Nonlinear Instability of Evolution Equations

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Master's thesis
2020:E5

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Abstract

In this thesis we explore the connection between spectral stability and nonlinear stability of evolution equations. Using semigroup theory a result on nonlinear instability from spectral instability is proven. This result is then applied to two examples: Travelling heteroclinic solutions of the Kuramoto-Sivashinsky equation and constant solutions of the Lugiato-Lefever equation.

Acknowledgement

I would like to thank my supervisor Erik Wahlén for introducing me to this topic, for his enthusiasm and patience during the project and for the advice and comments that helped and guided me while writing this thesis.

Populärvetenskaplig Sammanfattning

Ickelinjär stabilitet är en typ av beteende för ickelinjära evolutionsekvationer som beskriver om stationära lösningar förändras mycket under små störningar av startvärdet. Detta beteende är viktigt inom flera områden då det hjälper att förstå hur exempelvis vågor utvecklas med tid: Vissa vågor förändras inte trots små störningar medan andra förfaller. I denna masteruppsatts undersöks instabilitet av lösningar till två olika ekvationer: Kuramoto-Sivashinsky-ekvationen och Lugiato-Lefever-ekvationen. Kuramoto-Sivashinsky-ekvationen studerades av Yoshiki Kuramoto för att beskriva lösningar till reaktions-diffusions-ekvationer och av Gregory Sivahinsky för att beskriva laminära förbränningsfronter. Lugiato-Lefever-ekvationen har studerats av Luigi Lugiato och René Lefever för att beskriva ljusvågor i optiska kaviteter. Stabilitet av dessa lösningar är viktigt att förstå eftersom det förklarar varför vissa vågor uppstår spontant och ej faller samman.

Contents

1 Introduction

In the theory of linear ordinary differential equations it is well known that the spectrum of the matrix yields information about the asymptotic behaviour of solutions. If the nonlinear part behaves well it is also true for nonlinear ordinary differential equations. The purpose of this thesis is to investigate how the theorems relating spectral instability to nonlinear instability translates to partial differential equations. For this purpose we consider evolution equations as ordinary differential equations in infinite dimensional Banach spaces.

Linear autonomous evolution equations are studied using semigroups, and the theorems on nonlinear instability for nonlinear differential equations can then be stated as depending on two parts: The linear part generating a semigroup and the nonlinear part. A few different theorems are stated in the thesis, most of which were proved in the late 1990's. The different theorems make different assumptions on the spectrum of the linear part and the smoothness of the nonlinear part. The guiding principle of these theorems is that stronger conditions on the linear part can compensate for weaker conditions on the nonlinear part. There is no definite answer as to when spectral instability implies nonlinear instability and it remains an open question.

We focus on one of the theorems with relatively weak conditions on the linear part and consider two main examples. The Kuramoto-Sivashinsky equation, in which the nonlinear part has derivatives, but the semigroup is smoothing and the Lugiato-Lefever equation which has a simpler nonlinear term, but a less nice semigroup.

2 Preliminaries

2.1 Closed Linear Operators

It is often useful to consider differential operators as operators defined on a subspace rather than as bounded operators from one Banach space to another. In this context the operator is rarely bounded, as this would imply that it could be extended to the whole space. However in most cases however they will be closed.

Definition 2.1. Let X and Y be Banach spaces and $T : D(T) \subset X \to Y$ be a linear operator defined on a linear subspace $D(T)$ of X. We say that T is closed if $x_n \to x$ in X and $Tx_n \to y$ in Y imply that $x \in D(T)$ and $y = Tx$.

The sum of two closed operators is not necessarily closed, as can be seen with the example $T + (-T) = 0$ for any closed operator $(T, D(T))$ defined on a strict subset of X. In this case $(T + (-T), D(T))$ is not a closed operator since there exists a sequence $x_n \to x$ such that $x \notin D(T)$, but clearly $(T + (-T))x_n \to 0$. The sum of a closed operator with a bounded operator is however always closed. If T is injective we may consider the inverse of T, $T^{-1} : \mathrm{rg}(T) \subset Y \to D(T) \subset X$. The inverse of a closed operator is always closed. These two statements are summarized in the following lemma.

Lemma 2.1. *Let X and Y be Banach spaces and $T : D(T) \subset X \to Y$ be a closed linear operator and $B : X \to Y$ a bounded operator. Then $T + B : D(T) \subset X \to Y$ is closed. If T is injective then $T^{-1} : \mathrm{rg}(T) \subset Y \to X$ is closed.*

We mention the following classical result.

Theorem 2.1 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a closed linear operator defined on X. Then T is bounded.*

For an operator only defined on a subspace $D(T)$, the closed graph theorem says that if $D(T)$ is closed then T is a bounded operator on $D(T)$. The next lemma is an important result for characterizing the approximate point spectrum.

Lemma 2.2. *Let X and Y be Banach spaces and $T : D(T) \subset X \to Y$ be an injective closed linear operator. Then T^{-1} is bounded if and only if $\mathrm{rg}(T)$ is closed in Y.*

Proof. If $\mathrm{rg}(T)$ is closed in Y then $T^{-1} : \mathrm{rg}(T) \subset Y \to X$ is closed in Y and defined on the Banach space $\mathrm{rg}(T)$, therefore T^{-1} is bounded by the closed graph theorem. Conversely, suppose T^{-1} is bounded and let $y \in \overline{\mathrm{rg}(T)}$ and $y_n \to y$ in Y, $y_n \in R(T)$. Write $y_n = Tx_n$ with $x_n \in D(T)$ and estimate

$$\|x_n - x_m\|_X = \|T^{-1}T(x_n - x_m)\|_X \leq C\|T(x_n - x_m)\|_Y = C\|y_n - y_m\|_Y.$$

It follows that x_n is a Cauchy sequence. Since X is complete there exists a limit x. The fact that T is closed now implies that $x \in D(T)$ and $y = Tx$. Hence $y \in \mathrm{rg}(T)$. Since y was arbitrary, $\mathrm{rg}(T)$ is closed. □

2.2 Spectral Theory

This section contains the necessary spectral theory for semigroups and stability. The decomposition of the spectrum is adapted for this purpose and is perhaps not standard. In particular, the residual spectrum may contain eigenvalues.

Definition 2.2. Let X be a Banach space and $T : D(T) \subset X \to X$ be a closed linear operator. The spectrum of T, $\sigma(T) \subset \mathbb{C}$ is defined as the set of complex numbers λ such that $\lambda - T : D(T) \subset X \to X$ is not bijective. The resolvent set $\rho(T)$ is defined as the complement of the spectrum. For $\lambda \in \rho(T)$ we define the resolvent

$$R(\lambda, T) = (\lambda - T)^{-1}.$$

Since $\lambda - T$ is bijective for $\lambda \in \rho(T)$ the resolvent can be defined on X and by Lemma 2.1 and the closed graph theorem we get that $R(\lambda, T) : X \to X$ is bounded. As for bounded operators, the spectrum of a closed linear operator is always closed. However for unbounded operators it is not necessarily bounded and may be empty.

Definition 2.3. Let X be a Banach space and $T : D(T) \subset X \to X$ be a closed linear operator. The point spectrum of T, $\sigma_p(T) \subset \mathbb{C}$ is defined as the set of $\lambda \in \mathbb{C}$ such that $\lambda - T : D(T) \subset X \to X$ is not injective. The approximate point spectrum of T, $\sigma_a(T) \subset \mathbb{C}$ is defined as the set of $\lambda \in \mathbb{C}$ such that $\lambda - T : D(T) \subset X \to X$ is not injective or $\mathrm{rg}(\lambda - T)$ is not closed in X. The residual spectrum of T, $\sigma_r(T) \subset \mathbb{C}$ is defined as the set of $\lambda \in \mathbb{C}$ such that $\mathrm{rg}(\lambda - T)$ is not dense in X.

By definition $\sigma_p(T) \subset \sigma_a(T)$. Furthermore we have

$$\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$$

since if $\lambda - T$ is injective and $\mathrm{rg}(\lambda - T)$ is both closed and dense then $\lambda - T$ is bijective and so $\lambda \in \rho(T)$. However, the decomposition is not necessarily disjoint. The approximate point spectrum is important for at least two reasons: It is the part of the spectrum where the spectral mapping theorem for semigroups fails and it contains the boundary of the spectrum. Fortunately there exists a characterization of the approximate point spectrum that says we can find approximate eigenvectors, which explains the name.

Lemma 2.3. *Let $T : D(T) \subset X \to X$ be a closed linear operator. Then the following statements are equivalent.*

(a) $\lambda \in \sigma_a(T)$.

(b) $\lambda - T$ *is not injective or* $(\lambda - T)^{-1} : \mathrm{rg}(T) \subset X \to D(T) \subset X$ *is unbounded.*

(c) There exists a sequence $\{x_n\} \subset D(T)$ *such that* $\|x_n\| = 1$ *and* $\|Tx_n - \lambda x_n\| \to 0$ *as* $n \to \infty$.

Proof. To prove (a) $\iff$ (b) observe that the case in which $\lambda - T$ is not injective is trivial. Hence it is sufficient to show that, when $\lambda - T$ is injective, $\mathrm{rg}(\lambda - T)$ is not closed if and only if $(\lambda - T)^{-1}$ is unbounded, but this is just Lemma 2.2.

To prove (b) $\iff$ (c) suppose first that such a sequence x_n exists. If some x_n satisfy $(\lambda - T)x_n = 0$ then $\lambda - T$ is not injective so (b) holds, otherwise we may define

$$y_n = \frac{Tx_n - \lambda x_n}{\|Tx_n - \lambda x_n\|}.$$

Then $\|y_n\| = 1$ and $\|(\lambda - T)^{-1} y_n\| \to \infty$. Hence $(\lambda - T)^{-1}$ is unbounded. Conversely, if $\lambda - T$ is not injective (c) holds trivially and if $\lambda - T$ is injective, but $(\lambda - T)^{-1}$ is unbounded then there exists y_n such that $\|y_n\| = 1$ and $\|(\lambda - T)^{-1} y_n\| \to \infty$. Define

$$x_n = \frac{(T - \lambda)^{-1} y_n}{\|(T - \lambda)^{-1} y_n\|}.$$

Then $\|x_n\| = 1$ and $\|Tx_n - \lambda x_n\| \to 0$. □

Lemma 2.4. *Let $T : D(T) \subset X \to X$ be a closed linear operator. If $\lambda \in \partial\sigma(T)$, the boundary of $\sigma(T)$, then $\lambda \in \sigma_a(T)$.*

Proof. Let $\lambda \in \partial\sigma(T)$ and $\lambda_n \in \rho(T)$, $\lambda_n \to \lambda$. Then

$$\|(\lambda_n - T)^{-1}\| \geq \mathrm{dist}(\lambda_n, \sigma(T))^{-1} \to \infty$$

where the inequality follows from the fact that if $\mu \in \mathbb{C}$ with

$$|\mu - \lambda_n| < 1/\|(\lambda_n - T)^{-1}\|$$

then we can define

$$(\mu - T)^{-1} = (\lambda_n - T)^{-1}(I - (\mu - \lambda_n)(\lambda_n - T)^{-1})^{-1} = \sum_{k=0}^{\infty} (\mu - \lambda_n)^k ((\lambda_n - T)^{-1})^{k+1}$$

and so $\mu \in \rho(T)$. Now by the uniform boundedness principle there exists $x \in X$ such that $\|(\lambda_n - T)^{-1} x\| \to \infty$. Let

$$y_n = \frac{(\lambda_n - T)^{-1} x}{\|(\lambda_n - T)^{-1}\|}.$$

Then $\|y_n\| = 1$ and $(\lambda - T)y_n = (\lambda - \lambda_n)y_n + (\lambda_n - T)y_n \to 0$ as $n \to \infty$. □

2.3 Sobolev Spaces

In this section we define the Sobolev spaces that are used in the thesis. There are no deviations from the standard definitions. Denote by $L^2 = L^2(\mathbb{R})$ the space of square integrable functions on $\mathbb{R}$ with respect to the Lebesgue measure and equip it with the standard inner product $(\cdot, \cdot)$.

Definition 2.4. By $H^n = H^n(\mathbb{R})$ for $n \in \mathbb{N}$ we mean the set of functions in L^2 that have n weak derivatives which are also in L^2. We define the inner product in H^n by

$$(f, g)_{H^n} = \sum_{k \leq n} (f^{(k)}, g^{(k)}).$$

We define the fractional Sobolev spaces H^s as usual using the Fourier characterization: H^s is the set of functions $u \in L^2$ such that $(1 + (\cdot)^{s/2})^2 \hat{u} \in L^2$ with the norm

$$\|u\|_{H^s} = \|(1 + (\cdot)^{s/2})^2 \hat{u}\|_{L^2}$$

It is well known that these norms are equivalent to the norm above for non-negative integers, see [4] Chapter 5.8 Theorem 8. For a Banach space X and $T \in (0, \infty]$ we define the Banach space $C([0, T), X)$ as all continuous functions $u : [0, T) \to X$ with norm

$$\|u\|_{C([0,T),X)} = \sup_{t \in [0,T)} \|u(t)\|$$

and the space $C^1([0, T), X)$ as all continuously differentiable functions, i.e. all functions u such that

$$u'(t) := \lim_{h \to 0} \frac{u(t + h) - u(t)}{h}$$

exists in the strong sense and $u' \in C([0, T), X)$.

The following theorem is very useful when dealing with nonlinear differential equations since it makes sure that multiplication of functions behaves well. It is stated in generality, but will only be applied with $s = 1$ and $n = 1$.

Theorem 2.2. *Consider $H^s = H^s(\mathbb{R}^n)$. For $s > n/2$ there exists a $C > 0$ such that for any $u \in H^s$ we have $u \in L^\infty$ with*

$$\|u\|_{L^\infty} \leq C \|u\|_{H^s}$$

For $u, v \in H^s$

$$\|uv\|_{H^s} \leq C \|u\|_{H^s} \|v\|_{H^s}.$$

For a proof see [9], Theorem 3.4.

3 Semigroup Theory

3.1 Strongly Continuous Semigroups

Consider the following initial value problem

$$\begin{cases} u_t = Au \\ u(0) = u_0. \end{cases} \tag{1}$$

Here $u : [0, \infty) \to X$ and $A : D(A) \subset X \to X$ is a possibly unbounded linear operator on the Banach space X such that $D(A)$ is a dense linear subspace of X. If we have a solution $u(t)$ of (1) then we can write $T(t)u_0 = u(t)$. This idea naturally leads to the definition of a semigroup:

Definition 3.1. Let X be a Banach space and $T(t) : X \to X$, $t \geq 0$ be a family of bounded linear operators satisfying:

$$T(0) = I \tag{2}$$

$$T(s+t) = T(s)T(t) \tag{3}$$

$$T(t) \xrightarrow{s} T(0) \quad \text{as} \quad t \to 0. \tag{4}$$

Then we say that $T(t)$ is a strongly continuous semigroup of operators. The generator $A : D(A) \subset X \to X$ of a strongly continuous semigroup is defined as

$$Ax = \lim_{h \searrow 0} \frac{T(h)x - x}{h}$$

where $D(A)$ is the set such that the limit exists.

