Abstract

We establish the central convergence properties of ordinary Dirichlet series, including the classical result by Bohr, providing uniform convergence of the series where it has a bounded analytic continuation ($\sigma_b = \sigma_u$). We also derive a lower bound for the supremum of Dirichlet polynomials using Kronecker's theorem, of which we see one proof. With this knowledge and some probability theory we can follow the work of Queffélec and Boas proving the existence of random series $\sum \pm n^{-s}$ with certain convergence properties. In particular Boas work is a probabilistic version of what Bohnenblust and Hille did, namely showing that estimate for $\sigma_a - \sigma_u \leq \frac{1}{2}$ is sharp. Here, σ_a (introduced in Chapter 1) denotes the abscissa $Re(s) = \sigma_a$ to the right of which the Dirichlet series converges and to the left of which it diverges absolutely. σ_u is the corresponding abscissa for uniform convergence.

Populärvetenskaplig introduktion

En funktion som har fått mycket uppmärksamhet bland matematiker är den så kallade *Riemanns zeta-funktion*, som definieras med hjälp av en summa av oändligt antal termer, en så kallad *serie*:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

Här är $s = \sigma + it$ en komplex variabel med en reell del σ och en imaginär del t, där i har egenskapen $i^2 = -1$. Värdet på den här serien kommer givetvis bero på vilket värde på s man använder, och det kan ge serien både ett ändligt och ett oändligt värde. T.ex. så har man lyckats visa att $\zeta(2) = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = \frac{\pi^2}{6}$ men också att $\zeta(1) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots = \infty$, d.v.s. blir oändligt stort. Man kan visa att om $s = \sigma + it$ har realdel $\sigma > 1$ så har serien ett ändligt värde. I övriga fall säger man att serien inte är definierad, och då behöver man ett annat sätt att ge mening till zeta-funktionen.

Anledningen till att den här funktionen är så uppmärksammad är att den har en nära koppling till primtalen, och det har länge varit ett olöst problem att hitta alla dess nollställen, dvs de s sådana att $\zeta(s) = 0$. Riemann själv formulerade en hypotes om att alla (*icke-triviala*) nollställen finns längs en linje $s = \frac{1}{2} + it$, och Hardy har bevisat att längs den här linjen finns det oändligt många nollställen, [12]. Däremot är det ingen hittills som har lyckats visa att hypotesen faktiskt är sann eller hittat ett motbevis genom ett nollställe utanför linjen. Många matematiker har funderat på det här problemet och The Clay Mathematics Institute har till och med utlyst en belöning på 1 miljon dollar till den som löser det, som en del av deras sju prisbelönta Milleniumproblem, [20].

Ett försök att bättre förstå den här funktionen har varit att undersöka en mer allmän serie

$$\frac{a_1}{1} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \ldots = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{2}$$

I fallet av zeta-funktionen så noterar vi att $a_n = 1$, för alla n. Den här mer allmänna serien kallas en *Dirichlet-serie*. Det visar sig att alla serier som har den formen delar en hel del gemensamma egenskaper, som man därför hoppas kan ge insikt om Riemann's zeta-funktion.

Det här arbetet går främst ut på att kartlägga de så kallade *konvergensegenskaperna* hos Dirichletserier. Det handlar om att undersöka vilka s man kan stoppa in i serien (2) för att den ska ha ett ändligt värde och kunna definieras. På vägen betraktar vi relevanta exempel med en nära koppling till Riemann's zeta-funktion och stöter på ett par matematiska resultat intressanta i sig.

Acknowledgements

Without the help from my supervisor Jan-Fredrik Olsen I would not have accomplished this much. I would like to thank him for his guidance on relevant subjects, for his disponibility and for persistently insisting on not making any problems harder than they have to be.

Contents

Chapter 1

Basic convergence properties of the Dirichlet Series

Let $s \in \mathbb{C}$ and $\{a_n\}_{n\in\mathbb{N}}$ a sequence of complex numbers. A series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s} = a_1 + a_2 2^{-s} + a_3 3^{-s} + \dots \tag{1.1}$$

is said to be a *Dirichlet series*. (To be clear, this is what is said to be an *ordinary* Dirichlet series. We will only consider this type and not the *general* Dirichlet series of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$.)

However we only want to define it when it converges. This is one reason to investigate its convergence properties.

1.1 Absolute convergence

The series (1.1) can converge in several ways. As a start, we might consider the easiest case and ask ourselves for which $s = \sigma + it$

$$\sum_{n=1}^{\infty} |a_n n^{-s}| = \sum_{n=1}^{\infty} |a_n n^{-\sigma} n^{-it}| = \sum_{n=1}^{\infty} |a_n| n^{-\sigma}$$

converges. For all s such that this happens, the series (1.1) is absolutely convergent, and it is clear that the absolute convergence does not depend on the choice of t but only on the real part σ. If the series converges absolutely in one point it also converges to the right of this point, since $|a_n| n^{-\sigma}$ is a decreasing function of σ, for all $n \in \mathbb{N}$.

Hence absolute convergence occurs in a half-plane of $\mathbb{C}$, which proposes the introduction of an *abscissa of convergence*:

Definition 1.1 (Abcissa of absolute convergence)**.** The real number

$$\sigma_a := \inf\{\sigma_0 : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \text{ converges for } \sigma > \sigma_0\}$$

is the abscissa of absolute convergence of the series (1.1).

Remark 1.1. We note further that where the series converges absolutely, it is bounded. Therefore we can be sure that $f(s) = \sum a_n n^{-s}$ is a bounded function in the half-plane $\{s : Re(s) > \sigma_a + \varepsilon\}$, for $\forall \varepsilon > 0$.

As mentioned above, the series converges absolutely for all $s = \sigma + it$ in the half-plane where $\sigma > \sigma_a$, and by this definition it must diverge (in the absolute sense) for all $\sigma < \sigma_a$. In general we do not know what happens when $\sigma = \sigma_a$. Moreover, it may occur that a series converges for all $\sigma \in \mathbb{R}$ and also for none.

Example 1.2. The series

$$\sum_{n=1}^{\infty} n^{-n} n^{-s}$$

converges absolutely $\forall s \in \mathbb{C}$. Hence $\sigma_a = -\infty$.

Example 1.3. Contrarily

$$\sum_{n=1}^{\infty} n^{n} n^{-s}$$

never converges absolutely. Hence $\sigma_a = \infty$.

Example 1.4 (Riemman's zeta-function)**.** Consider

$$\sum_{n=1}^{\infty} n^{-s}.$$

It is known that it converges absolutely iff $Re(s) > 1$, so we have $\sigma_a = 1$. At this half-plane of absolute convergence it defines the *Riemman's zeta function*, $\zeta(s)$.

1.2 Pointwise convergence

Abscissae turn out to be relevant also for other kinds of convergences, although it is less obvious. The next result investigates what can be said about the domain of convergence of a series, given that it converges for a fixed $s_0 \in \mathbb{C}$.

Proposition 1.5 (Convergence in half-plane)**.** *Let $\sum_{n=1}^{\infty} a_n n^{-s}$ be a series that is convergent for $s_0 = \sigma_0 + it_0$. Then it converges for all s with $Re(s) > \sigma_0$.*

Proof. Consider the sequence of partial sums

$$\left(\sum_{n=1}^{N} a_n n^{-s}\right)_{N=1}^{\infty},$$

which we will show is a Cauchy sequence in $\mathbb{C}$. $\mathbb{C}$ being a complete metric space, this is enough to conclude convergence of the sequence. Let $\varepsilon > 0$ be fixed, and we will prove that $\exists N \in \mathbb{N}$ s.t. for $p, q \geq N$

$$\left|\sum_{n=1}^{p} a_n n^{-s} - \sum_{n=1}^{q} a_n n^{-s}\right| = \left|\sum_{n=p}^{q} a_n n^{-s}\right| < \varepsilon,$$

assuming that $p < q$. We write the sum as

$$\sum_{n=p}^{q} a_n n^{-s} = \sum_{n=p}^{p} a_n n^{-s_0} n^{-(s-s_0)},$$

to next use Abel's summation formula for a finite sum:

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} \sum_{m=p}^{n} a_m (b_n - b_{n+1}) + b_q \sum_{m=p}^{q} a_m.$$

Applying this formula with $a_n = a_n n^{-s_0}$, $b_n = n^{-(s-s_0)}$, gives us

$$\sum_{n=p}^{q} a_n n^{-s_0} n^{-(s-s_0)} = \sum_{n=p}^{q-1} \sum_{m=p}^{n} a_m m^{-s_0} \left(n^{-(s-s_0)} - (n+1)^{-(s-s_0)}\right) + q^{-(s-s_0)} \sum_{m=p}^{q} a_m m^{-s_0}.$$

The last term can be made smaller than ε for p big enough, for $Re(s) > \sigma_0$. Considering now the first term, we let this choice of p also be such that $\left|\sum_{m=p}^{n} a_m m^{-s_0}\right| < \varepsilon$, since this is a tail of a convergent series. From this, we get the upper bound

$$\left|\sum_{n=p}^{q-1} \sum_{m=p}^{n} a_m m^{-s_0} (n^{-(s-s_0)} - (n+1)^{-(s-s_0)})\right| \leq \varepsilon \sum_{n=p}^{q-1} |n^{-(s-s_0)} - (n+1)^{-(s-s_0)}| = (*).$$

We can write the difference $s - s_0$ as $s - s_0 = \delta + it$ ($\delta > 0$). With $f(x) = x^{-(\delta+it)}$ and $f'(x) = -(\delta + it)x^{-(\delta+it)-1}$, by Mean Value Theorem, with $x_n \in (n, n+1)$, we get

$$(*) = \varepsilon \sum_{n=p}^{q-1} |f'(x_n)| = \varepsilon \sum_{n=p}^{q-1} |\delta + it| x_n^{-(\delta+1)} \leq \varepsilon |\delta + it| \underbrace{\sum_{n=p}^{q-1} n^{-(\delta+1)}}_{(**)}.$$

Here, the sum $(**)$ can be evaluated from above by comparison of sum and integral to

$$\sum_{n=p}^{q-1} n^{-(\delta+1)} \leq \frac{(p-1)^{-\delta} - (q-1)^{-\delta}}{\delta}.$$

That leaves us with

$$(*) \leq \varepsilon|\delta + it| \frac{(p-1)^{-\delta} - (q-1)^{-\delta}}{\delta} \leq \varepsilon \left|1 + i\frac{t}{\delta}\right|, \tag{1.2}$$

which we can make arbitrarily small (by choosing an ε small enough, whenever $\left|1 + i\frac{t}{\delta}\right|$ is bounded. In particular, that is true for a fixed s with $Re(s) > Re(s_0)$, which gives the derived conclusion. □

We remark that the bound in (1.2) is finite for any choice of t in a compact subset of the half-plane $\{s : Re(s) > s_0\}$. More generally, whenever

$$\left|\frac{t}{\delta}\right| \leq C \iff |t| \leq C\delta,$$

for some constant C, the series converges uniformly. We state this as a corollary but will discuss it more thoroughly in the next section.

Corollary 1.6. *If $\sum_{n=1}^{\infty} a_n n^{-s}$ converges at a point s_0, it converges uniformly in every sector $\{s = \sigma_0 + it : |t| \leq C\delta\}$ and in any compact subset of $\{s : Re(s) > s_0\}$.*

Now let us again consider the pointwise convergence, because Proposition 1.5 proves that a Dirichlet series converges in the pointwise sense also in a halfplane. We have reason to define:

Definition 1.7 (Abcissa of pointwise convergence)**.** The real number

$$\sigma_c := \inf \left\{\sigma_0 : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges for } \quad \sigma > \sigma_0 \right\}$$

is the abscissa of pointwise convergence of the series (1.1). Whenever a Dirichlet series converges in one point, we saw that it will converge in a halfplane, and the abcissa, and otherwise we are not interested in defining σ_c.

As for the absolute convergence, the series diverges in the halfplane where $\sigma < \sigma_c$.

Since an absolutely convergent series is convergent, we have that absolute convergence implies pointwise convergence. Thus, a Dirichlet series is pointwise-convergent anywhere where it is absolutely convergent, which it is in half-planes, we record the following proposition.

Proposition 1.8.

$$\sigma_c \leq \sigma_a.$$

Example 1.9. We saw in Example 1.4 that for $Re(s) > 1$ the Riemann zeta-function is absolutely convergent. But since $\zeta(1) = \sum \frac{1}{n} = \infty$, we must have that $\sigma_c \geq 1$. By Prop 1.8 we conclude $\sigma_c = 1$.

Example 1.10 (The alternating ζ-function)**.** Define

$$\zeta_a(s) := \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}.$$

By comparison to $\zeta(s)$ we conclude that $\sigma_a = 1$, as ζ and ζ_a coincide after taking absolute values. However we are allowed to define this series on a bigger set. Indeed, for $s = \sigma \in \mathbb{R}$, $\zeta_a(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-\sigma}$ is an alternating series convergent for $\sigma > 0$. Since pointwise convergence occurs in a halfplane by Proposition 1.5, we can conclude that $\sigma_c = 0$, and we define the series for all s with $Re(s) > 0$. We call it the *alternating zeta-function*, $\zeta_a(s)$.

1.3 Uniform convergence

Now let us return to Corollary 1.6, where we saw that when we had (conditional) convergence in one point, we also had uniform convergence in any sector to the right of the point. The uniform convergence is in some way a bit less intuitive in the case of Dirichlet series, but it is something in between the pointwise and the absolute. To be clear, by uniform convergence in a set S we mean that

$$\forall \varepsilon > 0, \exists N : \quad |f(s) - \sum_{n=1}^{N} a_n n^{-s}| < \varepsilon, \forall s \in S.$$

We introduce again an abscissa:

Definition 1.11 (Abscissa of uniform convergence)**.**

$$\sigma_u := \inf\{\sigma_0 : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges uniformly for } \quad Re(s) > \sigma_0\}.$$

As a first approach to the the uniform convergence, we want to relate it to the other two types of convergence that we have studied above.

Proposition 1.12.

$$\sigma_c \leq \sigma_u \leq \sigma_a.$$

Proof. ($\sigma_\mathbf{u} \leq \sigma_\mathbf{a}$). Assume that the series converges absolutely at a point s_0. Then trivially it converges uniformly along the line $s = \sigma_0 + it$. But it also converges absolutely in all of the halfplane to the right $\{Re(s) \geq \sigma_0\}$ – thus uniformly along all the lines to the right of $s = \sigma_0 + it$. Finally the series $\sum |a_n| n^{-\sigma}$ is monotonically decreasing with respect to σ, and thus the convergence at σ_0 is enough to ensure the convergence for any $\sigma \geq \sigma_0$. So, the uniform convergence along the line $s = \sigma_0 + it$ is enough to ensure the uniform convergence along all other lines to the right, which means the uniform convergence takes place in all of the half-plane $Re(s) > \sigma_a$.

($\sigma_{\mathbf{c}} \leq \sigma_{\mathbf{u}}$). Suppose that the series converges uniformly on $\{Re(s) > \sigma_u + \varepsilon\}$. In particular that means that it converges no matter the choice of $s \in \{Re(s) > \sigma_u + \varepsilon\}$. $\square$

To the right of σ_u, possibly with a margin of ε, the series converges uniformly, so $\{Re(s) > \sigma_u + \varepsilon\}$ defines the largest half-plane where the series converges uniformly. By Proposition 1.12 we know that a half-plane like this exists whenever $\sigma_a < \infty$.

