Analysis of Cryptocurrency volatility and statistical distributions using ARMA and GARCH-type models

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Abstract

The study aims to investigate and model statistical properties of Bitcoin and other major cryptocurrencies. There were recent drastic changes in the level of Bitcoin prices as it moved from $740 in 2014 to $19,187 in 2017, and down to $3,830 in 2018. The current study aims to fill the gap in the analysis of cryptoccurencies, primarily Bitcoin returns statistical process. Specifically, the study selects and estimates a model that traces dynamics of returns using ARMA, and also volatility of the residual from the model. To my knowledge, there is gap in the academic literature for the case of Bitcoin to use such approach.

The analysis involves data on past daily prices of Bitcoin, in British Pounds and USD, as well as those of Ethereum and Litecoin, both in Pounds. The results obtained from the study provide evidence that ARMA model in combination with eGARCH volatility model can be used for analysis of statistical process of Bitcoin returns. Moreover, significant statistical differences are identified between Bitcoin prices in UK Pounds and US Dollars. Although there is no evidence that shows the price level has an effect on volatility, significant decline is identified in the volatility of Bitcoin log-returns since 2018 as compared to the previous period. For each considered cryptocurrency, the current study determines the optimal specification of ARMA model, as well as GARCH-type model and optimal statistical distribution of the residual. The set of results can be used to estimate statistical process behind the cryptocurrency historical prices. Moreover, relation between ARMA(p, q) lag order, and type of optimal volatility model and residual distribution is explored, no significant relation is identified.

Keywords: Cryptocurrency, Bitcoin, Litecoin, Ethereum, volatility, ARMA, GARCH-type models, eGARCH, Student’s t-distribution, Laplace distribution, statistical distributions

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List of Abbreviations

ARMA	Auto regressive moving average
ARCH	Autoregressive conditionally heteroscedastic
GARCH	Generalized ARCH
eGARCH	Exponential GARCH
GJR-GARCH	Glosten, Jagannathan and Runkle GARCH
VaR	Value-at-risk
AIC	Akaike's information criteria
BIC	Bayesian information criteria
ICO	Initial coin offering
GMLE	Gaussian maximum likelihood estimator
LADE	Least absolute deviation estimator
ADF	Augmented Dickey-Fuller
ACF	Auto correlation function
PACF	Partial auto correlation function
CDF	Cumulative distribution function
PDF	Probability density function

1 Introduction

Background

This study explores statistical properties of Bitcoin and few other major cryptocurrencies, such as Litecoin and Ethereum, which are the most widely traded cryptocurrencies, by their market capitalization. Bitcoin was first introduced and documented by Satoshi Nakamoto in 2008. Bitcoin is a form of Cryptocurrency- an "electronic payment system based on Cryptographic proof" instead of traditional trust (Nakamoto, 2008). According to Kristoufek (2013), the wild fluctuations in Bitcoin price cannot be explained by economic and financial theory. Factors such as interest rates and inflation do not exist, as there is no central bank overseeing the issuing of Crypotocurrencies. Following Bouri, Azzi and Dyhrberg (2017), the market for cryptocurrencies exceeds market capitalization of any large company, and could in the near future achieve the size of the stock market. The size of the cryptocurrency market calls for obtaining sufficient empirical evidence about the statistical and distributional properties of the returns on these assets.

Analysed properties are those of the type of volatility characteristics, as well as distributional properties of Bitcoin residuals. In fact during 2014-2018, the price of Bitcoin moved from $740 to $19,187 and then down to $3,830. Moreover, following the success of Bitcoin, quite a number of other cryptocurrencies were launched. Two other major cryptocurrencies, which still have much lower market capitalization, are Litecoin and Ethereum.

Research questions

In the course of extensive statistical testing and evaluation, the study aims to obtain answers to the following research questions:

- What is the best ARMA-type model to model the return on Bitcoin, Litecoin and Ethereum?
- Does price level or changes in price level affect volatility of Bitcoin returns?

- What distribution and which conditional volatility model provide the best suit for the Bitcoin residuals?
- Are statistical properties of Bitcoin the same for Bitcoin prices in GBP and USD?
- Are there relations between ARMA(p, q) lag order and choice of optimal GARCH-type model and distribution of residuals?

The third research question is also addressed to Litecoin and Ethereum, in addition to Bitcoin. Moreover, Bitcoin price properties in GBP are compared to those of USD-priced Bitcoin, to evaluate whether the exchange rate has an effect on the Bitcoin behaviour. Such investigations of statistical properties of cryptocurrencies are of relevance for risk management of investors into cryptocurrencies. This is of high relevance as cryptocurrencies are used by a number of institutional investors, and also used on some major exchanges. That is appropriate levels of risk can be set and managed in a more effective way, when statistical properties of the cryptocurrencies are better understood.

Contribution of the study

The study relates to the literature that explores statistical properties of Bitcoin and other cryptocurrencies (Bouri et al., 2017; Chan et al., 2017) as well as studies that explore distributional properties of random variables (Devi et al., 2013; Huang et al., 2008). In fact a system to model Bitcoin returns is suggested, that includes three components, to model return, conditional volatility, and distribution of model residuals.

The main contributions are as follows. First, properties of cryptocurrencies returns are explored by combining the ARMA model for returns modelling and their residuals obtained are subsequently fitted in various relevant distributions. Then the log-returns from ARMA model are explored by using GARCH-type models. Second, statistical properties of Bitcoin in GBP are compared to those of Bitcoin in USD. The obtained results are different between the two, as different optimal ARMA models are identified, as well as different optimal distributions are determined, namely Student’s t-distribution for GBP price of Bitcoin and Laplace distribution for USD price. Third, the relation between ARMA (p, q) lag order, and type of optimal volatility model and residual distribution is explored, no significant relation is identified. The implemented analysis results in the better understanding of the statistical process underlying Bitcoin prices. Specific models and distributions are selected among considered alternatives, and can be applied for modelling future returns and volatility. The investigation of this study makes it possible that relevant statistical process behind movements in Bitcoin prices can be estimated.

Structure

The study involves a number of sections. Thus, section 2 provides brief overview of relevant literature. Then, section 3 discusses methods and models that are used for the analysis, including ARMA models, conditional volatility models, and estimation of distributions as well as evaluation of their fit. Section 4 provides the results of the findings from statistical analysis, which suggests a system to model residuals that includes ARMA(p,q) model to predict Bitcoin returns, as well as conditional volatility GARCH-type model, and estimation of the best fit distribution to model residuals. The last section explores whether there is relation between the three components. Finally, the main findings are summarized in the concluding section, as well as recommendations for the future research.

Data source

All the Cryptocurrencies' prices are obtained since 1st January 2014 up to 31st December 2018. All the data were obtained from the international investment portal "Investing.com".

All the three cryptocurrencies' prices are obtained in Pounds, while for Bitcoin also US Dollar prices are obtained. Then the Bitcoin Dollar prices are translated into the UK Pound with actual daily exchange rate.

Estimation of the considered models, as well as obtaining MLE estimates of the distributions, is performed using statistical package R (R Core Team, 2016). Relevant software packages are those that aim to investigate time series analysis, including *'tseries'* (Trapletti and Hornik, 2017), ‘*rugarch’* (Ghalanos, 2017), *‘roll’* (Foster, 2019), as well as ‘*propagate’* (Speiss, 2018). Previous researchers noted high relevance of R for time series analysis (Podgorski, 2015).

2 Literature Review

As cryptocurrencies appeared only in 2000s there has not been thorough investigation of statistical properties of Bitcoin and other cryptocurrencies. In fact, Bitcoin is known to be more volatile and more risky in comparison with traditional financial instruments such as stocks and bonds (Barker, 2019). In late 2010s, most trades in Bitcoin and other cryptocurrencies were implemented by Japan and the US. In fact, according to Jones (2017), Japan accounted for over 51% of Bitcoin transactions in 2017. In some countries, like China, issuing of new cryptocurrencies, so called ‘initial coin offerings’ (ICOs) are forbidden, due to lack of information about the behaviour and statistical properties of cryptocurrencies (Jones, 2017).

Engle (1982) introduced a model in which the variance at time *t* is modeled as a linear combination of past squared residuals and called it an ARCH (autoregressive conditionally heteroscedastic) process. Bollerslev (1986) introduced a more general structure in which the variance model looks more like an ARMA than an AR and called this a GARCH (generalized ARCH) process. Nelson (1991) suggested an even modified version of GARCH model, with the resulting process being called eGARCH (exponential GARCH). These approaches allow the standard deviation to eGARCH change with each observation. Glosten, Jagannathan and Runkle (1993) introduced GARCH with differing effects of negative and positive shocks taking into account the leverage phenomenon, which is called GJR-GARCH. All those mentioned GARCH-type models are the most well-known and used models for forecasting conditional variance and volatility (Hayashi, 2000).

