $0 < a_n < b_n$, $c_n$ is a signed non-zero integer and $c_0 > 0$.

## 6.2 Two-term formula

From the expression of the Machin-like formula, equation (45), the two-term formula can be written as

$$\arctan \frac{a_1}{b_1} + \arctan \frac{a_2}{b_2} = \arctan \left(\frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}\right),$$
$$\text{for } \frac{-\pi}{2} < \arctan \frac{a_1}{b_1} + \arctan \frac{a_2}{b_2} < \frac{\pi}{2}$$

and the proof follows.

*Proof.* Using tangent of difference, equation (38), the two-term formula is proved. Let $x = \arctan u$ and $y = \arctan v$, where $u = \frac{a_1}{b_1}$ and $v = \frac{a_2}{b_2}$ then

$$\begin{aligned}
\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \\
&= \frac{\tan(\arctan u) + \tan(\arctan v)}{1 - \tan(\arctan u)\tan(\arctan v)} = \frac{u+v}{1-uv} = \\
&= \frac{\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 - (\frac{a_1}{b_1} \cdot \frac{a_2}{b_2})} = \frac{\frac{a_1}{b_1} \cdot \frac{b_2}{b_2} + \frac{a_2}{b_2} \cdot \frac{b_1}{b_1}}{(1 - \frac{a_1}{b_1} \cdot \frac{a_2}{b_2}) \cdot \frac{b_1 b_2}{b_1 b_2}} = \frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}.
\end{aligned}$$

This gives

$$\arctan u + \arctan v = \arctan \left(\frac{u+v}{1-uv}\right) = \arctan \left(\frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}\right)$$

and proves the two-term Machin-formula. $\square$

### 6.3 Complex factors

As stated earlier all Machin-type formulas can be expressed with complex factors. Studying complex numbers then the inverse tangent function can be written, up to an integer multiple of $\pi$, as

$$\arg((b_n + ia_n)^{c_n}) = c_n \arctan \frac{a_n}{b_n}. \tag{46}$$

Further, to observe the relationship between complex numbers and Machin-like formulas two factors with complex numbers and $c_n = 1$ are multiplied, that is, their associated arguments are added and this results in

$$\begin{aligned}(b_1 + ia_1) \cdot (b_2 + ia_2) &= b_1b_2 + ia_2b_1 + ia_1b_2 - a_1a_2 \\ &= b_1b_2 - a_1a_2 + i \cdot (a_1b_2 + a_2b_1).\end{aligned} \tag{47}$$

Now, using equation (46) with (47) the two-term Machin-formula is obtained when $n = 1, 2$ and $c_n = 1$ as

$$\arg\Big(b_1b_2 - a_1a_2 + i \cdot (a_1b_2 + a_2b_1)\Big) = \arctan\left(\frac{a_1b_2 + a_2b_1}{b_1b_2 - a_1a_2}\right)$$

and it has now been established that the two-term Machin formula can be represented with complex factors.

In general, for two complex numbers to fulfil Machin-like formulas it is necessary that the complex numbers, when multiplied with each other, can be factorized as $1 + i$ times an integer. In other words, the multiplication should result in

$$k(1+i)^{c_0} = \prod_{n=1}^{N} (b_n + ia_n)^{c_n}$$

where $k$ is an arbitrary integer. If this equation holds it is possible to achieve more than a two-term Machin-like formula and instead create a formula with more terms which is not only limited to the four unique two-term formulas.

As stated earlier, by using Gaussian integers and prime factors the single-term formula can be proved. For the lemma, corollary and the proofs below $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

*Lemma* 1. Let $z = a + bi \neq 0$ be a Gaussian integer. There is a natural number $n$ such that $z^n \in \mathbb{Z}$ if and only if $a = 0, b = 0$ or $a = \pm b$.