To the left of σ_u it can be a bit complicated. If $\sigma_u = \sigma_c$, then the series necessarily diverge to the left of the abscissa, by the properties of σ_c. However, if $\sigma_u > \sigma_c$, we saw in Corollary 1.6 that to the right of σ_c there was uniform convergence in sectors.

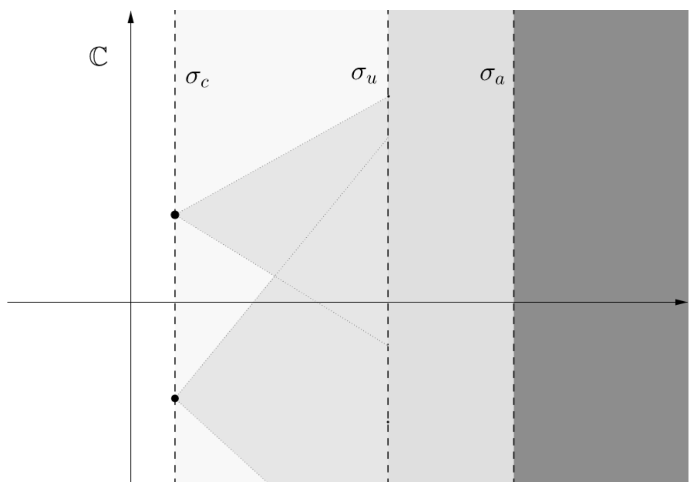

Figure 1.1: An illustration of the abcissas of and the domains of convergence.

What does this discussion tell us? Only that the uniform convergence does not divide the complex plane into one convergent half and one "divergent" half, separated by a line, as was the case with the absolute and the pointwise convergence. However, we can still study the largest halfplane where it surely converges uniformly, i.e. $\{s : Re(s) > \sigma_u + \varepsilon\}$.

Example 1.13. For $\zeta(s)$ we simply have $\sigma_u = 1$, since $\sigma_a = \sigma_c = 1$.

Remark 1.2 (Uniform convergence and boundedness). Moreover, whenever $f_N \to f$ uniformly, as $N \to \infty$, and the the Dirichlet polynomials f_N are bounded, then the f_N:s are also uniformly bounded (that is, there is one constant bounding all of them). The polynomials, f_N, are bounded on the half-plane $\{Re(s) \geq \sigma_u\}$, being finite sums of bounded functions. They converge uniformly to f on the half-plane $\{Re(s) > \sigma_u + \varepsilon\}$, so the polynomials f_N are uniformly bounded on this last half-plane.

Moreover, we remark that in the same situation also f will be bounded. This is the fact that uniform convergence preserves boundedness of a sequence, for its limit.

1.4 Distances between the abscissae

Proposition 1.14.

$$\sigma_a - \sigma_c \leq 1$$

Proof. Let s be such that $\sum a_n n^{-s}$ converges. Then $\lim_{n\to\infty} a_n n^{-s} = 0$ and $|a_n n^{-s}|$ is in particular bounded by a constant for all n. This gives

$$\sum |a_n n^{-(s+1+\epsilon)}| = \sum |a_n| n^{-\sigma} n^{-(1+\epsilon)} \leq C \sum n^{-(1+\epsilon)}.$$

The series on the right-hand side is convergent, thus the original series converges absolutely for $s+1+\epsilon$ whenever it converges pointwise for s. □

Example 1.15. When considering non-coinciding abscissae, we are not so interested in the usual zeta-function, but the more in its alternating relative. For ζ_a we already know that $\sigma_c = 0$, $\sigma_a = 1$, which is the biggest possible gap, and it proves that the inequality in Proposition 1.14 is sharp. With help of a result to come, Proposition 1.17, we will be able to conclude that $\frac{1}{2} \leq \sigma_u \leq 1$.

We will next relate the abscissae of absolute and uniform convergence, but in order to do that we need to use another result, due to Fritz Carlson, see [9].

Proposition 1.16 (Carlson's formula)**.** *For a Dirichlet series with a uniform limit f and the abscissa of uniform convergence $Re(s) = \sigma_u$, it holds that*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}, \quad \text{for } \sigma > \sigma_u.$$

Proof. We have that $f_N \to f$, $N \to \infty$ uniformly, where $f_N(s) = \sum_{n=1}^{N} a_n n^{-s}$ are Dirichlet polynomials. Remark 1.2 gives that the Dirichlet polynomials f_N thus are uniformly bounded.

We initially calculate

$$
\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}|f_N(\sigma+it)|^2dt &= \frac{1}{2T}\int_{-T}^{T}|\sum_{n=1}^{N}a_n n^{-\sigma}n^{-it}|^2dt \\
&= \frac{1}{2T}\int_{-T}^{T}\sum_{n=1}^{N}a_n n^{-\sigma}n^{-it}\sum_{k=1}^{N}\overline{a_k}k^{-\sigma}k^{it}dt \\
&= \frac{1}{2T}\int_{-T}^{T}\sum_{n=1}^{N}\sum_{k=1}^{N}a_n\overline{a_k}(nk)^{-\sigma}n^{-it}k^{it}dt \\
&= \frac{1}{2T}\sum_{n=1}^{N}\sum_{k=1}^{N}a_n\overline{a_k}(nk)^{-\sigma}\int_{-T}^{T}\left(\frac{n}{k}\right)^{-it}dt,
\end{aligned}
\tag{1.3}
$$

(k is never 0). Consider, for $n \neq k$, the integral

$$
\begin{aligned}
\int_{-T}^{T}\left(\frac{n}{k}\right)^{-it}dt = \int_{-T}^{T}e^{-it\log(n/k)}dt = \left[\frac{e^{-it\log(n/k)}}{-i\log(n/k)}\right]_{-T}^{T} = \frac{\left(\frac{n}{k}\right)^{-iT}-\left(\frac{n}{k}\right)^{iT}}{-i\log(n/k)} \\
= \frac{i}{\log\frac{n}{k}}\left(\left(\frac{n}{k}\right)^{-iT}-\left(\frac{n}{k}\right)^{iT}\right) = \frac{2}{\log\frac{n}{k}}\sin\left(T\log\frac{n}{k}\right),
\end{aligned}
$$

the last equality by Euler's formula. This implies that (1.3) becomes

$$
\frac{1}{2T}\sum_{n=1}^{N}\sum_{k=1}^{N}a_n\overline{a_k}(nk)^{-\sigma}\frac{2\sin\left(T\log\frac{n}{k}\right)}{\log\frac{n}{k}} = \sum_{n=1}^{N}\sum_{k=1}^{N}a_n\overline{a_k}(nk)^{-\sigma}\frac{\sin\left(T\log\frac{n}{k}\right)}{T\log\frac{n}{k}}.
$$

When $n \neq k$, the terms have limit 0 as $T \to \infty$. However for $n = k$, the integral becomes $\int_{-T}^{T}dt = 2T$, which leaves us with the limit

$$
\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_N(\sigma+it)|^2dt = \sum_{n=1}^{N}\sum_{k=1}^{N}a_n\overline{a_k}(nk)^{-\sigma} = \sum_{n=1}^{N}|a_n|^2n^{-2\sigma}.
$$

Now we investigate the integral of (1.3) with $f(s) = \sum_1^\infty a_n n^{-s}$, for s such that $Re(s) > \sigma_u$:

$$
\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}|f|^2dt = \\
=&\frac{1}{2T}\int_{-T}^{T}|f_N+(f-f_N)|^2dt \\
=&\frac{1}{2T}\int_{-T}^{T}|f_N|^2dt+\frac{1}{2T}\int_{-T}^{T}|f-f_N|^2dt+ \\
&+\underbrace{\frac{1}{2T}\int_{-T}^{T}f_N\overline{(f-f_N)}dt+\frac{1}{2T}\int_{-T}^{T}\overline{f_N}(f-f_N)dt}_{\leq\frac{1}{T}\sqrt{\int_{-T}^{T}|f_N|^2dt}\sqrt{\int_{-T}^{T}|f-f_N|^2dt}}.
\end{aligned}
$$

Next, we know that $|f-f_N| = |f(\sigma+it)-f_N(\sigma+it)| = o(1)$, as $N\to\infty$, uniformly in t. That gives $\frac{1}{2T}\int_{-T}^{T}|f-f_N|^2dt = o(1)$, $N\to\infty$. The same holds for the last terms, since moreover f_N is bounded by a constant. That allows us to conclude that

$$
\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^2dt \xrightarrow[T\to\infty]{} \sum_{n=1}^{N}|a_n|^2n^{-2\sigma}+o(1) \xrightarrow[N\to\infty]{} \sum_{n=1}^{\infty}|a_n|^2n^{-2\sigma}. \tag{1.4}
$$

This final series is a limit of increasing partial sums. Moreover, we have established the equality $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_N(\sigma+it)|^2dt = \sum_{n=1}^{N}|a_n|^2n^{-2\sigma}$, and since the f_N:s are bounded, so are these partial sums. Hence (1.4) is convergent. □

Proposition 1.17.

$$
\sigma_a-\sigma_u\leq\frac{1}{2}
$$

Proof. Let now s be such that $\sum a_nn^{-s}$ converges uniformly to a function f. We show that the series converges absolutely in $s+\frac{1}{2}+\varepsilon$. By the inequality of Cauchy-Schwarz,

$$
\sum|a_n|n^{-\sigma-1/2-\epsilon} = \sum|a_n|n^{-\sigma}\cdot n^{-(1/2+\epsilon)} \leq \left(\sum|a_n|^2n^{-2\sigma}\right)^{1/2}\cdot\left(\sum n^{-(1+2\epsilon)}\right)^{1/2},
$$

The second factor is convergent and to the first we apply Carlson's formula:

$$
\sum|a_n|^2n^{-2\sigma} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^2dt \leq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}||f||_\infty^2dt = ||f||_\infty^2,
$$

where $||f||_\infty^2 = \sup_{t\in\mathbb{R}}|f(\sigma+it)|$. The sup-norm is bounded since the function f is a uniform limit of bounded partial sums. See Remark 1.2.

□

Chapter 2

Analytic continuation and Bohr's Theorem

We have now covered three usual types of convergences of a function series, but we will consider one other property that turns out to be relevant. For complex valued analytic functions and series defined on an initial domain, it is of interest to investigate the existence of a possible extension: finding another function defined on a larger domain that coincides with the initial function on the initial domain. By assuming that this function and its continuation should be analytic, the *continuation* is unique.

It turns out that the property of analytic continuation of a Dirichlet series is closely related to its convergence properties. We will for that reason specifically consider the analytic continuations that are bounded, for which the key result was proved by Harald Bohr, [8]. Let us first introduce the relevant abscissa.

Definition 2.1.

$$\sigma_b := \inf\{\sigma_0 : \sum_{n=1}^{\infty} a_n n^{-s} \text{ can be analytically continued to a bounded function f for } \sigma > \sigma_0\}$$

It may not be obvious why we should look at analytic continuations in halfplanes, but let us see what it gives. Here is an example:

Example 2.2 (Analytic continuation of $\zeta(s)$)**.** Finding an analytic continuation of $\zeta(s)$ is highly relevant. One of the most famous (unsolved) mathematical problems is to determine all zeros of this function, and none of them is in the domain where it is defined by the Dirichlet series.

Let s have $Re(s) > 1$. Since $\zeta(s)$ converges absolutely, we can compute

$$\begin{aligned}(1-2^{1-s})\zeta(s) &= (1-2^{1-s})(1+2^{-s}+3^{-s}+4^{-s}+\ldots) \\ &= 1+2^{-s}+3^{-s}+4^{-s}+\ldots-2\cdot 2^{-s}-2\cdot(2\cdot 2)^{-s}-2\cdot(2\cdot 3)^{-s}+\ldots \\ &= 1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s} \\ &= \zeta_a(s).\end{aligned}$$

But $\zeta_a(s)$ is defined for $Re(s) > 0$, so $\zeta(s)$ must coincide with $\frac{\zeta_a(s)}{1-2^{1-s}}$ on this larger half-plane.

We now show that $\zeta_a(s)$ is analytic in the strip $0 < Re(s) \leq 1$. We can deduce it from the pointwise convergence for $Re(s) > 0$ together with Corollary 1.6. Let's go to the details.

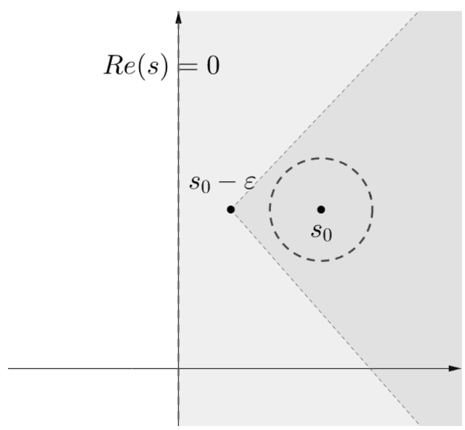

Figure 2.1: The Dirichlet Series converges in $s_0 - \varepsilon$, and by Cor 1.6 it converges uniformly in a compact subset to the right of that point or, as illustrated here, in a sector.

$\zeta_a(s)$ is analytic in a point s_0 with $Re(s_0) > 0$ if and only if it can be developed in a power series in a small disc around s_0. Since it converges in $\{s : Re(s) > 0\}$, it converges in particular at $s_0 - \varepsilon$ (> 0 when ε is taken small enough). By Corollary 1.6 the series converges uniformly in a compact subset of $\{s : Re(s) > Re(s_0) - \epsilon\}$, e.g., in a small disc centered at s_0. Thus the series is a uniform limit of the partial sums $\sum_{n=1}^{N}(-1)^{n+1}n^{-s}$ (and these are analytic since they are a finite sum of analytic functions). Hence its limit is also analytic. See Figure 2.1.

This provides the analytic continuation $(1-2^{1-s})^{-1}\zeta_a(s)$, to $\{s : Re(s) > 0\} \setminus \{1 + i\frac{2\pi n}{\log 2}, n \in \mathbb{Z}\}$, of $\zeta(s)$.

Remark 2.1. This tells more about the zeta-function. We already know that it is defined for $Re(s) > 1$ and that $\sigma_a = \sigma_c = 1$, since we have $\zeta(1) = \sum \frac{1}{n} = \infty$. But the relation to the analytic continuation tells that the singularity of $\zeta(s)$ at $s = 1$ is a pole of order 1, since $1 - 2^{1-s}$ has a zero of order 1 there.

There exists again another analytic continuation for $\zeta(s)$ to the half-plane $Re(s) \leq 0$, defined by a functional equation, which we will not consider here. Riemann himself conjectured that all of the complex zeros lie on the line $Re(s) = \frac{1}{2}$. This is known to be the, still unproved, *Riemann hypothesis* [16, p.23]. The function also have what is called the *trivial* zeros, lying on the negative real line.

Example 2.3 (σ_b for $\zeta(s)$)**.** In fact, the analytic continuation that we derived is not defined in a half-plane, but only a punctured one. The pole at $s = 1$ actually provides that $\sigma_b \geq 1$. Indeed, we can already observe that $\sigma_b \leq \sigma_a$ (see Remark 1.1), so we get $\sigma_b = 1$ here.