Brooks (2014) tells that in finance it makes sense to work with returns. Few recent studies explored some statistical aspects of Bitcoin returns. These include Chan et al. (2017) and Chu et al. (2015), who investigated volatility of log-returns, and also distributional properties. First, Chu et al. (2015) evaluated the relevance of different GARCH-type models to better model the properties of cryptocurrencies volatility. Chu et al. (2015) determined model 'GJR-GARCH' as the one with the highest fit. Moreover, GJR-GARCH aims to capture asymmetry of innovations effect on conditional volatility.

Later, Chan et al. (2017) explored the suitability of a number of statistical distributions, including Laplace and Gaussian for explaining the returns. In case with both, Bitcoin and Litecoin, generalized hyperbolic distribution provided the best fit. The determined distributions of the best fit were subsequently used to evaluate value-at-risk (VaR), and expected shortfall measures of risk. In the above mentioned studies, researchers used 2.5 years of historical data on Bitcoin and other cryptocurrencies. The current study extends the time spans of available data, and chooses to evaluate five main strain distributions, where generalized hyperbolic distribution is not included.

This study aims to combine two parts, using GARCH-type models to model clustering volatility nature, and also investigating the goodness of fit of a number of alternative distributions to evaluate the residual of returns on the considered cryptocurrencies.

A review of GARCH-type models and approaches to their estimation was performed by Huang et al. (2008). The study considered two types of estimation methodologies, namely likelihood based on Gaussian Likelihood (GMLE), and another one that is log-transform-based estimator known as least absolute deviation estimator (LADE). The researchers concluded that the LADE estimation is a special case of GMLE estimator.

3 Modelling Methodology

This chapter reviews the methodology involved in the considered statistical models. These are the ARMA time series model for modelling returns, as well as GARCH-type models, as well as a set of considered distributions for modelling the residuals from ARMA models. In this study, the return R_t is defined as the relative difference of the prices, $R_t = \frac{P_t}{P_{t-1}}$ - 1 and the log-return is then calculated as $r_t = \log(P_t) - \log(P_{t-1})$, where P_t is the price at time t and P_{t-1} is the price at time $t-1$, i.e. one time unit before. One time unit is usually considered to be on basis of trading days.

3.1 Stationarity tests

With time series analysis, variables are needed to be stationary, according to Brooks (2014). Otherwise, regression modelling would be affected by spurious regression bias. Spurious regression bias is present when regression modelling shows significant relation between variables that do not have actual casual relation. For example, if two unrelated series grow similarly over time, regression can show an impact of one of these variables on the other.

A stationary series has constant mean, constant variance and constant auto-covariance to each given lag (Brooks, 2014). An autoregressive time series of order one is stationary when its auto-regression coefficient is smaller than unit. The stationarity of time series of higher orders depends on the value of coefficients in the model. Otherwise, the process is known to have a unit root, when the autoregressive coefficient is 1. When the autoregressive coefficient is greater than 1, the process is considered as "explosive". For this purpose, the study applies the Dickey-Fuller test to evaluate whether the time series is stationary or not. The relevant regression model for augmented Dickey-Fuller test (ADF) is as follows.

$$\Delta y_t = \psi y_{t-1} + \mu + \lambda t + u_t \qquad (1)$$

Under the null hypothesis the series *y* is non-stationary, which is equivalent to ψ=1. Equation (1) also includes the time invariant intercept μ and the time trend rate λ. The left hand-side variable is used in difference, that is why under H_0 of non-stationarity $\psi = 1$. When the null hypothesis is rejected, the time series is considered stationary. The coefficient is tested for significance using a standardized test-quantity with a nonstandard distribution. Critical values were obtained by Dickey and Fuller and are used for the purpose of this test in most statistical packages (Brooks, 2014).

3.2 ARMA Models

Mean values of a single variable are predicted using ARMA model. This type of univariate models is also conveniently used for forecasting purposes. ARMA model contains auto-regressive components (AR(p)) and moving average components (MA(q)). The considered time-series variable y_t therefore is modelled as a function of its AR(p) and MA(q) components as

$$y_t \ = \ \mu + \sum_{i=1}^{p} \varphi_i \, y_{t-i} + \sum_{j=1}^{q} \theta_j \, u_{t-j} + u_t \tag{2}$$

so that the time series is a linear combination of its past realisations (AR-part), and a linear combination of the past realisation of the model errors (MA-part).

The autocorrelation function (ACF) and partial autocorrelation function (PACF) are useful in determining the type of model for the given series. According to Brooks (2014), the following are peculiarities for AR, MA and ARMA processes. In case of an AR-process, ACF is geometrically decaying, and there are a number of non-zero points with PACF, which equal to the order of AR process. In case of a MA-process, there is geometric decay process with PACF, and the number of non-zero points with ACF indicates the order of MA process. For the ARMA process, which combines AR and MA, both functions, ACF and PACF, show geometric decay. In relation to ARMA model, researcher Devi, Sundar and Alli (2013) reported its high forecasting performance, and its ability to deliver high performance in comparison with modern data mining algorithms.

Box-Jenkins (1976) approach to building ARMA model involves three steps - identification, estimation, and diagnostic testing. Identification can be tested based on the use of ACF and PACF, or the use of information criteria. Estimation is in fact obtaining model coefficients, and diagnostic testing involves adding further components to test whether they are insignificant, furthermore there is also autocorrelation test of residuals.

3.3 Volatility GARCH models

Due to changing nature of volatility of financial series, relevant models are developed to capture such characteristics. The relevant models include ARCH, as well as GARCH model.

Subsequently a number of further models are suggested, such as eGARCH and GJR-GARCH models.

An ARCH model defines the current period variance as a function of the past realised residual. The formula below shows ARCH(1) model, but it can have higher order of parameters, ARCH(p) for p≥1.

$$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 \quad (3)$$

GARCH models further expand the above ARCH model. In addition to the ARCH component, it also includes a 'GARCH' component, which is the lagged conditional variance. The example below shows GARCH(1, 1) model, although it can be more general with *p* and *q* orders, respectively. Most widely used is GARCH (1, 1) model that fits most financial series (Brooks, 2014).

$$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (4)$$

A number of other models provide enhancement of standard GARCH model. Specifically, there are models that strive to capture asymmetry in the actual data. For examples eGARCH and GJR-GARCH models are used to capture asymmetric effects in the volatility of returns.

Magnitude of upwards and downwards volatilities are different, so that such effects are captured by these types of models. A generalized approach towards non-symmetric GARCH-type models was developed by Javed and Podgórski (2015). A popular model to capture non-symmetry is eGARCH. For this model, the asymmetry effect is captured by gamma (γ_1).

$$\log(\sigma_t^2) = \omega + \alpha_1 u_{t-1} + \gamma_1(|u_{t-1}| - E|u_{t-1}|) + \beta_1 \log(\sigma_{t-1}^2) \quad (5)$$

In the eGARCH model, the coefficient α_1 captures a size effect, while γ_1 captures 'sign' effect (Ghalanos, 2018).

The GJR-GARCH model faces the asymmetry between the positive and negative innovations in straightforward manner. This is captured with coefficient γ_1, which is the slope coefficient for the interaction variable that includes the innovation u_{t-1} and an indicator for a negative innovation I_{t-1},

$$\sigma_t^2 = \omega + \alpha_1 u_{t-1}^2 + \gamma_1 I_{t-1} u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (6)$$

where I_{t-1}=0, if $u_{t-1} \geq 0$ and $I_{t-1} = 1$, if $u_{t-1} < 0$.

Evaluation of the GARCH models is considered based on Akaike's information criteria (AIC) and Bayesian information criteria (BIC). Such model selections are implemented for each of the considered three cryptocurrencies- Bitcoin, Litecoin, and Ethereum. The analysis of volatility models in this study is performed for log-returns since volatility is defined as the annualized standard deviation of log-returns (Bennett et al., 2012).

3.4 Distribution Fitting

While returns on Bitcoin, and also other cryptocurrencies (Litecoin and Ethereum) are modelled using ARMA, the residuals from the model are obtained and their distribution is fitted using a number of alternative distributions. Most of the considered distributions in this section are symmetric, because the distributions of the residuals of cryptocurrencies shown in Figure 9 are rather more symmetric.

Residuals of returns gained from ARMA model on Bitcoin can be explored by considering what distribution is closest to the actual distribution of returns on Bitcoin and other cryptocurrencies. Such distributions are explored as Normal distribution, Skewed normal distribution, Student's t-distribution, Laplace distribution and Weibull distribution. The cumulative density functions (CDF) of these distributions are explored below, as well as their parameters are mentioned. The summary of the estimated parameters and evaluation of the fit is provided in section 4.4 below.