Since semigroups are supposed to describe solutions to autonomous differential equations it is should be clear why properties (2) and (3) are part of the definition. The reason for choosing strong continuity is not obvious at first glance, but it turns out to be the perfect setting for a general theory: It is surprisingly equivalent to weak continuity. Furthermore if one requires uniform continuity instead, A will be bounded and the semigroup can be expressed as the exponential e^{tA} in the sense of an absolutely convergent series. There are however many interesting cases between strong continuity and uniform continuity, for example analytic semigroups which are described below. Now we collect some important properties of the generator of a strongly continuous semigroup:

Theorem 3.1. *Let $T(t)$ be a strongly continuous semigroup with generator A. Then*

(i) A is a linear operator.

(ii) If $x \in D(A)$ then $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \quad \forall t \geq 0.$$

(iii) For $x \in X$

$$T(t)x - x = A\int_0^t T(s)xds$$

and for $x \in D(A)$

$$T(t)x - x = \int_0^t T(s)Axds.$$

(iv) A is closed and $D(A)$ is dense in X. Furthermore, $T(t)$ is the unique strongly continuous semigroup with generator A.

(v) There exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)x\| \leq Me^{t\omega}, \quad t \geq 0.$$

Remark 3.1. *The integrals in Theorem* 3.1 *can be taken in the Riemann sense since the function is continuous. Throughout the thesis, any integral should be regarded in the same way.*

A proof of Theorem 3.1 can be found in any text on strongly continuous semigroups, for example [3] Chapter II Lemma 1.3, Theorem 1.4 and Theorem 1.10. The second property tells us that $u(t) = T(t)u_0$ solves (1) for the generator A, so long as the initial value is in $D(A)$. This solution will be continuously differentiable in the sense that $u \in C^1([0, \infty), X)$ since

$$\frac{du}{dt}(t) = T(t)Au_0 \to T(s)Au_0 = \frac{du}{dt}(s).$$

as $t \to s$. If $u_0 \notin D(A)$ we don't have a classical solution, but by the third property we have a "mild solution" $u(t) = T(t)u_0$ satisfying

$$u(t) = A\int_0^t u(s)ds + u_0.$$

Usually mild solutions are easier to work with and can be generalized to the nonlinear case without assuming too much regularity. Conversely one can ask, if given an initial value problem of the form (1) which has a unique strong solution $u \in C^1([0, \infty), X)$ for each $u_0 \in D(A)$, does A generate a semigroup on X? The short answer is no, but if one instead considers the space $X_1 = (D(A), \|\cdot\|_A)$ where $\|x\|_A = \|Ax\| + \|x\|$ then the restriction of A to this space will generate a strongly continuous semigroup on X_1. In fact, the initial value problem is uniquely solvable for each $u_0 \in D(A)$ if and only if A generates a strongly continuous semigroup on this space.

Given an initial value problem, we can ask whether A generates a strongly continuous semigroup on X. By the discussion above this will immediately give existence and uniqueness for each $u_0 \in D(A)$. This is answered by the Hille-Yosida theorem below. A proof can be found in [3] Chapter II Theorem 3.5.

Theorem 3.2. *Let $A : D(A) \subset X \to X$ be a linear operator, $M \geq 1$ and $\omega \in \mathbb{R}$. Then A is the generator of a strongly continuous semigroup satisfying*

$$\|T(t)\| \leq Me^{t\omega} \tag{5}$$

if and only if

1. *$D(A)$ is dense in X and A is closed,*
2. *For each $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ we have $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\Re\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

Definition 3.2. If we can choose $M = 1$ and $\omega \leq 0$ in (5) we say that $T(t)$ is a contraction semigroup. If we only have $M = 1$ we say that the semigroup is a quasicontraction.

Remark 3.2. *In the case of a contraction semigroup condition 2 can be replaced with $\|(\lambda - A)x\| \geq |\lambda|\|x\|$ and the condition that $\lambda - A$ is surjective for $\lambda > 0$. This is a useful reformulation since it does not require any knowledge of the resolvent. An operator that satisfies $\|(\lambda - A)x\| \geq |\lambda|\|x\|$ for all $x \in D(A)$ is called dissipative. If X is a real Hilbert space then it is equivalent to $(Ax, x) \leq 0$.*

Since any semigroup satisfies an estimate of the form (5) this classifies all generators of strongly continuous semigroups on X. However, it should be noted that a linear operator may generate a strongly continuous semigroup on one Banach space, but not on another. Therefore one has to be careful when choosing X. The general form Theorem 3.2 of the Hille-Yosida theorems is interesting for the reason that it tells exactly when an operator generates a strongly continuous semigroup, but it is impractical since each power of the resolvent has to be estimated. The following theorem for contraction semigroups is of more practical use:

Theorem 3.3. *Let $A : D(A) \subset X \to X$ be a linear operator. Then A is the generator of a strongly continuous semigroup satisfying*

$$\|T(t)\| \leq 1 \tag{6}$$

i.e. a contraction semigroup, if and only if

1. *$D(A)$ is dense in X and A is closed.*
2. *For each $\lambda > 0$ we have $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Although this can be seen as a special case of Theorem 3.2 the proof of Theorem 3.2 requires Theorem 3.3 so a proof this way would not be very interesting. The proof can instead be found in [3].

3.2 Analytic Semigroups

In this section we consider semigroups which are analytic in the time variable. This is one of the strongest requirements of semigroups that can not be written as a power series, that is, which is not uniformly continuous. The sector is defined as

$$\Sigma_{\delta,a} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \delta, \quad \lambda \neq a\}$$

$$\Sigma_\delta := \Sigma_{\delta,0}$$

Definition 3.3. Let X be a Banach space, $0 < \delta \leq \pi/2$ and $T(z)$, $z \in \Sigma_{\delta,a}$ be a family of bounded linear operators satisfying:

$$\begin{aligned}
&T(0) = I \\
&T(z_1 + z_2) = T(z_1)T(z_2), \quad z_1, z_2 \in \Sigma_\delta \\
&T(z) \xrightarrow{s} T(0) \quad \text{as} \quad z \to 0 \\
&z \mapsto T(z) \text{ is analytic in } \Sigma_\delta.
\end{aligned}$$

Then we say that $T(t)$ is an analytic semigroup of operators of angle δ.

There are many equivalent definitions, for example, we may define an analytic semigroup as a strongly continuous semigroup such that $t \mapsto T(t)$ is real analytic. An analytic semigroup is of course also a strongly continuous strongly semigroup if it is restricted to $\mathbb{R}_+$ and the generator is defined in the same way. However for analytic semigroups one can show that the generator will satisfy

$$T(z) = \frac{1}{2\pi i} \int_\gamma e^{\mu z} R(\mu, A) d\mu$$

for any curve γ that goes from $\infty e^{-i(\pi/2+\delta')}$ to $\infty e^{i(\pi/2+\delta')}$ where $\delta' \in (|\arg z|, \delta)$. We now attempt to characterize these semigroups as we did for strongly continuous semigroups.

Definition 3.4. Let X be a Banach space and $A : D(A) \subset X \to X$ a closed densely defined linear operator. Suppose there exists $\delta > 0$, $a \in \mathbb{R}$ such that $\rho(A)$ contains the sector $\Sigma_{\pi/2+\delta,a}$ and for each $0 < \epsilon < \delta$ there exists M_ϵ

$$\|R(\lambda, A)\| \leq \frac{M_\epsilon}{|\lambda - a|}, \quad 0 \neq \lambda \in \Sigma_{\pi/2+\delta-\epsilon,a}.$$

Then we say that A is sectorial of angle δ.

It is clear that any sectorial operator generates a strongly continuous semigroup. However, it also holds that the semigroup is analytic, see [3] Chapter II Theorem 4.6 for a proof.

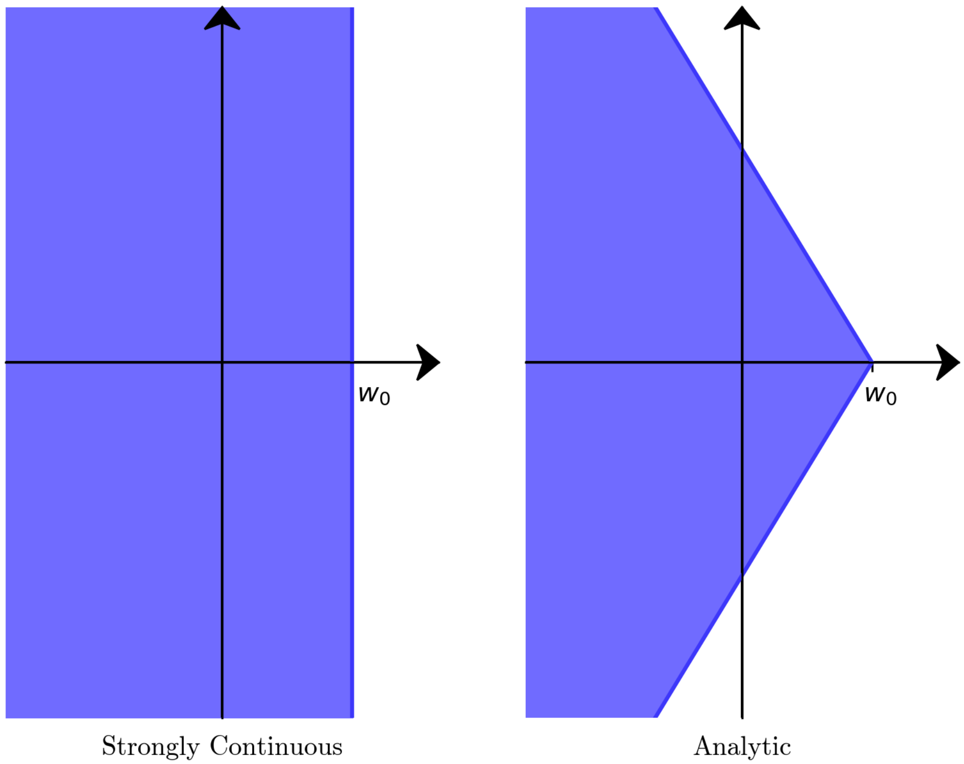

Figure 1: The difference between the spectrum of the generator of a strongly continuous semigroup and an analytic semigroup.

Theorem 3.4. *Let $A : D(A) \subset X \to X$ be a sectorial operator of angle δ generating a strongly continuous semigroup $T(t)$. Then $T(t)$ is real analytic in t and can be extended to an analytic semigroup $T(z)$ of angle δ.*

Figure 1 illustrates the difference between the spectrum of a generator of a strongly continuous semigroup and an analytic semigroup. Analytic semigroups are much nicer than strongly continuous semigroup, for example they always satisfy the spectral mapping theorem, see Theorem 3.9. It is however a strong condition and many important differential operators do not generate analytic semigroups. An example of such an equation is the Lugiato-Lefever equation, which is studied below.

For a sectorial operator we can define fractional operators. It turns out that the semigroup will behave well on the domain of the these fractional spaces with the graph norm, and one can formulate many theorems, for example on existence, uniqueness and stability, using these spaces. This is partly due to the fact that for nonlinearities to not affect the overall behaviour of a differential equation, they must behave well with respect to the linear part. More specifically they need to satisfy some bound in this norm, see for example the conditions on F and G in Theorem 4.8.

Definition 3.5. Let $A : D(A) \subset X \to X$ be a sectorial operator of angle δ with $\Re\sigma(A) < 0$. For any $\alpha > 0$ define the fractional operator

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt$$

and $(-A)^\alpha = ((-A)^{-\alpha})^{-1}$. If A is a sectorial operator, not necessarily with $\Re\sigma(A) < 0$, we can find a such that $A_1 = A + a$ satisfies $\Re\sigma(A_1) < 0$. For $\alpha \geq 0$ we define the fractional space $X^\alpha = D(A_1^\alpha)$ with norm $\|x\|_\alpha = \|(-A_1)^\alpha x\|$ where a is chosen so that the fractional operator is defined. The norms are equivalent for any such a.

For a more in depth theory of analytic semigroups and their applications, see [6].

3.3 Perturbation of Semigroups

It is often difficult to determine directly if an operator generates a semigroup, since the spectrum is generally hard to determine. For these cases, it is sometimes easier to divide the the operator in two parts: A main part whose spectrum can be determined and a perturbation whose spectrum can't be determined, but is expected to not contribute too much to the entire operator. This idea is illustrated with the Kuramoto-Sivashinsky equation in section 5.2 where the main part has constant coefficients and the perturbation has lower order derivatives and coefficients that behave well at infinity.

Definition 3.6. Let X be a Banach space and $A : D(A) \subset X \to X$, $B : D(B) \subset X \to X$ linear operators. We say that B is A-bounded if $D(A) \subset D(B)$ and there exist non negative constants a, b such that

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in D(A). \tag{7}$$

The A-bound a_0 of B is defined as the infimum of all a for which there exists a b such that (7) holds.

The following theorem applies when the main operator generates a contraction semigroup and the perturbation is both dissipative and A-bounded with a small enough A-bound.

Theorem 3.5. *Let X be a Banach space. Suppose that $A : D(A) \subset X \to X$ generates a contraction semigroup on X and $B : D(B) \subset X \to X$ is dissipative and A-bounded with A-bound $a_0 < 1$. Then $(A + B, D(A))$ generates a strongly continuous semigroup on X. Furthermore we have the relation*

$$e^{t(A+B)}x = e^{tA}x - \int_0^t e^{(t-s)A}Be^{s(A+B)}xds.$$

Even though the theorem is stated with fairly strong conditions on A and B it can be used in more general situations as demonstrated in Lemma 5.3. The Theorem is proved in [3] Chapter III Theorem 2.7.

We mention another perturbation result for analytic semigroups which will be used in section 5.3. Proofs of Theorem 3.5 and Theorem 3.6 can be found in [3].

Theorem 3.6. *Let X be a Banach space. Suppose that $A : D(A) \subset X \to X$ generates an analytic semigroup on X. Then there exists $\alpha > 0$ such that if $B : D(B) \subset X \to X$ is A-bounded with A-bound $a_0 < \alpha$ then $(A+B, D(A))$ generates an analytic semigroup on X. In particular, if B has A-bound $a_0 = 0$ then $(A+B, D(A))$ generates an analytic semigroup on X.*

3.4 The Growth Bound

In the previous sections, we gathered enough evidence to write $T(t) = e^{tA}$ for a semigroup with generator A. One should be careful with this notation however: We are not claiming the existence of an exponential function on the set of unbounded operators. In other words the exponential function depends on the generator and should only be used together in the form e^{tA} for $t \in \mathbb{R}_+$. This notation is useful as it highlights the exponential behaviour of the semigroup. In this section we examine the growth bound of a strongly continuous semigroup, which is an example of this exponential behaviour. The growth bound of a semigroup describes how the semigroup grows when $t \to \infty$. It is connected to the spectral bound of the generator and the spectral radius of the semigroup, as proved in Theorem 3.7.

