Systems of linear nonautonomous differential equations

Instability and eigenvalues with negative real part

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Abstract

For an autonomous system of linear differential equations we are able to determine stability and instability with classical criteria, by looking at the eigenvalues. If the system is stable, all the eigenvalues have negative real part and if the system is unstable, there exist at least one eigenvalue with positive real part.

However, if it were to be nonautonomous, the criterion fails. There exist examples where the systems are stable, yet the eigenvalues have real part with different or positive signs. Also for the unstable systems there exist examples where the matrices can have eigenvalues with strictly negative real part.

In this thesis we examine the instability of linear nonautonomous systems of differential equations, following the article of Josić and Rosenbaum [1]. They discuss a unified method for constructing two dimensional examples which we'll review and attempt to generalize to higher dimensions.

Populärvetenskaplig sammanfattning

Differentialekvationer är en matematisk modell som används för att beskriva olika förändringsprocesser och en central fråga inom detta ämne är om system av differentialekvationer är stabila eller instabila. Ett vanligt förekommande exempel för att illustrera detta är att titta på en pendel som hänger rakt ner. Skulle man putta till pendeln så kommer den till sist hitta tillbaka till sitt utgångsläge, rakt ner som den hängde innan knuffen, ett så kallat stabilt system. Om man istället vänder på pendelen, håller den upp och ned och knuffar till den, då kommer pendelen inte kunna hitta tillbaka till sitt utgångsläge. Den faller istället runt och detta kallar vi då ett instabilt system. Med detta exemplet kan man förstå att ett stabilt system inte störs av små rubbningar, men det gör däremot ett instabilt system.

När man arbetar med autonoma linjära system av differentialekvationer, där systemet ej beror på en tidsvariabel, kan man undersöka stabiliteten genom att titta på egenvärdena. Om egenvärdena till den beskrivande matrisen i systemet har strikt negativa realdelar så är systemet stabilt. Om däremot endast ett eller fler egenvärden skulle ha en strikt positiv realdel så är systemet instabilt.

När ett system beror på en tidsvariabel så kallas detta för ett ickeautonomt system. Det kan till exempel handla om att pendeln utsätts för en tidsberoende yttre kraft, vilket gör att även system som ska vara stabila, enligt stabilitetsteorin, kan bli känsliga för små rubbningar. Därav säger vi att stabilitetsegenskapen har blivit påverkad. Det har då visat sig att kriterierna för stabilitet i de autonoma fallen inte kan appliceras på de ickeautonoma fallen eftersom det finns exempel med instabila system där matrisen i fråga har strikt negativa egenvärden och vice versa.

I detta arbetet undersöks metoder för att konstruera exempel på hur vi kan bestämma instabilitet hos ickeautonoma linjära system av differentialekvationer trots egenvärden med negativ realdel. Vi undersöker först fallet då systemet av differentialekvationer är tvådimensionellt bestående av en tidsberoende matris. Sedan kommer vi övergå till det mer allmänna fallet när systemet är tredimensionellt eller högre.

Acknowledgements

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Contents

1 Introduction

Recall that the autonomous system $\vec{x}'(t) = A\vec{x}(t)$ is said to be stable if the solutions are bounded for $t \geq 0$ and it is asymptotically stable if all solutions converge to 0 as $t \to \infty$. The system is automatically stable if it's asymptotically stable and is called unstable if it's not stable. In order to check if the system is stable or not, we have a look at the eigenvalues corresponding to the matrix A. Then the system is asymptotically stable if and only if all eigenvalues have strictly negative real part. It is unstable if there is an eigenvalue with strictly positive real part.

Theorem 1.1 ([5]). *If λ is an eigenvalue of A and $\vec{y}$ is a corresponding eigenvector, then $\vec{x} = e^{\lambda t}\vec{y}$ is a solution of the system $\vec{x}'(t) = A\vec{x}(t)$.*

If $\mathrm{Re}(\lambda) > 0$, then the solution diverges when $t \to \infty$, if $\mathrm{Re}(\lambda) \leq 0$ the solution remains bounded as $t \to \infty$. We always look at $t > 0$ since we are only interested in future time. Thus it's enough that one eigenvalue has a strictly positive real part for the system to turn unstable.

Theorem 1.2 ([8]). *The linear system $\vec{x}'(t) = A\vec{x}(t)$ is asymptotically stable if and only if all eigenvalues α_j of A satisfy $Re(\alpha_j) < 0$.*

The linear system $\vec{x}'(t) = A\vec{x}(t)$ is unstable if one or more eigenvalues α_j of A satisfy $Re(\alpha_j) > 0$.

In figure 1, on the next page, we are able to see four initial values, the black lines,

$$(x(0) = 0,\ y(0) = 2)$$
$$(x(0) = 0,\ y(0) = 4)$$
$$(x(0) = 0,\ y(0) = -4)$$
$$(x(0) = 0,\ y(0) = -2)$$

to a system of the form $\vec{x}'(t) = A\vec{x}(t)$ with the corresponding eigenvalues $\lambda_1 = -2 + 2i$ and $\lambda_2 = -2 - 2i$. Even though the matrix has complex eigenvalues, the imaginary part does not decide nor affect the stability of the system. Since the eigenvalues have negative real part, the solutions will converge to the chosen equilibrium $(0, 0)$.

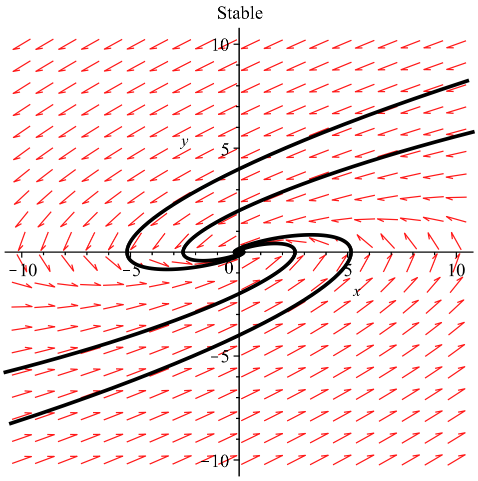

Figure 1: *An example of a stable system of autonomous linear differential equations. The matrix has two eigenvalues with negative real part.*

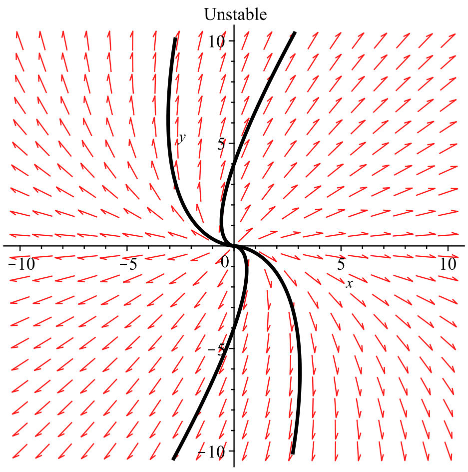

Figure 2: *An example of an unstable system of autonomous linear differential equations. The matrix has two eigenvalues with positive real part.*

In figure 2 we have the following four initial values

$$(x(0) = 0,\ y(0) = 4)$$
$$(x(0) = 0,\ y(0) = -4)$$
$$(x(0) = 3,\ y(0) = -8)$$
$$(x(0) = -3,\ y(0) = 8).$$

The corresponding two eigenvalues to the matrix in the system are $\lambda_1 = 2$ and $\lambda_2 = 3$. With real and positive eigenvalues the solutions diverge from the equilibrium $(0, 0)$ to infinity, hence the system is unstable.