Example 2.4 (σ_b for $\zeta_a(s)$)**.** What about the alternating zeta-function? It is known that the zeta-function $\zeta(\sigma + it)$ is unbounded when $t \to \infty$ along all lines $\sigma \in (0, 1)$, [11, p.184]. So in the region $0 < \sigma < 1$ that is called the *critical strip*, the analytic continuation is not bounded. Thus, since $(1 - 2^{1-s})\zeta(s) = \zeta_a(s)$ in this region, ζ_a cannot be bounded either. They will consequently share the same σ_b. On the same subject, Lindelöf also had an hypothesis about the critical line $Re(s) = \frac{1}{2}$, see for example [16, p.162], namely that $\zeta(\frac{1}{2} + it) = \mathcal{O}(|t|^{\varepsilon})$, $\forall \varepsilon > 0$, as $t \to \infty$. If Riemann's conjecture holds, so will Lindelöf's, however it remains neither proved nor rejected.

The following section will provide us with more information about the abscissa σ_b.

2.1 Bohr's theorem

We are now ready to connect the notions of analytic continuation and convergence. We do it by a classical but not so obvious theorem of Bohr (published in [8]), which states that the halfplane of uniform convergence coincides with the halfplane of analytic continuability to a bounded function. This also explains why we decided to consider only continuations to bounded functions.

The proof is rather long and contains a lot of computations. The outlines are explained in the beginning to be followed by the details, with techniques from especially complex analysis but also Fourier analysis.

Theorem 2.5. *(Bohr's Theorem)* $\sigma_u = \sigma_b$.

Proof. **(**$\sigma_u \geq \sigma_b$**).** Suppose that the Dirichlet series converges uniformly on $\{Re(s) \geq \sigma_u + \varepsilon\}$. The partial sums are bounded there (also on the slightly bigger set $\{Re(s) \geq \sigma_u\}$). Since uniform convergence preserves boundedness, see Remark 1.2, the uniform limit is also bounded on $\{Re(s) \geq \sigma_u + \varepsilon\}$. We conclude that $\sigma_u \geq \sigma_b$.

($\sigma_u \leq \sigma_b$). Outline of proof: By the above, we know that $\sigma_u \geq \sigma_b$. Let f denote the analytic continuation of $\sum_{n=1}^{\infty} a_n n^{-s}$, that is bounded in the open half-plane to the right of σ_b, and suppose that f is bounded by a constant K. To show that $\sigma_u = \sigma_b$ it remains to prove that the partial sums

$$S_N(s) = \sum_{n=1}^{N} a_n n^{-s}$$

converge uniformly to $f(s)$ on the half-plane $\{s : Re(s) > \sigma_b + \delta, \, \delta > 0\}$. Indeed, then $\sigma_u = \inf\{s : Re(s) > \sigma_b + \delta, \, \delta > 0\} = \sigma_b$. Note that we can assume that $\delta \in (0, 1)$ such that $\sigma_b + \delta < \sigma_a$. If it is not possible (if $\sigma_b = \sigma_a$) we have nothing to prove, since to the right of σ_a we automatically converge uniformly.

We will use Cauchy's integral formula with a certain choice of function and contour. This integral is by construction equal to f but by calculations also the limit of the partial sums S_N.

Proof: Consider s with $Re(s) > \sigma_b + \delta$. We introduce

$$g_s(\xi) = \frac{f(\xi)}{\xi - s} \left(N + \frac{1}{2}\right)^{\xi - s}, \quad Re(\xi) > \sigma_b,$$

and the contour γ (see figure 2.2).

We define the integral

$$I := \frac{1}{2\pi i} \int_\gamma g_s(\xi) d\xi = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)(N + \frac{1}{2})^{\xi - s}}{\xi - s} d\xi = \frac{(N + \frac{1}{2})^{-s}}{2\pi i} \int_\gamma \frac{f(\xi)(N + \frac{1}{2})^{\xi}}{\xi - s} d\xi,$$

and by Cauchy's formula ($f(\xi)(N + \frac{1}{2})^\xi$ being holomorphic)

$$I = (N + \frac{1}{2})^{-s} f(s)(N + \frac{1}{2})^s = f(s)\text{Ind}_\gamma(s) = f(s).$$

We split I into 4 integrals, following the contours $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, respectively. We first consider I_2:

$$2\pi i \cdot I_2 = \int_{\gamma_2} g(\gamma) d\gamma = \int_{\gamma_2} \frac{f(\xi)(N + \frac{1}{2})^{\xi - s}}{\xi - s} d\xi = \begin{bmatrix} \gamma_2(t) = s + t + iN^{\alpha+2}, \quad \text{with } \alpha = \sigma_a - \sigma_b \\ \gamma_2'(t) = 1 \\ t \in [\alpha, -\delta] \end{bmatrix}$$
$$= \int_\alpha^{-\delta} \frac{f(s + t + iN^{\alpha+2})(N + \frac{1}{2})^{t + iN^{\alpha+2}}}{t + iN^{\alpha+2}} dt$$

so that

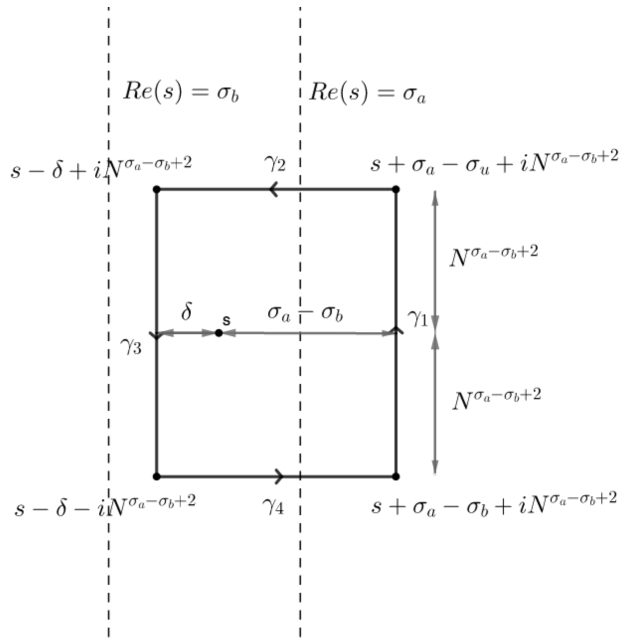

Figure 2.2: The contour of integration γ, consisting of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

$$
\begin{aligned}
|2\pi i I_2| &\leq \int_{-\delta}^{\alpha} \frac{K(N+\frac{1}{2})^t}{N^{\alpha+2}} dt \leq \frac{K(\alpha+\delta)(N+\frac{1}{2})^{\alpha}}{N^{\alpha+2}} \\
&= K(\alpha+\delta)(1+\frac{1}{2N})^{\alpha+2} \cdot \frac{1}{(N+\frac{1}{2})^2} \to 0, \text{ as } N \to \infty.
\end{aligned}
$$

More specifically $I_2 = \mathcal{O}\left(\frac{1}{N^2}\right)$ as $N \to \infty$. This holds also for I_4 by similar computations. We next consider I_3:

$$
\begin{aligned}
2\pi i \cdot I_3 &= \int_{\gamma_3} g(\xi) d\xi = \int_{\gamma_3} \frac{f(\xi)(N+\frac{1}{2})^{\xi - s}}{\xi - s} d\xi \\
&= \begin{bmatrix} \gamma_3(t) = s - \delta + it, \quad t \in [N^{\alpha+2}, -N^{\alpha+2}] \\ \gamma_3'(t) = i \end{bmatrix} = \\
&= \int_{N^{\alpha+2}}^{-N^{\alpha+2}} \frac{f(s-\delta+it)(N+\frac{1}{2})^{-\delta+it}}{it-\delta} i dt,
\end{aligned}
$$

so that

$$|2\pi i \cdot I_3| \leq K \int_{-N^{\alpha+2}}^{N^{\alpha+2}} \frac{(N+\frac{1}{2})^{-\delta}}{\sqrt{t^2+\delta^2}} dt = \frac{K}{(N+\frac{1}{2})^{\delta}} \cdot 2 \underbrace{\int_0^{N^{\alpha+2}} \frac{dt}{\sqrt{t^2+\delta^2}}}_{=:J}.$$

Naming the last integral J, we observe

$$J = \int_0^1 \frac{dt}{\sqrt{t^2+\delta^2}} + \int_1^{N^{\alpha+2}} \frac{dt}{\sqrt{t^2+\delta^2}} \leq C + \int_1^{N^{\alpha+2}} \frac{dt}{t} = C + \log N^{\alpha+2},$$

and so

$$|2\pi i \cdot I_3| \leq 2K \left(\frac{C}{(N+\frac{1}{2})^{\delta}} + \frac{(\alpha+2)\log N}{(N+\frac{1}{2})^{\delta}} \right) = \mathcal{O}\left(\frac{\log N}{N^{\delta}} \right), \; N \to \infty.$$

To sum up, the three first integrals considered are all $o(1)$, which we do not expect from I_1, but we hope to find something that is close to the partial sums of the series. This will need more careful computations. We thus investigate finally the behaviour of I_1, with contour in the region where the series converges uniformly:

$$\begin{aligned} 2\pi i \cdot I_1 &= \int_{\gamma_1} g(\xi) d\xi = \begin{bmatrix} \gamma_1(t) = s + \sigma_a - \sigma_p + it, \quad t \in [-N^{\alpha+2}, N^{\alpha+2}], \\ \gamma_1'(t) = i \end{bmatrix} \\ &= \int_{-N^{\alpha+2}}^{N^{\alpha+2}} \frac{f(s+\alpha+it)(N+\frac{1}{2})^{\alpha+it}}{\alpha+it} i dt \end{aligned} \tag{2.1}$$

Thanks to the uniform convergence f can be replaced by the series, and we can permute the integration and the summation, so that

$$\begin{aligned} 2\pi i \cdot I_1 &= i \int_{-N^{\alpha+2}}^{N^{\alpha+2}} \frac{\sum_{n=1}^{\infty} a_n n^{-(s+\alpha+it)}(N+\frac{1}{2})^{\alpha+it}}{\alpha+it} dt \\ &= i \sum_{n=1}^{\infty} a_n n^{-s} \left(\frac{N+\frac{1}{2}}{n} \right)^{\alpha} \underbrace{\int_{-N^{\alpha+2}}^{N^{\alpha+2}} \frac{(\frac{n}{N+\frac{1}{2}})^{-it}}{\alpha+it} dt}_{=:I_n}. \end{aligned}$$

We will look at the first N terms and the corresponding tail of this series separately. In fact the integral I_n turns out to behave differently for n smaller and greater than N. To see this, notice that the integral closely resembles

$$
\begin{aligned}
\lim_{T\to\infty}\int_{-T}^{T}\frac{\left(\frac{n}{N+\frac{1}{2}}\right)^{-it}}{\alpha+it}dt &= \lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-it\log\left(\frac{n}{N+\frac{1}{2}}\right)}}{\alpha+it}dt \\
&= 2\pi\mathcal{F}^{-1}\left(\frac{1}{2\pi(\alpha+it)}\right)\left(-\log\left(\frac{n}{N+\frac{1}{2}}\right)\right) = (*).
\end{aligned} \tag{2.2}
$$

The limit here is in L^2 *sense*, meaning a limit in norm. What we formally do here is to use the inversion formula for Fourier transforms (that is the equality between f and $\mathcal{F}^{-1}\hat{f}$) in $x = -\log(n/(N+1/2))$, and in order to do that we need to be cautious. For a function in L^2, the Fourier transform and its inverse is defined by such a limit in norm. Furthermore, if a function f is in L^1 (even though its transform is not) and if it is piecewise C^1, the inversion formula holds at any point where f is continuous. This is the *Dirichlet condition*, in e.g. [10, p.144].

To start at the beginning, we have the Fourier transform of a function in $L^2 \cap L^1$:

$$
\begin{aligned}
(\mathcal{F}f)(t) = \mathcal{F}(\chi_{(0,\infty)}(x)e^{-2\pi\alpha x})(t) := \int_{-\infty}^{\infty}\chi_{(0,\infty)}(x)e^{-2\pi\alpha x}e^{-2\pi ixt}dx = \int_{0}^{\infty}e^{-2\pi x(\alpha+it)}dx \\
= \left[\frac{e^{-2\pi x(\alpha+it)}}{-2\pi(\alpha+it)}\right]_0^{\infty} = \frac{1}{2\pi(\alpha+it)}.
\end{aligned}
$$

The transform $\mathcal{F}f$ is not in L^1, but only in L^2. Now, by the Dirichlet condition, we may apply the inversion formula as in (2.2), because the function $f = \chi_{(0,\infty)}(x)e^{-2\pi\alpha x}$ is continuous and C^1 with exception from one point, $x = 0$. We use the inversion formula in $x = -\log(\frac{n}{N+\frac{1}{2}}) \neq 0$, since both $n, N \in \mathbb{Z}$. Thus (2.2) becomes

$$
\begin{aligned}
(*) &= \begin{cases} 2\pi e^{-\alpha\left(-\log\frac{n}{N+\frac{1}{2}}\right)}, & n < N+\frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi\left(\frac{n}{N+\frac{1}{2}}\right)^{\alpha}, & n < N+\frac{1}{2} \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

and this is also what we would like to estimate I_n with. To do this, we need to investigate if and how it tends to the inverse Fourier transform as $N \to \infty$.

They differ by

$$2\pi\mathcal{F}^{-1}\left(\frac{1}{2\pi(\alpha+it)}\right)\left(-\log(\frac{n}{N+\frac{1}{2}})\right)-I_n=\lim_{T\to\infty}\int_{-T}^{T}\frac{(\frac{n}{N+\frac{1}{2}})^{-it}}{\alpha+it}dt-\int_{-N^{\alpha+2}}^{N^{\alpha+2}}\frac{(\frac{n}{N+\frac{1}{2}})^{-it}}{\alpha+it}dt$$
$$=\lim_{T\to\infty}\int_{T>|t|>N^{\alpha+2}}\frac{(\frac{n}{N+\frac{1}{2}})^{-it}}{\alpha+it}dt,$$

and we denote this difference by R_n.