Definition 3.7. For a strongly continuous semigroup e^{tA} with generator A we define the growth bound

$$\begin{aligned} w_0 = w_0(e^{tA}) = w_0(A) &= \inf\{w \in \mathbb{R} : \exists M_w \geq 1 \text{ such that } \|e^{tA}\| \leq M_w e^{tw}\} \\ &= \inf\{w \in \mathbb{R} : e^{-tw}\|e^{tA}\| \to 0 \text{ as } t \to \infty\}, \end{aligned}$$

the spectral bound

$$s(A) = \sup\{\Re\lambda : \lambda \in \sigma(A)\},$$

with $s(A) = -\infty$ if $\sigma(A) = \emptyset$, and the spectral radius

$$r(e^{tA}) = \sup\{|\lambda| : \lambda \in \sigma(e^{tA})\}.$$

The following important theorem relates the three values values defined above.

Theorem 3.7. *For a strongly continuous semigroup e^{tA} with generator A we have*

$$-\infty \leq s(A) \leq w_0 = \inf_{t>0} \frac{1}{t} \log \|e^{tA}\| = \lim_{t\to\infty} \frac{1}{t} \log \|e^{tA}\| = \frac{1}{t_0} \log r(e^{t_0 A}) < \infty$$

for all $t_0 > 0$. In particular

$$e^{t_0 w_0} = r(e^{t_0 A})$$

for all $t_0 > 0$.

To prove Theorem 3.7 we need this simple lemma regarding subadditive functions. For a proof of the lemma see [3], Chapter IV, Lemma 2.3.

Lemma 3.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a subadditive function that is bounded on compact intervals. Then*

$$\inf_{t>0} \frac{f(t)}{t} = \lim_{t\to\infty} \frac{f(t)}{t}.$$

Proof of theorem. If $\Re\lambda > w > w_0$ then $\lambda \in \rho(A)$ so $s(A) \leq w_0$. Apply Lemma 3.1 to the function $f(t) = \log \|e^{tA}\|$ and let

$$v = \inf_{t>0} \frac{1}{t} \log \|e^{tA}\| = \lim_{t\to\infty} \frac{1}{t} \log \|e^{tA}\|.$$

Then

$$e^{vt} \leq \|e^{tA}\|$$

so $v \leq w_0$. If $w > v$ then there exists t_0 such that if $t \geq t_0$ then

$$w > \frac{1}{t} \log \|e^{tA}\|$$

and

$$e^{wt} > \|e^{tA}\|.$$

Since $\|e^{tA}\|$ is bounded on the compact interval $[0, t_0]$ there exists an M such that

$$\|e^{tA}\| \leq M e^{wt}.$$

Hence $w_0 \leq w$ and since $w > v$ was arbitrary $v \geq w_0$. The final identity follows from the computation

$$r(e^{t_0 A}) = \lim_{m\to\infty} \|e^{mt_0 A}\|^{1/m} = \lim_{m\to\infty} e^{\frac{1}{m} \log \|e^{mt_0 A}\|} = e^{t \lim_{m\to\infty} \frac{1}{tm} \log \|e^{mt_0 A}\|} = e^{t_0 w_0}$$

□

3.5 Spectral Theorem for Semigroups

In order to relate the spectrum of A to the growth of the semigroup, it is very helpful if the spectral mapping theorem holds:

$$\sigma(e^{tA}) \setminus \{0\} = e^{t\sigma(A)}, \quad \forall t \geq 0. \tag{8}$$

If this equality holds then by Theorem 3.7 $s(A) = w_0(A)$. However, in general this is not true for strongly continuous semigroups. We always have one inclusion, but the other may fail. While this inclusion is not so useful to prove nonlinear stability it is often enough for instability.

Theorem 3.8 (Spectral Inclusion Theorem)**.** *Let e^{tA} be a strongly continuous semigroup with generator A. Then we have the inclusion*

$$\sigma(e^{tA}) \supset e^{t\sigma(A)}, \quad \forall t \geq 0. \tag{9}$$

Moreover (9) *holds for $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_a(A)$.*

Proof. If we apply Theorem 3.1 (ii) to the semigroup $S(t) = e^{-\lambda t}e^{tA}$ with generator $A - \lambda$ we get the relation

$$e^{-\lambda t}e^{tA}x - x = (A - \lambda)\int_0^t e^{-\lambda s}e^{sA}x ds = \int_0^t e^{-\lambda s}e^{sA}(A - \lambda)x ds$$

for $x \in D(A)$. Multiplying both sides by $e^{\lambda t}$ shows that if $A - \lambda$ is not bijective then neither is $e^{tA} - e^{t\lambda}$. Indeed, if $A - \lambda$ is not injective, then $(A - \lambda)x = 0$ for some $x \in D(A)$ and hence $(e^{tA} - e^{\lambda t})x = 0$ by the second equality. If instead $A - \lambda$ is not surjective then the range of the operator

$$(A - \lambda)\int_0^t e^{\lambda(t-s)}e^{sA}x ds$$

is not X and therefore the range of $e^{tA} - e^{t\lambda}$ can't be X. This also shows that $\sigma_p(e^{tA}) \supset e^{t\sigma_p(A)}$. To see that this also holds for the approximate spectrum, let x_n be a sequence of approximate eigenvectors of A. Then

$$\|e^{tA}x_n - e^{\lambda t}x_n\| = \left\| \int_0^t e^{\lambda(t-s)}e^{sA}(A - \lambda)x_n ds \right\| \leq C\|(A - \lambda)x_n\| \to 0.$$

If $\lambda \in \sigma_r(A)$ then the range of $\lambda - A$ is not dense in X and we have $\text{rg}(e^{tA} - e^{\lambda t}) \subset \text{rg}(\lambda - A)$, therefore the range of $e^{tA} - e^{\lambda t}$ can't be dense in X. □

An alternative proof of the spectral inclusion proof using Gelfand's theory of commutative Banach algebras can be found in [8]. The spectral mapping theorem holds for both the point spectrum and the residual spectrum and hence only fails on the approximate point spectrum. This is the spectral mapping theorem for strongly continuous semigroups. The idea of the proof is to restrict the semigroup to the space of eigenvectors. Then (10) follows from the theory of periodic semigroup. After this (11) follows from considering the dual semigroup restricted to the set where it is strongly continuous. The proofs of these details are lengthy and complicated in some parts, but for the interested reader a proof can be found in [3]. It should be noted that for an analytic semigroup it follows from the spectral mapping theorem in functional analysis.

Theorem 3.9 (Spectral Mapping Theorem). *Let e^{tA} be a strongly continuous semigroup with generator A. Then we have that the spectral mapping theorem holds for the point spectrum and residual spectrum:*

$$\sigma_p(e^{tA}) \setminus \{0\} = e^{t\sigma_p(A)}, \quad \forall t \geq 0 \tag{10}$$

$$\sigma_r(e^{tA}) \setminus \{0\} = e^{t\sigma_r(A)}, \quad \forall t \geq 0. \tag{11}$$

4 Stability and Instability

4.1 Statement of the Problem

Consider the following initial value problem

$$\begin{cases} u_t = A(u) = Lu + F(u) \\ u(0) = u_0 \end{cases} \tag{12}$$

where we have divided the operator into two parts: The linear part L and the nonlinear part F such that $F(0) = 0$. It makes sense to assume that $F'(0) = 0$ since we put all the linear parts in L, but we do not assume that F is differentiable so this will instead be expressed in the form $\|F(u)\| \leq \|u\|^\alpha$ for $\alpha > 1$. The other problem is to decide linear spaces on which we define L and F respectively. There are different approaches to this, but one of the main issues is that there may be some derivatives in F, so that $F : X \to Z$ where $X \subset Z$. It is useful to think of the case $X = H^1$, $Z = L^2$ and $F(u) = uu_x$. For now, we remain in the abstract case and define the terms that will be used in the following sections. First we define what is meant by a solution to (12) and what is meant by stability of a solution. While one can define classical solutions as well, we focus on mild solutions since they fit the theory of semigroups well.

Definition 4.1. Consider the nonlinear initial value problem (12) with two Banach spaces $X \subset Z$ where L generates a strongly continuous semigroup e^{tL} on Z, an initial value $u_0 \in Z$ and $F : X \to Z$ is continuous with $F(0) = 0$. We say that $u : [0, T) \to X$ is a solution of (12) if $u \in C([0, T), X)$ and

$$u(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L}F(u(s))ds, \quad 0 \leq t \leq T.$$

We say that u is a maximal solution of (12) if $u : [0, T) \to X$ is a solution and

$$T = \sup_{t\geq 0}\{t : \text{there exists a solution in } C([0, t), X)\}.$$

We define stability only for the zero solution. Stability for stationary solutions, that is solutions that do not depend on t, and travelling solutions, that is solutions of the form $u(x, t) = \phi(x - ct)$, can be defined as the stability of the zero solution after a change of variable. We leave this for the examples since it makes the abstract theory messier.

Definition 4.2. We say that the zero solution of (12) is nonlinearly stable in X if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\|u_0\|_X < \delta$ there exists a unique solution $u \in C([0, \infty), X)$ of (12) such that $\sup_t \|u(t)\|_X < \epsilon$.

Stability in this sense mean that if the solution is sufficiently close to the stationary solution then it will remain close to it at all times. With this definition if no solutions exist or a solution exists only for finite time then this is also a case of instability, therefore it is useful to combine an instability theorem with an existence theorem, at least for finite time. Such a theorem exists for semilinear equations, see Theorem 4.1. If a solution exists only for finite time then it still makes sense to say that the zero solution is unstable, due to the last statement in Theorem 4.1. Note that there are three choices of space above: The space for which the initial condition is small, the space for which the solution is small and the space in which u, u_0 exists.

The linearization of (12) is the problem

$$\begin{cases} u_t = Lu \\ u(0) = u_0 \end{cases} \tag{13}$$

and the linear stability of (12) is defined by the stability of (13):

Definition 4.3. We say that the zero solution of (12) is linearly stable in X if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\|u_0\|_X < \delta$ there exists a unique solution $u \in C([0, \infty), X)$ of (13) such that $\sup_t \|u(t)\|_X < \epsilon$.

Stability of a linear problem is closely related to the spectrum of the generator. This is well known for ordinary differential equations and one can show a similar theorem in the infinite dimensional case using semigroup theory. There are however some complications due to the spectral mapping theorem not necessarily being true and the fact that unbounded linear operators can have spectrum arbitrarily close to the imaginary line without intersecting it, this is explained further in the following sections. The purpose of this chapter is to examine when instability of the linear problem implies instability of the nonlinear problem, or equivalently when a large spectral bound of L implies nonlinear instability.

4.2 Well-Posedness of Semilinear Evolution Equations

In this section we prove well-posedness for semilinear equations when the semigroup compensates for the derivatives in the nonlinear part. We only discuss existence and uniqueness as this is the most interesting for the applications.

Theorem 4.1. *Let X, Z be Banach spaces such that $X \subset Z$ and there exists $C_1 > 0$ such that $\|x\|_Z \leq C_1 \|x\|_X$ for all $x \in X$. Suppose that L generates a strongly continuous semigroup e^{tL} on X and Z, that e^{tL} maps Z into X for $t > 0$,*

$$\int_0^1 \|e^{tL}\|_{Z \to X} dt = C_4 < \infty.$$

Moreover suppose that $F : X \to Z$ with $F(0) = 0$ is continuous and that for all $\rho > 0$ there exists $C_3 > 0$ such that $\|F(u) - F(v)\|_Z \leq C_3(\rho)\|u - v\|_X$ for $\|u\|_X < \rho$

and $\|v\|_X < \rho$. *Then for* $u_0 \in X$ *there exists* $T > 0$ *and a unique maximal solution of* (12) *in* $C([0,T),X)$. *If* $T < \infty$ *then*

$$\lim_{t \to T} \|u(t)\|_X = \infty.$$

Remark 4.1. *If we have two operators* $(A, D(A))$ *and* $(B, D(B))$ *which agree on* $D(B) \subset D(A)$ *and generate two strongly continuous semigroups* e^{tA} *and* e^{tB} *on* Z *and* X *respectively, then the semigroups must agree on* X. *This can be proven by examining the proof of the Hille-Yosida theorem. There the semigroups are constructed by limits of* $R(n, A)$. *Since the resolvents of* A *and* B *agree on* X *and the limits in* Z *and* X *are the same when they exist, the semigroups also agree on* X.

Proof. For any $\rho > 0$ let $\|u_0\|_X < \rho$ and set

$$M = \sup_{t \in [0,1]} \|e^{tL}\|_{X \to X}$$

and $r = 1 + M\rho > 1$. Choose $b_0(\rho) \leq 1$ such that

$$C(b_0) := \int_0^{b_0} \|e^{sL}\|_{Z \to X} ds \leq \frac{1}{C_3(r) r}$$

and let $b \leq b_0$. Define the closed subset $E(b, r) = \{u \in C([0, b], X) : \|u\|_{C([0,b],X)} \leq r\}$ and the map $\Phi : C([0, b], X) \to C([0, b], X)$

$$\Phi(u)(t) = e^{tL} u_0 + \int_0^t e^{(t-s)L} F(u(s)) ds.$$

Now we wish to show that Φ maps $E(b, r)$ into itself and is a contraction on $E(b, r)$. Let $u \in E(b, r)$, it is clear that $\Phi(u) \in C([0, b], X)$. That $\Phi(u) \in E(b, r)$ follows from

$$\begin{aligned}
&\|\Phi(u)(t)\|_X \leq \|e^{tL} u_0\|_X + \int_0^t \|e^{(t-s)L}\|_{Z \to X} \|F(u(s))\|_Z ds \\
&\leq M\rho + C_3(r) \|u\|_{C([0,b],X)} \int_0^t \|e^{(t-s)L}\|_{Z \to X} ds \\
&\leq M\rho + C_3(r) C(b_0) r \leq r
\end{aligned}$$

and Φ is a contraction since

$$\begin{aligned}
&\|\Phi(u)(t) - \Phi(v)(t)\|_X \leq \int_0^t \|e^{(t-s)L}\|_{Z \to X} \|F(u(s)) - F(v(s))\|_Z ds \\
&\leq C_3(r) \|u - v\|_{C([0,b],X)} \int_0^t \|e^{(t-s)L}\|_{Z \to X} ds \\
&\leq C(b) C_3(r) \|u - v\|_{C([0,b],X)} < \|u - v\|_{C([0,b],X)}.
\end{aligned}$$

Thus the Banach fixed point theorem implies that for each $u_0 \in$ there exists a unique solution $u \in C([0,b],X)$ such that $\|u\|_{C([0,b],X)} \leq 1 + M\rho$.