Normal distribution (Gaussian)

The following is the CDF of the general normal distribution where $x \sim N\,(\mu, \sigma^2)$:

$$F(x) = \Phi(\frac{x-\mu}{\sigma}) \quad (7)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt \quad (8)$$

is the CDF for the standardized normal distribution, $N(0,1)$.

The mean of normal distribution is defined on the whole set of real numbers, while variance has to be finite, and also non-negative. In fact, the use of Q-Q plots in this study is visually explored in order to evaluate whether the cryptocurrencies log-returns follow normal distribution.

Skewed normal distribution

A version of normal distribution that is not symmetric is referred to as skewed normal.

Its CDF is as follows:

$$F(x) = \Phi(\frac{x-\lambda}{\delta}) - 2T(\frac{x-\lambda}{\delta}, \alpha) \quad (9)$$

where λ is a location parameter, δ is a scale parameter, and α is a shape parameter. Here $T(\frac{x-\lambda}{\delta}, \alpha)$ is Owen's T-function (Owen, 1956), which is a function involving an integral of a weighted normal density function.

Relevance of this distribution is related to its applicability to a number of processes that are skewed and therefore cannot be fitted with normal distribution. In fact, the distributions of residuals from cryptocurrencies are further shown in the Figure 9 to be rather symmetric but not perfectly, so that this distribution could turn out to be relevant.

Student's t-distribution

Another alternative is Student's t-distribution, which entails relatively fatter tails, compared to the normal distribution. Student's t-distribution CDF is provided below. It involves the Gamma function $\Gamma(.)$ and hypergeometric function ${}_2F_1(.)$.

$$F(x) = \frac{1}{2} + \frac{x-\mu}{\sigma} \frac{\Gamma[\frac{1}{2}(\nu+1)]}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})} {}_2F_1[\frac{1}{2}, \frac{1}{2}(\nu+1); \frac{3}{2}; \frac{-(x-\mu)^2/\sigma^2}{\nu}] \quad (10)$$

Student's t-distribution involves location parameter μ, scale parameter σ (here, σ^2 is not the variance) and the shape parameter ν which is commonly known as "degree of freedom".

Laplace distribution

The Laplace distribution, also called the double exponential distribution, is the distribution of differences between two independent variates with identical exponential distributions (Abramowitz and Stegun, 1972).

It has CDF given by:

$$F(x) = \frac{1}{2}\ [1 + sgn(x-\mu)(1-e^{-|x-\mu|/b})] \tag{11}$$

Parameters of Laplace distribution involve location (μ, real number) and scale (*b,* positive real number). The Laplace distribution has been proved to be more relevant than the Gaussian for modelling in topography (Johannesson et al., 2017).

Weibull distribution

The Weibull distribution was first identified by Fréchet in 1927 and is named after Waloddi Weibull who was first to promote the usefulness of this distribution by modelling data sets from various disciplines (Murthy, Xie, and Jiang 2004).

The CDF is defined as:

$$F(x)= 1- exp[-(\frac{x-a}{\beta})^{\gamma}], \quad \text{for } x > a \tag{12}$$

$F(x) = 0$, for $x \leq a$. Here a represents the location parameter, γ represents the shape parameter and β represents scale parameter. The three-parameter Weibull distribution can be used to model both positive and negative observations.

3.5 Model selection

There are several steps for the analysis of Bitcoin, Litecoin, and also Ethereum. First, the dynamics of returns on cryptocurrencies is modelled using ARMA model. Second, volatility is modelled using clustered volatility models, and their performance is compared. Third, residual returns from ARMA model are obtained and their statistical properties are modelled using statistical distributions. Goodness of fit of these distributions is evaluated based on information criteria; the Akaike's information criterion (AIC) and the Bayesian information criterion (BIC). AIC (Akaike, 1973) is a popular method for comparing the adequacy of multiple, possibly nonnested models. An alternative model selection criterion is BIC, which penalizes more on the additional parameters of the model (Schwarz, 1978). Formulas for the information criteria are provided as follows:

$$\mathrm{AIC} = 2k - 2\ln L(\hat{\boldsymbol{\Theta}})$$

$$\mathrm{BIC} = k\ln n - 2\ln L(\hat{\boldsymbol{\Theta}})$$

where n is the number of data points in the training set, k is the number of parameters in the model, and L is the maximized likelihood of a model $\hat{\boldsymbol{\Theta}}$. $\hat{\boldsymbol{\Theta}}$ is an estimator of $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}$ is the vector of the relevant parameters of distribution. The most appropriate distribution results in the minimum information criterion.

4 Analysis of Results

This chapter provides statistical analysis results. Cryptocurrencies' returns are explored by combining the ARMA model for returns modelling and their residuals obtained are subsequently fitted in various relevant distributions. Then the approach is to fit different statistical distributions and then implement goodness of fit based on a number of errors and information criteria. The log-returns from ARMA model are explored by using GARCH-type models for modelling conditional variance. At last, to explore and evaluate the relations between ARMA(p,q) lag orders and choice of optimal GARCH-type models and the residual distributions.

4.1 Descriptive statistics

Firstly, statistical features of the distribution of the considered cryptocurrencies are considered. These include the analysis of the prices and daily log-returns. Among the three cryptocurrencies, the price (in £) is the highest for Bitcoin, then Ethereum, and on average, the lowest price is for Litecoin. All price level series are non-stationary, which is proved by augmented Dickey-Fuller test in Table 1.

Table 1 – Summary statistics of Cryptocurrency prices

	Bitcoin (£)	Bitcoin ($)	Litecoin (£)	Ethereum (£)
Min	116.9	183.0	2.8	5.0
1st Q	262.2	402.2	7.9	9.5
Median	476.7	652.9	39.4	156.8
Mean	1,990.4	2,677.4	49.4	189.5
3rd Q	3,372.4	4,371.9	64.1	292.4
Max	14148.9	18,934.0	268.6	948.1
St dev	2,624.2	3,520.6	49.2	200.2
ADF test*	-1.975	-1.935	-1.747	-1.294

** ADF Test significance: * - 10%, ** - 5%, *** - 1%*

Furthermore, basic statistical properties of log-returns are evaluated in Table 2. The min daily log-return of -12.36% is the lowest for Ethereum, and the highest daily log-return is 48.69% in case of Litecoin. On average, log-return is 0.02% for Bitcoin, 0.21% for Litecoin, and 0.13% for Ethereum. Among the three cryptocurrencies, the one with the highest volatility is Litecoin (sd=3.19%).

Table 2 – Summary statistics of Cryptocurrency daily log-returns

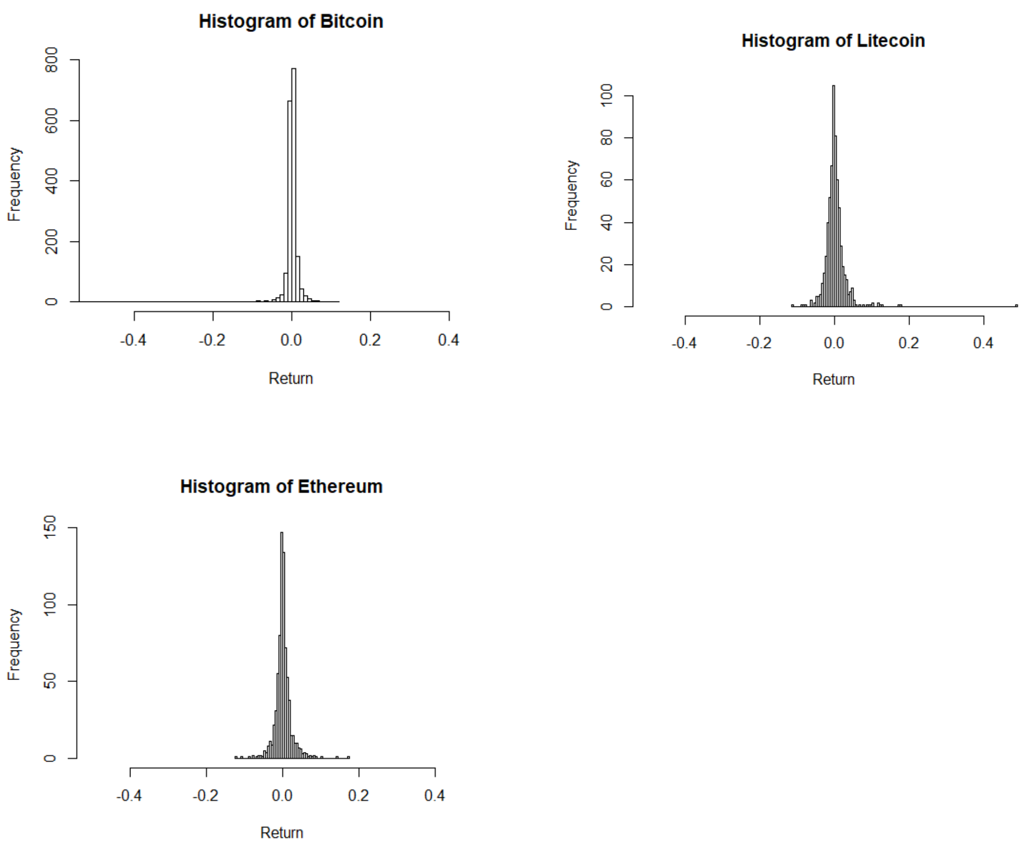
	Bitcoin, £ (%)	Bitcoin, $ (%)	Litecoin, £ (%)	Ethereum, £ (%)
Min	-6.85	-6.82	-11.36	-12.36
1st Q	-0.26	-0.24	-1.03	-0.72
Median	0.04	0.02	-0.12	-0.00
Mean	0.02	0.02	0.21	0.13
3rd Q	0.32	0.29	1.02	0.89
Max	4.43	4.07	48.69	17.15
St dev	0.72	0.67	3.19	2.23
ADF test*	-38.02***	-38.19***	-26.37***	-25.98***