By calling $\beta := \frac{n}{N+\frac{1}{2}}$ let $R_n = R_n^+ + R_n^-$, where

$$\begin{aligned}
R_n^+ &= \lim_{T\to\infty}\int_{N^{\alpha+2}}^{T}\frac{\beta^{-it}}{\alpha+it}dt \\
&= \lim_{T\to\infty}\left[\frac{\beta^{-it}}{-i\log\beta}\cdot\frac{1}{\alpha+it}\right]_{N^{\alpha+2}}^{t=T}-\lim_{T\to\infty}\int_{N^{\alpha+2}}^{T}\frac{\beta^{-it}}{-i\log\beta}\cdot\frac{-i}{(\alpha+it)^2}dt \\
&= \mathcal{O}\left(\frac{1}{\log(\beta)N^{\alpha+2}}\right)=\mathcal{O}\left(\frac{1}{\log(\frac{n}{N+\frac{1}{2}})N^{\alpha+2}}\right), \text{ as } N\to\infty.
\end{aligned}$$

We see in after the partial integration that our limit will converge. Similarly

$$R_n^- = \mathcal{O}\left(\frac{1}{\log(\frac{n}{N+\frac{1}{2}})N^{\alpha+2}}\right),\ N\to\infty.$$

To simplify the errors we estimate from below:

$$\begin{aligned}
n\le N: &\quad \log\left(\frac{n}{N+\frac{1}{2}}\right)\ge\log\left(\frac{1}{N+\frac{1}{2}}\right)\ge\frac{C}{N}, \\
n> N: &\quad \log\left(\frac{n}{N+\frac{1}{2}}\right)\ge\log\left(\frac{N+1}{N+\frac{1}{2}}\right)=\log\left(1+\frac{1}{2N+1}\right)\ge\frac{C}{2N+1},
\end{aligned}$$

so that, for all n,

$$\mathcal{O}\left(\frac{1}{\log(\frac{n}{N+\frac{1}{2}})N^{\alpha+2}}\right)=\mathcal{O}\left(\frac{1}{N^{\alpha+1}}\right),\ N\to\infty.$$

Hence we can replace I_n by the estimation and get

$$
\begin{aligned}
2\pi i \cdot I_1 &= i\sum_{n=1}^{N} a_n n^{-s}\left(\frac{n}{N+\frac{1}{2}}\right)^{-\alpha} I_n + i\sum_{n=N+1}^{\infty} a_n n^{-s}\left(\frac{n}{N+\frac{1}{2}}\right)^{-\alpha} I_n \\
&= i\sum_{n=1}^{N} a_n n^{-s}\left(\frac{n}{N+\frac{1}{2}}\right)^{-\alpha}\left(2\pi\left(\frac{n}{N+\frac{1}{2}}\right)^{\alpha} + \mathcal{O}\left(\frac{1}{N^{\alpha+1}}\right)\right) + \\
&+ i\sum_{n=N+1}^{\infty} a_n n^{-s}\left(\frac{n}{N+\frac{1}{2}}\right)^{-\alpha}\left(0 + \mathcal{O}\left(\frac{1}{N^{\alpha+1}}\right)\right) \\
&= 2\pi i\sum_{n=1}^{N} a_n n^{-s} + i\sum_{n=1}^{N} a_n n^{-(s+\alpha)}\mathcal{O}\left(\frac{1}{N}\right) + i\sum_{n=N+1}^{\infty} a_n n^{-(s+\alpha)}\mathcal{O}\left(\frac{1}{N}\right).
\end{aligned}
\tag{2.3}
$$

Further, $\sum_1^\infty a_n n^{-(s+\alpha)} \leq C$, for $Re(s) > \sigma_b + \delta$. One way of seeing that this bound is uniform in s is by the absolute convergence of the series in $s + \alpha = s + \sigma_a - \sigma_b$, for $Re(s) > \sigma_b + \delta$. By the Remark 1.1, the series is bounded in the same set, and the functions $a_n n^{-(\sigma+\alpha)}$ are monotonously decreasing. We thus conclude that we can pick a uniform bound. Finally

$$
2\pi i \cdot I_1 = \int_{\gamma_1} g(\xi)d\xi = 2\pi i\sum_{n=1}^{N} a_n n^{-s} + \mathcal{O}\left(\frac{1}{N}\right),\ N \to \infty.
$$

In conclusion

$$
f(s) = \frac{1}{2\pi i}\int_{\gamma} g_s(\xi)d\xi = \frac{1}{2\pi i}\left(2\pi i\sum_{n=1}^{N} a_n n^{-s} + \mathcal{O}\left(\frac{1}{N}\right) + o(1)\right),
$$

as N gets big, which implies

$$
\lim_{N\to\infty}\sum_{n=1}^{N} a_n n^{-s} = f(s),
$$

independent of the choice of s with $Re(s) > \sigma_b + \delta$. Thus the series converges uniformly in the half-plane $\{Re(s) > \sigma_b + \delta\}$, and $\sigma_b = \sigma_u$. □

Remark 2.2. In fact, we can deduce another result along the way. If we instead of the integral I_1 in (2.1) change the endpoints and calculate

$$
\frac{1}{2\pi}\int_{-T}^{T} \frac{f(\sigma+it)}{\sigma+it}\left(N+\frac{1}{2}\right)^{\sigma+it} dt,
$$

the expression in (2.3) becomes slightly different:

$$
2\pi i \cdot I_1 = 2\pi\sum_{n=1}^{N} a_n n^{-s} + \sum_{n=1}^{\infty} a_n n^{-(s+\alpha)}\mathcal{O}\left(\frac{N^\alpha}{T}\right).
$$

In this case we consider N to be fixed, and the limit as $T \to \infty$ then yields the following result.

Corollary 2.6 (Perron's formula)**.**

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{f(\sigma+it)}{\sigma+it} \left(N+\frac{1}{2}\right)^{\sigma+it} dt = \sum_{n=1}^{N} a_n n^{-s},$$

where $f(s) = \sum a_n n^{-s}$ *for all* s *where the series can be analytically continued to a bounded function.*

Example 2.7 ($\zeta(s)$, $\zeta_a(s)$, revisited)**.** We saw already in Example 2.3 that $\sigma_b = 1$ for $\zeta(s)$, and we can summarize the quite one-sided properties of $\zeta(s)$:

$$\sigma_c = \sigma_u = \sigma_b = \sigma_a = 1.$$

Concerning $\zeta_a(s)$, we found in Example 2.4 that $\zeta_a(s)$ and $\zeta(s)$ share the same σ_b. By Bohr's theorem, we know that it also coincides with σ_u, so for the alternating zeta-function we again see that we have:

$$\sigma_c = 0,$$
$$\sigma_u = \sigma_b = \sigma_a = 1.$$

Chapter 3

Bohr's Inequality for Dirichlet Series and Kronecker's theorem

In Chapter 4 we will construct an example of a Dirichlet series with certain properties. We will then use an inequality that is not at all obvious but a special property of Dirichlet series, that we present in Theorem 3.1. The idea behind this lies in the connection between Dirichlet series and Taylor polynomials of infinitely many variables. It is based on the unique prime factorization of integers, $n = p_1^{\nu_1} \cdot \ldots \cdot p_l^{\nu_l}$. Thus, n^{-s} can be factorized into $(p_1^{\nu_1} \cdot \ldots \cdot p_l^{\nu_l})^{-s} = (p_1^{-s})^{\nu_1} \cdot \ldots \cdot (p_l^{-s})^{\nu_l}$, which can be thought of as a product of l complex variables due to Kronecker's theorem.

This idea was introduced by Bohr already in 1913, see [7], and it is still highly relevant in recent research. For example it was a central idea used by Hedenmalm, Lindqvist and Seip in their article from 1997, [13].

Concerning the inequality that we are going to derive, we will not consider the general case but only the series with terms of the form $\pm k^{it}$ (we denote their coefficients by ε_k). The reason for this is that we will consider random series related to the ζ-functions $\sum \pm n^{-s} = \sum \varepsilon_n n^{-s}$ in the following section.

Theorem 3.1 (Bohr)**.** *For the partial sums of a Dirichlet series with coefficients ε_k the following property holds:*

$$\sup_{t\in\mathbb{R}} U_n(t) := \sup_{t\in\mathbb{R}} \left| \sum_{k=1}^{n} \varepsilon_k k^{-it} \right| \geq \sum_{p \leq n, p\, prime} |\varepsilon_p|.$$

As one may imagine, this inequality relates Dirichlet series to prime numbers. For example, Theorem 3.1 could give an upper bound of the number of primes $\leq N$, by studying the supremum of the partial sums of $\zeta(s)$ more closely

A difficulty in proving the theorem lies in the fact that this is a 1-dimensional supremum that is hard to evaluate. Playing with the thought, we could have evaluated: $\sup_{t_1,\ldots,t_n} |\sum_{k=1}^{n} \varepsilon_k k^{-it_k}| = \sum_{k=1}^{n} |\varepsilon_k| \geq \sum_{p\leq n,\, p\, prime} |\varepsilon_p|$. But since we can only choose one value of t for all the terms, we can not immediately evaluate the supremum. What we can do is to consider the Dirichlet polynomial as a Taylor polynomial of several variables – thanks to the unique prime factorization of every integer. Once here, a number theoretic result by Kronecker is what helps out us of the tricky situation with the supremum of one variable. Let's go to the details.

Proof of Theorem 3.1. We assume that the integers $\leq n$ have at most m prime factors, and we can factorize it as $k = p_1^{\nu_1} \cdot \ldots \cdot p_l^{\nu_l}, l \leq m$. We use this prime factorization to translate our Dirichlet polynomial into a Taylor polynomial of m variables:

$$\sum_{k=1}^{n} \varepsilon_k k^{-it} = \sum_{k=1}^{n} \varepsilon_k (p_1^{-it})^{\nu_1} \cdot \ldots \cdot (p_l^{-it})^{\nu_l},$$

where we can consider p_j^{-it} as a complex variable z_j, belonging to the unit circle. It is then natural to instead investigate

$$\sum_{k=1}^{n} \varepsilon_k z_1^{\nu_1} \ldots z_l^{\nu_l} = f(z_1,\ldots,z_m), \quad z_1,\ldots,z_m \in \partial\mathbb{D}. \tag{3.1}$$

What will now be our key, and what we need to establish, is the equality

$$\sup_{t\in\mathbb{R}} U_n(t) = \sup_{z_1,\ldots,z_m\in\partial\mathbb{D}} |f(z_1,\ldots,z_m)|, \tag{3.2}$$

but for this we will need one more tool.

Let us just introduce a more readable multi-notation. What was before $\varepsilon_k k^{-it}$, is in the Taylor polynomial $\varepsilon_k z_1^{\nu_1} \ldots z_l^{\nu_l}$, but we will simply denote it by $\varepsilon_\nu z^\nu$ where $z = (z_1, z_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_l, 0, \ldots)$. Each term in this sum has a different amount of variables z_j (namely $l = l(k)$, depending on the prime factorization of k and thus on ν), and we sort the sum into different parts according to this amount:

$$f(z) = \varepsilon_1 + \underbrace{\sum_{\nu:\sum\nu_i=1} \varepsilon_\nu z^\nu}_{\text{1 prime factor}} + \underbrace{\sum_{\nu:\sum\nu_i=2} \varepsilon_\nu z^\nu}_{\text{2 prime factors, counting multiplicity}} + \ldots + \underbrace{\sum_{\nu:\sum\nu_i=m} \varepsilon_\nu z^\nu}_{\text{m prime factors, counting multiplicity}}.$$

This choice was not coincidental, since the first sum (besides ε_1) is exactly the sum corresponding to the prime numbers, and these are the integers k with exactly 1 prime factor, counting multiplicity. In addition we note that for this block $\varepsilon_\nu = \varepsilon_p$.

Kronecker's Theorem

The equality that we want to establish is the same as

$$\sup_{t\in\mathbb{R}} |f(2^{-it}, \dots, m^{-it})| = \sup_{z_1,\dots,z_m\in\partial\mathbb{D}} |f(z_1, \dots, z_m)|.$$

For $m = 1$, this is true, since $\sup_{t\in\mathbb{R}} |f(2^{-it})| = \sup_{z_1\in\partial\mathbb{D}} |f(z_1)|$.

Consider the next case, $m = 2$. z_1, z_2 being taken independently on the unit circle is the same as $e^{i2\pi\theta_1}, e^{i2\pi\theta_2}$ for θ_1, θ_2 taken arbitrarily in $\mathbb{R}$, see Figure 3.1. Consider them as fixed and arbitrary.

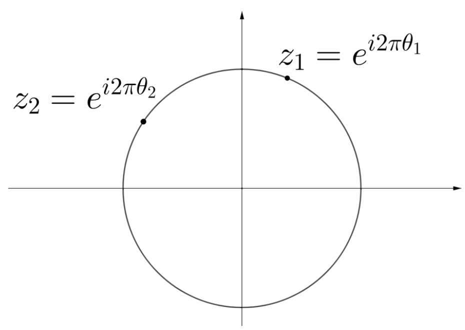

Figure 3.1: $z_1 = e^{i2\pi\theta_1}, z_2 = e^{i2\pi\theta_2}$ arbitrary in the unit circle $\partial\mathbb{D}$.

We want to convince ourselves that we can choose $t \in \mathbb{R}$ such that

$$\begin{aligned} &e^{-i2\pi t\log 2} \text{ is arbitrarily close to } z_1 = e^{i2\pi\theta_1}, \\ &e^{-i2\pi t\log 3} \text{ is arbitrarily close to } z_2 = e^{i2\pi\theta_2}. \end{aligned} \tag{3.3}$$

For a continuous function f, (3.3) is enough to have the desired equality (3.2). And we can be sure that such a t exists. It is here that our theorem comes into play.

Theorem 3.2 (Kronecker in 2 dimensions)**.** *Suppose that $\lambda_1, \lambda_2 \in \mathbb{R}$ are linearly independent with respect to $\mathbb{Z}$ and let $\theta_1, \theta_2 \in \mathbb{R}$ be arbitrary.*

Then $\forall\varepsilon > 0$, $\exists t \in \mathbb{R}$ s.t. $|e^{2\pi i(\lambda_n t-\theta_n)} - 1| < \varepsilon$, $n = 1, 2$.

Remark 3.1. *λ_1, λ_2 linearily independent w.r.t. $\mathbb{Z}$* means that they cannot be written as a linear combination with non-trivial integer coefficients, i.e. if $\exists m, n \in \mathbb{Z}$ s.t. $m\lambda_1 + n\lambda_2 = 0 \Rightarrow m, n = 0$.

Remark 3.2. To give an even more visual image of this problem, consider the equivalent one of the existence of $t \in \mathbb{R}$ s.t. the line

$$\begin{cases} x = t\log 2 \\ y = t\log 3 = \dfrac{\log 3}{\log 2}\cdot t\log 2 \end{cases} \quad \text{passes arbitrarily close to } (\theta_1,\theta_2) \qquad (\text{mod } 1),$$

see Figure 3.2.

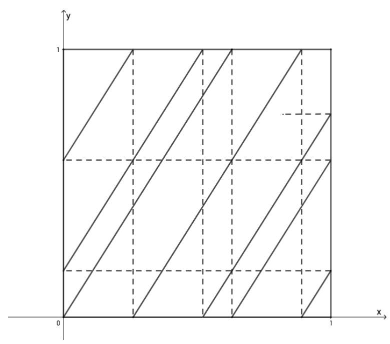

Figure 3.2: The line $(x, y) = (t \log 2, t \log 3) \pmod 1$. Whenever it arrives at the boundary of $[0, 1)^2$ it continues where it would have entered a neighbouring square. Will it pass arbitrarily close to any point (θ_1, θ_2)?

So what the result says is that this line in fact is dense in $[0, 1)^2$, and it will pass arbitrarily near every point. This holds because the numbers $\log 2, \log 3$ are linearly independent with respect to $\mathbb{Z}$. The proof of the theorem that we will now see is not as intuitive as this geometric discussion of its consequence, but it is very efficient. We thus leave our geometrical argumentation here.

Proof. We consider the case $m = 2$. The general case follows by slightly modifying this argument.

The strategy of this proof will be to construct the function

$$F(t) = 1 + e^{2\pi i(\lambda_1 t - \theta_1)} + e^{2\pi i(\lambda_2 t - \theta_2)}$$

and show that $\sup_{t\in\mathbb{R}} F(t) = 3 = 1 + 2$. This is sufficient for the existence of a t such that $|e^{2\pi i(\lambda_n t - \theta_n)} - e^0| < \varepsilon$, $n = 1, 2$.