For any initial value u_0 and a solution u on a closed interval $[0,b]$ there exists an extension of u to a maximal solution. This is because there exists a solution to the initial value problem $u_0 = u(b)$ and these solutions can be glued together. If we have two solutions u, v on intervals J_u, J_v respectively then u, v must agree on $J_u \cap J_v$. To see this, note that from the argument above the solutions must agree on some small interval so if they do not agree for all time then there exists a time τ such that $u(t) = v(t)$ on $[0,\tau]$ and a sequence $t_n \geq \tau$, $t_n \to \tau$ such that $u(t_n) \neq v(t_n)$. However by the argument above, both u and v can be extended, and for a short time the extensions must agree, which contradicts the existence of t_n. Thus we have shown uniqueness.

Existence of a maximal solution is now obvious as we can define it to be

$$u(t) = \lim_{n\to\infty} u_n(t)$$

where u_n is the solution on $[0,b_n]$ and $b_n \to T$.

If there exists a sequence $b_n \to T$ such that $\sup_n \|u(b_n)\|_X = C < \infty$ then we can choose n such that $b_n + b_0(C) > T$ and by extending the solution on $[0,b_n]$ to $[0, b_n + b_0(C)]$ we get a contradiction.

□

4.3 Linear Stability and Instability

The following theorem explains the relation between the spectrum of L and linear stability. An important detail is that instability does not rely on the spectral mapping theorem being true. The other detail is that it is not sufficient that $\sigma(L) \subset \{\Re\lambda < 0\}$ since the spectrum of L is not necessarily bounded so it can get arbitrarily close to the line $\{\Re\lambda = 0\}$ without intersecting it.

Theorem 4.2. *Consider the initial value problem* (1) *and suppose that A generates a strongly continuous semigroup e^{tA}. If $w_0(A) < 0$ then the zero solution is linearly stable and if $w_0(A) > 0$ then the zero solution is linearly unstable. In particular, if $s(A) > 0$ then the zero solution is unstable and if $s(A) < 0$ and e^{tA} satisfies the spectral mapping theorem* (8) *then the zero solution is stable.*

Proof. If $w_0 < 0$ then there exists $w < 0$ and $M_w \geq 1$ such that $\|T(t)\| \leq e^{tw} M_w$. This immediately implies that $\|T(t)x\| \leq e^{tw} M_w \|x\|$ for all $x \in X$. So if $\delta < \epsilon / M_w$ then $\|u_0\| < \delta \implies \sup_t \|u(t)\| = \sup_t \|T(t)u_0\| \leq \epsilon$.

On the other hand, if $w_0 > 0$, then there exists $w > 0$ such that $\sup_t \|e^{-tw}T(t)\| = \infty$. Then by the uniform boundedness principle, there exists an $x \in X$ with $\|x\| = 1$ such that $\sup_t \|e^{-tw}T(t)x\| = \infty$. For any $\delta > 0$ we let $u_0 = x\delta/2$ so that $\|u_0\| < \delta$, but $\sup_t \|u(t)\| = \infty > \epsilon$.

The last statements follows from the fact that $s(A) \leq w_0$ and $s(A) = w_0$ if (8) holds. □

In fact, one can define exponential stability as the existence of a negative exponential bound. In this context the previous theorem tells us that exponential stability implies stability. Since the stability theorem only depends on the spectral bound and not on the spectrum itself, the condition can be weakened to a weak spectral mapping theorem:

$$\sigma(T(t)) \setminus \{0\} = \overline{e^{t\sigma(A)}} \setminus \{0\}, \quad \forall t \geq 0.$$

4.4 Instability in Finite Dimensions

Now we attempt to answer the question of whether linear stability implies nonlinear stability. We first examine the finite dimensional case because it provides some insight into the techniques used in the infinite dimensional case. It is easier in the finite dimensions because the spectrum is discrete, consisting of a finite number of eigenvalues. The proof of stability is left out as the focus is on instability.

Theorem 4.3. *Let $X = \mathbb{R}^n$. Suppose that all eigenvalues λ_k of L satisfy $\Re\lambda_k < 0$. Suppose furthermore that there exists $\rho > 0$, $D > 0$ such that $\|F(u)\| \leq D\|u\|^2$ for $\|u\| < \rho$. Then L is nonlinearly stable at 0.*

Theorem 4.4. *Let $X = \mathbb{R}^n$. Suppose there exists an eigenvalue λ of L with $\Re\lambda > 0$. Suppose furthermore that there exists $\rho > 0$, $D > 0$ such that $\|F(u)\| \leq D\|u\|^2$ for $\|u\| < \rho$. Then L is nonlinearly unstable at 0.*

Proof. We will use the fact that for any $a > 0$ there exists $C_a > 0$ such that

$$e^{t\Re\lambda} \leq \|e^{tL}_X\|_{X\to X} \leq C_a e^{t(\Re\lambda + a)}.$$

This is trivial in the finite dimensional case, but will be proven to hold more generally in Lemma 4.2. Assume that the zero solution is stable, so for all $\epsilon > 0$ there exists $\delta_0 > 0$ such that if $\|v\| = \delta < \delta_0$ then $\sup_t \|u(t)\| < \epsilon$. Let $\epsilon < \rho$ and v be an eigenvector corresponding to the eigenvalue λ with largest real part, $\Re\lambda > 0$ with $\|v\| = \delta$. Let

$$T = \sup_t \{t : \|u(s) - e^{sL}v\| < \frac{\delta}{2} e^{s\Re\lambda} \quad \forall s \leq t\}.$$

Since $u(0) = v$ we have $T > 0$. Moreover if $T = \infty$ then

$$\|u(t)\| \geq \|e^{tL}v\| - \|e^{tL}v - u(t)\| \geq \delta e^{t\Re\lambda} - \frac{\delta}{2} e^{t\Re\lambda} = \frac{\delta}{2} e^{t\Re\lambda}$$

for all $t \geq 0$ contradicting stability. Hence $0 < T < \infty$ and

$$\|u(T) - e^{TL}v\| \leq \int_0^T \|e^{(T-t)L}\|\|F(u(t))\|dt \leq DC_a \int_0^T e^{(T-t)(\Re\lambda+a)}\|u(t)\|^2 dt$$
$$\leq DC_a \int_0^T e^{(T-t)(\Re\lambda+a)}(\|e^{tL}v\| + \|u(t) - e^{tL}v\|)^2 dt$$
$$\leq DC_a \int_0^T e^{(T-t)(\Re\lambda+a)}(\frac{3\delta}{2}e^{t\Re\lambda})^2 dt \leq \frac{9DC_a\delta^2}{4}e^{T(\Re\lambda+a)}\int_0^T e^{t(\Re\lambda-a)}dt$$
$$\leq \frac{9DC_a\delta^2}{4(\Re\lambda-a)}e^{T(\Re\lambda+a)}(e^{T(\Re\lambda-a)}-1) \leq \delta^2 Ce^{2T\Re\lambda}.$$

Hence

$$\frac{\delta}{2}e^{T\Re\lambda} = \|u(T) - e^{TL}v\| \leq \delta^2 Ce^{2T\Re\lambda}$$
$$\delta e^{T\Re\lambda} > \frac{1}{2C}.$$

Since C only depends on the constants D, $\Re\lambda$ and a which was arbitrary we may set $\epsilon < \frac{1}{4C}$ and then

$$\|u(T)\| \geq \|e^{TL}v\| - \|e^{TL}v - u(T)\| \geq \delta e^{T\Re\lambda} - \frac{\delta}{2}e^{T\Re\lambda} = \frac{\delta}{2}e^{T\Re\lambda} > \frac{1}{4C} > \epsilon.$$

□

Remark 4.2. *The proof uses two main properties of λ: The existence of an eigenvector satisfying $e^{tL}v = e^{t\Re\lambda}v$ and that λ is the maximal eigenvalue, so that $\|e^{tL}\| \leq C_a e^{t(\Re\lambda+a)}$ for any $a > 0$.*

4.5 Nonlinear Instability in Infinite Dimensions

We now attempt to generalize the previous section to the infinite dimensional case. The first difference from the linear case is the important choice of linear space and norm. In the following theorem, we consider two linear spaces, one large space Z and a smaller subspace X. The nonlinear part is only defined on the subset, while the linear part is smoothing, so that the semigroup maps Z into X for $t > 0$. The linear and nonlinear part compensate for each other to get instability. The theorem also works for $X = Z$ in which case it reduces to the theorem in [12]. This section closely follows the method in [14], although the assumptions are slightly different due to an omission in the original article about the spaces on which L generates a strongly continuous semigroup: In this theorem we assume that we have a strongly continuous semigroup on both X and Z. Note that by Remark 4.1 we can consider this as one semigroup. Throughout this section, we denote by $e_X^{tL} : X \to X$ the restriction of e^{tL} to X.

Theorem 4.5. *Let X, Z be Banach spaces such that $X \subset Z$ and there exists $C_1 > 0$ such that $\|x\|_Z \leq C_1\|x\|_X$ for all $x \in X$. Suppose that L generates a strongly continuous semigroup e^{tL} on X and Z, that e^{tL} maps Z into X for $t > 0$, that*

$$\int_0^1 \|e^{tL}\|_{Z\to X} dt = C_4 < \infty$$

and $r(e^L) > 1$ on X. Moreover suppose that $F : X \to Z$ is continuous and that there exists $\rho_0 > 0$, $C_3 > 0$, $\alpha > 1$ such that $\|F(u)\|_Z \leq C_3\|u\|_X^\alpha$ for $\|u\|_X < \rho_0$. Then the zero solution of (12) is nonlinearly unstable in X.

Remark 4.3. *By the spectral inclusion theorem Theorem 3.8 we may strengthen the assumption on the spectrum to $s(L) > 0$, which is in most cases an easier condition to prove.*

In infinite dimensions our λ is no longer necessarily an eigenvalue, so there might not be an eigenvector. However λ lies on the boundary of the spectrum, so there exists an approximate eigenvector. The purpose of the following lemma is to give, for each integer m, an approximate eigenvector of e^{mL}. This allows the idea to fix an integer time, T^*, at which we expect the linear solution to be large, yet close enough to the nonlinear solution that we can say that the nonlinear solution is large. Hence in the nonlinear case we have a different approximate eigenvector at every integer time. This is a crucial difference from the linear case where the same eigenvector was used for all times.

Lemma 4.1. *Let $\mu \in \partial\sigma(e^L_X)$ with $|\mu| = r(e^L_X)$ and write $\mu = e^\lambda$. For all $\gamma > 0$ and all positive integers m there exists $v \in X$ with $\|v\|_X = 1$ such that*

$$\|(e^{mL} - e^{m\lambda})v\|_X \leq \gamma$$
$$\|e^{tL}v\|_X \leq 2Ke^{t\Re\lambda}$$

for all $0 < t \leq m$. Here $K = \sup\{\|e^{sL}\|_{X\to X} : s \in [0, 1]\}$.

Proof. By Lemma 2.3 and Lemma 2.4 there exists a sequence $v_n \in X$ with $\|v_n\|_X = 1$ and $(e^\lambda - e^L)v_n \to 0$ in X. It follows that

$$(e^{mL} - e^{m\lambda})v_n = \sum_{j=0}^{m-1} e^{jL}e^{(m-1-j)\lambda}(e^L - e^\lambda)v_n \to 0 \quad \text{as } n \to \infty,$$

so we can choose n large enough that $v := v_n$ satisfies $\|(e^{mL} - e^{m\lambda})v\|_X \leq \gamma$ and $\|(e^{jL} - e^{j\lambda})v\|_X < 1$ for $0 \leq j \leq m$. For $t \leq m$, let j be the integer part of t, then

$$\|e^{tL}v\|_X \leq \|e^{jL}v\|_X\|e^{(t-j)L}\|_{X\to X} \leq K(1 + \|e^{j\lambda}v\|_X) \leq 2Ke^{t\Re\lambda}$$

since $e^{t\Re\lambda} \geq 1$. □

Lemma 4.2. *Let $\mu \in \partial\sigma(e^L_X)$ with $|\mu| = r(e^L_X)$ and write $\mu = e^\lambda$. $\forall a > 0$ there exists C_a such that*
$$e^{t\Re\lambda} \leq \|e^{tL}_X\|_{X\to X} \leq C_a e^{t(\Re\lambda + a)}$$
for all $t \geq 0$.

Proof. Since $r(e^L) = e^{\Re\lambda}$ we have the spectral bound $w_0 = \Re\lambda$ by Theorem 3.7. This implies that for every a there exists C_a such that
$$\|e^{tL}_X\|_{X\to X} \leq C_a e^{t(\Re\lambda+a)}.$$
Since
$$\|e^{mL}\|_X^{1/m} \to e^{\Re\lambda}$$
we can find S_a such that
$$e^{\Re\lambda - a} \leq \|e^{mL}\|_X^{1/m}$$
for integers $m \geq S_a$. For $t > S_a$ let m be the integer part of t and write
$$K\|e^{tL}\|_X \geq \|e^{(m+1)L}\|_X \geq e^{(m+1)(\Re\lambda - a)} \geq e^{t(\Re\lambda - a)}.$$
Now for any $t > 0$ let m be an integer such that $mt > S_a$. Then
$$\|e^{tL}\|_X \geq \|e^{mtL}\|_X^{1/m} \geq K^{-1/m} e^{t(\Re\lambda - a)} \to e^{t(\Re\lambda - a)}, \quad \text{as} \quad m \to \infty.$$
Now we can simply let $a \to 0$. □