** ADF Test significance: * - 10%, ** - 5%, *** - 1%*

Augmented Dickey-Fuller tests are used to evaluate the stationarity of prices, returns and log-returns for Bitcoin, Litecoin and Ethereum. The results indicate that prices are non-stationary, whereas the returns and log-returns are stationary and therefore suitable for time series regression analysis.

Histograms of the three cryptocurrencies' returns are provided in Figure 1. The following plot indicates that actual returns of Bitcoin do not have fat tails, and most of returns distribution is close to zero. In general, the distributions of cryptocurrencies are in fact narrower in comparison to normal distribution.

Figure 1 - Histogram of returns on cryptocurrencies



In addition to the distributional analysis, descriptive analysis is performed to evaluate relation between volatility of Bitcoin returns and Bitcoin price. It can be expected that price levels may cause effect on the volatility of Bitcoin returns. The relation between Bitcoin prices (P), as changes of Bitcoin prices (ΔP) with volatility is explored using scatter plot and pairwise correlation. Figure 2 indicates the absence of definitive relation between volatility of Bitcoin returns and Price, as well as between volatility of Bitcoin returns and change in Price. Correlation relation is statistically tested, and the results are not significantly different from zero.

Figure 2 – Bitcoin prices and volatility

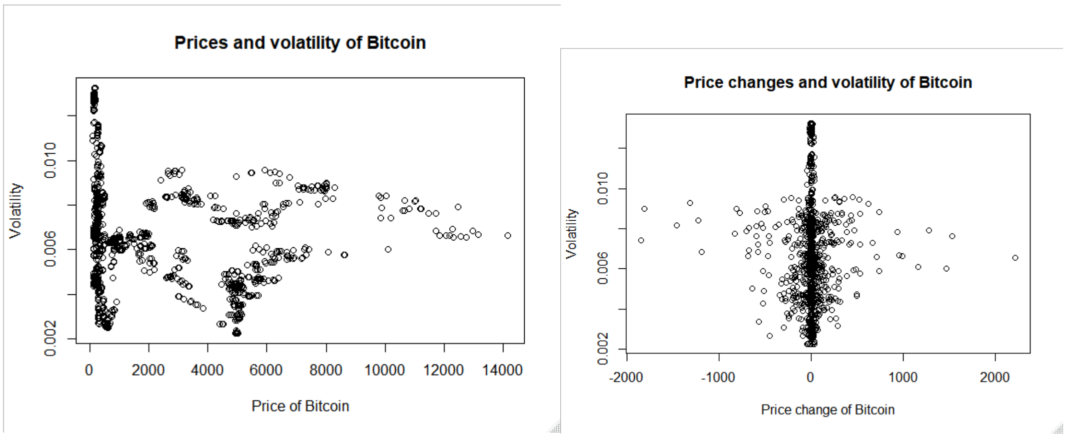
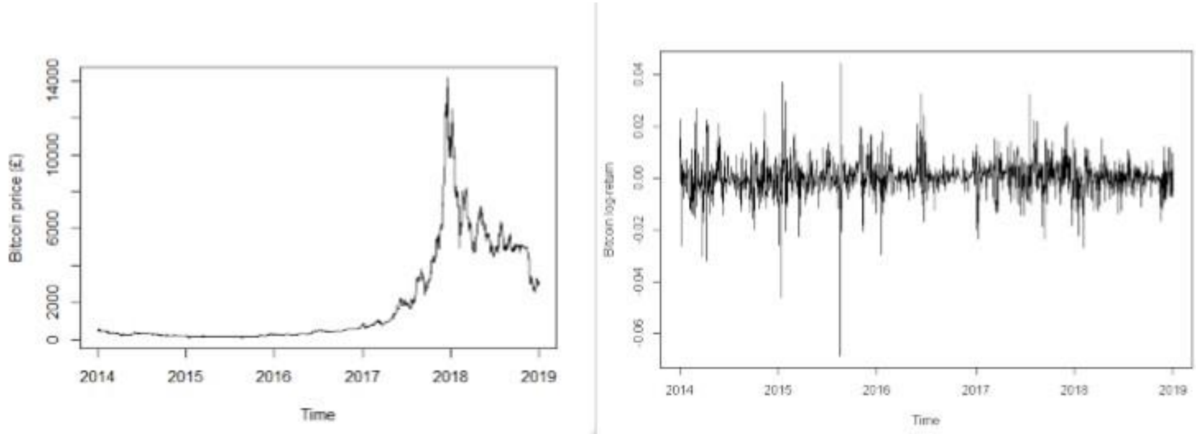


Figure 3 below aims to illustrate the considerable changes Bitcoin price has gone through during the years 2014 to 2019. This study divides the time interval of log-returns into three periods in order to test whether there are any differences in volatilities under these periods. The first period involves relatively low price of Bitcoin and lasted till May 2017. The second period involves considerable acceleration of Bitcoin price and can be considered to last till April 2018. The third period involves some decline of Bitcoin prices, after April 2018. The F-test shows that there is no significant difference in the variances between Period 1 and Period 2 (F = 0.969, p=0.74), while the variance in the third period is much lower in comparison to the second period (F = 2.902, p <0.01). The results from comparing variances in different time periods of log-returns can vary depending on how the time periods are identified. The plot of log-returns indicates that the log-return is almost zero, and the volatility seems to be nervous since there exist many very large fluctuations.

Figure 3 - Time series of Bitcoin prices (left) and log-returns (right) in 2014 - 2019



4.2 Modeling of returns with ARMA

On this stage, the goal is to identify and estimate the ARMA model, for estimation of the daily returns. Forecasting of returns is relevant for investors as they need to predict future returns in order to earn returns. Following the three step Box-Jenkins methodology, the steps include identification, estimation and evaluation. Identification is performed using ACF and PACF (Figures 4 - 6). The analysis of the ACF and PACF functions for Bitcoin, Litecoin and Ethereum return shows that both functions experience some intricate dependent structure of ARMA characteristics. Therefore, the relevant ARMA model has to involve both AR and MA components for all three cryptocurrencies.

Figure 4 - ACF and PACF for Bitcoin (£)