For the nonautonomous case

$$\vec{\boldsymbol{x}}'(t) = A(t)\vec{\boldsymbol{x}}(t) \tag{1.1}$$

there exist examples where all eigenvalues have strictly negative real part but still the system is unstable. When does this happen and what conclusions can we draw from there?

In [1] Krešimir Josić and Robert Rosenbaum characterize a class of 2×2 time-dependent matrices for which the eigenvalues have a strictly negative real part, yet the system is unstable. In this thesis we recall their analysis and look at the more general case of $n \times n$ matrices with $n \geq 3$ and a number of examples with some questions raised in [1] will be investigated.

1.1 History and Background

Henri Poincaré was one of France's greatest mathematicians, born in 1854. In 1908 he wrote about Chance [3] where he formulates a problem with unstable equilibrium. A cone is balanced up side down and its evident that it will fall, but at which side? "If the cone were perfectly symmetrical, if its axis were perfectly vertical, if it were subject to no other force but gravity, it would not fall at all.". Poincaré was also the first mathematician discovering the chaotic deterministic system which later on became the foundation of modern Chaos Theory. He researched about the Three Body Problem, a study about three, alone, bodies in the universe who affect each other through Newtons laws. With their location, speed and directions of movement known, the problem is about deciding these three properties in a given future time. A full analytic solution to this problem has not yet been found. Whereas the Two Body Problem generally have stable solutions, the Three Body Problem usually have chaotic solutions.

In 1857, Aleksandr Liapunov was born in Russia. In 1882 he published the monograph 'The general problem of the stability of motion' which later on led to his doctorial thesis of this title. In the thesis he writes "The problem that I have posed to myself, in starting the present study, can be formulated as follows: to indicate cases where the first approximation really solves the stability question, and to give procedures which would allow to solve it, at least in some cases, when the first approximation is no more sufficient". Liapunov can be linked to Chaos Theory through the characteristic timescale Liapunov Time where a dynamical system is chaotic. One of the simplest example is the two rod pendulum struggling to find it's equilibrium, with different initial values resulting in different outcomes.

In [1] they mention examples made by different mathematicians who studied nonautonomous linear differential equations such as R. E. Vinograd, L. Markusand, H. Yamabe, D. Hinrichsen and M. Y. Wu. These mathematicians had different examples with varying focus areas such as complex eigenvalues with negative real part, a single negative eigenvalue of geometric multiplicity 1 and where the eigenvalues have opposite signs. The first two examples where supposed to be stable and the third where supposed to be unstable, yet it was the other way around.

1.2 A Family of Matrices

In contrast to autonomous systems, there's no general solution formula for nonautonomous systems. This makes it difficult to determine if the system is stable or not. However we can find a necessary condition for the system to be unstable by considering the norm,

$$||\vec{\boldsymbol{x}}(t)|| = \sqrt{\sum_{j=1}^{n} (x_j(t))^2},$$

of the solution. If the system is unstable, there must exist a solution $\vec{\boldsymbol{x}}(t)$ such that $||\vec{\boldsymbol{x}}(t)||$ is unbounded. In particular $\frac{d}{dt}||\vec{\boldsymbol{x}}(t)||^2 = 2\vec{\boldsymbol{x}}'(t) \cdot \vec{\boldsymbol{x}}(t)$ should increase over time for some $t = t_0$. Setting $B = A(t_0)$, we get $B\vec{\boldsymbol{x}}(t_0) \cdot \vec{\boldsymbol{x}}(t_0) > 0$. Since we also want the eigenvalues to have a strictly negative real part, it is natural to introduce the class

$$\mathcal{B} = \{B \text{ is a real } n \times n \text{ matrix, with } Re(\lambda_j) < 0,$$

$$j = 1, 2, .., n,\ \vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}} > 0 \text{ for some } \vec{\boldsymbol{x}} \in \mathbb{R}^n\}.$$

Note that a matrix $B \in \mathcal{B}$ can't be symmetric since symmetric matrices with negative eigenvalues are negative definite.

2 Nonautonomous Systems when $n = 2$

K. Josić and R. Rosenbaum construct specific nonautonomous systems that are unstable, where the eigenvalues real part are strictly negative. This can be done by taking one of the matrices from class $\mathcal{B}$ and rotating the corresponding vector field at a constant angular velocity to see how it evolves in time. With a skew-symmetric matrix $G(\omega)$ of the form

$$G(\omega) = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

we can rotate the system without affecting the norm of the solution. Since

$$\begin{aligned} ||e^{tG(\omega)}\vec{\boldsymbol{x}}||^2 &= e^{tG(\omega)}\vec{\boldsymbol{x}} \cdot e^{tG(\omega)}\vec{\boldsymbol{x}} \\ &= (e^{tG(\omega)})^T e^{tG(\omega)}\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}}. \end{aligned}$$

$G(\omega)^T = -G(\omega)$ gives $G(\omega)^T + G(\omega) = -G(\omega) + G(\omega) = 0$, therefore what is left is

$$I\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}} = ||\vec{\boldsymbol{x}}||^2,$$

with I being the identity-matrix. We can therefore say that $e^{tG(\omega)}$ does not affect the length. Then

$$R(t,\omega) = e^{tG(\omega)} = \begin{bmatrix} \cos t\omega & -\sin t\omega \\ \sin t\omega & \cos t\omega \end{bmatrix}$$

rotates the plane by an angle $t\omega$. Take $B \in \mathcal{B}$ and let $A(t) = R(t,\omega)B[R(t,\omega)]^{-1}$ so that $A(t)$ is obtained from B by a rotation through the angle $t\omega$. Rotating the coordinates will help us solve the system $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$. Define $\vec{\boldsymbol{y}} = [R(t,\omega)]^{-1}\vec{\boldsymbol{x}}$ and note that

$$\vec{\boldsymbol{y}} = e^{-tG(\omega)}\vec{\boldsymbol{x}} \Leftrightarrow e^{tG(\omega)}\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}.$$

We have

$$\begin{aligned} \vec{\boldsymbol{y}}' &= -G(\omega)e^{-tG(\omega)}\vec{\boldsymbol{x}} + e^{-tG(\omega)}\vec{\boldsymbol{x}}' \\ &= -G(\omega)\vec{\boldsymbol{y}} + e^{-tG(\omega)}A(t)\vec{\boldsymbol{x}} \\ &= -G(\omega)\vec{\boldsymbol{y}} + e^{-tG(\omega)}A(t)e^{tG(\omega)}\vec{\boldsymbol{y}}. \end{aligned}$$

On the other hand

$$A(t) = R(t,\omega)B[R(t,\omega)]^{-1} \Leftrightarrow e^{-tG(\omega)}A(t)e^{tG(\omega)} = B.$$

This means that we get

$$\vec{\boldsymbol{y}}' = -G(\omega)\vec{\boldsymbol{y}} + B\vec{\boldsymbol{y}} = [B - G(\omega)]\vec{\boldsymbol{y}}. \tag{2.1}$$

Here $\vec{\boldsymbol{y}}$ solves this system and we get

$$\vec{\boldsymbol{y}}(t) = e^{[B-G(\omega)]t}\vec{\boldsymbol{y}}(0).$$

Inverting the coordinates back gives:

$$\vec{\boldsymbol{x}}(t) = R(t,\omega)e^{[B-G(\omega)]t}\vec{\boldsymbol{x}}(0). \tag{2.2}$$

Since $R(t,\omega)$ is a rotation matrix, as shown earlier, it does not affect the norm of $\vec{\boldsymbol{y}}(t)$. The conclusion is that if the system in (2.1) is unstable then the system $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ will be unstable as well. Therefore, given a matrix $B \in \mathcal{B}$, we should find a rotation matrix $G(\omega)$ such that $(B - G(\omega))$ has at least one strictly positive eigenvalue.