We can already deduce that $\sup_{t\in\mathbb{R}} F(t) \leq 3$, since, by the triangle inequality,

$$|F(t)| \leq 1 + 1 + 1 = 3.$$

This leaves us to show that $\sup_{t\in\mathbb{R}} F(t) \geq 3$. We will use the *Fejér kernel.* It is a trigonometric polynomial with certain properties that can be used to prove Fejér Theorem in Fourier analysis. See for example [17, p.28]. It can be expressed using any one of the formulas:

$$K_n(t) = \sum_{j=-n}^{n} \frac{n-|j|}{n} e^{ijt} = \frac{1}{n}\left(\frac{\sin(\frac{1}{2}nt)}{\sin(\frac{1}{2}t)}\right)^2.$$

For example, from the second expression, we note that it is a positive function. We will need the composite kernel

$$\begin{aligned}\mathbb{K}_n(t) &= K_n\big(2\pi(\lambda_1 t-\theta_1)\big)K_n\big(2\pi(\lambda_2 t-\theta_2)\big)\\ &= \left(\sum_{j=-n}^{n} \frac{n-|j|}{n} e^{ij2\pi(\lambda_1 t-\theta_1)}\right)\left(\sum_{j=-n}^{n} \frac{n-|j|}{n} e^{ij2\pi(\lambda_2 t-\theta_2)}\right).\end{aligned}$$

Multiplying out gives

$$\begin{aligned}\mathbb{K}_n(t) &= \left(1+\sum_{j=-n,j\neq 0}^{n} \frac{n-|j|}{n} e^{ij2\pi(\lambda_1 t-\theta_1)}\right)\left(1+\sum_{j=-n,j\neq 0}^{n} \frac{n-|j|}{n} e^{ij2\pi(\lambda_2 t-\theta_2)}\right)\\ &= 1+\frac{n-1}{n}\left(e^{-i2\pi(\lambda_1 t-\theta_1)}+e^{-i2\pi(\lambda_2 t-\theta_2)}\right)+R(t),\end{aligned} \tag{3.4}$$

where $R(t) = \sum c_{m,n} e^{i2\pi t(m\lambda_1+n\lambda_2)}$ is a trigonometric polynomial with exponents $i2\pi t(m\lambda_1+n\lambda_2)$ that are all different from $0, -i2\pi t\lambda_1, -i2\pi t\lambda_2$. Thus we can express the product

$$\begin{aligned}F(t)\mathbb{K}_n(t) &= 1+\frac{n-1}{n}\left(e^{i2\pi(\lambda_1 t-\theta_1)}e^{-i2\pi(\lambda_1 t-\theta_1)}+e^{i2\pi(\lambda_2 t-\theta_2)}e^{-i2\pi(\lambda_2 t-\theta_2)}\right)+S(t)\\ &= 1+\frac{n-1}{n}\cdot 2+S(t),\end{aligned}$$

where $S(t)$ is a trigonometric polynomial with exponents all different from 0. We further note that Fejér's kernel K_n has mean 1 over one period, meaning $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$, which follows from straight-forward calculations using the definition. The composite kernel will instead have the property of $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbb{K}_n(t)dt = 1$. This can again be obtained by calculations, using the expression of $\mathbb{K}_n$ in (3.4). The calculations are identical to what we will see in (3.5), so we don't go through the details here.

A key for the proof is to note that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\left|\int_{-T}^{T} F(t)\mathbb{K}_n(t)dt\right| &\leq \sup_{t\in\mathbb{R}}|F(t)|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\mathbb{K}_n(t)|dt\\ &= \sup_t |F(t)|\underbrace{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbb{K}_n(t)dt}_{=1} = \sup_t|F(t)|,\end{aligned}$$

since this gives a lower bound to our supremum. To compute this lower bound we evaluate the limit

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\left|\int_{-T}^{T}F(t)\mathbb{K}_n(t)dt\right| &= \lim_{T\to\infty}\frac{1}{2T}\left|\int_{-T}^{T}\left(1+2\cdot\frac{n-1}{n}+S(t)\right)dt\right|\\ &= \lim_{T\to\infty}\frac{1}{2T}\left|2T\left(1+2\frac{n-1}{n}\right)+\int_{-T}^{T}S(t)dt\right|\\ &= \left|1+2\frac{n-1}{n}+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}S(t)dt\right|.\end{aligned}$$

As mentioned above, S(t) is a trigonometric polynomial and can be expressed as

$$S(t)=\sum_k c_k e^{i\gamma_k t},\ \gamma_k\neq 0.$$

Thus

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}S(t)dt &= \frac{1}{2T}\int_{-T}^{T}\sum_k c_k e^{i\gamma_k t}dt = \frac{1}{2T}\sum_k c_k\int_{-T}^{T}e^{i\gamma_k t}dt\\ &= \frac{1}{2T}\sum_k c_k\frac{1}{i\gamma_k}\left(e^{i\gamma_k T}-e^{-i\gamma_k T}\right)=\sum_k\frac{c_k}{\gamma_k}\frac{e^{i\gamma_k T}-e^{-i\gamma_k T}}{2iT} \qquad (3.5)\\ &= \sum_k\frac{c_k}{\gamma_k}\frac{\sin(\gamma_k T)}{T}\xrightarrow[T\to\infty]{}0.\end{aligned}$$

Hence

$$\lim_{T\to\infty}\frac{1}{2T}\left|\int_{-T}^{T}F(t)\mathbb{K}_n(t)dt\right|=1+2\frac{n-1}{n},\forall n\in\mathbb{N},$$

so in particular this tends to $1+2=3$ as $n\to\infty$, and this ends the proof. □

Remark 3.3. In the proof of the Theorem 3.1, we apply this result in the general m-dimensional case, which is still provided by Kronecker's theorem, as mentioned in the beginning of the proof.

Now, we conclude that

$$\sup_{t\in\mathbb{R}}U_n(t)=\sup_{z_1,\dots,z_m\in\partial\mathbb{D}}|f(z_1,\dots,z_m)|.$$

Let's again look at the Taylor polynomial in (3)

$$f(z_1,\dots,z_m)=\varepsilon_1+\sum_{\nu:\sum\nu_i=1}\varepsilon_p z^\nu+\sum_{\nu:\sum\nu_i=2}\varepsilon_\nu z^\nu+\ldots+\sum_{\nu:\sum\nu_i=m}\varepsilon_\nu z^\nu$$

for which we are interested in the supremum over $z_j\in\partial\mathbb{D}$. By choosing $z_j=r\cdot\text{sgn}(\varepsilon_{p_j})$, $|r|=1$ (we remind that $\varepsilon_k=\varepsilon_{p_j}$ when k is the j:th prime), we estimate from below that

$$\sup_{z_1,\dots,z_m\in\partial\mathbb{D}}|f(z_1,\dots,z_m)|\geq|\varepsilon_1+\sum_{\nu:\sum\nu_i=1}\varepsilon_p(r\text{sgn}(\varepsilon_p))^\nu+\ldots+\sum_{\nu:\sum\nu_i=m}\varepsilon_\nu(r\text{sgn}(\varepsilon_p))^\nu|$$
$$=|\varepsilon_1+r\sum_{\nu:\sum\nu_i=1}|\varepsilon_p|+\ldots+r^m\sum_{\nu:\sum\nu_i=m}\varepsilon_\nu(\text{sgn}(\varepsilon_p))^\nu|.$$

This is in fact (the absolute value of) a polynomial of r, that we note as

$$F(r)=c_0+c_1r+\ldots+c_mr^m,\ \text{defined for } |r|=1.$$

But such a function can without any problem be defined for all $r\in\mathbb{C}$, in particular all $|r|\leq 1$, and we will use this extention on the whole disc $\overline{\mathbb{D}}$. We will use this polynomial to find the lower bound of $\sup_{z_1,\dots,z_m\in\partial\mathbb{D}}|f(z_1,\dots,z_m)|$. The supremum is a finite number, since the Dirichlet polynomial f is bounded on $\overline{\mathbb{D}}$. Thus we have

$$K=:\sup_{z_1,\dots,z_m\in\partial\mathbb{D}}|f(z_1,\dots,z_m)|\geq|F(r)|,\ \forall|r|\leq 1.$$

From here, we can by Cauchy's integral formula prove that

$$|F'(0)|=c_1=\sum_{p\leq m,\,p\,prime}|\varepsilon_p|\leq K.$$

Indeed, this holds because, for $r=1$

$$|F'(0)|=\left|\frac{1}{2\pi i}\int_{|\xi|=1}\frac{F(\xi)}{\xi^2}d\xi\right|\leq\frac{1}{2\pi}\int_{|\xi|=1}\frac{|F(\xi)|}{|\xi|^2}d\xi\leq K.$$

Thus $|F'(0)|\leq K$. But that means exactly that

$$|F'(0)|=\sum_{p\leq m,\,p\,prime}|\varepsilon_p|\leq\sup_{z_1,\dots,z_m\in\partial\mathbb{D}}|f(z_1,\dots,z_m)|=\sup_{t\in\mathbb{R}}U_n(t),$$

and we are done. □

Chapter 4

Studies of the Random Series $\sum \pm n^{-s}$

We have up until now examined general Dirichlet series and their convergence behaviour in different senses. We have seen some important examples, namely $\zeta(s)$ and $\zeta_a(s)$. $\zeta(s)$ turned out to be quite one-sided in its convergence with all the abscissae coinciding, whereas the abscissae of $\zeta_a(s)$ are not all the same.

We will in the remaining chapters focus on constructing certain examples. In particular we wish to see a series of maximal strip width $\sigma_a - \sigma_u = \frac{1}{2}$. We will not reach there until Chapter 5, but for all the examples we will have the same starting point. That is considering series with coefficients ± 1 (thus resembling the zeta-functions) and also taking a step into the probabilistic world, by allowing the choice of these coefficients to be stochastic. What we will see in this chapter is some of the work done by Hervé Queffeléc in [19].

Now, consider the series of the form

$$\sum_{n=1}^{\infty} \varepsilon_n n^{-s}. \tag{4.1}$$

We choose ε_n to be either 1 och -1, with equal probability $\frac{1}{2}$. Formally we introduce the sample space $\Omega = \{-1, +1\}^{\mathbb{N}}$, equipped with the product topology $\mathcal{F}$ of the discrete topology in each factor. Further we equip this topological space with the product probability distribution P of $\mathbb{P}_n(\varepsilon = 1) = \mathbb{P}_n(\varepsilon = -1) = \frac{1}{2}$ in each factor. A point (an event) $\omega \in \Omega$ is of the form $\omega = (\varepsilon_1, \dots, \varepsilon_n, \dots)$ with components $\varepsilon_n(\omega)$ that are independent stochastic variables. They are said to be *Rademacher distributed* [15].

4.1 Almost Surely and Quasi Surely

Let $(\Omega, \mathcal{F}, P)$ be the probability space introduced above. We make the following definitions:

Definition 4.1 (Almost surely)**.** An event $E \in \mathcal{F}$ is said to or to occur *almost surely* (denoted a.s.) if $\mathbb{P}(E) = 1$.

This definition is rather standard. Another notion not as frequent is the following:

Definition 4.2 (Quasi sure)**.** A property that occurs on a countable intersection of open dense sets is said to be *quasi sure* (q.s.).

This is a topological concept and does not involve a probability measure (and thus not a probability space, but only a topological space). The theory of probability and topology is not meant to be a central part of this work, so we will only treat and explain the necessary parts briefly.

Let us just look at an example of a quasi sure property:

Example 4.3. Consider for example all the real numbers, $\mathbb{R}$. Each subset $\mathbb{R} \setminus \{q\}$, $q \in \mathbb{Q}$, with one rational number removed, is a dense, open set in $\mathbb{R}$. Since $\mathbb{Q}$ is countable, the intersection over all rational numbers of all such subsets is $\bigcap_{q\in\mathbb{Q}} \mathbb{R} \setminus \{q\} = \mathbb{R} \setminus \mathbb{Q}$. This means that a real number is *quasi surely* irrational.

These are two interpretations of what we mean by a big event, and – as will be clear – different events will be big, depending on the choice of interpretation. What is important for us is that the non-zero probability of a certain event provides the existence of Dirichlet series with certain properties. This is the content of Queffélec's theorem that we will now consider.

4.2 Queffélec's Example

The following results tell us the typical properties of $\sum \pm n^{-s}$. Of course these typical properties depend on our notion of "typical".

Theorem 4.4 (Queffélec)**.** *The series* $f(s) = \sum_{n=1}^{\infty} \varepsilon_n n^{-s}$ (4.1) *has q.s. the properties*

$$\sigma_u = \sigma_c = 1,$$

and the line $\sigma = 1$ is a natural boundary for f, i.e. there is no possible analytic continuation to the left of that line.

Theorem 4.5 (Kahane-Queffelec)**.** *The series* (4.1) *has a.s. the properties*

$$\sigma_u = 1, \quad \sigma_c = \frac{1}{2},$$

and the line $\sigma = \frac{1}{2}$ is a natural boundary for f.

The first theorem provides a series $\sum \pm n^{-s}$ such that all abscissae coincide at $\sigma = 1$, just as for the series representation of $\zeta(s)$, but there is no analytic continuation possible. The second theorem provides the existence of a sequence of ± 1 such that its corresponding series has a shifted abscissa σ_c – just in between the abscissa for $\zeta(s)$ and $\zeta_a(s)$. We are going to prove these properties, following the approach of Queffélec did.

Remark 4.1. We know that since

$$\sum_{n=1}^{\infty} |\varepsilon_n n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$$

converges for s such that $Re(s) = \sigma > 1$, the the series (4.1) converges absolutely for these s, and $\sigma_a = 1$, not depending on the choice of $(\varepsilon_n)_{n\in\mathbb{N}}$.

Proof of Theorem 4.4

Proof. **($\sigma_c = \sigma_u = 1$, $\sigma = 1$ is a natural boundary, q.s.)**: Since $\sigma_a = 1$, and if we can show that $\sigma = 1$ is q.s. a natural boundary, we will automatically have the rest. But what does it mean that this line would be a natural boundary, quasi surely? By definition of q.s., this means that

The set of all points $\omega \in \Omega$ for which the series $\sum_{1}^{\infty} \varepsilon_n n^{-s}$ has

$Re(s) = 1$ as a natural boundary is a countable intersection of open dense sets.

In other words, the set $\{\omega \in \Omega : Re(s) = 1$ is a natural boundary for $\sum \varepsilon_n n^{-s}\}$ is of the form $\bigcap_{j\in\mathbb{N}} \mathcal{O}_j$, where $\mathcal{O}_j$ are open, dense sets. We will instead consider its complement for which an equivalent statement is

The set of all points $\omega \in \Omega$ for which the series $\sum_{1}^{\infty} \varepsilon_n n^{-s}$ does

<u>not</u> have $Re(s) = 1$ as a natural boundary is a countable union of closed sets with

empty interior, that is $\left(\bigcap_{j\in\mathbb{N}} \mathcal{O}_j\right)^{\complement} = \bigcup_{j\in\mathbb{N}} \mathcal{O}_j^{\complement}$, $\mathcal{O}_j^{\complement}$ closed, of empty interior.