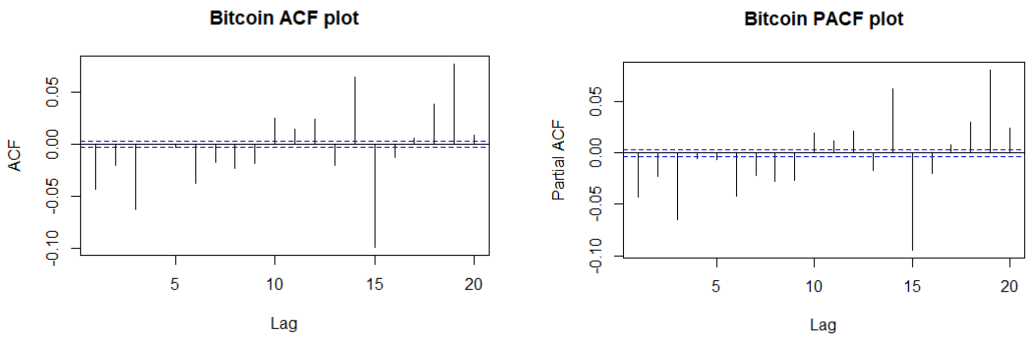


Figure 5 - ACF and PACF for Litecoin

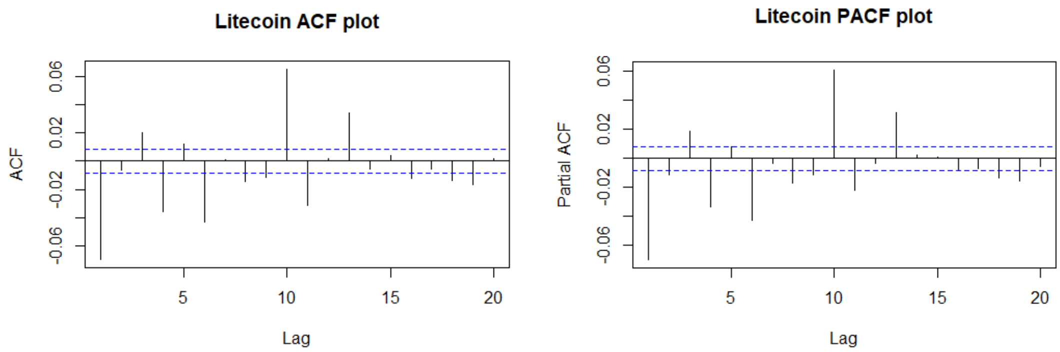
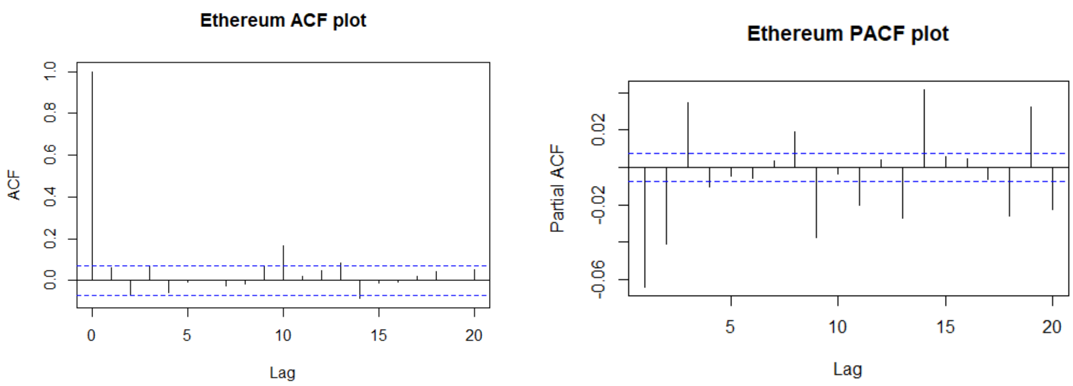


Figure 6 - ACF and PACF for Ethereum residuals



While there is no definitive conclusion from visual analysis of ACF and PACF, additional analysis is needed. There are additional analysis based on obtaining and comparison of the information criteria. The iterative analysis is implemented in order to consider the full set of all possible models with combination of AR(p), where p is an integer in the interval [0, 10] and MA(q), where q is an integer in the interval [0, 10]. Therefore, the testing set includes 121 models.

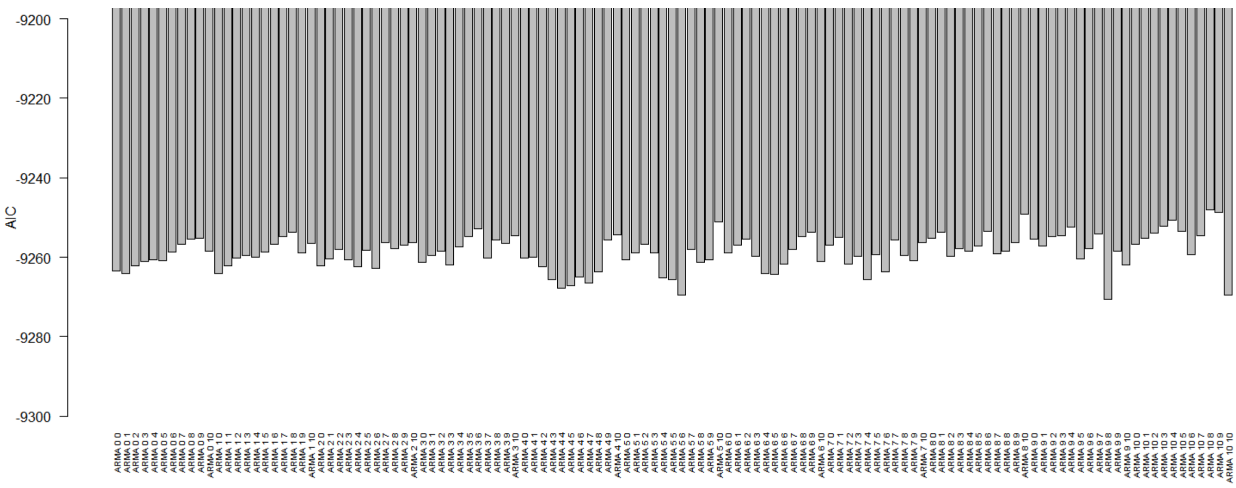
For the Bitcoin prices, the result shows that in fact the specification of optimal ARMA model is different for the case of GBP prices and in case of USD prices for Bitcoin (Table 3). The models provided in the table are the optimal models selected based on the criterion of AIC minimization.

Table 3 – Optimal ARMA models summary

	Bitcoin £	Bitcoin US$	Litecoin £	Ethereum £
Model	ARMA(9, 8)	ARMA(10, 10)	ARMA(10, 4)	ARMA(8, 8)
AIC	-9,273.1	-9,454.0	-2,626.8	-3,653.1

Figure 7 shows the plot of all possible ARMA models starting with ARMA(0, 0) and going up to ARMA(10, 10). The lowest value of AIC is that for ARMA (9, 8). But there are a number of close competitor models of lower order, for example ARMA (5, 6) has AIC = -9269.6, which is higher by less than 0.03%. Moreover, most of models to the right of the considered spectrum have relatively poorer performance, as compared to the models in the left part of the spectrum. This might suggest that with modelling financial asset returns, such as Bitcoin, simpler models might be generally preferred, since the results turned out to be relatively close to each other. Over-parameterization can also be avoided by applying simpler models. Since ARMA (9, 8) shows top performance among the considered models, it is retained for the subsequent analysis as the optimal one. Comparison with lower lag order ARMA models is implemented in further parts.

Figure 7 - Dynamics of AIC from ARMA(0, 0) to ARMA(10,10) (Bitcoin £)



4.3 Conditional volatility models Estimation

The conditional variance is frequently modelled using the GARCH-type models, which are capable of predicting variance, and also modelling its clustering nature (Tsay, 2010). Four models from the GARCH family of models are considered, and their parameters are estimated and provided in Table 4.

Table 4 summarises the considered GARCH-type models for Bitcoin and two other cryptocurrencies. The analysis for Bitcoin is performed based on its GBP and USD prices. Based on AIC and BIC criteria, it can be concluded that eGARCH is the best model specification for Bitcoin UK price, Litecoin and Ethereum. Even though for Bitcoin US dollar price, the GARCH model provides slightly lower AIC and BIC, but since the difference between the information criterion of two models is smaller than 0.1% and all components' coefficients are highly significant in eGARCH model, the study concludes that eGARCH is the best model for Bitcoin US price as well. According to Brooks (2014), ARCH (1) and GARCH (1,1) are sufficiently rich for financial and economics. This study has also evaluated some of the ARCH and GARCH models by including more lags but there were no improvements in terms of information criteria.

Moreover, from the eGARCH model, relevance of the model components is different for the different cryptocurrencies. For the Bitcoin case, all coefficients are significant, including the constant term, as well as the ARCH(1) and GARCH(1,1) components, and also the sign component ($\alpha_{1)}$. The effect of the sign component is about of the same magnitude for the all three cryptocurrencies. Unlike for Bitcoin, for Litecoin and Ethereum the ARCH(1) component is not significant. Relevance of GARCH(1,1) component is highly significant for all three cryptocurrencies.