Proof of theorem. Suppose that the the zero solution is stable. If we pick $\epsilon < \min(\frac{\rho_0}{2}, \frac{1}{4k})$ where k will be chosen later then there exists $\delta_0 > 0$ such that if $\|v\|_X = \delta < \delta_0$ then there exists a unique solution u such that $\sup_t \|u(t)\|_X < \epsilon$, $u(0) = v$. Let $\mu \in \partial\sigma(e^L_X)$ with $|\mu| = r(e^L_X)$. We first choose $T*$ such that
$$\frac{1}{k} < \delta e^{T^*\Re\lambda} \leq \frac{|\mu|}{k}.$$
By Lemma 4.1 we can then choose v with $\|v\|_X = \delta$ such that
$$\|(e^{T^*L} - e^{T^*\lambda})v\|_X \leq \frac{\|v\|_X}{4k},$$
$$\|e^{tL}v\|_X < 2Ke^{t\Re\lambda}\|v\|_X$$
for $t \leq T^*$. Let
$$T = \sup_t\{t : \|u(s) - e^{sL}v\|_X < \frac{\delta}{2|\mu|}e^{s\Re\lambda} \quad \forall s \leq t\}.$$
Clearly, $T > 0$. The idea of the following computation is this: At the point T the difference between the linear and nonlinear solution depends linearly on δ, but

we now show that this implies that it will depend linearly on δ^α. Thus $\delta e^{T\Re\lambda}$ is bounded below by some constant, as in the finite dimensional case. However, in the infinite dimensional case there is no eigenvector for the time T, instead we have to choose T^* so that $\delta e^{T^*\Re\lambda}$ is smaller than this constant. Then $T > T^*$ and hence T^* is a time for which there exists an approximate eigenvector and the linear solution is close to the nonlinear solution. For $t \le \min(T, T^*)$ and $t \ge 1$ we have

$$
\begin{aligned}
&\|u(t) - e^{tL}v\|_X \le \int_0^t \|e^{(t-s)L}\|_{Z\to X}\|F(u(s))\|_Z ds \le C_3 \int_0^t \|e^{(t-s)L}\|_{Z\to X}\|u(s)\|_X^\alpha ds \\
&\le C_3 \int_0^t \|e^{(t-s)L}\|_{Z\to X}(\|e^{sL}v\|_X + \|u(s) - e^{sL}v\|_X)^\alpha ds \\
&\le C_3 \int_0^t \|e^{(t-s)L}\|_{Z\to X}(2Ke^{s\Re\lambda}\delta + \frac{\delta}{2|\mu|}e^{s\Re\lambda})^\alpha ds \\
&\le C_3\delta^\alpha(2K + \frac{1}{2|\mu|})^\alpha \int_0^t \|e^{(t-s)L}\|_{Z\to X}e^{s\Re\lambda\alpha} ds \\
&\le C_3\delta^\alpha(2K + \frac{1}{2|\mu|})^\alpha \left(\int_0^{t-1} \|e^{(t-s)L}\|_{Z\to X}e^{s\Re\lambda\alpha} ds + \int_{t-1}^t \|e^{(t-s)L}\|_{Z\to X}e^{s\Re\lambda\alpha} ds \right) \\
&\le C_3\delta^\alpha(2K + \frac{1}{2|\mu|})^\alpha \left(\int_0^{t-1} \|e^{(t-s-1)L}\|_{X\to X}\|e^L\|_{Z\to X}e^{s\Re\lambda\alpha} ds + C_4 e^{t\Re\lambda\alpha} \right) \\
&\le C_3 C_a\delta^\alpha(2K + \frac{1}{2|\mu|})^\alpha \left(\int_0^{t-1} e^{(t-s-1)(\Re\lambda + a)} C_5 e^{s\Re\lambda\alpha} ds + C_4 e^{t\Re\lambda\alpha} \right) \\
&\le C_3 C_a\delta^\alpha(2K + \frac{1}{2|\mu|})^\alpha \left(\frac{C_5 e^{(t-1)\Re\lambda\alpha}}{\Re\lambda\alpha - \Re\lambda - a} + C_4 e^{t\Re\lambda\alpha} \right) \\
&\le C_3 C_a\delta^\alpha(2K + \frac{1}{2|\mu|})^\alpha \left(\frac{C_5}{\Re\lambda\alpha - \Re\lambda - a} + C_4 \right) e^{t\Re\lambda\alpha} \\
&\le C_3 C_a\delta^\alpha(2K + \frac{1}{2|\mu|})^\alpha \left(\frac{2C_5}{\Re\lambda\alpha - \Re\lambda} + C_4 \right) e^{t\Re\lambda\alpha}
\end{aligned}
$$

by setting $a = (\alpha - 1)\Re\lambda/2$. Note that we get a smaller bound if $0 \le t < 1$ so the above bound holds for all $t \le \min(T, T^*)$. We now choose k by setting

$$k^{\alpha-1} = 2|\mu|^\alpha C_3 C_a(2K + \frac{1}{2|\mu|})^\alpha \left(\frac{2C_5}{\Re\lambda\alpha - \Re\lambda} + C_4 \right).$$

If $T \le T^*$ then

$$\frac{\delta}{2|\mu|}e^{T\Re\lambda} = \|u(T) - e^{TL}v\|_X < \delta^\alpha \frac{k^{\alpha-1}}{2|\mu|^\alpha} e^{T\alpha\Re\lambda}.$$

Hence

$$(\delta e^{T\Re\lambda})^{\alpha-1} > \Big(\frac{|\mu|}{k}\Big)^{\alpha-1} \geq (\delta e^{T^*\Re\lambda})^{\alpha-1} \implies T > T^*$$

which is a contradiction. It follows that $T^* \leq T$ and also

$$\|u(T^*) - e^{T^*L}v\|_X < \frac{k^{\alpha-1}}{2|\mu|^\alpha}(\delta e^{T^*\Re\lambda})^\alpha \leq \frac{k^{\alpha-1}}{2|\mu|^\alpha}(\frac{|\mu|}{k})^\alpha = \frac{1}{2k}.$$

Finally, this shows that the solution is bounded below independently of δ, contradicting stability:

$$\|u(T^*)\|_X \geq \|e^{T^*L}v\|_X - \|u(T^*) - e^{T^*L}v\|_X > \|e^{T^*\lambda}v\|_X - \frac{\delta}{4k} - \frac{1}{2k} > \frac{2-\delta}{4k} \geq \frac{1}{4k}.$$

□

4.6 More Instability Results

In this section we collect some other results on nonlinear instability. One of the issues with applying the theorem is that either F can not contain any derivatives or the semigroup has to be regularizing. Another theorem exists, which demands less of F, but which requires that there exists a spectral gap. The theorem is also stated in terms of a strongly continuous group and not a semigroup, which means that e^{tL} is defined for all $t \in \mathbb{R}$.

Theorem 4.6. *Let X, Z be Banach spaces such that $X \subset Z$, X is dense in Z and there exists $C_1 > 0$ such that $\|x\|_Z \leq C_1\|x\|_X$ for all $x \in X$. Suppose that L generates a strongly continuous group e^{tL} on Z and that e^{tL} maps X into X for $t \in \mathbb{R}$. Also assume that for every $t \in \mathbb{R}$ we can write $\sigma(e^{tL}) = \sigma_+ \cup \sigma_-$ where $\sigma_+ \neq \emptyset$ and*

$$\sigma_+ \subset \{z \in \mathbb{C} : e^{tM} < |z| < e^{t\Lambda}\}$$
$$\sigma_- \subset \{z \in \mathbb{C} : e^{t\lambda} < |z| < e^{t\mu}\}$$

for some $-\infty < \lambda < \mu < M < \Lambda < \infty$ and $M > 0$. Moreover suppose that $F : X \to Z$ is continuous and that there exists $\rho_0 > 0$, $C_3 > 0$ such that $\|F(u)\|_Z \leq C_3\|u\|_X\|u\|_Z$ for $\|u\|_X < \rho_0$. Then the zero solution is nonlinearly unstable in X.

The proof is very different from the proof of Theorem 4.5, the idea is that due to the spectral decomposition one can define spectral projections corresponding to σ_+ and σ_-. Then the operator can be divided into a growing part and decaying part. The proof can be found in [5]. Another closely related theorem from [5] requires the existence of an eigenvalue close to the spectral radius.

Theorem 4.7. *Let X, Z be Banach spaces such that $X \subset Z$, X is dense in Z and there exists $C_1 > 0$ such that $\|x\|_Z \leq C_1 \|x\|_X$ for all $x \in X$. Suppose that L generates a strongly continuous group e^{tL} on Z and that e^{tL} maps X into X for $t \in \mathbb{R}$. Moreover suppose that $F : X \to Z$ is continuous and that there exists $\rho_0 > 0$, $C_3 > 0$ and $0 < \alpha \leq 1$ such that $\|F(u)\|_Z \leq C_3 \|u\|_X^{1-\alpha} \|u\|_Z^{1+\alpha}$ for $\|u\|_X < \rho_0$. Also assume that there exists $C_4 > 0$, $C_5 > 0$, $v \in X$ and*

$$\lambda > \frac{w_0}{1+\alpha}$$

such that

$$C_4 e^{t\lambda} \|v\|_Z \leq \|e^{tL} v\|_Z \leq C_5 e^{t\lambda} \|v\|_Z.$$

Then the zero solution is nonlinearly unstable in X.

For analytic semigroups we have the following theorem on instability in the fractional spaces X^α. These abstract spaces can be related to the regular Sobolev spaces using an embedding theorem or, as in the example below, they can be determined explicitly. A proof can be found in [6], corollary 5.1.6.

Theorem 4.8. *Let X be Banach space and $L : D(L) \subset X \to X$ be a sectorial operator. Let $0 < \alpha < 1$, $p > 1$, u_0 be a stationary solution of (12) and suppose that $F(u_0 + u) = F(u_0) + Bu + G(u)$ for some bounded linear map $B : X^\alpha \to X$ and $\|G(u)\| = O(\|u\|_\alpha^p)$. Moreover suppose that $F : X^\alpha \to X$ is continuous and that there exists $\rho_0 > 0$, $C_3 > 0$ such that $\|F(x) - F(y)\| \leq C_3 \|x - y\|_\alpha$ for $\|x\|_\alpha, \|y\|_\alpha < \rho_0$. Finally assume $\sigma(L + B) \cap \{\Re\lambda > 0\}$ is nonempty. Then the zero solution of*

$$u_t = (L + B)u + G(u)$$

is nonlinearly unstable in X^α.

5 The Kuramoto-Sivashinsky Equation

5.1 Instability of Constant Solutions

The Kuramoto-Sivashinsky equation

$$u_t = -u_{xxxx} - u_{xx} - uu_x. \tag{14}$$

was derived independently by Kuramoto [7] and Sivashinsky [13] in the context of reaction-diffusion equations and laminar flame fronts respectively. The equation has gained much interest due to its many applications and its chaotic behaviour. The instabilities we study are examples of such behaviour. It is clear that any constant function $\phi = b$ is a solution to (14). In this section we show that they are all unstable. If we linearize (14) around the constant solution by writing $w(x,t) = u(x,t) - b$ we get the equation

$$w_t = -w_{xxxx} - w_{xx} - bw_x - ww_x \tag{15}$$

As expected we get a differential equation with constant coefficients so that we can apply Fourier transform methods.

Theorem 5.1. *If $\phi = b \in \mathbb{R}$ is a constant solution of* (14) *then the zero solution of* (15) *is nonlinearly unstable.*

Write

$$\begin{aligned} w_t &= L_0 w + F(w), \\ L_0 &= -\partial_x^4 - \partial_x^2 - b\partial_x, \\ F(w) &= -w_x w. \end{aligned}$$

We will apply Theorem 4.5 with $Z = L^2$ and $X = H^1$. We deal with the nonlinear term first and show that it behaves well close to 0. Note that we only need the second estimate to use Theorem 4.5, but we need the first estimate to show well posedness with Theorem 4.1.

Lemma 5.1. *The nonlinear part of* (17), *$F(w) = -ww_x$, is continuous from X to Z and satisfies*

$$\begin{aligned} \|F(w_1) - F(w_2)\|_{L^2} &\le C(\rho)\|w_1 - w_2\|_{H^1}^2, \\ \|F(w_1)\|_{L^2} &\le C(\rho)\|w_1\|_{H^1}^2. \end{aligned}$$

for all $w_1, w_2 \in X$ with $\|w_1\|_{H^1} \le \rho$, $\|w_2\|_{H^1} \le \rho$.

Proof. The first estimate follows from Theorem 2.2 with the computation

$$\begin{aligned} \|F(w_1) - F(w_2)\|_{L^2} &= \|w_1 w_1' - w_2 w_2'\|_{L^2} \le \|w_1 w_1' - w_1 w_2'\|_{L^2} + \|w_1 w_2' - w_2 w_2'\|_{L^2} \\ &\le \|w_1\|_{H^1}\|w_1' - w_2'\|_{L^2} + \|w_2\|_{H^1}\|w_1 - w_2\|_{L^2} \le C(\rho)\|w_1 - w_2\|_{H^1}^2. \end{aligned}$$

This also shows continuity. The second estimate follows from the first by letting $w_2 = 0$. ☐

The next lemma shows that L_0 generates a strongly continuous semigroup on L^2 and H^1 and gives the proper estimate so that we can apply Theorem 4.5. The constant coefficients are easily handled using the Fourier transform.

Lemma 5.2. *The linear operator* $(L_0, D(L_0) = H^{s+4})$ *generates a strongly continuous semigroup* e^{tL_0} *on* H^s, $s \geq 0$. *The semigroup* e^{tL_0} *maps* L^2 *into* H^1 *for* $t > 0$ *and satisfies*

$$\|e^{tL_0}\|_{H^s \to H^s} \leq e^{\frac{t}{4}}, \quad s \in \mathbb{R}, t \geq 0,$$
$$\|e^{tL_0}\|_{L^2 \to H^1} \leq 4t^{-\frac{1}{4}}, \quad 0 < t \leq 1.$$

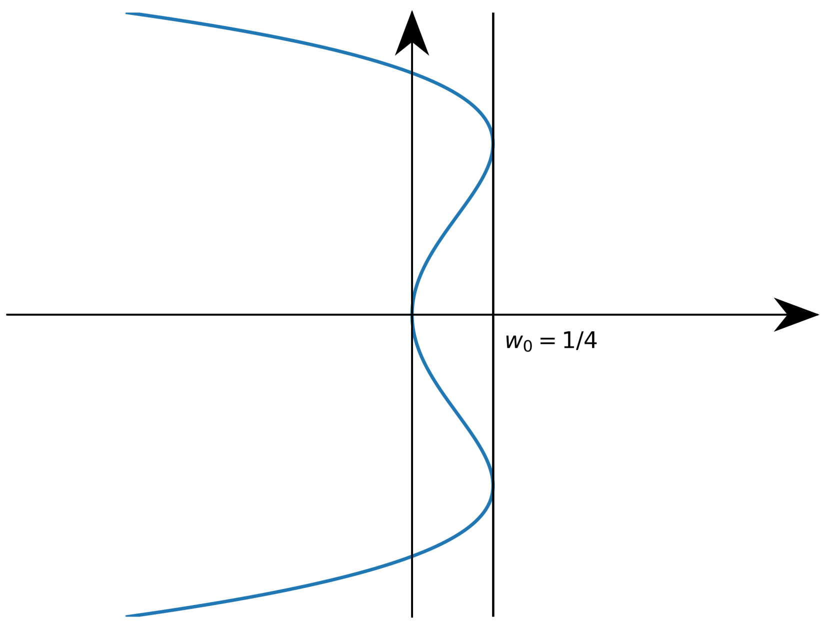

Figure 2: The curve defined by the symbol of L_0 for $b \neq 0$. If $b = 0$ the curve is just the real line with $h(\xi) \leq 1/4$.