We will prove that the last statement is true. That the line $\sigma = 1$ is not a natural boundary means that there is an analytic continuation beyond the line. Denote E the set of all such ω. Note that this set can be expressed as

$$E = \{\omega \in \Omega | \sum_1^\infty \varepsilon_n(\omega) n^{-s} \text{ has an analytic continuation beyond } Re(s) = 1\}$$
$$= \{\omega \,|\, \exists a \in \mathbb{C}, Re(a) = 1, \exists r > 0 \text{ s.t. } \sum_1^\infty \varepsilon_n(\omega) n^{-s} \text{ has an analytic cont. on } D_{a,r}\}.$$

By $D_{a,r} = \{s \in \mathbb{C} : |s - a| < r\}$ we mean the disc of center a and radius r. This is a minimal requirement for analytic continuation.

To be able to caracterize this set even more, we note that we may have such an analytic continuation that is not bounded on all of $D_{a,r}$. But in this case it would be bounded on $D_{a,\frac{r}{2}}$, so in fact we can without loss of generality assume that the continuation is bounded on the disc, i.e.

$$E = \{\omega \in \Omega \,|\, \exists a \in \mathbb{C}, Re(a) = 1, \exists r > 0, \exists N \text{ s.t. } \sum_1^\infty \varepsilon_n(\omega) n^{-s} = f_\omega(s)$$
$$\text{has an analytic cont. on } D_{a,r}, \text{ and } |f_\omega| \le N \text{ on } D_{a,r}\}.$$

But this is a union over all a, r, N for which an ω is in E, so that E can be written

$$E = \bigcup_{a,r,N} E_{a,r,N}.$$

This union is in particular countable if $a, r \in \mathbb{Q}$, $N \in \mathbb{N}$. We thus have to show that the sets $E_{a,r,N}$ are closed and of empty interior. We take one of them, and fix thus a, r, N, and write $a = 1 + it$.

($E_{a,r,N}$ is closed): As the standard procedure for this kind of result, we prove that $\overline{E_{a,r,N}} = E_{a,r,N}$, where $\overline{E_{a,r,N}}$ is the closure of $E_{a,r,N}$. We di it by taking a point in $\overline{E_{a,r,N}}$ and showing that it is also in $E_{a,r,N}$ (since the converse inclusion is always true).

So, let $\omega \in \overline{E_{a,r,N}}$. Since this set is closed there exists a sequence in $E_{a,r,N}$ converging to this point, i.e.

$$\exists\, (\omega^i)_{i\in\mathbb{N}} \text{ s.t. } \omega^i \in E_{a,r,N} \text{ and } \omega^i \to \omega,\ i \to \infty.$$

For each $\omega^i \in E_{a,r,N}$ there is a corresponding analytic continuation f_{ω^i}, by the definition of $E_{a,r,N}$ and since each $\omega^i \in E_{a,r,N}$. We denote it f_i.

We have that $|f_i| \le N$ on $D_{a,r}$, which means that the f_i:s forms a family of uniformly bounded holomorphic functions on the compact discs. Thus, by Montel's theorem [18, p.225], the functions in $(f_i)_{i\in\mathbb{N}}$ form a *normal family* on $D_{a,r}$. By definition, this means that there exists a uniformly converging subsequence on every compact subset of $D_{a,r}$. Differently formulated, $\exists i_k$, $k \in \mathbb{N}$ and $\exists g$ s.t. $f_{i_k} \to g$, $k \to \infty$, uniformly on every compact of $D_{a,r}$. Moreover, g is holomorphic and $|g| \leq N$ on $D_{a,r}$.

We will show that this g defines an analytic continuation corresponding to ω, which is enough to conclude that $\omega \in E_{a,r,N}$.

We use the fact that the subsequence $\omega^{i_k} \to \omega$, $k \to \infty \iff \varepsilon_n^{i_k} \to \varepsilon_n$, $k \to \infty$, $\forall n \in \mathbb{N}$. Then, for $Re(s) > 1$,

$$\left| \underbrace{\sum_1^\infty \varepsilon_n^{i_k} n^{-s}}_{\text{has analytic cont. } f_{i_k}} - \underbrace{\sum_1^\infty \varepsilon_n n^{-s}}_{\text{has analytic cont. } f_\omega} \right| = \left| \sum_1^\infty (\varepsilon_n^{i_k} - \varepsilon_n) n^{-s} \right| \leq \sum_1^\infty |\varepsilon_n^{i_k} - \varepsilon_n| n^{-\sigma}$$

is arbitrarily small for k big enough.

Thus $g \equiv f_\omega$ on $D_{a,r} \cap \{Re(s) > 1\}$, so g is an analytic continutaion of f_ω on $D_{a,r}$, and $|g| \leq N$. Hence ω fulfills all the criterion of being in $E_{a,r,N}$, which is then closed.

($E_{a,r,N}$ is of empty interior): Note that

$$\begin{aligned} & [(E_{a,r,N})^{\circ} = \varnothing] \\ \iff & [\nexists V_M \subset E_{a,r,N}, V_M \text{ open}] \\ \iff & [\forall \omega \in E_{a,r,N}, \forall V_M \ni \omega, \text{ s.t. } V_M \text{ open} . \Rightarrow V_M \not\subseteq E_{a,r,N}] \end{aligned}$$

We will try to prove the last statement by contradiction.

Let $\omega = (\varepsilon_1, \varepsilon_2, \ldots) \in E_{a,r,N}$. A neighbourhood of ω (open set containing ω) is

$$V_M = \{\omega' = (\varepsilon'_1, \varepsilon'_2, \ldots) | \varepsilon'_n = \varepsilon_n, n \leq M\}.$$

These neighbourhoods form a basis for the product topology on Ω, so any open set of Ω (and hence any open set included in $E_{a,r,N}$) can be written as a union of V_M:s. By assuming that

$$V_M \subset E_{a,r,N}, \tag{4.2}$$

we seek a contradiction. This is done in several steps. We will initially find an analytic function on the disc $D_{a,r}$. Its Taylor expansion will yield a converging series that we know must diverge, and we will then have reached our contradiction.

Step 1: (taylor series argument): Let $\omega'' \in \Omega$ be an arbitrary point. It defines a series with coefficients ε''_n. We want to find out if it has an analytic continuation on $D_{a,r}$.

$$\begin{aligned}\sum_{n=1}^{\infty} \varepsilon''_n n^{-s} &= \sum_{n=1}^{M} \varepsilon''_n n^{-s} + \sum_{n=M+1}^{\infty} \varepsilon''_n n^{-s} + \sum_{n=1}^{M} (\varepsilon_n - \varepsilon_n) n^{-s} \\ &= \underbrace{\sum_{n=1}^{M} \varepsilon_n n^{-s} + \sum_{n=M+1}^{\infty} \varepsilon''_n n^{-s}}_{f_{\omega'}(s),\, \omega' \in V_M} + \underbrace{\sum_{n=1}^{M} (\varepsilon''_n - \varepsilon_n) n^{-s}}_{\text{analytic on } \mathbb{C}} .\end{aligned} \tag{4.3}$$

The last sum is a polynomial and analytic on all of $\mathbb{C}$. The first two define a series corresponding to a point $\omega' \in V_M$. We have assumed that $V_M \subset E_{a,r,N}$, so the point ω' is such that its series has a bounded analytic continuation to $D_{a,r}$, and we denote it by $f_{\omega'}(s)$. Thus the series in (4.3) does have an analytic continuation on the disc. We denote this continuation by $f_{\omega''}$.

In light of the above, we can Taylor expand $f_{\omega''}$ on the disc $D_{a,r}$. To stay in the region where $f_{\omega''}$ can be expressed by its Dirichlet series, we expand it in a smaller disc of center $a + \varepsilon$, with radius e.g. 3ε (we can do this if ε is sufficiently small), such that the disc $D_{a+\varepsilon,3\varepsilon} \subset D_{a,r}$, see Figure 4.1.

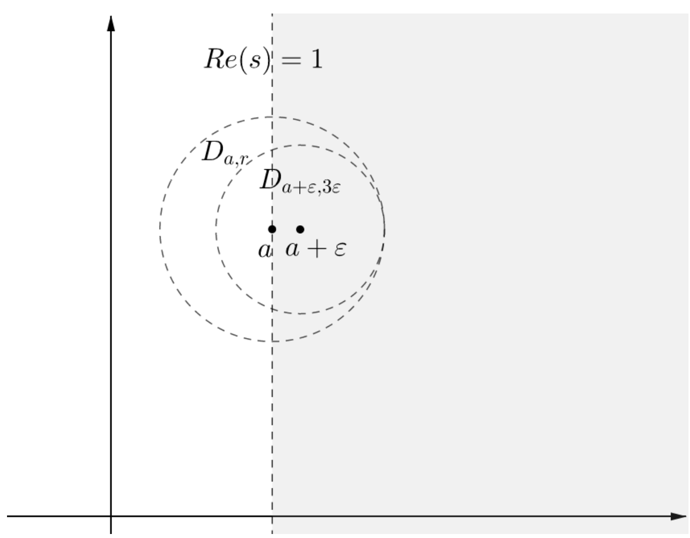

Figure 4.1: $f_{\omega''}$ has an analytic continuation in $D_{a,r}$ that we can express in a Taylor expansion. If $\varepsilon > 0$ is chosen small enough for the disc $D_{a+\varepsilon,3\varepsilon}$ to be included in $D_{a,r}$, the expansion can be expressed using derivatives of the Dirichlet series.

The Taylor expansion of $f_{\omega''}$ centered at $a+\varepsilon$ is

$$f_{\omega''}(s) = \sum_{n=0}^{\infty} \frac{f_{\omega''}^{(n)}(a+\varepsilon)(s-(a+\varepsilon))^n}{n!}. \tag{4.4}$$

Since

$$f_{\omega''}^{(n)}(a+\varepsilon) = \sum_{m=1}^{\infty} (-1)^n (\log m)^n \varepsilon''_m m^{-(a+\varepsilon)},$$

(4.4) becomes

$$\begin{aligned} f_{\omega''}(s) &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (-1)^n (\log m)^n \varepsilon''_m m^{-(a+\varepsilon)} \frac{(s-(a+\varepsilon))^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(s-(a+\varepsilon))^n}{n!} \sum_{m=1}^{\infty} (\log m)^n \varepsilon''_m m^{-(a+\varepsilon)}, \text{ for } s \text{ close to } a. \end{aligned}$$

We now use this expansion to evaluate $f_{\omega''}$ at $s = a - \varepsilon = 1 - \varepsilon + it \in D_{a+\varepsilon,3\varepsilon}$. We evaluate the Taylor expansion:

$$f_{\omega''}(a-\varepsilon) = \sum_{n=0}^{\infty} (-1)^n \frac{(-2\varepsilon)^n}{n!} \sum_{m=1}^{\infty} (\log m)^n \varepsilon''_m m^{-(a+\varepsilon)}.$$

Furthermore we make a choice for ε''_m (which is possible, since $\omega'' = (\varepsilon'_1, \varepsilon'_2, \ldots)$ was arbitrary), namely $\varepsilon''_m := \text{sgn}(\cos(t \log m))$. This choice is in order to get positive terms in the Taylor series, when considering the real part of $f_{\omega''}(a-\varepsilon)$:

$$\begin{aligned} Re f_{\omega''}(a-\varepsilon) &= \sum_{n=0}^{\infty} \frac{2\varepsilon^n}{n!} \sum_{m=1}^{\infty} (\log m)^n m^{-(1+\varepsilon)} \varepsilon''_m Re(m^{-it}) = \\ &= \sum_{n=0}^{\infty} \frac{2\varepsilon^n}{n!} \sum_{m=1}^{\infty} (\log m)^n m^{-(1+\varepsilon)} \varepsilon''_m \cos(t \log m) \\ &= \sum_{n=0}^{\infty} \frac{2\varepsilon^n}{n!} \sum_{m=1}^{\infty} (\log m)^n m^{-(1+\varepsilon)} |\cos(t \log m)|. \end{aligned}$$

Since all terms are now positive, we can permute the sum signs, and we recognize another well-known expansion

$$\begin{aligned} Ref_{\omega''}(a-\varepsilon) &= \sum_{m=1}^{\infty} |\cos(t\log m)| m^{-(1+\varepsilon)} \underbrace{\sum_{n=0}^{\infty} \frac{(2\varepsilon \log m)^n}{n!}}_{=e^{2\varepsilon \log m}=m^{2\varepsilon}} \\ &= \sum_{m=1}^{\infty} |\cos(t\log m)| m^{-(1-\varepsilon)} < \infty. \end{aligned} \tag{4.5}$$

This series is finite since $f_{\omega''}$ is bounded on $D_{a,r}$.

The final step of the proof is to use this to conclude by this that also

$$\sum m^{-(1-\varepsilon)} < \infty, \tag{4.6}$$

which would obviously be a contradiction.

Step 2: ((4.5) $\Rightarrow$ (4.6)) If the implication does not hold, and the series in (4.6) diverges but (4.5) converges, this factor would be small enough, often enough, to stop the divergence. However, it turns out that most of the time the cosine-factor only scales the terms with a constant. And we know that scaling the terms of a divergent series with a constant is not enough to stop its divergence. We will now see exactly how it works.

The contrapositive implication claims that

$$\sum m^{-(1-\varepsilon)} \quad \text{diverges} \Rightarrow \sum_{m=1}^{\infty} |\cos(t\log m)| m^{-(1-\varepsilon)} \quad \text{diverges}. \tag{4.7}$$

We use the idea described above, that for some $t \log m$ the cosine term is very small (denote this set by $M^{\complement}$), and the rest of the time it is greater than a constant, say $\frac{1}{2}$ (denote it by M)

$$\sum_{m=1}^{\infty} |\cos(t\log m)| m^{-(1-\varepsilon)} \geq \sum_{m\in M} \frac{1}{2} m^{-(1-\varepsilon)} + \sum_{m \in M^{\complement}} 0 \cdot m^{-(1-\varepsilon)} = \frac{1}{2} \sum_{m\in M} m^{-(1-\varepsilon)}.$$

The set M is a union of "periodic" intervals where $|\cos(t\log m)| \geq \frac{1}{2}$,

$$M = \bigcup_{n\in\mathbb{N}} (a_n, b_n).$$

Let us find out what the intervals (a_n, b_n) exactly look like.

$$|\cos(t\log m)| \geq \frac{1}{2} \iff \begin{cases} \cos(t\log m) \geq \frac{1}{2}, \text{ or} \\ \cos(t\log m) \leq -\frac{1}{2} \end{cases}$$

$$\iff \begin{cases} t\log m \in [-\frac{\pi}{3}, \frac{\pi}{3}] + 2\pi n, \quad \text{or} \\ t\log m \in [-\frac{2\pi}{3}, \frac{4\pi}{3}] + 2\pi n, \quad n \in \mathbb{N} \end{cases}$$

$$\iff t\log m \in [-\frac{\pi}{3}, \frac{\pi}{3}] + \pi n, \quad n \in \mathbb{N}.$$

We are interested in for which m this occurs, so we consider t as fixed. Note that we can assume that $t > 0$, since $\cos(-t\log m) = \cos(t\log m)$, and we exclude the case $\cos(0) = 1$. Thus we are interested in intervals

$$m \in [e^{-\frac{\pi}{3t}+\frac{\pi n}{t}}, e^{\frac{\pi}{3t}+\frac{\pi n}{t}}] = [a_n, b_n], n \in \mathbb{N}.$$

We note further that

$$m \in [a_n, b_n] \iff m \in [\lceil a_n \rceil, \lfloor b_n \rfloor] = [a_n + r_a, b_n - r_b],$$

since a_n, b_n need not to be integers. Initially, we hope that we can ignore r_a, r_b, to take a closer look later.