Theorem 2.1 ([1])**.** *If there exists a unit vector $\vec{\boldsymbol{x}}$ such that $B\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}} > 0$, where B is a given 2×2 matrix with strictly negative eigenvalues, then we can find an ω such that $(B - G(\omega))\vec{\boldsymbol{x}} = \eta\vec{\boldsymbol{x}}$ for a positive η.*

Proof. Glance at figure 3 on page 7 where we for instance have the set $U = \{\vec{\boldsymbol{y}} \in \mathbb{R}^2, \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} > 1\}$. The unit-vector $\vec{\boldsymbol{x}}$ points in to U from the origin and lies on the line l_1. The line l_3 is tangent to the unit-circle at the top of $\vec{\boldsymbol{x}}$ and orthogonal to l_1. Parallel to l_3 and at the end of $B\vec{\boldsymbol{x}}$ we find the line l_2. $B\vec{\boldsymbol{x}}$ has its origin at the top of $\vec{\boldsymbol{x}}$ and ends somewhere on l_2. To be able to go from $B\vec{\boldsymbol{x}}$ to l_1 we use the direction vector for l_2, $-G(\omega)\vec{\boldsymbol{x}}$. It can be seen through calculations that $-G(\omega)\vec{\boldsymbol{x}}$ is orthogonal to $\vec{\boldsymbol{x}}$, therefore we know it lies on l_2. With the right ω we can get the right length needed for $-G(\omega)\vec{\boldsymbol{x}}$ to go from the end of $B\vec{\boldsymbol{x}}$ to l_1. Then there exist a vector $\eta\vec{\boldsymbol{x}}$, $\eta > 0$, that starts at the end of $\vec{\boldsymbol{x}}$, lies on l_1 and stretches to the line l_2. We then have that $B\vec{\boldsymbol{x}} - G(\omega)\vec{\boldsymbol{x}} = \eta\vec{\boldsymbol{x}}$ through the parallelogram rule. □

Next we'll give an example of this theorem for how it can be applied.

Example 2.2. As an example we can construct a matrix $A(t) = R(t,\omega)A(0)[R(t,\omega)]^{-1}$ where $A(0)$ has $Re(\lambda_1), Re(\lambda_2) < 0$ but $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ is unstable. Let

$$A(0) = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

and

$$\omega = -\frac{3}{2}$$

then

$$A(t) = \begin{bmatrix} -\frac{3}{2} + \frac{1}{2}\cos(3t) + \frac{3}{2}\sin(3t) & -\frac{1}{2}\sin(3t) + \frac{3}{2}\cos(3t) + \frac{3}{2} \\ -\frac{1}{2}\sin(3t) - \frac{3}{2} + \frac{3}{2}\cos(3t) & -\frac{3}{2} - \frac{1}{2}\cos(3t) - \frac{3}{2}\sin(3t) \end{bmatrix}.$$

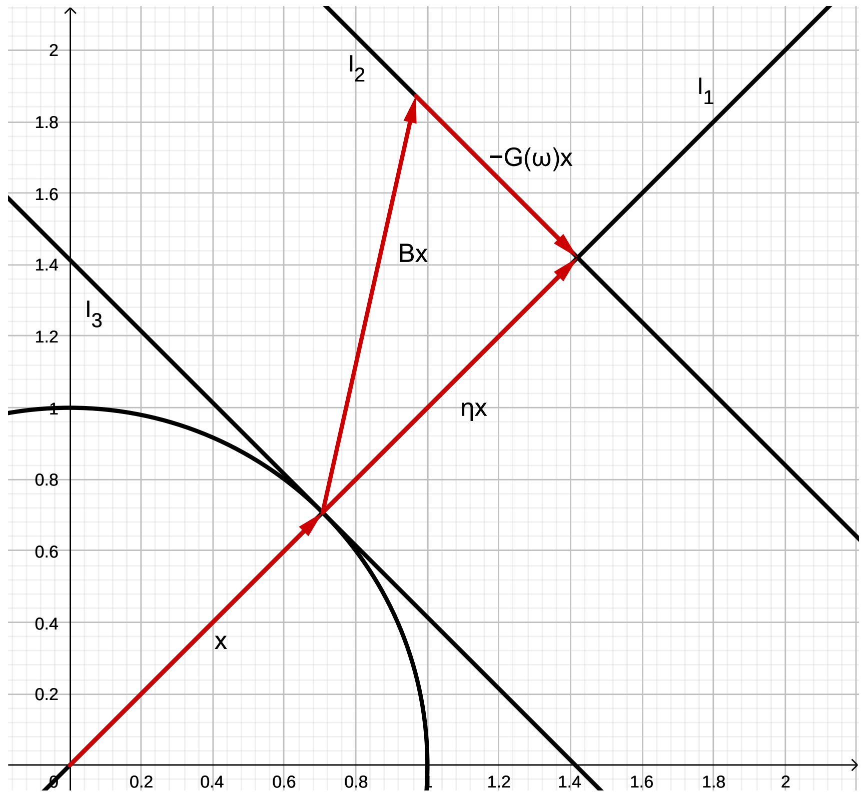

Figure 3: *An illustration of the proof in theorem* 2.1.

$A(t)$ and $A(0)$ have the same eigenvalues, -1 and -2. According to the above steps, $\vec{\boldsymbol{x}} = R(t, -\frac{3}{2})e^{[A(0)-G(-\frac{3}{2})]t}\vec{\boldsymbol{x}}(0)$ where the eigenvalues of $A(0) - G(-\frac{3}{2})$ are $\lambda_1 = \frac{\sqrt{10}-3}{2} > 0$ and $\lambda_2 = -\frac{\sqrt{10}+3}{2} < 0$. With one eigenvalue greater than zero, it's clear to see that the system is unstable. If we instead were to choose, for example, $\omega = 2$, then $A(0)$ would still have the same eigenvalues but the system would have become stable, since the real part of the eigenvalues of $A(0) - G(2)$ is then $Re(\lambda_{1,2}) = -\frac{3}{2}$.
As we can see, depending on how we choose ω, $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ can either be stable or unstable.

□

It's also possible to replace $G(\omega)$ with any matrix A_1, but it can be problematic if A_1 diminishes the vector $e^{(B-A_1)t}\vec{\boldsymbol{x}}(0)$. This can be seen through the following theorem and proof.

Theorem 2.3 ([1])**.** *Let* $A(t) = e^{A_1 t}A(0)e^{-A_1 t}$ *where* A_1 *and* $A(0)$ *are constant matrices. Then the solution to the system* $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ *will be* $\vec{\boldsymbol{x}} = e^{A_1 t}e^{(A(0)-A_1)t}\vec{\boldsymbol{x}}(0)$.