Proof. Let $h(\xi) = -\xi^4 + \xi^2 - ib\xi$, shown in Figure 2, and define

$$e^{tL_0}u_0 := \mathcal{F}^{-1}(e^{th(\cdot)}\hat{u}_0).$$

which maps H^s into H^s since, by the Fourier characterisation of H^s,

$$\|e^{tL_0}u_0\|_{H^s}^2 = \int_{-\infty}^{\infty}(1+\xi^2)^s|\widehat{e^{tL_0}u_0}|^2 d\xi = \int_{-\infty}^{\infty}(1+\xi^2)^s|e^{-t(\xi^4-\xi^2+ib\xi)}\hat{u}_0|^2 d\xi$$
$$\leq \sup_{\xi\in\mathbb{R}} e^{-2t(\xi^4-\xi^2)}\int_{-\infty}^{\infty}(1+\xi^2)^s|\hat{u}_0|^2 d\xi \leq e^{\frac{t}{2}}\|u_0\|_{H^s}^2$$

which also proves the first inequality. To show that it defines a strongly continuous semigroup, note that (2) and (3) are obvious and (4) follows from

$$\|e^{tL_0}u_0 - u_0\|_{H^s} = \|(1+(\cdot)^2)^{\frac{s}{2}}(e^{th(\cdot)}\hat{u}_0 - \hat{u}_0)\|_{L^2} \to 0$$

by the dominated convergence theorem together with the estimate $e^{th} \leq e^{t/4}$ which is bounded for small t. Now let $u_0 \in H^{s+4}$, then $(1+(\cdot)^2)^{\frac{s}{2}}h\hat{u}_0 \in L^2$ and we have the estimate

$$\Big|\frac{e^{th}\hat{u}_0 - \hat{u}_0}{t}\Big| \leq |he^{th}u_0|$$

which is uniformly bounded by $|hu_0|$ close to $t=0$. The next limit now follows from another application of the dominated convergence theorem:

$$\Big\|\frac{e^{tL_0}u_0 - u_0}{t} - L_0u_0\Big\|_{H^s} = \Big\|(1+(\cdot)^2)^{\frac{s}{2}}(\frac{e^{th(\cdot)}\hat{u}_0 - \hat{u}_0}{t} - h\hat{u}_0)\Big\|_{L^2} \to 0.$$

This shows that L_0 is the generator of e^{tL_0}. To prove the second inequality we again use the Fourier characterization, but estimate differently: We move one power of the $1+\xi^2$ out and instead get the supremum over

$$f(\xi) = (1+\xi^2)e^{-2t(\xi^4-\xi^2)}$$

which can be shown to satisfy

$$f(\xi) \leq \Big(\frac{3}{2} + \frac{t^{-\frac{1}{2}}}{2}\Big)e^{\frac{t}{2}}.$$

Thus

$$\|e^{tL_0}u_0\|_{H^1} \leq \Big(\frac{3}{2} + \frac{t^{-\frac{1}{2}}}{2}\Big)^{\frac{1}{2}}e^{\frac{t}{4}}\|u_0\|_{L^2} \leq 4t^{-\frac{1}{4}}$$

for $0 < t \leq 1$. □

It is also clear that $\sigma(L_0)$ on H^1 contains $\{-\xi^4+\xi^2-ib\xi\}$ and thus meets the right half plane. Since all conditions in Theorem 4.5 are satisfied we have proved Theorem 5.1. It should be noted that the lemmas also show that the initial value problem has a unique solution in $C([0,T),H^1)$ for any initial value in H^1 by Theorem 4.1.

5.2 Instability of Heteroclinic Solutions

If we look for travelling solutions, $u(x,t) = \phi(x - ct)$ we find that it is equivalent to ϕ satisfying the ordinary differential equation

$$\phi''' + \phi' + \frac{1}{2}(\phi - c)^2 = k \tag{16}$$

for some constant $k \in \mathbb{R}$. In the case that $k = 1$ this equation has two fixed points at $b_\pm = c \mp \sqrt{2}$. The existence of both periodic solutions and heteroclinic solutions of (16) with $k = 1$ has been proven in [15]. We will focus on the heteroclinic solutions, satisfying $\lim_{x\to\pm\infty} \phi(x) = b_\pm$. If we linearize (14) around a travelling solution, by writing $w(x,t) = u(x,t) - \phi(x - ct)$ and also changing variables $y = x - ct$ we get the equation

$$w_t = -w_{yyyy} - w_{yy} + cw_y - \phi w_y - \phi' w - ww_y \tag{17}$$

for which $w = 0$ is a solution corresponding to the travelling solution of (14). We divide the linear part into two parts, one with constant coefficients and one perturbation. Write

$$\begin{aligned} w_t &= Lw + F(w) = L_0 w - (\phi - b_+)\partial_y w - \phi' w + F(w) \\ L_0 &= -\partial_y^4 - \partial_y^2 + (c - b_+)\partial_y \\ F(w) &= -w_y w. \end{aligned}$$

We now claim that under some conditions on ϕ the zero solution of (17) is nonlinearly unstable on H^1. To prove this we will again apply Theorem 4.5 with $X = H^1$ and $Z = L^2$. First we specify exactly which conditions on ϕ are necessary. These conditions are satisfied by the heteroclinic solutions found in [15], but since they conjectured the existence of infinitely many such solutions we show the nonlinear instability of all of them, should they exist.

Theorem 5.2. *If $\phi \in C^\infty$ is a solution of* (16) *such that $b_\pm = \lim_{x\to\pm\infty} \phi(x)$ exists, $\phi' \in L^2$, $\phi'' \in L^2$, $\phi - b_+ \in L^2$ then the zero solution of* (17) *is nonlinearly unstable.*

The above assumptions on ϕ imply that $\phi^{(k)} \in L^2$ and $\phi^{(k)}(x) \to 0$ as $x \to \pm\infty$. This can be seen by differentiating (16) and seeing that $\phi^{(4)} \in L^2$. Then $\phi^{(3)} \in L^2$ since both $\phi^{(2)} \in L^2$ and $\phi^{(4)} \in L^2$ by a Fourier transform argument. Although the other derivatives are not important they can be shown to also be in L^2 by an induction argument. The nonlinear term is the same as the one for the constant solution so Lemma 5.1 applies in this case as well. Since L_0 is of the same form as for the constant solutions, Lemma 5.2 holds for the part with constant coefficients. This is the part which we expect will determine the spectrum since the coefficients of the other part has been chosen so that they are small. We have now shown that the main part generates a strongly continuous semigroup on both L^2 and H^1, to show that it holds for L we use the perturbation result Theorem 3.5.

Lemma 5.3. *Let $B_0 = -(\phi - b_+)\partial_y - \phi'$, the perturbation of L_0. Then $L = L_0 + B_0$ generates a strongly continuous semigroup on both L^2 and H^1.*

Proof. For some $\gamma > \sup(-\phi')/2$, let $B = B_0 - \gamma$ on the set $D(B) = D(A) = H^4$. The following computation shows that B is dissipative on L^2:

$$(Bu, u) = \int -(\phi - b_+)\partial_y \frac{u^2}{2} - \phi' u^2 - \gamma u^2 dy = \int \phi' \frac{u^2}{2} - \phi' u^2 - \gamma u^2 dy \leq 0.$$

From the previous lemma we have that $L_1 := L_0 - w$ generates a contraction semigroup on L^2 if $w \geq \frac{1}{4}$. We now show that B is L_1-bounded with L_1-bound equal to 0.

$$\|Bu\|_{L^2} \leq \|\phi - b_+\|_{L^\infty} \|\partial_y u\|_{L^2} + (\|\phi'\|_{L^\infty} + \gamma)\|u\|_{L^2}$$

The second term is already of the correct form and the first term can be estimated using Young's inequality with ϵ:

$$\begin{aligned}
\|\partial_y u\|_{L^2}^2 = \int \xi^2 |\hat{u}|^2 dy &\leq \frac{3}{4\epsilon^{1/3}} \int |\hat{u}|^2 dx + \frac{\epsilon}{4} \int \xi^8 |\hat{u}|^2 dy \\
&\leq \frac{3}{4\epsilon^{1/3}} \int |\hat{u}|^2 dy + \epsilon C \int (1 + |h(\xi) - w|^2)|\hat{u}|^2 dy \\
&= (\frac{3}{4\epsilon^{1/3}} + C\epsilon)\|u\|_{L^2}^2 + \epsilon C \|(L_0 - w)u\|_{L^2}^2.
\end{aligned}$$

The fact that $L - w - \gamma$ generates a contraction semigroup on L^2 now follows from Theorem 3.5 and then L also generates a strongly continuous semigroup on L^2. To show that it also holds on H^1 we first show that B is dissipative on H^1:

$$\begin{aligned}
(Bu, u)_{H^1} &= (\partial_y Bu, \partial_y u) + (Bu, u) \\
&= -\int \Big(\phi'(\partial_y u)^2 + (\phi - b_+)\partial_y^2 u \partial_y u + \phi'' u \partial_y u + \phi'(\partial_y u)^2 + \gamma(\partial_y u)^2 dy\Big) \\
&\quad - \int \Big(\frac{\phi'}{2} + \gamma\Big) u^2 dy \\
&= -\int (\gamma + 2\phi')(\partial_y u)^2 + (\phi - b_+)\partial_y \frac{(\partial_y u)^2}{2} + \phi'' \partial_y \frac{u^2}{2} dx - \int (\frac{\phi'}{2} + \gamma) u^2 dy \\
&= -\int (\gamma + \frac{3}{2}\phi')(\partial_y u)^2 dy - \int (\frac{\phi' + \phi''}{2} + \gamma) u^2 dy.
\end{aligned}$$

From this it is clear that we can choose γ large enough so that B is dissipative on H^1. $L_1 = L_0 - w$ is still a contraction so we must show that B is L_1-bounded with L_1-bound equal to 0 in H^1. The estimate on L^2 is the same as above, although with a different γ, so it remains to estimate the derivative:

$$\|\partial_y Bu\|_{L^2} \leq \|\phi''\|_{L^\infty} \|u\|_{L^2} + (\|2\phi'\|_{L^\infty} + \gamma)\|\partial_y u\|_{L^2} + \|\phi - b_+\|_{L^\infty} \|\partial_y^2 u\|_{L^2}.$$

The first term is again of the correct form, the second has already been estimated above, so only the last term remains:

$$\begin{aligned}\|\partial_y^2 u\|_{L^2}^2 = \int \xi^4 |\hat{u}|^2 dx &\leq \frac{1}{4\epsilon}\int |\hat{u}|^2 dx + \epsilon \int \xi^8 |\hat{u}|^2 dx \\ &\leq \frac{1}{4\epsilon}\int |\hat{u}|^2 dx + \epsilon C \int (1 + |h(\xi) - w|^2)|\hat{u}|^2 dx \\ &= (\frac{1}{4\epsilon} + C\epsilon)\|u\|_{L^2}^2 + \epsilon C \|(L_0 - w)u\|_{L^2}^2.\end{aligned}$$

Another application of Theorem 3.5 gives that L generates a strongly continuous semigroup on H^1. □

The following lemma is a quick generalisation of Grönwall's inequality which is very useful for parabolic equations.

Lemma 5.4. *Let $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$, $a > 0$, $b > 0$ and $0 < T < \infty$. Then there exists an $M > 0$ such that for any integrable $f : [0, T] \to \mathbb{R}$ satisfying*

$$0 \leq f(t) \leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta} f(s) ds, \quad 0 < t \leq T$$

we have

$$f(t) \leq aMt^{-\alpha}, \quad 0 < t \leq T.$$

Proof. Iterate the inequality and change the order of integration to get

$$\begin{aligned}f(t) &\leq at^{-\alpha} + b\int_0^t (t-s)^{-\beta} f(s) ds \\ &\leq at^{-\alpha} + ab\int_0^t (t-s)^{-\beta} s^{-\alpha} ds + b^2 \int_0^t (t-s)^{-\beta} \int_0^s (s-r)^{-\beta} f(r) dr ds \\ &= at^{-\alpha} + t^{1-\beta-\alpha} ab \int_0^1 (1-x)^{-\beta} x^{-\alpha} dx + b^2 \int_0^t f(r) \int_r^t (t-s)^{-\beta}(s-r)^{-\beta} ds dr \\ &= at^{-\alpha} + t^{1-\beta-\alpha} ab \int_0^1 (1-x)^{-\beta} x^{-\alpha} dx \\ &+ b^2 \int_0^t f(r)(t-r)^{1-2\beta} \int_0^1 (1-x)^{-\beta} x^{-\beta} dx dr \\ &\leq aCt^{-\alpha} + D\int_0^t f(r) dr.\end{aligned}$$

The lemma now follows from Grönwall's inequality □

Now we apply the previous lemma to show that the norm of the semigroup from L^2 to H^1 does not grow to fast close to $t = 0$.

Lemma 5.5. *The strongly continuous semigroup e^{tL} maps L^2 into H^1 for $t > 0$ and satisfies*

$$\|e^{tL}\|_{L^2\to H^1} \leq Dt^{-\frac{1}{4}}, \quad 0 < t \leq 1.$$

Proof. By Theorem 3.5

$$u = e^{tL}u_0 = e^{tL_0}u_0 - \int_0^t e^{(t-s)L_0}((\phi - b_+)\partial_y u + \phi' u)ds$$

so that we can estimate the semigroup

$$\begin{aligned}
\|e^{tL}u_0\|_{H^1} &\leq \|e^{tL_0}u_0\|_{H^1} + \int_0^t \|e^{(t-s)L_0}((\phi - b_+)\partial_y u + \phi' u)\|_{H^1}ds \\
&\leq \|e^{tL_0}\|_{L^2\to H^1}\|u_0\|_{L^2} \\
&+ \int_0^t \|e^{(t-s)L_0}\|_{L^2\to H^1}(\|\phi - b_+\|_{L^\infty}\|\partial_y u\|_{L^2} + \|\phi'\|_{L^2}\|u\|_{L^\infty})ds \\
&\leq \|e^{tL_0}\|_{L^2\to H^1}\|u_0\|_{L^2} \\
&+ (\|\phi - b_+\|_{L^\infty} + \|\phi'\|_{L^2})\int_0^t \|e^{(t-s)L_0}\|_{L^2\to H^1}\|u\|_{H^1}ds \\
&\leq 4t^{-\frac{1}{4}}\|u_0\|_{L^2} + 4(\|\phi - b_+\|_{L^\infty} + \|\phi'\|_{L^2})\int_0^t (t-s)^{-\frac{1}{4}}\|u\|_{H^1}ds.
\end{aligned}$$

Applying Lemma 5.4 with $T = 1$, $\alpha = 1/4$ and $a = 4\|u_0\|_{L^2}$ finishes the proof of the inequality and then clearly e^{tL} maps L^2 into H^1.

□

Finally, we show that the spectrum is not changed by too much when going from L_0 to L so that it still contains the spectrum of L_0.