Thus we have

$$\sum_{m \in M} m^{-(1-\varepsilon)} = \sum_{n\in\mathbb{N}} \left(\sum_{m \in \{\lceil a_n \rceil, \ldots, \lfloor b_n \rfloor\}} m^{-(1-\varepsilon)} \right) \geq \sum_{n\in\mathbb{N}} \int_{\lceil a_n \rceil}^{\lfloor b_n \rfloor} \frac{dx}{x^{1-\varepsilon}}.$$

We estimate the integral from below

$$\int_{\lceil a_n \rceil}^{\lfloor b_n \rfloor} \frac{dx}{x^{1-\varepsilon}} \geq \int_{\lceil a_n \rceil}^{\lfloor b_n \rfloor} \frac{dx}{\lfloor b_n \rfloor^{1-\varepsilon}} \geq \frac{1}{b_n^{1-\varepsilon}} \left(\lfloor b_n \rfloor - \lceil a_n \rceil\right) \geq \frac{(b_n - 1) - (a_n + 1)}{b_n^{1-\varepsilon}}$$

$$= \frac{e^{-\frac{\pi}{3t}+\frac{\pi n}{t}} - e^{\frac{\pi}{3t}+\frac{\pi n}{t}} - 2}{\left(e^{\frac{\pi}{3t}+\frac{\pi n}{t}}\right)^{1-\varepsilon}} = \frac{e^{\frac{\pi n \varepsilon}{t}}}{e^{\frac{\pi n}{t}(1-\varepsilon)}} \left(\frac{e^{\frac{\pi}{3t}} - e^{-\frac{\pi}{3t}}}{e^{\frac{\pi}{3t}(1-\varepsilon)}} \right) - \frac{2}{e^{\frac{\pi n}{t}(1-\varepsilon)} e^{\frac{\pi}{3t}(1-\varepsilon)}}$$

$$\geq e^{\frac{\pi n \varepsilon}{t}} \left(e^{\frac{\pi\varepsilon}{3t}} - e^{-\frac{\pi}{3t}(2-\varepsilon)} \right) - \frac{2}{e^{\frac{\pi n}{t}(1-\varepsilon)}},$$

to obtain

$$\sum_{n\in\mathbb{N}} \int_{\lceil a_n \rceil}^{\lfloor b_n \rfloor} \frac{dx}{x^{1-\varepsilon}} \geq \left(e^{\frac{\pi\varepsilon}{3t}} - e^{-\frac{\pi}{3t}(2-\varepsilon)} \right) \sum_{n\in\mathbb{N}} (\underbrace{e^{\frac{\pi\varepsilon}{t}}}_{>1})^n - 2 \sum_{n\in\mathbb{N}} (\underbrace{e^{-\frac{\pi(1-\varepsilon)}{t}}}_{<1})^n.$$

The first series diverges, because $e^{\frac{\pi\varepsilon}{t}} > e^0 = 1$ are terms of a divergent geometric series, whereas the second converges, since $e^{-\frac{\pi(1-\varepsilon)}{t}} < 1$. Recall that ε is a choice of ours (although we have to choose it small).

We have then reached the conclusion of (4.7) that

$$\sum m^{-(1-\varepsilon)} \quad \text{diverges} \Rightarrow \sum_{m=1}^{\infty} |\cos(t \log m)| m^{-(1-\varepsilon)} \quad \text{diverges},$$

and we can apply the contrapositive to conclude that

$$\sum_{m=1}^{\infty} m^{-(1-\varepsilon)} < \infty,$$

which we know is certainly not true, and we have reached the contradiction in our proof.

In conclusion, our assumption of the neighbourhoods in (4.2) were wrong and we must have

$$V_M \nsubseteq E_{a,r,N}.$$

As we noticed before, any open set of Ω is a union of V_M:s. Thus $E_{a,r,N}$ cannot include any open set, and that proves its empty interior. □

Proof of Theorem 4.5

To prove the second theorem we will need more tools, more specifically Theorem 3.1 that we proved in Section 3 and the two following results from probability theory:

Theorem 4.6. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Then*

$$\sum_{1}^{\infty} \varepsilon_n u_n \text{ converges almost surely} \iff \sum_{1}^{\infty} |u_n|^2 < \infty$$

Theorem 4.7. *Let*

$$\sum_{n=1}^{\infty} \varepsilon_n(\omega) n^{-s}$$

be a Dirichlet series of independent and symmetric ($\varepsilon_n \sim -\varepsilon_n$) coefficients. Then the abscissa of pointwise convergence $\sigma_c(\omega)$ is a.s. a constant, e.g. $\exists c$ s.t. $\sigma_c(\omega) = c$ a.s. Moreover, if $\sigma_c > -\infty$, then the line $Re(s) = \sigma_c$ is a.s. a natural boundary for the series.

The first of the two theorems is a special case of Theorem 22.6 in [3, p.289]. Theorem 4.7 can be found in [15, p. 44], and we will look closer into it later. First we prove Theorem 4.5.

Proof of Theorem 4.5. Theorem 4.6 says in our case that

$$\sum_{1}^{\infty} \varepsilon_n n^{-s} \text{ converges almost surely } \iff \sum_{1}^{\infty} |n^{-s}|^2 = \sum_{1}^{\infty} n^{-2Re(s)} < \infty.$$

That is true whenever $Re(s) > \frac{1}{2}$, and from this we conclude that $\sigma_c = \frac{1}{2}$ almost surely.

From Theorem 4.7 we know that, since $\sigma_c > -\infty$, the line $Re(s) = \sigma_c = \frac{1}{2}$ is a.s. a natural boundary for the series.

To determine σ_u we use the formula presented in [2] among others (with credits given to Bohr):

$$\sigma_u = \limsup_{n\to\infty} \frac{1}{\log n} \log\left(\sup_{t\in\mathbb{R}} \left| \sum_{k=1}^{n} \varepsilon_k k^{it} \right| \right).$$

By Theorem 3.1,

$$\sup_{t\in\mathbb{R}} \left| \sum_{k=1}^{n} \varepsilon_k k^{it} \right| \geq \sum_{p\leq n,\, p\, prime} |\varepsilon_p|,$$

and the last sum is just the number of primes p, that are less than n. This is the Prime-counting function $\pi(n)$, and by the Prime Number Theorem (see for example [1, p.74]) it is asymptotic to $n/\log n$, meaning that

$$\frac{\pi(n)}{n/\log n} \to 1,\ n \to \infty$$

Thus

$$\frac{1}{\log n} \log\left(\sup_{t\in\mathbb{R}} \left| \sum_{k=1}^{n} \varepsilon_k k^{it} \right| \right) \geq \frac{\log \pi(n)}{\log n} = \frac{\log\left(\frac{\pi(n)}{n/\log n} \cdot n/\log n\right)}{\log n} = \frac{\log \frac{\pi(n)}{n/\log n} + \log n - \log(\log n)}{\log n}$$
$$\to 0 + 1 + 0 = 1,\ n \to \infty$$

and finally

$$\sigma_u = 1.$$

□

Proof of Theorem 4.7

We recall that we are considering the series of symmetrically distributed and independent coefficients as follows:

$$\sum_{n=1}^{\infty} \varepsilon_n(\omega) n^{-s} = F(s). \tag{4.8}$$

When it converges we denote its limit by F. It might or it might not be continuable beyond the domain of convergence of the series.

Proof. Let $\sigma_c(\omega)$ be the lower bound of $\sigma \in \mathbb{R}$ such that $\sum_1^\infty \varepsilon_n(\omega)n^{-\sigma}$ converges. Then the series (4.8) converges for $\sigma > \sigma_c(\omega)$ and diverges for $\sigma < \sigma_c(\omega)$ (recall that a Dirichlet series converges pointwise in a half-plane, see Proposition 1.5). We will here use a result from probability theory (see [15, p.7]):

Theorem 4.8 (Zero-one law)**.** *Let the random variable ε be defined on the sample space $\Omega = \prod \Omega_n$. If*

$$\varepsilon(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) = \varepsilon(\omega_1', \ldots, \omega_n', \omega_{n+1}, \ldots),$$

for any $\omega_1, \ldots, \omega_n, \omega_1', \ldots, \omega_n', \omega_{n+1}, \ldots$, then ε is a.s. a constant.

If a Dirichlet series $\sum \varepsilon_n(\omega)n^{-s}$ only has a finite number of coefficients different from another $\sum \varepsilon_n'(\omega)n^{-s}$, they will converge for the same s (indeed, their difference is a polynomial). This implies that $\sigma_c(\omega) = \sigma_c(\omega')$, when only a finite number of the ω_n:s differ from the ω_n':s. The zero-one law then implies that $\sigma_c(\omega)$ is a.s. a constant. Denote this constant by σ_c. We suppose that $-\infty < \sigma_c < \infty$.

Now we have two possible cases. Either $Re(s) = \sigma_c$ is a.s. a natural boundary for $F(s)$, or it can be analytically continued beyond this line, for ω in a set of positive measure, say $\omega \in E$. In the latter case there would exist a disc with center c on the line $Re(s) = \sigma_c$, into which F is continuable, for $\omega \in E$. We will, by a contradiction, show that this is not possible. The approach is the same as in the proof of Theorem 4.4, where we used a Taylor expansion argument to find a contradiction.

What we need to do here is to Taylor expand F at a disc centered at $c + \delta$, containing c. We try to do this in the most straight forward way, by using the Taylor series. At $c + \delta$, the Dirichlet series converges absolutely to F, a.s. The Taylor series can thus be evaluated at $c - \delta$, for $\delta > 0$ small enough:

$$\begin{aligned}\sum_{n=0}^{\infty} F^{(n)}(c+\delta)\frac{((c-\delta)-(c+\delta))^n}{n!} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \varepsilon_m(\omega)(-1)^n(\log m)^n m^{-(c+\delta)}\frac{(-2\delta)^n}{n!} \\ &\overset{(*)}{=} \sum_{m=1}^{\infty} \varepsilon_m(\omega)m^{-(c-\delta)}, \text{ for } \omega \in E.\end{aligned} \tag{4.9}$$

This is certainly a contradiction, but it is due incautious calculations. More precisely, we are not allowed to change the summation signs in $(*)$, since the double sum is not absolutely convergent (the terms are not absolutely summable in the product space).

To deal with this, we can instead consider the Taylor polynomial, to securely permute the sum signs. We get that

$$\begin{aligned}\sum_{n=0}^{N} F^{(n)}(c+\delta)\frac{((c-\delta)-(c+\delta))^n}{n!} &= \sum_{n=0}^{N}\sum_{m=1}^{\infty}\varepsilon_m(\omega)(-1)^n(\log m)^n m^{-(c+\delta)}\frac{(-2\delta)^n}{n!} \\ &= \sum_{m=1}^{\infty}\varepsilon_m(\omega)m^{-(c-\delta)}\underbrace{m^{-2\delta}\sum_{n=0}^{N}(\log m)^n\frac{(2\delta)^n}{n!}}_{a_{mN}}\end{aligned} \tag{4.10}$$

converges for all N, for $\omega \in E$. We name the last part a_{mN} to apply a last result of Kahane in [15, Thm. 1, p.13]:

Theorem 4.9. *Let v_m be independent, symmetric r.v. If $\sum_{m=1}^{\infty} v_m a_{mN}$ converges a.s. and if $a_{mN} \to 1$, $N \to \infty$, $\forall m$, then*

$$\sum_{m=1}^{\infty} v_m \quad \textit{converges a.s.}$$

We see that in our case $a_{mN} = 1$, $\forall m$. We note further that for $\omega \in E$ the Taylor expansion in (4.10) converges a.s. (but not in the whole space Ω). That is enough to be able to apply Theorem 4.9 (by assumption the coefficients are symmetric) and conclude that also the Taylor series $\sum_{m=1}^{\infty} \varepsilon_m(\omega)m^{-(c-\delta)}$ converges, a.s. for $\omega \in E$. But that is a contradiction, since this is the Dirichlet series evaluated in a point to the left of σ_c, and it does *not* converge there on any set of Ω of positive measure.

Hence we can reject the possibility of analytic continuation beyond $Re(s) = \sigma_c$, and this line must be a.s. a natural boundary.

□

Chapter 5

Proving the Maximal Width of the Strip $\sigma_a - \sigma_u$

In the preceding chapter we saw how Queffélec proved the existence of a Dirichlet series with $\sigma_a - \sigma_c = \sigma_u - \sigma_c = \frac{1}{2}$, by constructing a probabilistic example. In that example $\sigma_a - \sigma_u = 0$, but we know since Proposition 1.17 that we possibly may be able attain a distance between these two abscissae of $\frac{1}{2}$.

Harold Boas points out in [4] that proving that this inequality is sharp is a real challenge. Bohr who otherwise had done great progress in the area of Dirichlet series did not come up with an example. It was finally Bohnenblust and Hille found one example with $\frac{1}{2}$, [5]. Their work was however advanced and we shall not go into it. Instead we will look at the example that Boas presents in [4], also proving the existence using a probabilistic approach.

5.1 Construction of Boas' Series

The series resembles a bit Queffélec's, with coefficients of ± 1 assigned at random, but some of them will also be chosen as 0. We will introduce the coefficients $\varepsilon_n \in \{-1, 0, 1\}$ to the series $\sum \varepsilon_n n^{-s}$ depending on the prime factorization of n.

Consider initially the first 16 primes:

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53
p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}

From here we group, for each k, the 2^k consecutive primes starting at the 2^k:th. For example, for $k = 0$, we pick $2^0 = 1$ prime starting at the $2^0 = 1$:st prime. That gives: $p_1 = 1$. For $k = 1$, we pick $2^1 = 2$ primes starting at $p_{2^1} = p_2 = 3$. Those will be 3, 5. When $k = 2$, we pick $2^2 = 4$ primes, starting at $p_{2^2} = p_4 = 7$. In this group we will thus have 7, 11, 13, 17. In this way, the first 16 primes will be grouped as

$(k = 0)$	2								
	p_1								
$(k = 1)$	3	5							
	p_2	p_3							
$(k = 2)$	7	11	13	17					
	p_4	p_5	p_6	p_7					
$(k = 3)$	19	23	29	31	37	41	43	47	53
	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}

Using such groups, we then form different integers n. The idea is to, for each k, form products using the primes in the k:th group. However, we only allow products of k factors.

For example, when $k = 2$, we can choose between the primes 7, 11, 13, 17, but we only pick 2 primes (perhaps the same twice). This gives us n of the form $n = 7^{\alpha_1} \cdot 11^{\alpha_2} \cdot 13^{\alpha_3} \cdot 17^{\alpha_4}$, where $\sum \alpha_i = 2$. For example we can form $n = 7^2$, $7 \cdot 11$, $7 \cdot 13$, etc.