Proof. Just as in the example, we change the coordinates and define $\vec{\boldsymbol{y}} = e^{-A_1 t}\vec{\boldsymbol{x}}$ so that $\vec{\boldsymbol{x}}' = \frac{d\vec{\boldsymbol{y}}}{dt} e^{A_1 t}\vec{\boldsymbol{y}} = A_1 e^{A_1 t}\vec{\boldsymbol{y}} + e^{A_1 t}\vec{\boldsymbol{y}}'$, from this we can derive

$$\vec{\boldsymbol{y}}' = e^{-A_1 t}\vec{\boldsymbol{x}}' - A_1\vec{\boldsymbol{y}}.$$

Using $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ we get

$$\vec{\boldsymbol{y}}' = e^{-A_1 t}A(t)\vec{\boldsymbol{x}} - A_1\vec{\boldsymbol{y}}.$$

Use $\vec{\boldsymbol{x}} = e^{A_1 t}\vec{\boldsymbol{y}}$

$$\vec{\boldsymbol{y}}' = (e^{-A_1 t}A(t)e^{A_1 t} - A_1)\vec{\boldsymbol{y}}.$$

Use $A(0) = e^{-A_1 t}A(t)e^{A_1 t}$

$$\vec{\boldsymbol{y}}' = (A(0) - A_1)\vec{\boldsymbol{y}}.$$

If we set $A_2 = A(0) - A_1$ we get the system $\vec{\boldsymbol{y}}' = A_2\vec{\boldsymbol{y}}$ and the general solution to this will then be $\vec{\boldsymbol{y}} = e^{A_2 t}\vec{\boldsymbol{y}}(0)$. Inverting this back to $\vec{\boldsymbol{x}}$ by using $\vec{\boldsymbol{y}} = e^{-A_1 t}\vec{\boldsymbol{x}}$ gives us the general solution to $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$,

$$e^{-A_1 t}\vec{\boldsymbol{x}} = e^{A_2 t}\vec{\boldsymbol{x}}(0)$$

$$\Leftrightarrow \vec{\boldsymbol{x}} = e^{A_1 t}e^{(A(0)-A_1)t}\vec{\boldsymbol{x}}(0).$$

□

2.1 Main Result

We have constructed the solution to $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ and found that

$$\vec{\boldsymbol{x}} = R(t,\omega)e^{[B-G(\omega)]t}\vec{\boldsymbol{x}}(0).$$

We want B to have eigenvalues with negative real part, but $B - G(\omega)$ should have an eigenvalue with positive real part. Since $R(t,\omega)$ is a rotation matrix, it doesn't effect the length of the vectors. Therefore it's enough to require $e^{[B-G(\omega)]t}\vec{\boldsymbol{x}}(0)$ to grow over time for the system to be unstable.

From example 2.2 we could see that $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ turned unstable when $\omega = -\frac{3}{2}$, but got stable when $\omega = 2$. For what ω is the system unstable? Through the following theorem we find the interval for ω.

Theorem 2.4 ([1]). *Let B be a real-valued 2×2 matrix of the form*

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$$

and

$$G(\omega) = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix},$$

a skew-symmetric matrix. If $B \in \mathcal{B}$*, then*

$$\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}} = (R(t,\omega)B[R(t,\omega)]^{-1})\vec{\boldsymbol{x}} \tag{2.3}$$

is an unstable system if and only if ω *is in the nonempty interval*

$$I = \left(D - \sqrt{D^2 - \det(B)}, D + \sqrt{D^2 - \det(B)}\right)$$

where

$$D = \frac{b_{2,1} - b_{1,2}}{2}$$

and $D^2 - \det(B)$ *is the discriminant of* ω*.*

In example 2.2 we showed that depending on how we choose our ω we will be able to find an unstable system. From theorem 2.4 we can calculate $D = -\frac{3}{2}$ and this leads to

$$I = \left(-\frac{3}{2} - \sqrt{\frac{9}{4} - 2}, -\frac{3}{2} + \sqrt{\frac{9}{4} - 2}\right)$$

$$\Leftrightarrow I = (-2, -1).$$

It's an open interval and choosing $\omega = 2$ did not give us a strictly positive eigenvalue. However choosing $\omega = -\frac{3}{2}$ did.

Proof. We start by finding the characteristic equation of $B - G(\omega)$, $\det(B - G(\omega) - \eta I) = 0$. We get $\eta^2 - \text{tr}(B)\eta + (\det(B) + (b_{1,2} - b_{2,1})\omega + \omega^2) = 0$. Solving for η gives us the eigenvalues

$$\eta_1, \eta_2 = \frac{\text{tr}(B)}{2} \pm \sqrt{\frac{\text{tr}(B)^2}{4} - \det(B) + 2D\omega - \omega^2}.$$

D can also be expressed in the coordinate invariant form

$$D = \frac{B\vec{\boldsymbol{v_1}} \cdot \vec{\boldsymbol{v_2}} - B\vec{\boldsymbol{v_2}} \cdot \vec{\boldsymbol{v_1}}}{2\det[\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}]}$$

where $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}$ are arbitrary linearly independent vectors. This can be seen by expanding $\vec{\boldsymbol{v_1}}$ and $\vec{\boldsymbol{v_2}}$ in the basis $\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}$. In particular, when $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}$ is an orthonormal basis, we obtain $D = \frac{\tilde{b}_{2,1} - \tilde{b}_{1,2}}{2}$ if $\tilde{B}$ is the matrix for B with respect to $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}$.

We only need one eigenvalue to be strictly positive for the system to be unstable, let

$$\eta_1 = \text{tr}(B) + \sqrt{\text{tr}(B)^2 - 4\det(B) + 8D\omega - 4\omega^2}.$$

If the discriminant of η_1 is less than or equal to zero, then the real part of η_1 will also be less than zero since $\mathrm{tr}(B) = \lambda_1 + \lambda_2 < 0$, $B \in \mathcal{B}$. This is not what we want. However, if the discriminant is positive then $\eta_1 > 0$ if and only if

$$\mathrm{tr}(B)^2 - 4\det(B) + 8D\omega - 4\omega^2 > (\mathrm{tr}(B))^2$$

which is equivalent to

$$\omega^2 - 2D\omega + \det(B) < 0.$$

The interval I of ω where $Re(\eta_1) > 0$ is therefore

$$D - \sqrt{D^2 - \det(B)} < \omega < D + \sqrt{D^2 - \det(B)}$$

given that the discriminant of ω is strictly positive. If it were not to be, I would be empty. We now show that $D^2 - \det(B) > 0$ no matter if B has complex, real or equal eigenvalues. The proof is divided into three cases.

Case 1: B has real distinct eigenvalues

Consider a 2×2 matrix with real eigenvalues $\lambda_1 < \lambda_2 < 0$, distinct from each other and both less than zero. By making a rotation, we can assume that B has one eigenvector $(1, 0)$ on the positive x-axis and that the angle δ between the eigenvectors satisfies $\delta \in (0, \pi)$. Therefore, B has the form

$$B = T\tilde{B}T^{-1} \Leftrightarrow \begin{bmatrix} 1 & \cot\delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\cot\delta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & (\lambda_2 - \lambda_1)\cot\delta \\ 0 & \lambda_2 \end{bmatrix}.$$

We have $B \in \mathcal{B}$ if and only if $r(\vec{\boldsymbol{x}}) > 0$ for some $\vec{\boldsymbol{x}} \neq 0$, where

$$\begin{aligned} r(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & (\lambda_2 - \lambda_1)\cot\delta \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^2\lambda_1 + x_1x_2(\lambda_2 - \lambda_1)\cot\delta + x_2^2\lambda_2. \end{aligned}$$