Lemma 5.6. *The spectrum of L on H^1 contains $\{-\xi^4 + \xi^2 + i(c - b_+)\xi,\ \xi \in \mathbb{C}\}$.*

Proof. Let $\lambda = P(\xi) = -\xi^4 + \xi^2 + i(c - b_+)\xi$. We will construct an approximate eigenvector corresponding to λ. Let $\zeta \neq 0$ be any smooth function with compact suppport in $\mathbb{R}_+$ and set

$$\zeta_n(x) = c_n \frac{e^{i\xi x}\zeta(x/n)}{\sqrt{n}}$$

where $c_n > 0$ is chosen so that $\|\zeta_n\|_{H^1} = 1$. The sequence c_n is bounded from above and below:

$$c_n = \frac{\sqrt{n}}{\|\zeta(\cdot/n)\|_{H^1}} \leq \frac{\sqrt{n}}{\|\zeta(\cdot/n)\|_{L^2}} = \frac{1}{\|\zeta\|_{L^2}},$$

$$c_n \geq C\frac{n^{\frac{3}{2}}}{n\|\zeta(\cdot/n)\|_{L^2} + \|\zeta'(\cdot/n)\|_{L^2}} \geq C\frac{1}{\|\zeta(\cdot)\|_{H^1}}.$$

We wish to show that $\|(L-\lambda)\zeta_n\|_{H^1} \to 0$. The first part $L_0 - \lambda$ has only constant coefficients and can estimated by explicitly computing the derivative

$$(L_0 - \lambda)\zeta_n = c_n e^{i\xi x} \sum_{k=1}^{4} \frac{P^k(\xi)\zeta_0^{(k)}(x/n)}{k! n^{1/2+k}}$$

$$\partial_y (L_0 - \lambda)\zeta_n = c_n e^{i\xi x} \sum_{k=1}^{4} \frac{P^k(\xi)\zeta^{(k+1)}(x/n)}{k! n^{3/2+k}} + i\xi c_n e^{i\xi x} \sum_{k=1}^{4} \frac{P^{(k)}(\xi)\zeta^{(k)}(x/n)}{k! n^{1/2+k}}$$

$$\|(L_0-\lambda)\zeta_n\|_{H^1} \le (1+|\xi|)c_n \sum_{k=1}^{4} \frac{|P^{(k)}(\xi)| \|\zeta^{(k)}(\cdot/n)\|_{L^2}}{k! n^{1/2+k}} + c_n \sum_{k=1}^{4} \frac{|P^{(k)}(\xi)| \|\zeta^{(k+1)}(\cdot/n)\|_{L^2}}{k! n^{3/2+k}} \to 0$$

since $c_n/n \to 0$. The next part is estimated by using that $\phi - b_+$ is small in L^2 for positive x where ζ_n has support:

$$\|(\phi - b_+)\partial_y \zeta_n\|_{L^2} \le \|\partial_y \zeta_n\|_{L^\infty} \|\phi - b_+\|_{L^2} \to 0$$

since $\|\zeta_n^{(k)}\|_{L^\infty} \to 0$ for any $k \ge 0$. The derivative is estimated similarly:

$$\begin{aligned} \|\partial_y((\phi - b_+)\partial_y \zeta_n)\|_{L^2} &\le \|\phi' \partial_y \zeta_n\|_{L^2} + \|(\phi - b_+)\partial_y^2 \zeta_n\|_{L^2} \\ &\le \|\partial_y \zeta_n\|_{L^\infty} \|\phi'\|_{L^2} + \|\partial_y^2 \zeta_n\|_{L^\infty} \|\chi_{[0,\infty)}(\phi - b_+)\|_{L^2} \to 0 \end{aligned}$$

These two estimates now gives that

$$\|(\phi - b_+)\partial_y \zeta_n\|_{H^1} \to 0.$$

Finally the last part is estimated by

$$\|\phi' \zeta_n\|_{L^2} \le \|\phi'\|_{L^2} \|\zeta_n\|_{L^\infty} \to 0$$

and the derivative of it by

$$\|\partial_y(\phi' \zeta_n)\|_{L^2} \le \|\phi'' \zeta_n\|_{L^2} + \|\phi' \partial_y \zeta_n\|_{L^2} \le \|\phi''\|_{L^2} \|\zeta_n\|_{L^\infty} + \|\phi'\|_{L^2} \|\partial_y \zeta_n\|_{L^\infty} \to 0$$

which gives that

$$\|\phi' \zeta_n\|_{H^1} \to 0.$$

□

We have shown that the equation satisfies all the conditions of Theorem 4.5 and thus Theorem 5.2 is proven. Again we remark that we have also showed all conditions of Theorem 4.1 so that the equation around the heteroclinic solution is well-posed.

5.3 Instability by Analyticity

In this section we provide an alternative method of proving instability using analyticity. Since the generator in the Kuramoto-Sivashinsky equation is sectorial the semigroup is analytic and one can apply the instability result for analytic semigroups, Theorem 4.8. Although this will give instability in an abstract space, we are able to identify the space explicitly if we consider only the part with constant coefficients. We deal with the heteroclinic solution directly after changing variable $y = x - ct$ and write

$$\begin{aligned} w_t &= L_0 w + F(w), \\ L_0 &= -\partial_y^4 - \partial_y^2 + c\partial_y, \\ F(w) &= -w_y w. \end{aligned}$$

To begin we show that L_0 generates an analytic semigroup.

Lemma 5.7. *The linear operator* $(L_0, D(L_0) = H^4)$ *generates an analytic semigroup* e^{tL_0} *on* L^2.

Proof. Let $a > 1/4$ and choose $\delta > 0$ such that the resolvent of L_0 contains the sector $\Sigma_{\pi/2+\delta,a}$. This is possible since the spectrum of L_0 is given by the curve $-\xi^4 + \xi^2 + ic\xi$ so

$$R(\lambda, L_0)f = \mathcal{F}^{-1}\Big(\frac{\mathcal{F}(f)}{\lambda - h}\Big).$$

defines an inverse from L^2 to $H^4 \subset L^2$. Furthermore for $0 < \epsilon < \delta$ and $\lambda \in \Sigma_{\pi/2+\delta-\epsilon,a}$

$$\|R(\lambda, L_0)f\|_{L^2} = \|\mathcal{F}(R(\lambda, L_0)f)\|_{L^2} \leq \Big\|\frac{1}{\lambda - h}\Big\|_{L^\infty} \|\mathcal{F}(f)\|_{L^2} \leq \frac{M_\epsilon}{|\lambda - a|}\|f\|_{L^2}$$

which shows that L_0 generates an analytic semigroup on L^2. □

Lemma 5.8. *If we consider* e^{tL_0} *on* L^2 *with* $D(L_0) = H^4$ *then* $X^\alpha = H^{4\alpha}$ *for* $0 < \alpha < 1$.

Proof. Choose $a < -1/4$ so that $L_1 = L_0 + a$ has negative spectrum and set $\tilde{h} = h + a$ with $h(\xi) = -\xi^4 + \xi^2 + ic\xi$. The computation

$$\begin{aligned} \mathcal{F}((-L_1)^{-\alpha}u_0) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{t\tilde{h}(\xi)}\hat{u}_0(\xi)dt = \frac{\tilde{h}(\xi)^{-\alpha}\hat{u}_0(\xi)}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}e^s ds \\ &= \tilde{h}(\xi)^{-\alpha}\hat{u}_0(\xi) \end{aligned}$$

shows that $(-L_1)^\alpha = \mathcal{F}^{-1}(\tilde{h}^\alpha \hat{u}_0)$ and the rest follows from the fact that the Fourier transform is an isometry on L^2 and the estimate

$$C(1 + \xi^2)^{2\alpha} \leq \tilde{h}(\xi)^\alpha \leq C'(1 + \xi^2)^{2\alpha}.$$

Indeed, the norm can be estimated from above by

$$\|(-L_1)^\alpha u_0\|_{L^2} \le C\|h^\alpha \hat{u}_0\|_{L^2} \le C\|(1+\xi^2)^{2\alpha}\hat{u}_0\|_{L^2} = C\|u_0\|_{H^{4\alpha}}$$

and similarly from below. □

We will now consider the case $\alpha = 1/4$ so that $X^\alpha = H^1$.

Lemma 5.9. *The nonlinear part $F(w)$ is locally lipschitz from H^1 to L^2 and can be written as*

$$F(\phi + w) = F(\phi) + Bw + g(w)$$

where B is a bounded linear map from H^1 to L^2 and $\|g(w)\|_{L^2} = O(\|w\|_{H^1}^2)$.

Proof. A simple computation shows that

$$F(\phi + w) = F(\phi) - (\phi\partial_y + \phi')w - ww'.$$

If $B = -(\phi\partial_y + \phi')$ then clearly B is bounded from $X^\alpha = H^1$ to L^2. Moreover, if $g(w) = -ww'$ then $\|g(w)\|_{L^2} = O(\|w\|_{H^1}^2)$ is just a restatement of Lemma 5.1. That F is locally Lipschitz was also shown in Lemma 5.1. □

Lemma 5.10. *The spectrum of L_0+B on L^2 contains $\{-\xi^4+\xi^2+i(c-b_+)\xi,\ \xi \in \mathbb{C}\}$.*

Proof. The same construction as in Lemma 5.6 yields an approximate eigenvector on L^2 corresponding to $\lambda = -\xi^4 + \xi^2 + i(c - b_+)\xi$. □

Thus we have showed that all conditions of Theorem 4.8 are satisfied and the zero solution is nonlinearly unstable.

Remark 5.1. *We could also have used the perturbation theorem Theorem* 3.6 *to show that $L_0 + B$ is analytic and then apply Theorem* 4.8 *directly on the linearized equation, but this would also have made it harder to identify X^α.*

In the following section we consider an example where the semigroup is not analytic and this technique fails, but we can still use Theorem 4.5 as in the previous section.

6 The Lugiato-Lefever Equation

6.1 Instability of Constant Solutions

The Lugiato-Lefever equation is a nonlinear wave equation that shows up in optics. The equation is used to model light in optical cavities when applying an electric field. It was first derived by Lugiato and Lefever [10] and can be written as

$$u_t = -i\beta\partial_x^2 u - (1+i\alpha)u + F + iu|u|^2 \tag{18}$$

for some real numbers α, β and $F > 0$. We are of course looking for a function u that is complex-valued, but we will consider the problem as a system of equations for the real and imaginary part. By rescaling the equation with $\tilde{u}(x,t) := u(\pm x/\sqrt{|\beta|}, t)$, we may assume that $|\beta| = 1$, which means that there are two cases, $\beta = -1$ and $\beta = 1$. To begin, we attempt to find the simplest kind of stationary solutions, constant solutions. Then we get an algebraic equation

$$(1+i\alpha)u - iu|u|^2 = F.$$

Multiplying with $\overline{u}$ we get the equation

$$(1+i\alpha)|u|^2 - i|u|^4 = F\overline{u}. \tag{19}$$

Write $u = u_r + iu_i$ and define $\rho = |u|^2$. Then by taking real and imaginary parts, and taking the absolute value squared, we see that (19) is equivalent to the system of equations

$$\begin{aligned} u_r &= \frac{\rho}{F} \\ u_i &= \frac{\rho(\rho-\alpha)}{F} \\ \rho((\rho-\alpha)^2+1) &= F^2. \end{aligned}$$

Thus all the constant solutions are determined by the solutions of the last equation. If $\alpha \leq \sqrt{3}$ then the left hand side is strictly increasing in ρ and hence for each $F > 0$ there exists exactly one constant solution. If $\alpha > \sqrt{3}$ then there exists two values $F_-(\rho_-)$ and $F_+(\rho_+)$, $F_-(\rho_-) < F_+(\rho_+)$, the local minima and maxima, respectively, attained at the points ρ_- and ρ_+. There is one solution for $F > F_+$ or $F < F_-$, two solutions for $F = F_\pm$ and three solutions for $F_- < F < F_+$. The solutions for different ρ as well as their stability properties are shown in Figure 3 and Figure 4.

To linearize (18) around a constant solution u^* write $u = u^* + v$. Then we get

$$
\begin{aligned}
\partial_t v &= -i\beta\partial_x^2 v - (1+i\alpha)(u^*+v) + F + i(u^*+v)|u^*+v|^2 \\
&= -i\beta\partial_x^2 v - (1+i\alpha)v + F - (1+i\alpha)u^* + iu^*(|u^*|^2 + |v|^2 + 2\Re\overline{u^*}v) \\
&\quad + iv(|u^*|^2 + |v|^2 + 2\Re\overline{u^*}v) \\
&= -i\beta\partial_x^2 v - (1+i\alpha)v + iu^*(|v|^2 + 2\Re\overline{u^*}v) + iv(|u^*|^2 + |v|^2 + 2\Re\overline{u^*}v) \\
&= -i\beta\partial_x^2 v - (1+i\alpha)v + iu^*2\Re\overline{u^*}v + iv|u^*|^2 + iv|v|^2 + 2iv\Re\overline{u^*}v + iu^*|v|^2 \\
&= -i\beta\partial_x^2 v - (1+i\alpha)v + iu^*2u_r^*v_r + iu^*2u_i^*v_i + iv|u^*|^2 + iv|v|^2 + 2iv\Re\overline{u^*}v \\
&\quad + iu^*|v|^2 \\
&= -i\beta\partial_x^2 v - (1+i\alpha)v + iu^*2u_r^*v_r + iu^*2u_i^*v_i + iv|u^*|^2 + O(|v|^2)
\end{aligned}
$$

By considering the real and imaginary parts separately the equation can be written

$$
\partial_t \begin{pmatrix} v_r \\ v_i \end{pmatrix} = A_* \begin{pmatrix} v_r \\ v_i \end{pmatrix} + N(v_r, v_i)
$$

where the linear part is

$$
A_* = \begin{pmatrix} -1 - 2u_i^*u_r^* & \beta\partial_x^2 + \alpha - 3u_i^{*2} - u_r^{*2} \\ -\beta\partial_x^2 - \alpha + 3u_r^{*2} + u_i^{*2} & -1 + 2u_r^*u_i^* \end{pmatrix}
$$

and the nonlinear part

$$
N(v_r, v_i) = \begin{pmatrix} -v_i(v_r^2 + v_i^2) - 2v_rv_iu_r^* - 3v_i^2u_i^* - v_r^2u_i^* \\ v_r(v_r^2 + v_i^2) + 2v_rv_iu_i^* + 3v_r^2u_r^* + v_i^2u_r^* \end{pmatrix}.
$$

Lemma 6.1. *The nonlinear part N satisfies*

$$
\|N(v_r, v_i)\|_{H^1\times H^1} \le C\|(v_r, v_i)\|_{H^1\times H^1}^2.
$$

for $\|(v_r, v_i)\|_{H^1\times H^1} < 1$.

Proof. Throughout the proof, denote by C any constant possibly depending on u^*. All the estimates follow from Theorem 2.2. Indeed, we can estimate each part independently:

$$
\begin{aligned}
\|2v_rv_iu_r^*\|_{H^1} &\le C\|v_r\|_{H^1}\|v_i\|_{H^1} \le C(\|v_r\|_{H^1}^2 + \|v_i\|_{H^1}^2) \le C\|(v_r, v_i)\|_{H^1\times H^1}^2 \\
\|3v_i^2u_i^*\|_{H^1} &\le C\|v_i\|_{H^1}\|v_i\|_{H^1} \le C\|(v_r, v_i)\|_{H^1\times H^1}^2 \\
\|v_i(v_r^2 + v_i^2)\|_{H^1} &\le C\|v_i\|_{H^1}\|(v_r, v_i)\|_{H^1\times H^1}^2 \le C\|(v_r, v_i)\|_{H^1\times H^1}^2. \\
\|v_r^2u_i^*\|_{H^1} &\le C\|v_r^2\|_{H^1} \le C\|v_r\|_{H^1}^2 \le C\|(v_r, v_i)\|_{H^1\times H^1}^2.
\end{aligned}
$$

The other four inequalities follow by symmetry. □

For the linear part, we proceed as in [2]. In order to apply Theorem 4.5 we need to check that A_* generates a strongly continuous semigroup. With these definitions it follows from a Fourier transform argument.