We note that for each k we construct integers using different primes, so we will never construct the same integer twice. In general we get terms of the form

$$\begin{aligned} n^{-s} &= (p_{2^k}^{-s})^{\alpha_1} \dots (p_{2^{k+1}-1}^{-s})^{\alpha_{2^{k+1}-1}} \\ &= z_{2^k}^{\alpha_{2^k}} \dots z_{2^{k+1}-1}^{\alpha_{2^{k+1}-1}}, \quad (\alpha_{2^k} + \dots + \alpha_{2^{k+1}-1} = k). \end{aligned}$$

To all these terms we assign $\varepsilon_n = \pm 1$ randomly. For all the integers n that will never be constructed throughout this process we set $\varepsilon_n = 0$.

This process characterizes what the Dirichlet series looks like. We can now investigate its convergence properties. Throughout the proofs, we will consider the series as a summation over blocks indexed by k

$$\sum_{n=1}^{\infty} \varepsilon_n n^{-s} = \sum_{k=0}^{\infty} \left(\sum_{\sum \alpha_i = k} \pm z_{2^k}^{\alpha_{2^k}} \dots z_{2^{k+1}-1}^{\alpha_{2^{k+1}-1}} \right).$$

5.2 Properties of Boas' Series

Proposition 5.1 (Boas). *There exists a series constructed as above with the following properties:*

$$\sigma_a = 1, \sigma_u = \frac{1}{2}.$$

Throughout the proof we will apply several estimation bounds. One comes from inequalities with close relation to the the prime number theorem. It is an application of [1, Thm 4.7, p.84]. We will in this proof not talk about the distribution of $\pi(n)$ but the size of p_n:

Lemma 5.2. *For $n \geq 2$ there exists a positive constant c_1 such that*

$$\frac{1}{c_1} < \frac{p_n}{n \log n} < c_1.$$

Next we discuss a counting argument about how many terms of our series that are non-zero, which is the number of terms constructed as described above. We form monomials of type

$$z_1^{\alpha_1} z_2^{\alpha_2} \cdot \ldots \cdot z_n^{\alpha_n}, \quad \alpha_1 + \alpha_2 + \ldots + \alpha_n = m.$$

In our case $n = 2^k, m = k$, but we denote them by n, m to keep it readable.

The question we ask ourselves is: how many monomials can we form, for each m, n? That is in fact the same problem as choosing m elements from n possible, allowing repetitions, without respecting the order. The exact number of ways to do this is $\binom{n+m-1}{m}$. However we will settle for the estimation

$$\binom{n+m-1}{m} \geq \frac{n^m}{m^m}. \tag{5.1}$$

Finally, we will also need to use some theory of random trigonometric polynomials. We use the following result of Kahane in [15, Thm 3, p.70], without proof:

Theorem 5.3 (Kahane). *Consider a random trigonometric polynomial in n variables*

$$P(t_1, t_2, \dots, t_n) = \sum \varepsilon_i f_i(t_1, t_2, \dots, t_n),$$

where the f_i are complex trigonometric polynomials of degree less than or equal to m, $\varepsilon_i = \pm 1$ and $\sum$ a finite sum. Then we have

$$\mathbb{P}\left(||P||_\infty \geq C\big(n \sum ||f_i||_\infty^2 \log m\big)^{1/2} \right) \leq m^{-2} e^{-n},$$

for some absolute constant C.

We deal with a polynomial $P(z_1,\dots,z_n) = \pm z_1^{\alpha_1}\cdot\ldots\cdot z_n^{\alpha_n}$, of degree m, with the variables lying in the unit disc and ± 1 assigned at random. By this result (considering the complementary event) there is a constant c_2 so that with high probability ($\geq 1-\frac{1}{m^2e^2}$) the maximum modulus of the polynomial is less than

$$\sup_{z_i\in\mathbb{D}} |\pm z_1^{\alpha_1}\cdot\ldots\cdot z_n^{\alpha_n}| \leq c_2 n^{(m+1)/2}\sqrt{\log m}. \tag{5.2}$$

In particular this result provides the existence of at least one sequence of ± 1 such that the above inequality holds, for each n, m. That gives us a choice of a Dirichlet series to which we can apply the inequality on each of the k-blocks.

Proof. ($\sigma_{\mathbf{a}} = \mathbf{1}$). We start by showing the divergence for $\sigma < 1$, by estimating from below. For each k we get monomials of the form $\pm z_{2^k}^{\alpha_{2^k}}\cdot\ldots\cdot z_{2^{k+1}-1}^{\alpha_{2^{k+1}-1}}$, $\alpha_{2^k}+\ldots+\alpha_{2^{k+1}-1} = k$, just as described above. The estimation in (5.1) gives that for each k, we will never have less than $\frac{2^{k^2}}{k^k}$ monomials. By estimating the modulus of these monomials we can evaluate the original series by a series of k. By Lemma 5.2, the 2^{k+1}:th prime is bounded by

$$p_{2^{k+1}} < c_1 2^{k+1}\log 2^{k+1} < c_1 2^{k+1}(k+k) = 4c_1 2^k k.$$

The integers n formed for each k are then bounded above by

$$|n| = |p_{2^k}^{\alpha_{2^k}}\cdot\ldots\cdot p_{2^{k+1}-1}^{\alpha_{2^{k+1}-1}}| < |p_{2^{k+1}}^k| < (4c_1 2^k k)^k = (4c_1 k)^k 2^{k^2}.$$

We put this together and estimate from below

$$\sum_{n=1}^{\infty}\varepsilon_n n^{-s} > \sum_{k=2}^{\infty}\underbrace{\left|\frac{\pm 1}{((4c_1k)^k 2^{k^2})^s}\right|}_{\text{size of each term}}\cdot\underbrace{\frac{2^{k^2}}{k^k}}_{\text{number of terms}} = \sum_{k=2}^{\infty}\frac{2^{k^2(1-\sigma)}}{(4c_1)^{k\sigma}k^{k(1+\sigma)}}.$$

By observing that $(4c_1)^k \geq 1, \forall k$ (c_1 must be larger than 1), we can make one more estimation from below, obtaining

$$\sum_{n=1}^{\infty}\varepsilon_n n^{-s} > \sum_{k=2}^{\infty}\frac{2^{k^2(1-\sigma)}}{(4c_1k)^{k(1+\sigma)}} = \sum_{k=2}^{\infty}\frac{e^{k^2(1-\sigma)\log 2}}{e^{k(1+\sigma)\log 4c_1 k}}.$$

When $\sigma < 1$, we conclude by e.g. the root-test that this is a divergent series, since

$$\left(\frac{e^{k^2(1-\sigma)\log 2}}{e^{k(1+\sigma)\log(4c_1k)}}\right)^{1/k} = \frac{e^{k(1-\sigma)\log 2}}{e^{(1+\sigma)\log(4c_1k)}} = e^{k\left((1-\sigma)\log 2-(1+\sigma)\log(4c_1k)/k\right)} \to \infty, \text{ as } k\to\infty.$$

However we know that for $\sigma > 1$: $\sum|\varepsilon_n n^{-s}| \leq \sum n^{-\sigma} = \zeta(\sigma)$ converges absolutely. Hence $\sigma_a = 1$.

$(\sigma_{\mathbf{u}} = \mathbf{\frac{1}{2}})$. We want to estimate the sup of the modulus of the k:th block, that is

$$\sup_{z_i = p_i^{-s}} \left| \sum_{\sum \alpha_i = k} \pm z_{2^k}^{\alpha_{2^k}} \cdot \ldots \cdot z_{2^{k+1}-1}^{\alpha_{2^{k+1}-1}} \right|. \tag{5.3}$$

Each of the variables $z_i = p_i^{-s}$ takes its value in the disc $D(0, p_i^{-\sigma})$, and we estimate the sup from above by letting all the variables instead take values in the largest of these discs, the one with radius corresponding to the smallest prime, p_{2^k}. Then the k:th block (5.3) is bounded by (using the estimation in (5.2))

$$\begin{aligned} \sup_{z_i \in D(0, p_i^{-\sigma})} \left| \sum_{\sum \alpha_i = k} \pm z_{2^k}^{\alpha_{2^k}} \cdot \ldots \cdot z_{2^{k+1}-1}^{\alpha_{2^{k+1}-1}} \right| &\leq \frac{1}{p_{2^k}^{k\sigma}} \sup_{z_i \in \mathbb{D}} \left| \sum_{\sum \alpha_i = k} \pm z_{2^k}^{\alpha_{2^k}} \cdot \ldots \cdot z_{2^{k+1}-1}^{\alpha_{2^{k+1}-1}} \right| \\ &\leq \frac{1}{p_{2^k}^{k\sigma}} \cdot c_2 (2^k)^{\frac{k+1}{2}} \sqrt{\log k}. \end{aligned}$$

We can estimate this again from above by estimating the size of the smallest prime $p_{2^k} > \frac{2^k k \log 2}{c_1} > \frac{2^k k}{4c_1}$, to obtain that the supremum is less than

$$\frac{c_2 2^{\frac{k(k+1)}{2}} \sqrt{\log k}}{\left(\frac{2^k k}{4c_1}\right)^{k\sigma}} =: b_k.$$

To study the convergence of $\sum_k b_k$ we apply the root-test:

$$\sqrt[k]{|b_k|} = \left| \frac{c_2 2^{\frac{k(k+1)}{2}} \sqrt{\log k}}{\left(\frac{2^k k}{4c_1}\right)^{k\sigma}} \right|^{\frac{1}{k}} \sim \frac{\sqrt{2}^k (\log k)^{\frac{1}{2}}}{(2^\sigma)^k k^\sigma} = \left(\frac{\sqrt{2}}{2^\sigma}\right)^k \cdot \frac{(\log k)^{\frac{1}{2k}}}{k^\sigma} =: L_k.$$

We conclude

$$L_k \xrightarrow[k \to \infty]{} \begin{cases} 0, \text{ if } \sigma \geq \frac{1}{2} \\ \infty, \text{ if } \sigma < \frac{1}{2} \end{cases}$$

By the root-test, $\sum |b_k|$ converges for $\sigma \geq \frac{1}{2}$. Furthermore, b_k was the supremum of the k:tk block, so being terms of a convergent series, we can almost apply Weierstrass M-test to conclude the uniform convergence of our Dirichlet series.

Why do we say "almost"? We need to clarify one last point of this argument. The convergence we established bases on the order of summation we chose (when dividing the series into blocks). If we are to have convergence we must accept the natural summation order. To come around this problem we can in fact use Bohr's Theorem (Theorem 2.5). The fact that the series converges with one specific summation order is enough to establish the bounded analytic continuation for $\sigma > \frac{1}{2}$. Since $\sigma_b = \sigma_u$, we must also have $\sigma_u = \frac{1}{2}$.

□

Chapter 6

Concluding remarks

In this work we have introduced ordinary Dirichlet series and we have closely studied its convergence properties, most of it thanks to the work of Harald Bohr. We saw that one can consider several types of convergences and that we can divide the complex plane into half-planes where the series will be convergent and divergent respectively. We established relations between these planes. We got a glimpse of how to relate a Dirichlet series to a Taylor series of infinitely many variables, and we established one inequality using this approach. We finally considered a probabilistic approach to generate examples that can be considered as the "extreme cases".

Throughout the text we saw a lot of Riemann's zeta-function and it is not a coincidence. The function itself offers a very important relation to number theory, and the theory of Dirichlet series was to some extent developed to better understand $\zeta(s)$. One thing that we however have not mentioned, is the Euler product, which is only a small step away. Especially the equality between the Euler product and Riemann's zeta function

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p,\, prime}^{\infty} \frac{1}{1-p^{-s}} \tag{6.1}$$

is particularly important, relating the prime numbers to the Riemann zeta-function. Euler studied initially $\zeta(s)$ as a generating function of number theory and proved that $\sum p^{-1}$ diverges in order to moreover provide another proof that there exist infinitely many primes (see [16]). The equality provides also information about the distribution of the prime numbers. Moreover, the inequality in Theorem 3.1 provides another connection to the distribution of primes.

A relation between $\log \zeta(s)$ and an integral transform of the prime counting function $\pi(x)$ can be derived using Stiltjes integration, see for example [16]. Kahane presented a Fourier analysis approach to the same problem in [14]. The point of the Riemann hypothesis is that having all non-trivial zeros on the line $Re(s) = \frac{1}{2}$ implies the smallest (in some sense) possible error term in the asymptotic estimate for $\pi(x)$.

Bibliography

[1] T. Apostol, *Introduction to Analytic Number Theory*, Undergraduates Texts in Mathematics, Springer-Verlag, New-York, 1976.

[2] V. Bernstein, *Leçons sur les progrès récents de la théorie des series de Dirichlet*, Collection de monographies sur la théorie des fonctions, Gauthier-Villars, Paris, 1933.

[3] P. Billingsley, *Probability and Measure*, Wiley Series in Probability and Mathmatical Statistics, 3rd edition, John Wiley and Sons, 1995.

[4] H. P. Boas, *The Football Player and the Infinite Series*, Notice of the AMS, vol. 42, no. 11, 1997, pp. 1430-1435, arXiv:math/9705204.

[5] H. F. Bohnenblust, E. Hille, *On the Absolute Convergence of Dirichlet Series*, Annals of Mathematics, vol. 32, no. 3, pp. 600–622, 1931.

[6] H. Bohr, B. Jessen, *One More Proof of Kronecker's Theorem*, Journal of the London Mathematical Society, vol. s1-7, no. 4, pp. 274–275, 1932.

[7] H. Bohr, *Uber die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen reihen* $\sum a_n/n^s$, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, vol. 1913, pp. 441-488.

[8] H. Bohr, *Über die gleichmäßige Konvergenz Dirichletscher Reihen*, Journal für die reine und angewandte Mathematik, vol. 143, pp. 203-211, Copenhagen, 1913.

[9] F. Carlson, *Contributions à la théorie des Séries de Dirichlet*, Note IV. Arkiv för Matematik, Band 2, no. 13, p.293–298, Uppsala, 1952.

[10] M. Chevalier, *Analyse de Fourier et fonctionelle*, Unpublished lecture notes, 2001.

[11] H. M. Edwards, *Riemann's Zeta Function*, Academic Press Inc., New York, 1974.

[12] G. H. Hardy, *Sur les zéros de la fonction* $\zeta(s)$ *de Riemann*, Comptes Rendus de l'Académie des Sciences, vol. 158, pp. 1012-1014, 1914.

[13] H. Hedenmalm, P. Lindqvist, K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in* $L^2(0,1)$, Duke Mathematical Journal, vol. 86, pp. 1-37, 1997.

[14] J.-P. Kahane, *A Fourier Formula for Prime Number*, Canadian Mathematical Society, Conference Proceedings, vol. 21, pp. 89-102, 1997.

[15] J.-P. Kahane, *Some Random Series of Functions*, 2nd edition, Cambridge University Press, Cambridge, 1985.

[16] A. A. Karatsuba, S. M. Voronin, *The Riemann Zeta-function*, Walter de Gruyter & Co., Berlin, 1992.

[17] B. M. Levitan, V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, 1982.

[18] E. M. Stein, Rami Shakarchi, *Complex Analysis*, Princeton Lectures in Analysis, No. II, Princeton University Press, Princeton, N. J., 2003.

[19] H. Queffélec, *Propriétés presque sûres et quasi-sûres des séries de Dirichlet et des produits d'Euler*, Can. J. Math., vol. 32, No. 3, 1980, pp. 531-558.

[20] Riemann Hypothesis. (no date). Retrieved from: http://www.claymath.org/millennium-problems/riemann-hypothesis