To check whether $r(\vec{\boldsymbol{x}})$ is positive, it's enough to locate its maxima on the unit circle. Parameterize $\vec{\boldsymbol{x}}(\theta) = (\cos\theta, \sin\theta)$ so that

$$
\begin{aligned}
r(\vec{\boldsymbol{x}}(\theta)) &= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta + (\lambda_2 - \lambda_1)\cos\theta \sin\theta \cot\delta \\
&= \lambda_1(1-\sin^2\theta) + \lambda_2 \sin^2\theta + (\lambda_2-\lambda_1)\frac{\sin(\theta-\theta)+\sin(\theta+\theta)}{2}\cot\delta \\
&= \lambda_1 + (\lambda_2-\lambda_1)\left(\sin^2\theta + \frac{\sin 2\theta}{2}\cot\delta\right) \\
&= \lambda_1 + (\lambda_2-\lambda_1)\left(\frac{1-\cos 2\theta}{2} + \frac{\sin 2\theta}{2}\cot\delta\right) \\
&= \frac{2\lambda_1}{2} + \frac{(\lambda_2-\lambda_1)}{2} + \frac{(\lambda_2-\lambda_1)}{2}\left(-\cos 2\theta + \sin 2\theta \frac{\cos\delta}{\sin\delta}\right) \\
&= \frac{(\lambda_2+\lambda_1)}{2} + \frac{(\lambda_2-\lambda_1)}{2}\frac{1}{\sin\delta}\left(-\cos 2\theta \sin\delta + \sin 2\theta\cos\delta\right) \\
&= \frac{(\lambda_2+\lambda_1)}{2} + \frac{(\lambda_2-\lambda_1)}{2}\frac{1}{\sin\delta}\left(\sin(2\theta-\delta)\right).
\end{aligned}
$$

$r(\vec{\boldsymbol{x}}(\theta))$ attains it's maximum when $2\theta - \delta = \frac{\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4} + \frac{\delta}{2}$. This gives us $\sin(2(\frac{\pi}{4}+\frac{\delta}{2}) - \delta) = \sin\frac{\pi}{2} = 1$. We then have

$$r(\vec{\boldsymbol{x}}(\theta_{max})) = \frac{(\lambda_2+\lambda_1) + (\lambda_2-\lambda_1)\csc\delta}{2} \cdot 1.$$

$r(\vec{\boldsymbol{x}}(\theta_{max}))$ is greater than zero when

$$\lambda_2 - \lambda_1 + (\lambda_1+\lambda_2)\sin\delta > 0. \tag{2.4}$$

We will now show that this is exactly the condition for I to be nonempty.
We start by calculating D. Notice that the rotation performed earlier does not effect the formula for D, hence

$$D = \frac{1}{2}(\lambda_1 - \lambda_2)\cot\delta$$

and

$$D^2 - \det(B) = \frac{1}{4}(\lambda_1-\lambda_2)^2\cot^2\delta - \lambda_1\lambda_2.$$

Since $D^2 - \det(B)$ is the discriminant for ω, we want this expression to be positive. Otherwise I will be empty hence there exist no ω such that $\eta_1 > 0$. Therefore we want

$$\frac{1}{4}(\lambda_1-\lambda_2)^2\cot^2\delta - \lambda_1\lambda_2 > 0.$$

Writing $\cot^2\delta = \frac{1-\sin^2\delta}{\sin^2\delta}$ we get the following inequality

$$(\lambda_1-\lambda_2)^2 > (\lambda_1+\lambda_2)^2\sin^2\delta.$$

Taking the square root of both sides given the equivalent inequality

$$|\lambda_1 - \lambda_2| > |\lambda_1 + \lambda_2||\sin\delta|.$$

Since $\delta \in (0, \pi)$, this is equivalent to

$$\begin{aligned} &-(\lambda_1 - \lambda_2) > -(\lambda_1 + \lambda_2)\sin\delta \\ &\Leftrightarrow (\lambda_1 + \lambda_2)\sin\delta + \lambda_2 - \lambda_1 > 0. \end{aligned} \tag{2.5}$$

Which is precisely the condition (2.4) for B to belong to $\mathcal{B}$.

Case 2: B has complex eigenvalues

Let

$$\vec{\boldsymbol{w}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

be a complex vector in $\mathbb{C}^2$ where $u_j, v_j \in \mathbb{R}$, $j = 1, 2$. Then by multiplying $\vec{\boldsymbol{w}}$ with a complex number $z = e^{i\phi}$, $\phi \in [0, 2\pi]$, we can choose an angle ϕ for z such that $\mathrm{Im}(z\vec{\boldsymbol{w}}) \cdot \mathrm{Re}(z\vec{\boldsymbol{w}}) = 0$. We start by rewriting $w = r_j e^{i\theta_j}$, $r_j > 0$, $\theta_j \in [0, 2\pi]$, $j = 1, 2$. Recall that $|e^{i\phi}| = 1$ which implies that z does not effect the length of $\vec{\boldsymbol{w}}$. Furthermore

$$z\vec{\boldsymbol{w}} = e^{i\phi} \begin{bmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{bmatrix} = \begin{bmatrix} r_1(\cos(\theta_1 + \phi) + i \sin(\theta_1 + \phi)) \\ r_2(\cos(\theta_2 + \phi) + i \sin(\theta_2 + \phi)) \end{bmatrix}.$$

Then

$$\begin{aligned} \mathrm{Im}(z\vec{\boldsymbol{w}}) \cdot \mathrm{Re}(z\vec{\boldsymbol{w}}) &= r_1^2 \cos(\theta_1 + \phi) \sin(\theta_1 + \phi) + r_2^2 \cos(\theta_2 + \phi) \sin(\theta_2 + \phi) \\ &= \frac{1}{2}(r_1^2 \sin[2(\theta_1 + \phi)] + r_2^2 \sin[2(\theta_2 + \phi)]) \\ &= f(\phi). \end{aligned}$$

For what ϕ is it possible for $f(\phi) = 0$? We get the following two cases:

$$f(-\theta_1) = \frac{r_2^2 \sin[2(\theta_2 - \theta_1)]}{2}$$

and

$$f(-\theta_1 + \frac{\pi}{2}) = \frac{r_2^2 \sin[2(\theta_2 - \theta_1 + \frac{\pi}{2})]}{2} = -\frac{r_2^2 \sin[2(\theta_2 - \theta_1)]}{2}.$$

If $\theta_2 - \theta_1 = \frac{\pi}{2}n$ for some $n \in \mathbb{Z}$ then both $f(-\theta_1)$ and $f(-\theta_1 + \frac{\pi}{2})$ will be equal to zero, otherwise they have opposite signs. With the intermediate value theorem we can say that there exists a ϕ in the interval $-\theta_1 \leq \phi \leq -\theta_1 + \frac{\pi}{2}$ such that $f(\phi) = 0$.

The same calculations can be used for $\vec{\boldsymbol{w}}$ and it won't change the outcome of the angles.

We now assume B is of the form

$$B = \begin{bmatrix} 1 & 1 \\ a+bi & a-bi \end{bmatrix} \begin{bmatrix} k+\sigma i & 0 \\ 0 & k-\sigma i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a+bi & a-bi \end{bmatrix}^{-1}$$

with eigenvalues $\lambda_{1,2} = k \pm \sigma i$ and eigenvectors

$$\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}} = \begin{bmatrix} 1 \\ a \pm bi \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \pm i \begin{bmatrix} 0 \\ b \end{bmatrix}.$$

Setting $a = 0$ is possible, from what we concluded earlier, since then the imaginary part of the eigenvectors is orthogonal to the real part. We get

$$B = \begin{bmatrix} k & \frac{\sigma}{b} \\ -b\sigma & k \end{bmatrix}.$$

Here k is the real part of the eigenvalues of B which implies $k < 0$. Just as in case 1, $B \in \mathcal{B}$ if $\vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}} > 0$. So we set $r(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}}$ for some $\vec{\boldsymbol{x}} \neq 0$. Using the formula above for B, we have

$$r(\vec{\boldsymbol{x}}) = x_1^2 k + x_2^2 k + x_1 x_2 \left(\frac{\sigma}{b} - \sigma b \right).$$