Table 4 - GARCH-type model

	Bitcoin £	Bitcoin $	Litecoin £	Ethereum £
ARCH	ω =0.000	ω = 0.000	ω = 0.000 ***	ω = 0.000 ***
	α_1 = 0.422 ***	α_1 = 0.096 ***	α_1 = 0.204 ***	α_1 = 0.090 ***
	AIC = -1.291	AIC = 22.84	AIC = 2.428	AIC = 41.410
	BIC = -1.212	BIC = 22.93	BIC = 2.546	BIC = 41.525
GARCH	ω = 0.000	ω = 0.000 *	ω = 0.000	ω = 0.00
	α_1 = 0.154 ***	α_1 = 0.185 ***	α_1 = 0.109 ***	α_1 = 0.101 ***
	β_1 = 0.841 ***	β_1 = 0.815 ***	β_1 = 0.891 ***	β_1 = 0.896 ***
	AIC = -7.164	AIC = -7.335	AIC = -4.559	AIC = -5.156
	BIC = -7.082	BIC = -7.241	BIC = -4.434	BIC = -5.034
EGARCH	ω = -0.724 ***	ω = -0.627 ***	ω = -0.147	ω = -0.097 ***
	α_1 = -0.056 ***	α_1 = -0.060 ***	α_1 = 0.073	α_1 = 0.004
	β_1 = 0.925 ***	β_1 = 0.935 ***	β_1 = 0.976 ***	β_1 = 0.985 ***
	γ_1 = 0.290 ***	γ_1 = 0.291 ***	γ_1 = 0.293 ***	γ_1 = 0.222 ***
	AIC = -7.194	AIC = -7.328	AIC = -4.584	AIC = -5.181
	BIC = -7.108	BIC = -7.230	BIC = -4.452	BIC = -5.053
GJRGARCH	ω = 0.000 **	ω = 0.000	ω = 0.000	ω = 0.000
	α_1 = 0.121 ***	α_1 = 0.130 ***	α_1 = 0.079 ***	α_1 = 0.078 ***
	β_1 = 0.826 ***	β_1 = 0.812 ***	β_1 = 0.896 ***	β_1 = 0.911 ***
	γ_1 = 0.067 **	γ_1 = 0.106 *	γ_1 = 0.050	γ_1 = 0.022
	AIC = -7.141	AIC = -7.301	AIC = -4.551	AIC = -5.162
	BIC = -7.055	BIC = -7.203	BIC = -4.419	BIC = -5.033

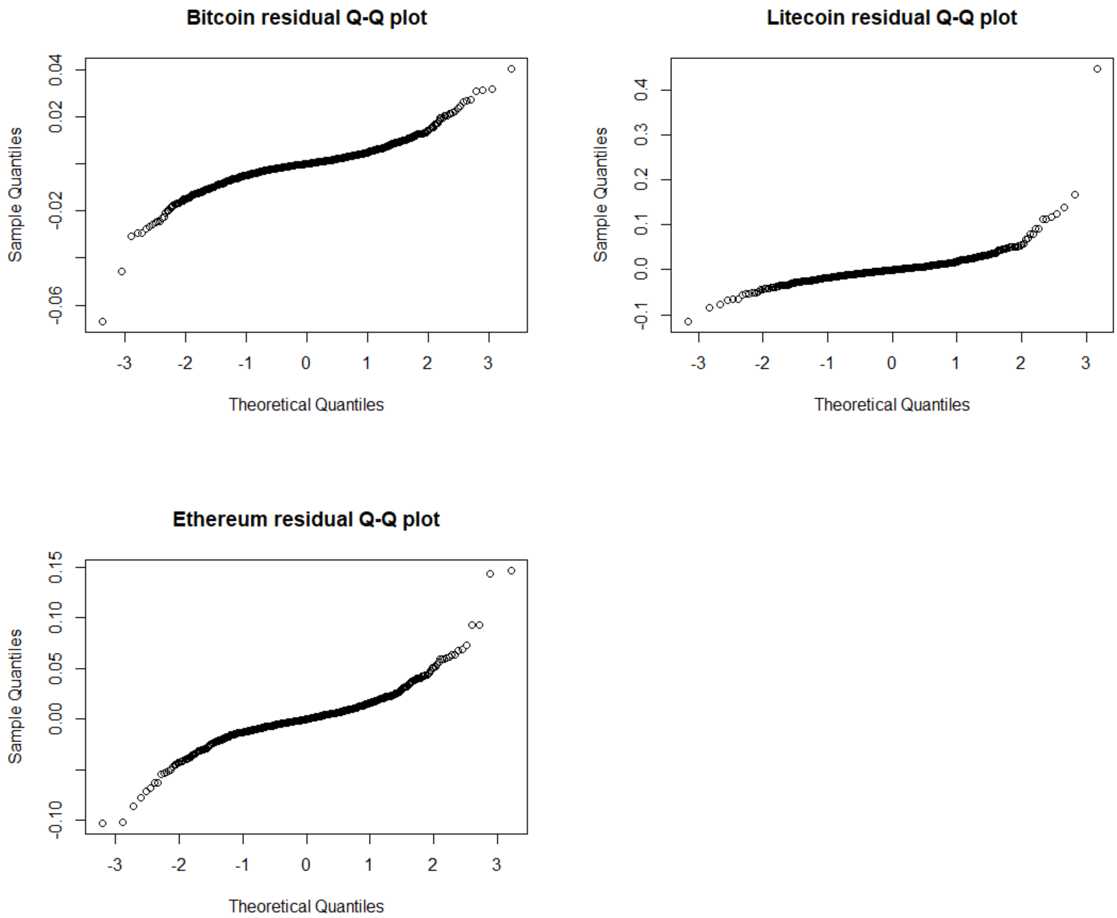
** Coefficients significance: * - 10%, ** - 5%, *** - 1%.*

4.4 Fitting the best distribution for the residuals

The residuals are obtained from estimated ARMA models of the considered cryptocurrencies. From ARMA models, residuals are obtained as the difference between actual and predicted by the model values. Now the distribution of these residuals is evaluated. Firstly, Q-Q plots are considered, in order to observe the nature of the residuals. The Q-Q plots are provided in Figure 8.

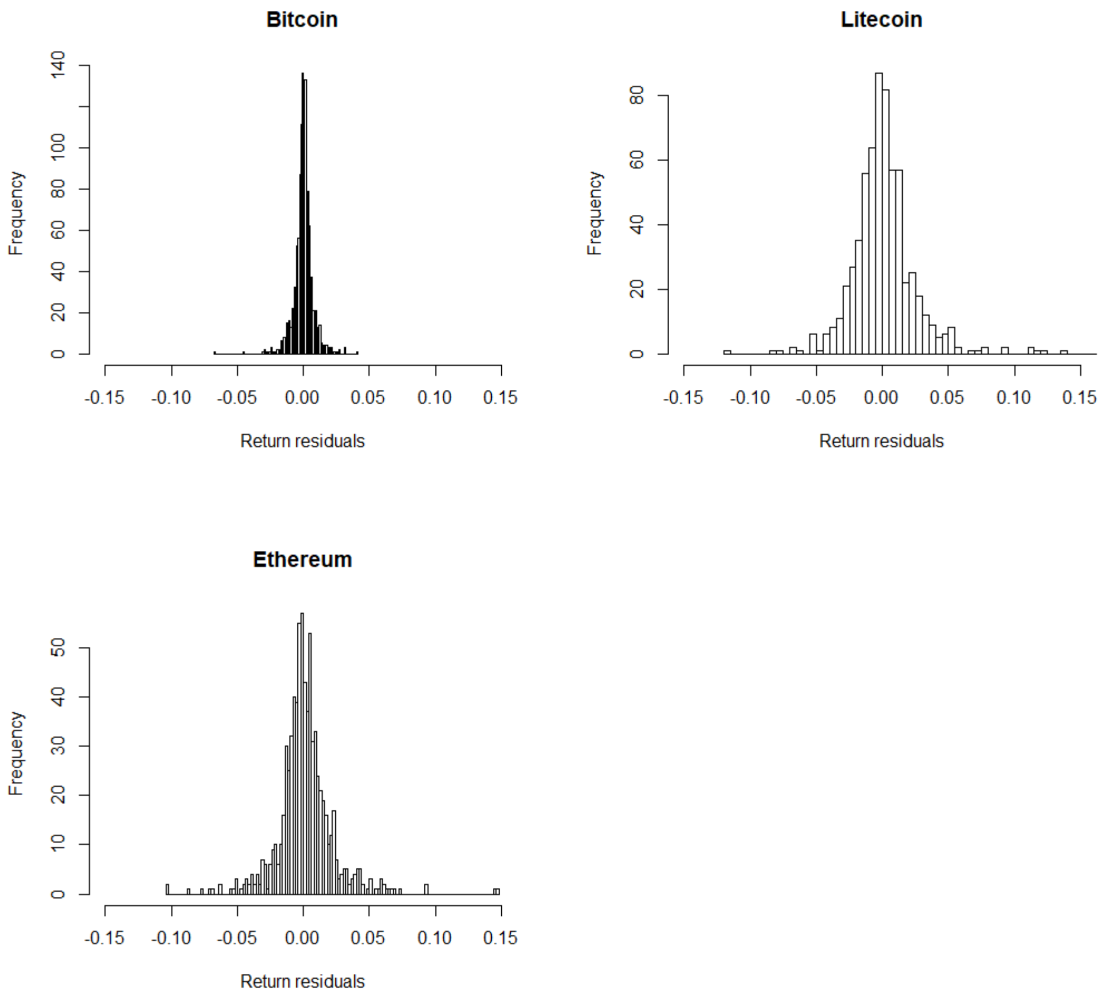
A Q-Q plot is a scatterplot conducted by plotting two sets of quantiles against one another. If the data is normally distributed, the points in the plot follow a straight diagonal line of 45-degrees. For all three cryptocurrencies, the Q-Q plot departs considerably from a 45-degree line, in particular in the lower and upper ends of the distribution. In this way, the Q-Q plot indicates that actual distribution is far from normal. Additional insight is that the three distributions are different from each other. For example, Litecoin plot exhibits greater asymmetry as compared to the two other cryptocurrencies (Figure 8).