*Proof.* The backward direction follows by setting $n = 1, 2$ or $4$. In order to prove the forward direction, assume that $z^n = m$, where $m \in \mathbb{Z}$. Since the Gaussian integers is a unique factorization domain (UFD) [17]

$$z = p_1 \cdot p_2 \cdots p_k \cdot q$$

where $p$ is a Gaussian prime factor, $k \in \mathbb{Z}$ and $q$ is any unit $(\pm 1, \pm i)$ [17]. It follows from laws of exponents that

$$z^n = p_1^n \cdot p_2^n \cdots p_k^n \cdot q^n.$$

Since $m = p_k \cdot q$, it follows that, $p_k|m$ and further $m = p_k \cdot q \Rightarrow \overline{p_k}|\overline{m} \Rightarrow \overline{p_k}|m$ since $m = \overline{m}$. This means that all Gaussian prime factors that are not $1+i$ comes in pairs. The factor $1+i$ will not come in a pair since $1+i$ and $1-i$ is the "same" Gaussian integer, $1+i = (+i)(1-i)$ and $i$ is a unit. Further, since $(a+bi)(a-bi) = a^2+b^2$ this means that the contribution will only be integers. Also, ordinary primes $\equiv 3(mod\, 4)$ gives integer values. To summarize, $z$ is a product of integer values and a possible non-negative integer power of $(1+i)$. That is, $z = p \cdot (\pm i)(1+i)^k$, $p \in \mathbb{Z}$, can be split up in three different cases: $z$ is either just an integer value $p$ or only an imaginary part $p(\pm i)$ or the whole expression and this gives the only solutions $a = 0$, $b = 0$ or $a = \pm b$, which proves our lemma. □

*Corollary* 1. The only rational values of $\tan \frac{k\pi}{n}$ are 0 and $\pm$ 1 when $k \in \mathbb{Z}$.

*Proof.* Suppose that $\tan \frac{k\pi}{n} = \frac{b}{a}$ where $b \in \mathbb{Z}$ and $a \in \mathbb{N}$. Then

$$\arg(a+bi)^n = n \arg(a+bi) = \frac{n(k\pi)}{n} = k\pi$$

which, modulo $2\pi$ equals 0 or $\pi$, depending if $k$ is odd or even. Notice that when $(a+bi)^n \in \mathbb{Z}$ the fact that $\tan \frac{k\pi}{n} = 0$ or $\tan \frac{k\pi}{n} = \pm 1$ follows from the lemma. □

*Corollary* 2. Identities of the form $k\pi = n \arctan x$ with $x$ rational have $x = 0$ or $x = \pm 1$. In particular, $\pi = 4 \arctan 1$ is the most efficient such identity for computing $\pi$ using Gregory's series.

*Proof.* By applying tan on $\frac{k\pi}{n} = \arctan x$ gives $\tan \frac{k\pi}{n} = x = \frac{a}{b}$ and applying Corollary 1 with the Lemma gives Corollary 2. □

Moreover, there exists several of multiple-angle identities, not only for $b_1 = b_2 = 1$, to compute $\pi$ and this can be written as

$$\frac{k\pi}{n} = m_1 \arctan \frac{b_1}{a_1} + m_2 \arctan \frac{b_2}{a_2}$$

and the following example will show how to achieve one of these formulas.

*Example.* Let's pick a Gaussian prime, for instance $2+3i$ and denote $z_1 = 2+3i$, $z_2 = 2-3i$ and $m_1 = m_2 = n = 1$. Then the corresponding arctangent identity is: $0\pi = \arctan \frac{3}{2} - \arctan \frac{3}{2}$ which is useless. However, by introducing a factor $(1-i)$ then

$$z_1 = (2+3i)(1-i) = 5-i.$$

For $n = 4$ the corresponding identity is

$$\frac{\pi}{4} = \arctan \frac{3}{2} - \arctan \frac{1}{5}$$

and by using the formula for multi-angle identities more identities can be found.

### 6.3.1 Euler's formula

This formula is

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

and it can be rewritten with complex numbers. According to equation (46) these factors can be determined to be $(2+i)$ and $(3+i)$. Multiplying these two factors gives

$$(2+i)\cdot(3+i) = (6+2i+3i+i^2) = (5+5i) = 5(1+i)$$

and $(5+5i)$ has the angle $(\pi/4)$ in the first quadrant of the complex plane. This proves that Euler's two-term formula can be re-written with complex factors.