Parameterizing $\vec{\boldsymbol{x}}(\theta) = (\cos\theta, \sin\theta)$ since we want to maximize $r(\vec{\boldsymbol{x}})$ on the unit circle, we get

$$\begin{aligned} r(\vec{\boldsymbol{x}}(\theta)) &= k(\cos^2\theta + \sin^2\theta) + \cos\theta\sin\theta \left(\frac{\sigma(1-b^2)}{b} \right) \\ &= k + \left(\frac{\sigma(1-b^2)}{b} \right) \frac{\sin 2\theta}{2}. \end{aligned}$$

This is maximized either when $\theta = \frac{\pi}{4}$ or when $\theta = -\frac{\pi}{4}$. Therefore, B belongs to $\mathcal{B}$ precisely when

$$k + \left| \frac{\sigma(1-b^2)}{2b} \right| > 0. \tag{2.6}$$

Next, we consider the discriminant for ω. We have

$$D = -\frac{\sigma(1+b^2)}{2b}$$

and

$$D^2 - \det(B) = \frac{\sigma^2(1+b^2)^2}{4b^2} - (k+\sigma i)(k-\sigma i) > 0$$

$$\Leftrightarrow \sigma^2\left(\frac{1+2b^2+b^4-4b^2}{4b^2}\right) = \left(\frac{\sigma}{2b}\right)^2 (1-b^2)^2 > k^2$$

$$\Leftrightarrow \left|\frac{\sigma}{2b}\right| |1-b^2| > |k|.$$

Since $k < 0$, this is equivalent to

$$k + \left|\frac{\sigma(1-b^2)}{2b}\right| > 0. \tag{2.7}$$

Which is precisely the condition (2.6) for B to belong to $\mathcal{B}$

Case 3: The eigenvalues of B are equal and real

Assume that the eigenvalues are real and equal such that $\lambda_1 = \lambda_2 < 0$. By a rotation we can assume that

$$B = \begin{bmatrix} \lambda & c \\ 0 & \lambda \end{bmatrix}.$$

Following the same steps as in the other cases gives us similar calculations to what we had in Case 1.

$\vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}} > 0$ is equivalent to $\lambda x_1^2 + cx_1x_2 + \lambda x_2^2 > 0$. As in the other cases, we want to maximize the expression on to the unit circle. Set $(x_1, x_2) = (\cos\theta, \sin\theta)$ implies

$$r(\vec{\boldsymbol{x}}(\theta)) = \lambda\cos^2\theta + c\cos\theta\sin\theta + \lambda\sin^2\theta > 0$$

$$\Leftrightarrow \lambda + \frac{\sin 2\theta}{2}c > 0.$$

This is maximized for $\theta = \pm\frac{\pi}{4}$ and

$$r(\vec{\boldsymbol{x}}(\theta_{max})) = \lambda + \left|\frac{c}{2}\right| > 0. \tag{2.8}$$

To see that also in this case, $D^2 - \det(B) > 0$ is equivalent to $\vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}} > 0$, note that $D = -\frac{c}{2}$ and

$$D^2 - \det(B) = (-\frac{c}{2})^2 - \lambda^2 > 0 \Leftrightarrow \left|\frac{c}{2}\right| > |\lambda|$$

$$\Leftrightarrow \lambda + \left|\frac{c}{2}\right| > 0, \tag{2.9}$$

which is precisely (2.8). The proof is complete. □

2.2 Other

Following examples are illustrations that shows it's possible to have different matrices instead of $G(\omega)$ and still be able to use theorem 2.1 and 2.3.

Example 2.5 ([1]). We can replace $G(\omega)$ with

$$H(\omega, b) = \begin{bmatrix} 0 & -\frac{\omega}{b} \\ \omega b & 0 \end{bmatrix}$$

where $e^{H(\omega,b)t}$ is an elliptic rotation. However, $H(\omega, b)$ is not skew-symmetric, which is what we demand in theorem 2.1. A change of basis makes it possible for us to follow theorem 2.4. Note that $H(\omega, b) = TG(\omega)T^{-1}$ where

$$T = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}.$$

Under this change of basis,

$$\tilde{B} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$$

changes to

$$B = T^{-1}\tilde{B}T = \begin{bmatrix} b_{1,1} & bb_{1,2} \\ \frac{b_{2,1}}{b} & b_{2,2} \end{bmatrix}.$$

We then have

$$\tilde{B} - H(\omega, b) = T(B - G(\omega))T^{-1} \Leftrightarrow B - G(\omega) = T^{-1}(\tilde{B} - H(\omega, b))T$$

which implies

$$D = \frac{b_{2,1} - b^2 b_{1,2}}{2b}.$$

Assuming that $B \in \mathcal{B}$ and $\omega \in I$, from theorem 2.4, gives us that the system

$$\vec{\boldsymbol{x}}' = (R(t, \omega)B[R(t, \omega)]^{-1})\vec{\boldsymbol{x}} = (e^{G(\omega)t}Be^{-G(\omega)t})\vec{\boldsymbol{x}} \tag{2.10}$$

is unstable. Then it follows that the system

$$\vec{\boldsymbol{y}}' = (e^{H(\omega,b)t}\tilde{B}e^{H(\omega,b)t})\vec{\boldsymbol{y}}$$

is also unstable, since it is equivalent to (2.10) under the change of variable

$$\vec{\boldsymbol{y}} = T\vec{\boldsymbol{x}}.$$

□

Example 2.6 ([1]). Replace $G(\omega)$ with

$$F(\mu_1, \mu_2) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

This matrix is not skew-symmetric or of the form $H(\omega, b)$, which means that we can't handle it with the same technique as in the last example or as in theorem 2.4. However, we can use the same approach as in theorem 2.3 to find the criterion for instability.
Assume $A(0)$ is of the form B from Case 1 and $A_1 = F(\mu_1, \mu_2)$, then

$$A(t) = e^{F(\mu_1,\mu_2)t} A(0) e^{-F(\mu_1,\mu_2)t} = \begin{bmatrix} \lambda_1 & e^{(\mu_1-\mu_2)t}(\lambda_2 - \lambda_1)\cot\delta \\ 0 & \lambda_2 \end{bmatrix},$$

$0 < \delta < \pi$. We then have

$$\begin{cases} x_1' = \lambda_1 x_1 + e^{(\mu_1-\mu_2)t}(\lambda_2 - \lambda_1)\cot(\delta) x_2 \\ x_2' = \lambda_2 x_2 \Leftrightarrow x_2 = e^{\lambda_2 t} x_2(0) \end{cases}$$

$$\implies x_1' - \lambda_1 x_1 = e^{(\mu_1-\mu_2)t}(\lambda_2 - \lambda_1)\cot(\delta) e^{\lambda_2 t} x_2(0).$$

Solving this equation with the integrating factor $e^{-\lambda_1 t}$ gives us

$$\int_0^\tau (e^{-\lambda_1 t} x_1(t))' dt = x_2(0)(\lambda_2 - \lambda_1)\cot\delta \int_0^\tau e^{(\lambda_2-\lambda_1+\mu_1-\mu_2)t} dt$$

$$\Leftrightarrow e^{-\lambda_1 \tau} x_1(\tau) - x_1(0) = x_2(0)(\lambda_2 - \lambda_1)\cot\delta \left[\frac{e^{(\lambda_2-\lambda_1+\mu_1-\mu_2)t}}{\lambda_2 - \lambda_1 + \mu_1 - \mu_2} \right]_0^\tau$$

$$\Leftrightarrow x_1(\tau) = e^{\lambda_1 \tau} x_1(0) + \frac{x_2(0)(\lambda_2 - \lambda_1)\cot\delta}{\lambda_2 - \lambda_1 + \mu_1 - \mu_2} (e^{(\lambda_2+\mu_1-\mu_2)\tau} - e^{\lambda_1 \tau}).$$

Now it's clear to see that $\vec{\boldsymbol{x}}'(t)$ is unstable when $\lambda_2 + \mu_1 - \mu_2 > 0$, since then $e^{\lambda_1 \tau} \to 0$ when $\tau \to \infty$ but $e^{(\lambda_2+\mu_1-\mu_2)\tau} \to \infty$.