Lemma 6.2. *$(A_*, D(A_*) = H^{s+2} \times H^{s+2})$ generates a strongly continuous semigroup on $H^s \times H^s$, $s \geq 0$.*

Proof. Let

$$A_*(\xi) = \begin{pmatrix} -1 - 2u_i^* u_r^* & -\beta\xi^2 + \alpha - 3u_i^{*2} - u_r^{*2} \\ \beta\xi^2 - \alpha + 3u_r^{*2} + u_i^{*2} & -1 + 2u_r^* u_i^* \end{pmatrix}.$$

For $t \in \mathbb{R}_+$ we define

$$e^{tA_*} u_0 := \mathcal{F}^{-1}(e^{tA_*(\cdot)} \hat{u}_0) \tag{20}$$

which maps $H^s \times H^s$ into $H^s \times H^s$ since $e^{tA_*(\xi)}$ is uniformly bounded in ξ for each $t \geq 0$. Indeed if λ is the largest eigenvalue of $(A_*(\xi) + A_*(\xi)^T)/2$, which clearly does not depend on ξ, we have the estimate

$$\begin{aligned} \frac{d}{dt}\|e^{tA_*(\xi)}x\|^2 &= (A_*(\xi)e^{tA_*(\xi)}x, e^{tA_*(\xi)}x) + (e^{tA_*(\xi)}x, A_*(\xi)e^{tA_*(\xi)}x) \\ &= ((A_*(\xi) + A_*(\xi)^T)e^{tA_*(\xi)}x, e^{tA_*(\xi)}x) \leq 2\lambda\|e^{tA_*(\xi)}x\|^2. \end{aligned}$$

By Grönwall's inequality we then have

$$\|e^{tA_*(\xi)}\| \leq e^{t\lambda}$$

which proves the claim. We will show that (20) defines a strongly continuous semigroup with generator A_*. Properties (2) and (3) are obvious. Strong continuity follows from the uniform boundedness of the exponential close to $t = 0$ and the dominated convergence theorem:

$$\|e^{tA_*}u_0 - u_0\|_{H^s \times H^s} = \|(1 + (\cdot)^2)^{s/2}(e^{tA_*(\cdot)}\hat{u}_0 - \hat{u}_0)\|_{L^2 \times L^2} \to 0, \quad t \to 0.$$

To show that the generator is A_* let $u_0 \in H^{s+2}$. Then for every $\xi \in \mathbb{R}$

$$\frac{e^{tA_*(\xi)}\hat{u}_0(\xi) - \hat{u}_0(\xi)}{t} \to A_*(\xi)\hat{u}_0(\xi)$$

and $\mathcal{F}^{-1}(A_*(\cdot)\hat{u}_0) \in H^s \times H^s$ since $A_*(\xi) = O(|\xi|^2)$. We also have the bound

$$\left\| \frac{e^{tA_*(\xi)} - I}{t} \right\| \leq C|\xi|^2$$

which together with the dominated convergence theorem gives that

$$\left\| \frac{e^{tA_*}u_0 - u_0}{t} - A_*u_0 \right\|_{H^s \times H^s} = \left\| (1 + (\cdot)^2)^{s/2} \left(\frac{e^{tA_*(\cdot)}\hat{u}_0 - \hat{u}_0}{t} - A_*(\cdot)\hat{u}_0 \right) \right\|_{L^2 \times L^2} \to 0$$

as $t \to 0$. □

We proceed to characterize spectral stability. The spectrum of A_* is given by

$$\sigma(A_*) = \bigcup_{\xi\in\mathbb{R}} \sigma(A_*(\xi)) \tag{21}$$

and a simple computation shows that the spectrum of $A_*(\xi)$ is given by the roots of the polynomial $\lambda^2 + 2\lambda + a(\xi)$ with

$$a(\xi) = \xi^4 + \xi^2 2\beta(2\rho - \alpha) + \alpha^2 - 4\alpha\rho + 3\rho^2 + 1.$$

The roots of this polynomial are

$$\lambda = -1 \pm \sqrt{1 - a(\xi)}.$$

From this it follows that $A_*(\xi)$ has a positive eigenvalue if and only if $a(\xi) < 0$. Combined with (21) this implies that there exists $\lambda \in \sigma(A_*)$ with $\Re\lambda > 0$ if and only if $a(\xi) < 0$ for some $\xi \in \mathbb{R}$. The derivative of a is

$$a'(\xi) = 4\xi(\xi^2 + \beta(2\rho - \alpha)) \tag{22}$$

and the derivative of F^2 is

$$\frac{d}{d\rho}F^2(\rho) = (2\rho - \alpha)^2 + 1 - \rho^2$$

which implies that the stationary points of F^2 satisfy

$$\rho_\pm = \frac{2\alpha}{3} \mp \frac{\sqrt{\alpha^2 - 3}}{3}$$

if $\alpha > \sqrt{3}$. Furthermore we have the identities

$$F_\pm^2 = F^2(\rho_\pm) = \frac{2\alpha^3}{27} + \frac{2\alpha}{3} \pm \frac{\sqrt{\alpha^2 - 3}}{27}(2\alpha^2 - 6)$$

and

$$F_1^2 = F^2(1) = \alpha^2 - 2\alpha + 2.$$

Consider first the equation with $\beta = -1$. From (22) it follows that if $2\rho - \alpha > 0$ then the minimum of a is $1 - \rho^2$ and if $2\rho - \alpha \leq 0$ the minimum is $(2\rho - \alpha)^2 + 1 - \rho^2$. Note that for $\alpha \leq 2$ this means that the solution is spectrally stable if and only if $\rho \leq 1$. In the case $\alpha \leq \sqrt{3}$ there exists exactly one constant solution for each F. For this solution we have spectral stability if and only if $\rho \leq 1$. In the second case $\sqrt{3} < \alpha < 7/4$ we have one, two or three solutions, depending on F, all of which are spectrally stable if and only if $\rho \leq 1$. Since $F_1^2 < F_-^2$ we have spectral instability for each F such that there exists more than one solution. For $\alpha = 7/4$ we have $F_-^2 = F_1^2$ and so when $F = F_-$ there is one stable solution and one unstable. Otherwise the situation is the same as the one above. For $7/4 < \alpha < 2$ we have $F_-^2 < F_1^2 < F_+^2$ so for F with $F_-^2 \le F^2 \le F_1^2$ we have spectral stability for the smallest solution, but not the other. For $\alpha \ge 2$ we have stability if and only if $\rho \le \rho_+$. This can be seen by noting that $\rho_+ < \alpha/2 < \rho_-$. Then for $\rho > \alpha/2$ instability is trivial and for $\rho \le \alpha/2$ we have the minimum $(2\rho-\alpha)^2+1-\rho^2 = F^{2\prime}(\rho)$ which implies stability if and only if $\rho \le \rho_+$ since this is the interval for which F^2 is increasing. This means that we always have stability for the smallest solution, but not the other if $F^2 \le F_+^2$. The situation is summarized in Figure 3.

The case $\beta = 1$ is opposite. If $2\rho-\alpha > 0$ then the minimum of a is $(2\rho-\alpha)^2+1-\rho^2$ and if $2\rho-\alpha \le 0$ the minimum is $1-\rho^2$. For $\alpha \le \sqrt{3}$ the only solution is stable. For $\sqrt{3} < \alpha \le 2$ we have instability if and only if $\rho_+ < \rho < \rho_-$. To see this, note that $2\rho_+ > \alpha$ so if $\rho > \rho_+$ then the minimum is $(2\rho-\alpha)^2+1-\rho^2$ which is equal to $F^{2\prime}$ and is therefore negative if and only if $\rho_+ < \rho < \rho_-$. If on the other hand $\rho \le \rho_+$ then the minimum is positive. Finally, if $\alpha > 2$ then we have instability if and only if $1 < \rho < \rho_-$. Indeed if $\rho \le 1$ the minimum is $1-\rho^2 \ge 0$ and if $1 < \rho < \alpha/2$ the minimum is $1-\rho^2 < 0$. The case $2\rho \ge \alpha$ is the same as for $\alpha < 2$. The two important cases are shown in Figure 4.

Whenever the constant solution is spectrally unstable, then Theorem 4.5 implies that the solution also is nonlinearly unstable in $H^1 \times H^1$, due to Lemma 6.1 and Lemma 6.2.

6.2 Homoclinic and Periodic Solutions

For $\alpha < 2$ and $\rho \le 1$, the Lugiato-Lefever equations also has homoclinic solutions, see [11]. Such a homoclinic solution converges to a stable constant solution, which does not necessarily mean that the homoclinic solution is stable, but the same method as for the Kuramoto-Sivashinsky equation does not apply since it relies on splitting the operator in two parts, one corresponding to the constant solution and one corresponding to the perturbation. Periodic solutions of the Lugiato-Lefever are discussed in both [1] and [2].

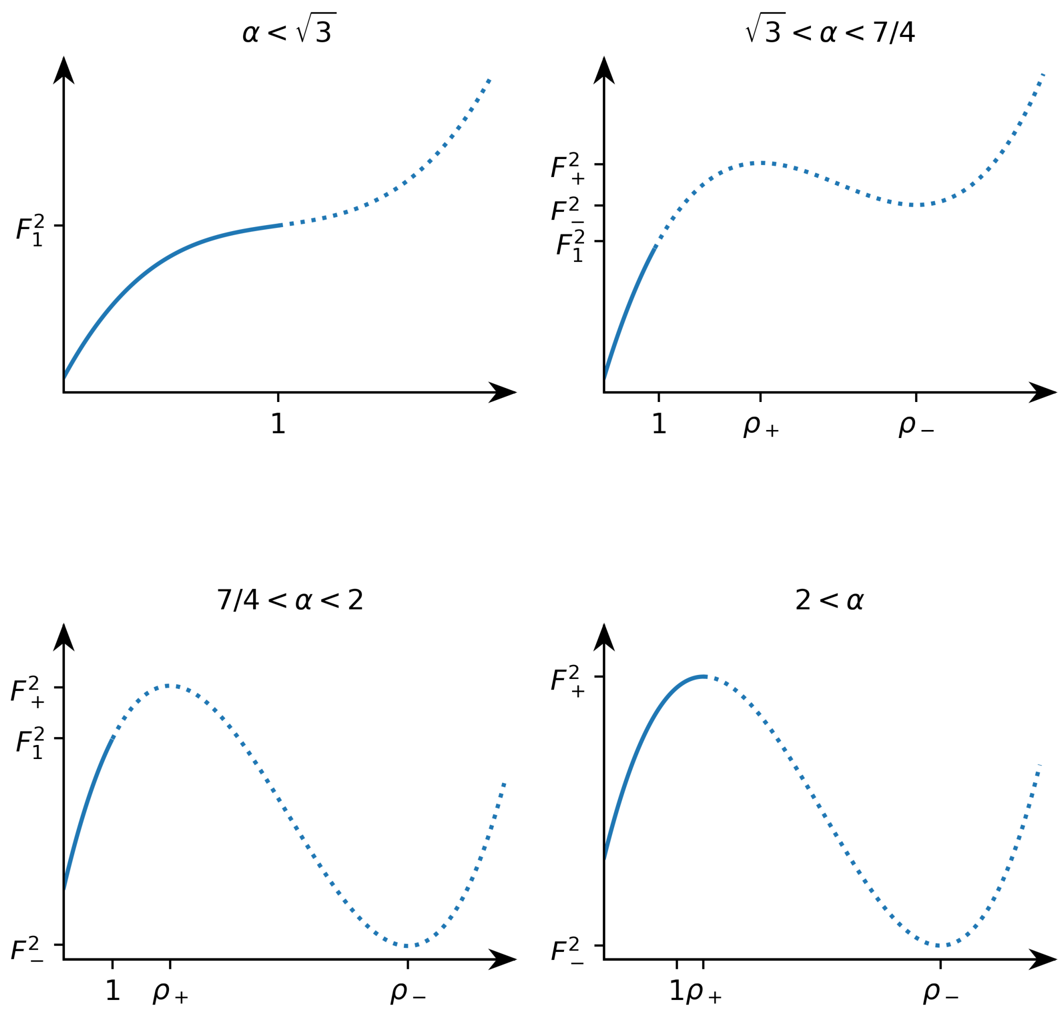

Figure 3: The existence and stability properties of constant solutions for $\beta = -1$. The dotted parts represent unstable solutions. The edge cases are left out, but can be derived from continuity. Note that this is a only a part of the entire polynomial and the graphs do not start at the origin.

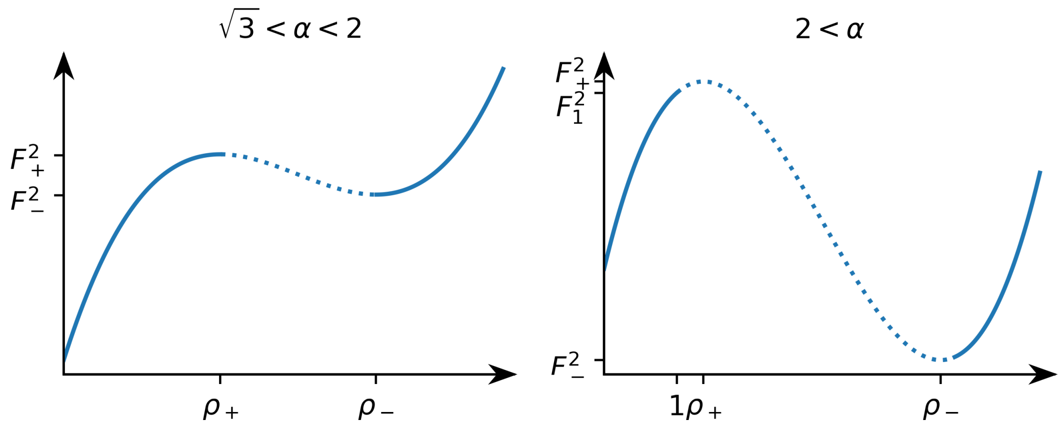

Figure 4: The existence and stability properties of constant solutions for $\beta = 1$. The dotted parts represent unstable solutions. The edge cases are left out, but can be derived from continuity. Note that this is a only a part of the entire polynomial and the graphs do not start at the origin.

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Master’s Theses in Mathematical Sciences 2020:E5
ISSN 1404-6342

LUNFMA-3113-2020

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