Just as in previous sections the convergence is of interest. In this calculation certain steps have been left out, but all steps are shown in the section where the convergence of Machin's formula is studied. Here, equations (29) and (30) gives

$$|4R_N| \leq \frac{4\cdot\left(\frac{1}{2}\right)^{2N+3}}{2N+3} + \frac{4\cdot\left(\frac{1}{3}\right)^{2N+3}}{2N+3} = \frac{1}{4^N\cdot 2(2N+3)} + \frac{4}{9^N\cdot 27(2N+3)}$$

as the error term. It can be seen directly from the approximated error term that this approximation will converge rather fast since the error decreases with $4^N$ in the first term and $9^N$ in the second. As an example, for $N=8$ then six correct decimals are obtained.

### 6.3.2 Machin's formula

As familiar, this formula is

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

and can be rewritten with factors of complex numbers following the same steps as for Euler's formula. Then

$$(5+i)^4\cdot(-239+i) = -2^2\cdot(13^4)\cdot(1+i)$$

and the estimated error for this formula is calculated in Section 5.3.

### 6.3.3 Hermann's formula

Hermman's formula is

$$\frac{\pi}{4} = 2\arctan\frac{1}{2} - \arctan\frac{1}{7}$$

and rewriting it with factors of complex numbers results in

$$(2+i)^2\cdot(-7+i) = (3+4i)\cdot(-7+i) = (-25-25i) = -25\cdot(1+i).$$

Now, the convergence is studied in the same way as in the section regarding Euler's two-term formula. The error term is

$$|4R_N| \leq \frac{4 \cdot 2 \cdot \left(\frac{1}{2}\right)^{2N+3}}{2N+3} - \frac{4 \cdot \left(\frac{1}{7}\right)^{2N+3}}{2N+3} = \frac{1}{4^N \cdot (2N+3)} - \frac{4}{49^N \cdot 343(2N+3)}$$

and here it can be seen that this formula will converge faster than Euler's formula, since $49^N > 9^N$. However, it does not converge faster than Machin's formula since $25^N$ and $239^{2N}$ is larger than those in Hermann's formula.

### 6.3.4 Hutton's formula

The fourth and final two-term formula is

$$\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}$$

and in complex form this is written as

$$(3+i)^2 \cdot (7+i) = (8+6i) \cdot (7+i) = (50+50i) = 50 \cdot (1+i).$$

Here, the error term is

$$|4R_N| \leq \frac{4 \cdot 2 \cdot \left(\frac{1}{3}\right)^{2N+3}}{2N+3} + \frac{4 \cdot \left(\frac{1}{7}\right)^{2N+3}}{2N+3} = \frac{8}{9^N \cdot 27(2N+3)} + \frac{4}{49^N \cdot 343(2N+3)}$$

and it can be concluded, by studying the denominator, that not even Hutton's formula converges faster than Machin's. Comparing all the four two-term formulas it can be concluded that Machin's formula has the fastest convergence rate. However, all of these formulas converge rather rapidly comparing them to the formulas of Viète's, Wallis's and Gregory–Leibniz formula.

## 7 Conclusion

To conclude, the number $\pi$ has fascinated mathematicians throughout history. Since $\pi$ is an irrational number the decimal expansion is infinite and the search for more correct digits is an ongoing process. To decide if an infinite series or product is good or not, one have to study the convergence. Faster convergence means that less terms are needed in the expansion. Several different formulas can be used to compute $\pi$, but the most commonly used formula is the arctan formula, established by Gregory and Leibniz. In order to achieve a rapid convergence the argument needs to be small.

Instead of using just a single-term formula Machin found an arctan formula for two terms and not long after him three more unique two-term formulas were established by Euler, Hutton and Hermann. All of these formulas gave a smaller error term than the one-term formulas and as a result the two-term formulas converged more rapidly. This way, less terms were needed in the infinite series to obtain more correct digits.

Today, the Machin-like formulas are widely used to compute more and more decimals to $\pi$. We have now found so many decimals that the search for more decimals no longer serves any real purpose other than the hunt for more decimals itself. 40 decimals is namely more than enough to describe the world around us with extreme accuracy.

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