$\square$

3 Nonautonomous Systems when $n \geq 3$

In [1] they mention a few starting points for further studies of greater dimensions. We'll continue one of them which consists of the $n-$dimensional proof, $n \geq 3$, of theorem 2.1.

Recall from Section 2 that e^{tG} is orthogonal to $\vec{\boldsymbol{x}}$ if G is a skew-symmetric matrix. It therefore does not effect any length when $A(t) = e^{tG}Be^{-tG}$ from the system $\vec{\boldsymbol{x}}' = A(t)\vec{\boldsymbol{x}}$ that gets unstable if $B \in \mathcal{B}$ and G is chosen as in the following theorem.

Theorem 3.1. *Given a matrix $B \in \mathbb{R}^{n\times n}$, it is possible to find a skew-symmetric matrix $G \in \mathbb{R}^{n\times n}$ such that*

$$(B - G)\vec{\boldsymbol{x}} = \eta\vec{\boldsymbol{x}}$$

for some $\vec{\boldsymbol{x}} \neq 0$ and $\eta > 0$ if and only if $\vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}} > 0$.

Proof. We begin by showing that the condition $\vec{\boldsymbol{x}} \cdot B\vec{\boldsymbol{x}} > 0$ is necessary. If $(B-G)\vec{\boldsymbol{x}} = \eta\vec{\boldsymbol{x}}$, $\vec{\boldsymbol{x}} \neq 0$ and $\eta > 0$ then $\eta|\vec{\boldsymbol{x}}|^2 = (B-G)\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}} = B\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}} - G\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}} = B\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}}$ since $G\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}} = 0$. This can also be seen with the help of a linear transformation T on M. Assume $T(\vec{\boldsymbol{x}}) = G\vec{\boldsymbol{x}}$, $T(\vec{\boldsymbol{y}}) = G\vec{\boldsymbol{y}}$ and the definition of a skew symmetric matrix: $G^t = -G$. Then it follows that a linear transformation T onto M is called skew symmetric if $T(\vec{\boldsymbol{x}})\cdot\vec{\boldsymbol{y}} = -\vec{\boldsymbol{x}}\cdot T(\vec{\boldsymbol{y}})$ for all vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ in M. This is equivalent to

$$T(\vec{\boldsymbol{x}})^t\vec{\boldsymbol{y}} = -\vec{\boldsymbol{x}}^tT(\vec{\boldsymbol{y}})$$

$$\Leftrightarrow (G\vec{\boldsymbol{x}})^t\vec{\boldsymbol{y}} = -\vec{\boldsymbol{x}}^tG\vec{\boldsymbol{y}}$$

$$\Leftrightarrow \vec{\boldsymbol{x}}^tG^t\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^t(-G)\vec{\boldsymbol{y}}$$

and this is independent of ON-basis. Hence, $B\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}} > 0$.

We next show that the condition $\vec{\boldsymbol{x}}\cdot B\vec{\boldsymbol{x}} > 0$ is sufficient. After rotating and normalizing, $\vec{\boldsymbol{x}} = \vec{\boldsymbol{e_1}}$ we let $\mathbf{U} = \{\vec{\boldsymbol{y}} \in \mathbb{R}^n,\ \vec{\boldsymbol{e_1}}\cdot\vec{\boldsymbol{y}} > 0\}$ and $\mathbf{M} = \{\vec{\boldsymbol{y}} \in \mathbb{R}^n,\ \vec{\boldsymbol{e_1}}\cdot\vec{\boldsymbol{y}} = 0\}$. Let $\vec{\boldsymbol{z}}$ be the orthogonal projection of $B\vec{\boldsymbol{e_1}}$ on $\mathbf{M}$ we get that $B\vec{\boldsymbol{e_1}} - \vec{\boldsymbol{z}} = \eta\vec{\boldsymbol{e_1}}$ for some $\eta \in \mathbb{R}$. Moreover, $\eta > 0$ since $\eta = \eta|\vec{\boldsymbol{e_1}}|^2 = B\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}} > 0$.

The unit vector $\vec{\boldsymbol{e_1}}$ is pointing out of the origin into the set $\mathbf{U}$ and the plane $\mathbf{M}$ is tangent to the unit sphere at the end of $\vec{\boldsymbol{e_1}}$. From the end of $\vec{\boldsymbol{e_1}}$ points the vector $B\vec{\boldsymbol{e_1}}$, still in $\mathbf{U}$. It is possible to move $\mathbf{M}$ to the end of $\vec{\boldsymbol{e_1}}$ and also to the end of $B\vec{\boldsymbol{e_1}}$, then we will be able to go from this vector to $\eta\vec{\boldsymbol{e_1}}$ with a skew symmetric matrix G. This can be done by showing that all vectors in $\mathbf{M}$ can be obtained by multiplying $\vec{\boldsymbol{e_1}}$ with G.

Note that the standard unit vectors $\vec{e_2}, ..., \vec{e_n}$ form a base for M. Let the vector $\vec{z} = (0, z_2, ..., z_n)$ belong to $\mathbf{M}$ then $G\vec{e_1} = \vec{z}$ if G has the skew-symmetric form

$$G = \begin{bmatrix} 0 & -z_2 & \cdots & -z_n \\ z_2 & \ddots & & \vdots \\ \vdots & & \ddots & \\ z_n & \cdots & & 0 \end{bmatrix}.$$

By choosing $\vec{z} = B\vec{e_1} - \eta\vec{e_1}$ it is possible to find a vector $-G\vec{e_1} \in M$ that goes from $B\vec{e_1}$ to $\eta\vec{e_1}$. □

3.1 Upper Triangular Matrices

We will now study a few examples with B being an upper triangular matrix in $\mathbb{R}^{n\times n}$, $n \geq 3$. Is it possible for us to use the same approach as in theorem 2.4 but for larger matrices?

Example 3.2. Since only one eigenvalue need to be positive for $\vec{x}' = A(t)\vec{x}$ to be unstable, we can choose an upper triangular matrix B_n where the negative eigenvalues is on the diagonal and a skew-symmetric matrix $G_n(\omega)$ where the first 2×2 sub-matrix looks exactly like $G(\omega)$ and the rest of the matrix consists of zeros. Let

$$B_n = \begin{bmatrix} \lambda_1 & b_{1,2} & \cdots & b_{1,n} \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix} \text{ and } G_n(\omega) = \begin{bmatrix} 0 & -\omega & \cdots & 0 \\ \omega & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}.$$

Then the eigenvalues of $B_n - G_n(\omega)$ will be

$$\left(\frac{\lambda_1 + \lambda_2}{2} \pm \sqrt{\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 - \omega(b_{1,2} + \omega)}, \lambda_3, ..., \lambda_n \right).$$

It's possible to find the interval for ω depending on $b_{1,2}$.