Figure 8 - Q-Q plots of cryptocurrencies



Further exploration is performed for the residuals of the three cryptocurrencies. As the first step, distributional characteristics of residuals for the three cryptocurrencies are shown in the histograms of the residuals in Figure 9. The results indicate that there are thin tails for Bitcoin residuals, while those are thick in case of Ethereum and especially for Litecoin.

Figure 9 - Return residual distributions



Fitting of the residuals for Bitcoin, Litecoin and Ethereum is implemented using the considered five distributions. The distribution fitting performance are summarised for each of the cryptocurrencies, and each of the distributions, in Tables 5, 6, 7 and 8. Then Table 9 provides the summary of the best fit distribution and summarises its parameters.

In case of Bitcoin, the results for prices in GBP and in USD are provided in Tables 5 and 6. The results for the two cases are different - while Student's t-distribution shows the top performance for Bitcoin prices in GBP, Laplace distribution is the top-performing one in case of the USD Bitcoin prices. With these mentioned distributions, the values of BIC are minimized for the two series of returns. Even though in case of GBP Bitcoin, Laplace distribution is not the top performing one, still there is solid performance of this distribution as it shows slightly higher value of BIC information criterion (Table 5). Vice versa, for the Student's t-distribution in the case of US Bitcoin, it provides as good a fit as the optimal one - Laplace distribution (Table 6). The weakest performance is in the case of Weibull distribution which provides the highest BIC and the largest error.

Table 5 – Residuals distribution fitting of Bitcoin (£)

Distribution	BIC	RSS	MSE
Normal	-322.33	5.739e-05	0.00267
Skewed normal	-319.96	5.739e-05	0.00262
Student's t *	-460.12	4.432e-05	0.00072
Laplace	-443.65	5.739e-05	0.00087
Weibull (3 pars)	-307.59	5.739e-05	0.00293

Note: '' denotes the best distribution among considered options*

Table 6 – Residuals distribution fitting of Bitcoin ($)

Distribution	BIC	RSS	MSE
Normal	-265.04	5.739e-05	0.00360
Skewed normal	-262.78	5.739e-05	0.00352
Student's *t*	-350.80	4.363e-05	0.00146
Laplace *	-362.33	5.739e-05	0.00136
Weibull (3 pars)	9.29	5.281e-05	0.05344

Note: '' denotes the best distribution among considered options*

In case with Litecoin (Table 7), the best fit is obtained in case of Laplace distribution. Also, solid fit is achieved with Student's t-distribution. These are the two leading distributions that are the same for Bitcoin. The weakest performing distribution is the Weibull distribution, as its fitting errors are the highest, and the value of BIC is in fact the largest as well (as opposed to be minimized). The same rationale is valid as for Bitcoin – the Weibull distribution provides the weakest performance for both the UK Pound price and the US Dollar price.

Table 7 – Residuals distribution fitting of Litecoin (£)

Distribution	BIC	RSS	MSE
Normal	-720.36	9.644e-06	9.313e-05
Skewed normal	-718.68	9.644e-06	9.067e-05
Student's t	-812.59	4.166e-06	3.978e-05
Laplace *	-820.84	8.332e-06	3.858e-05
Weibull (3 pars)	-699.42	9.645e-06	0.00011

Note: '' denotes the best distribution among considered options*

Finally, for the case of Ethereum, Table 8 shows that the best performing distribution is the Student's t-distribution (BIC= -663.60), followed by the Laplace distribution (BIC = -654.28). The weakest performing distribution is the Weibull distribution.

Table 8 – Residuals distribution fitting of Ethereum (£)

Distribution	BIC	RSS	MSE
Normal	-597.37	0.00017	0.00046
Skewed normal	-604.47	0.00017	0.00041
Student's t *	-663.60	0.00014	0.00026
Laplace	-654.28	0.00016	0.00029
Weibull (3 pars)	-577.31	0.00017	0.00051

Note: '' denotes the best distribution among considered options*

The best and the worst performing distribution seems to maintain its persistence for all cases. The Laplace distribution and the Student's t-distribution provide the best descriptions of the residuals in this study. The Weibull distribution provides the weakest fit in all cases. Among the considered three cryptocurrencies, the best fit also is somewhat varied. The value of BIC information criterion is the lowest for Litecoin (BIC = -820.84), suggesting that the use of its best fit distribution would be more effective than the use of the best fit distribution for Bitcoin (BIC = -460.12) or Ethereum (BIC = -663.60).

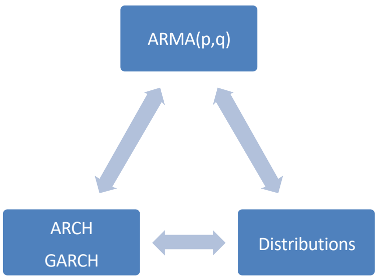
Table 9 – Best fitted distributions and their parameters

Cryptocurrency	Best Fitting Distribution	Parameter estimates
Bitcoin £	Student's *t*	$\mu = 0.0194$ $\sigma = 0.3359$ $\nu = 1.5007$
Bitcoin US$	Laplace	$\mu = 0.0408$ $b = 0.5667$
Litecoin £	Laplace	$\mu = -0.0668$ $b = 2.2768$
Ethereum £	Student's *t*	$\mu = -0.0258$ $\sigma = 1.0571$ $\nu = 1.8296$

4.5 Relation between models and distributions

This final section aims to explore and evaluate relations between the considered models and the residual distributions. The three components of the larger model are the ARMA model, as well as conditional volatility model, and specific distributions of residual. These components can be related to each other, in some way as it is schematically shown in Figure 10 below.

Figure 10 - Components of a larger model



Of the three components, the ARMA(p,q) model is used to predict future return on Bitcoin, with a related ARCH/GARCH model responsible to model conditional volatility of Bitcoin log-returns, and a specific distribution is then required to effectively evaluate risks and for hypotheses testing. Thus the three considered aspects can be considered as parts of a system that is used to model Bitcoin prices. It is possible that ARMA(p,q) models maintain persistence in terms of optimal GARCH-type model and residual distribution. Or, alternatively, there can be some changing patterns.

In order to test possible linkages, the following approach is considered using the case of Bitcoin. The optimal ARMA is ARMA(9, 8) that is a high order model with lowest AIC value, which has an accompanying optimal GARCH-type model eGARCH, and the distribution of residuals being Student's t-distribution. The approach is aiming to gradually decrease the level of ARMA(p,q) model for Bitcoin, and observe the change of optimal GARCH-type model and optimal residual distribution. For the purpose of such exercise, such models are considered as ARMA(7, 7), ARMA(5, 5), ARMA (3, 3) and ARMA(1, 1). The summary of ARMA models and their optimal GARCH models, as well as optimal residual distributions are summarized into Table 10.

Figure 11- AIC plot of five chosen ARMA models

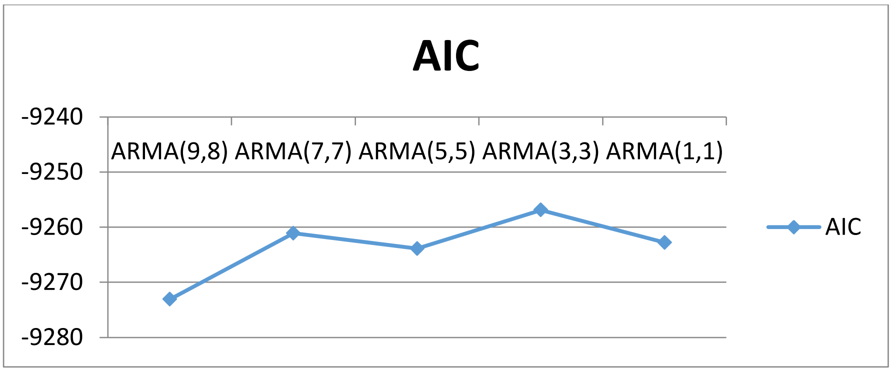


Table 10 - Optimal components comparison for ARMA(p, q) Bitcoin (£) models

	ARMA(9, 8)	ARMA(7, 7)	ARMA(5, 5)	ARMA(3, 3)	ARMA(1, 1)
AIC	-9273.1	-9261.1	-9263.9	-9256.9	-9262.8
Optimal GARCH:					
- ARCH (AIC)	-1.291	-3.772	-1.535	-6.976	-4.388
- GARCH (AIC)	-7.164	-7.287	-7.287	-7.287	-7.291
- EGARCH (AIC)	-7.194 *	-7.324 *	-7.304 *	-7.325 *	-7.332 *
- GJRGARCH (AIC)	-7.141	-7.305	-7.295	-7.297	-7.295
Optimal residuals distribution (BIC)	t-dist (-460.1)	t-dist (-498.9)	t-dist (-512.7)	Laplace (-570.1)	t-dist (-543.3)

The above results indicate that the optimal conditional volatility model is the same for all considered alternative ARMA(p,q) models. The eGARCH model is optimal in each of the considered cases. The lack of any relation between lag order of ARMA and the conditional distribution model hints towards the conclusion that there is not much relation between the type of model for modelling conditional volatility of Bitcoin and a model to estimate return of Bitcoin.