$$\frac{\lambda_1 + \lambda_2}{2} + \sqrt{\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 - \omega(b_{1,2} + \omega)} > 0$$

$$\Leftrightarrow (\lambda_1 - \lambda_2)^2 - 4\omega(b_{1,2} + \omega) > (-\lambda_1 - \lambda_2)^2$$
$$\Leftrightarrow \omega^2 + b_{1,2}\omega + \lambda_1\lambda_2 < 0$$

$$\implies I_n = \left(-\frac{b_{1,2}}{2} - \frac{1}{2}\sqrt{b_{1,2}^2 - 4\lambda_1\lambda_2} < \omega < -\frac{b_{1,2}}{2} + \frac{1}{2}\sqrt{b_{1,2}^2 - 4\lambda_1\lambda_2} \right).$$

The condition $b_{1,2}^2 > 4\lambda_1\lambda_2$ must be upheld, otherwise I_n will be empty and there will exist no ω such that $\vec{x}' = A(t)\vec{x}$ can be unstable.
More generally: If $b_{i,j}^2 > 4\lambda_i\lambda_j$, where $i, j = 1, 2, ..., n$ and $i < j$, then $-\omega$ will be found on place (i, j) and ω on place (j, i) in $G_n(\omega)$. Then $B_n - G_n(\omega)$ has a strictly positive eigenvalue if

$$\omega \in I_n = \left(-\frac{b_{i,j}}{2} \pm \frac{1}{2}\sqrt{b_{i,j}^2 - 4\lambda_i\lambda_j} \right).$$

In this way we can see how we should construct $G_n(\omega)$. However it's not the only criterion we have for $B \in \mathcal{B}$. It's still important to show $\vec{x} \cdot B\vec{x} > 0$ for some $\vec{x} \neq 0$. Choosing a vector $\vec{x} = (0, \ldots, x_i, \ldots, 0, \ldots, x_j, \ldots, 0)$ we'll end up with the same calculations as in Case 1, given that $b_{i,j} = (\lambda_j - \lambda_i)\cot\delta$ which implies equation (2.5) where in that case $j = 2$ and $i = 1$.

□

A different way of handling the condition $\vec{x} \cdot B\vec{x} > 0$ is to use Sylvester's Criterion. This theorem gives a necessary and sufficient condition for a Hermitian matrix to be positive (semi-) definite in terms of principal minors. Recall that the leading principal minors are the upper left sub-determinants of a matrix. The principal minors of an $n \times n$ matrix of order k are obtained by taking the determinants of the matrix obtained by deleting $n - k$ rows and columns with the same number.

Theorem 3.3 (Sylvester's Criterion [6])**.** *A Hermitian matrix H is positive definite if and only if all of the leading principal minors are strictly positive.*
A Hermitian matrix H is positive semi-definite if and only if all principal minors of H are positive.

Example 3.4. We want to construct a 3×3 matrix B that is upper triangular and has negative eigenvalues on the diagonal but satisfies $\vec{x} \cdot B\vec{x} > 0$ for some $\vec{x}$. Then it's also possible to construct the symmetric version of B, B_s:

$$B = \begin{bmatrix} \lambda_1 & b_{1,2} & b_{1,3} \\ 0 & \lambda_2 & b_{2,3} \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad B_s = \frac{B + B^T}{2} = \begin{bmatrix} \lambda_1 & \frac{b_{1,2}}{2} & \frac{b_{1,3}}{2} \\ \frac{b_{1,2}}{2} & \lambda_2 & \frac{b_{2,3}}{2} \\ \frac{b_{1,3}}{2} & \frac{b_{2,3}}{2} & \lambda_3 \end{bmatrix}.$$

We want to find a vector $\vec{x} \neq 0$ such that $\vec{x} \cdot B\vec{x} = \vec{x} \cdot B_s\vec{x} > 0$. If all the eigenvalues of B_s would have been strictly positive then $\vec{x} \cdot B_s\vec{x} > 0 \;\forall\; \vec{x} \neq 0$ which implies that B_s would be positive definite. This is however not possible since, following Sylvester's criterion, a symmetric matrix is positive semi-definite if and only if the principal minors are non-negative. The condition $\vec{x} \cdot B\vec{x}$ can not be positive semi-definite since the principal minors of order one will allways be equal $\lambda_i < 0$. If we however turn things around and try to find when $\vec{x} \cdot B\vec{x}$ is not negative semi-definite then it will be possible for us to find at least one vector $\vec{x} \neq 0$ such that $\vec{x} \cdot B\vec{x} > 0$. This is of course equivalent to finding out when $\vec{x} \cdot (-B)\vec{x}$ is not positive semi-definite.

With the 3×3 matrix, there will be seven principal minors to examine. The three first orders $\det(-\lambda_1)$, $\det(-\lambda_2)$ and $\det(-\lambda_3)$, all of which are strictly positive. The second orders are also three and are given by

$$\det \begin{pmatrix} -\lambda_2 & -\frac{b_{2,3}}{2} \\ -\frac{b_{2,3}}{2} & -\lambda_3 \end{pmatrix}, \quad \det \begin{pmatrix} -\lambda_1 & -\frac{b_{1,3}}{2} \\ -\frac{b_{1,3}}{2} & -\lambda_3 \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} -\lambda_1 & -\frac{b_{1,2}}{2} \\ -\frac{b_{1,2}}{2} & -\lambda_2 \end{pmatrix}.$$

The last minor is given by the full determinant, $\det(-B_s)$. The second order principal minors can be expressed as $\lambda_i\lambda_j - \frac{b_{i,j}^2}{4}$ where $1 \leq i < j \leq 3$. If $\lambda_i\lambda_j < \frac{b_{i,j}^2}{4}$ then B_s is not negative semi-definite. This is precisely the same condition as in example 3.2. It would therefore be interesting to find an example where $\lambda_i\lambda_j \geq \frac{b_{i,j}^2}{4}$ for all i, j but $\det(-B_s) < 0$.

□

The following example is a demonstration of the recently given example that Sylvester's Criterion is something that can be applied.

Example 3.5. Regard the matrix B and it's symmetric negative version $-B_s$:

$$B = \begin{bmatrix} -2 & -2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix}, \quad -B_s = -\frac{B + B^T}{2} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Following theorem 3.3 and example 3.4 we get the condition that at least one sub-determinant of $-B_s$ must be strictly negative. Only then can we know that it's possible to find a vector $\vec{x}$ such that $\vec{x} \cdot B\vec{x} > 0$. The seven principal minors of $-B_s$ consists of the three 1×1 principal minors: $\det(2) = 2$, $\det(1) = 1$ and $\det(3) = 3$, the three 2×2 principal minors:

$$\det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1, \quad \det \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} = 2 \quad \text{and} \quad \det \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} = 2$$

and the last principal minor, which is the determinant of the whole matrix, $\det(-B_s) = -7$. With at least one negative principal minor we can move on to the next step, to find the vector $\vec{x}$ that will make $\vec{x} \cdot B\vec{x} > 0$, this can be done by looking at the eigenvectors of B_s and choosing the one belonging to a positive eigenvalue. In this case we have

$$\begin{bmatrix}-0.596 & 0.635 & 0.492\end{bmatrix}\begin{bmatrix}-2 & -2 & -4\\ 0 & -1 & 2\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}-0.596\\ 0.635\\ 0.492\end{bmatrix} = 0.715 > 0.$$

An example of what the answer would have been if we would have chosen our vector $\vec{\boldsymbol{x}}$ with respect to a negative eigenvalue.

$$\begin{bmatrix}0.599 & -0.056 & 0.798\end{bmatrix}\begin{bmatrix}-2 & -2 & -4\\ 0 & -1 & 2\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}0.99\\ -0.056\\ 0.798\end{bmatrix} = -4.565 \leq 0.$$

□

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