Also, when the optimal distribution of residuals is considered, the results were also confirmed to be rather consistent. In fact, the Student's t-distribution appeared to be optimal in 4 out of 5 models. Only for ARMA (3,3) the Laplace distribution provides better fit. In fact, in all of the considered ARMA models, the Laplace distribution provides almost as good fit as that of the Student's t-distribution. For example, in case of ARMA(9,8) model, from Table 5, BIC for the Student's t-distribution is -460.1, while BIC for Laplace is -443.7. Thus both these distributions, namely Student's t-distribution and Laplace provide close fit for residuals of Bitcoin returns, for all considered lag orders of ARMA(p, q) model.

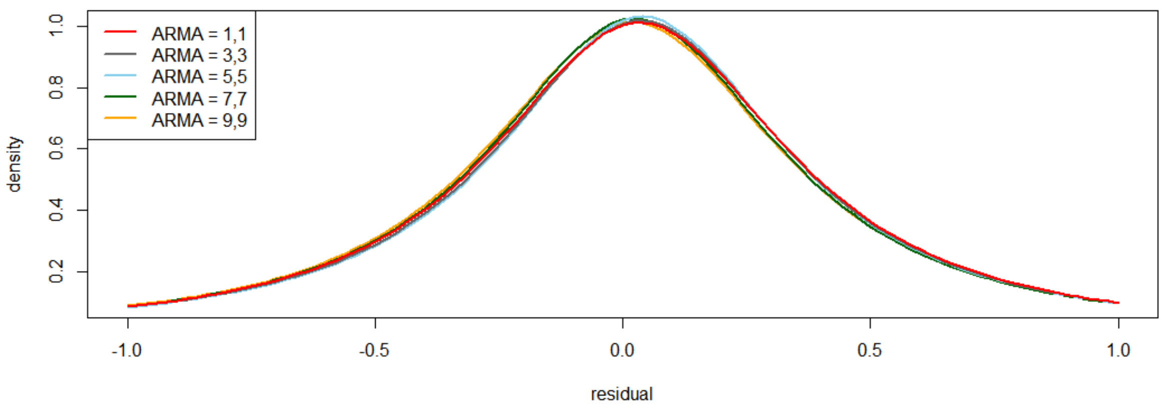
Furthermore, for each of considered five ARMA models, the estimated Student's t-distribution parameters are summarised in the Table 11. The results indicate that although these distributions are close to each other, they are not identical.

Table 11 – Parameters of Student's t-distribution of ARMA models (Bitcoin £)

ARMA Model	Parameter estimates (Student's t)		
	μ	σ	ν
ARMA(9, 8)	0.0194	0.3359	1.5007
ARMA(7, 7)	0.0243	0.3307	1.4601
ARMA(5, 5)	0.0386	0.3262	1.3939
ARMA(3, 3)	0.0368	0.3317	1.4331
ARMA(1, 1)	0.0337	0.3376	1.5299

Moreover, for the above distributions, the probability density functions (PDF) are plotted (Figure 12). These end up being highly close to each other. Visual inspection of the fitted distributions indicate that fitted distribution of the residuals from the five different lag order ARMA models are not very much different from each other. It thus probably makes sense to use lower lag order ARMA model, rather than a higher order one, since the differences are very small.

Figure 12 - Fitted t-distribution of residuals from ARMA models



5 Conclusion and future research

The study suggests a system to model Bitcoin returns, which includes three components. These three components include modeling logarithmic returns using ARMA model based on Box-Jenkins (1976) methodology. Volatility of the log-return, obtained from the optimal ARMA model, is modeled using GARCH-type models of conditional volatility, as well as using different statistical distributions to evaluate the residuals obtained from ARMA model, such as Normal, Laplace, and Student’s t-distribution. Also, analysis is performed to explore whether different lag orders of ARMA model has an effect on selection of optimal GARCH-type models and selection of optimal residual distributions. The answers to the research questions (RQs) are aimed to be obtained.

RQ1: What is the best ARMA-type model to model the return on Bitcoin, Litecoin and Ethereum?

In fact, model identification results in different models chosen: ARMA(9,8) for Bitcoin in GBP, ARMA (10,10) for Bitcoin in USD, ARMA(10,4) for Litecoin, and ARMA(8,8) for Ethereum.

RQ2: Does price level or changes in price level affect volatility of Bitcoin returns?

No significant relation is identified between the price level or the changes in price level and volatility of Bitcoin. In case of Bitcoin, returns distribution, as well as that of price changes is relatively symmetric around zero. Moreover, volatility of Bitcoin is shown not to be related to the returns on Bitcoin, so that in periods of high returns volatility is roughly the same as in periods of low returns. This feature makes Bitcoin to stand out of the traditional approach that viewed financial assets (such as stocks) to have expected return - volatility trade-off. Although the issue can be explored deeper in further research.

RQ3: What distribution, and which conditional volatility model provide the best suit for the Bitcoin residuals?

Student's t-distribution shows the top performance for Bitcoin prices in GBP, Laplace distribution is the top -performing one in case of the USD Bitcoin prices. The best models are eGARCH(1, 1) for all of the three cryptocurrencies. The fit of this model is significantly higher than that of standard GARCH, ARCH, or GJR-GARCH. This confirms relevance of eGARCH models for fitting the residual volatility of Bitcoin and other cryptocurrencies. These findings make up the tool set that can be used to estimate the underlying statistical process behind the Bitcoin price dynamics.

RQ4: Are statistical properties of Bitcoin the same for Bitcoin prices in GBP and USD?

The analysis is conducted separately for Bitcoin traded in GBP and that in USD. The obtained results suggests that the optimal models and distribution for Bitcoin prices in Pounds are ARMA (9,8), eGARCH and Student's t-distribution. An ARMA(10,10) model for the returns, an eGARCH for the volatility and a Laplace distribution for the Bitcoin returns in Dollar shown to be the best performing model. This suggests that statistical properties of cryptocurrencies do differ between the considered currencies. Since the statistical characteristics of Bitcoin are not the same depending on the choice of currency, this provides evidence that the exchange rates, which is another type of time dependent process, interfere with the statistical properties of Bitcoin. This finding enhances the research outcomes about exchange rates of Bitcoin, which was conducted by Chu, Nadarajah and Chan in 2015 (Chu et al., 2015).

RQ5: Are there relations between ARMA(p,q) lag order and choice of optimal GARCH-type model and distribution of residuals?

This study gradually decreases the level of ARMA(p,q) model for Bitcoin, and observes the change of optimal GARCH-type model and optimal residual distribution. The optimal conditional volatility models are the same for all considered alternative ARMA(p,q) models. Student' t-distribution appeared to be optimal in 4 out of 5 chosen models. Only for ARMA

(3, 3) the Laplace distribution provides better fit. In fact, in all of the considered ARMA models, the Laplace distribution provides almost as good fit as that of t-distribution. In conclusion, no significant relation is identified.

To apply spectral analysis to Bitcoin prices would be an alternative way to model the statistical properties of Bitcoin. Following Cryer and Chan (2008) spectral analysis aims to identify hidden repeating patterns within distribution of time series variables, involving sine and cosine functions. For example, spectral specification of ARMA model can be estimated to closely fit the observed history of cryptocurrency price. The performance of such model can be compared to that in the current study.

The further research could be extended to include more cryptocurrencies, and also to compare the statistical properties of cryptocurrencies to those of the major stock market indices. Also, intra-day frequency data could be used for such analysis. The analysis in this study is implemented using the Bitcoin exchange rates versus GBP, additional insights could be obtained from the analysis of Bitcoin and other cryptocurrencies prices versus other major currencies and further explore how exactly the exchange rates affect the statistical properties of Cryptocurrencies. Moreover, to research the rationale behind the drastic increase of Bitcoin prices during 2018 would also be interesting.

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