STEADY IDEAL FLOWS WITH VORTICITY IN TOROIDAL DOMAINS AND PERIODIC CYLINDERS

GUSTAV JILLBRATT

Master's thesis
2020:E7

LUND UNIVERSITY

Faculty of Science
Centre for Mathematical Sciences
Mathematics

CENTRUM SCIENTIARUM MATHEMATICARUM

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Abstract

In a paper from 1970, Lortz constructs rotational solutions to the steady Euler equation in toroidal domains using a fixed point method. We shall review this method and rewrite it using Banach's fixed point theorem. Using the ideas presented we shall also consider rotational flows in unbounded cylinder type domains, with a given periodicity condition. This is motivated in part by the study of three dimensional, doubly periodic water waves.

Populärvetenskaplig sammanfattning

För att matematiskt modellera beteendet av fluider används de välkända Navier-Stokes ekvationer. Dessa har visat sig vara relativt svåra att studera, och är ett aktivt forskningsområde i dagsläget. I denna uppsats studerar vi en förenklad version av Navier-Stokes ekvationer i tre rumsdimensioner, nämligen Eulers ekvationer. Vi tittar på det fallet där man antar att fluidens hastighet och tryck är tidsoberoende, tillsammans med antagandet att fluiden saknar viskositet och är inkompressibel. Fluidens vorticitet är ett mått på dess lokala rotation. I det fall där fluiden saknar vorticitet, och området där fluiden rör sig är tillräckligt enkelt ur en topologisk synvinkel så förenklas Eulers ekvationer till Laplace ekvation. Om man istället kräver att vorticiteten inte är identiskt lika med noll, eller att topologin av området är mer komplicerad, blir saker och ting svårare, och det finns relativt få resultat angående existensen av lösningar till Eulers ekvationer i detta fall. I denna uppsats visar vi, under vissa antaganden, att lösningar med vorticitet existerar i områden som är topologiska torusar. Eulers ekvationer sammanfaller med ekvationerna som beskriver magnetohydrostatiska jämviktstillstånd. Det senare har tillämpningar inom plasmafysik och fusionsenergi, och således kan studiet av Eulers ekvationer vara intressant också ur en icke-matematisk synvinkel.

Contents

1 Introduction

We shall consider the three-dimensional steady Euler equation, governing the motion of an inviscid incompressible ideal fluid, given by

$$\begin{aligned} (u \cdot \nabla) u &= -\nabla P, \\ \text{div } u &= 0. \end{aligned} \tag{1.1}$$

Here $u : \Omega \to \mathbb{R}^3$ is interpreted as the fluid's velocity vector field, defined in an open set Ω in $\mathbb{R}^3$ in which the fluid is moving. The scalar function $P : \Omega \to \mathbb{R}$ is interpreted as the pressure. We note that a solution to equation (1.1) is a pair (u, P), as also the pressure function is unknown.

It is natural to impose boundary conditions on the solution u to the Euler equation. We shall mainly consider flows in bounded domains, where we impose the constraint

$$n \cdot u = 0 \tag{1.2}$$

on the boundary $\partial\Omega$ of Ω. Here n denotes the exterior unit normal to the boundary, and we always assume enough regularity on the boundary for this to make sense. The constraint (1.2) says the field u is tangential to the boundary, which effectively means that there is no flow through the boundary of the domain Ω.

Using the identity

$$(u \cdot \nabla) u = \nabla\big(\frac{1}{2}|u|^2\big) - u \times \text{curl } u,$$

one may rewrite the Euler equation (1.1) in the form

$$\begin{aligned} u \times \omega &= \nabla H, \\ \text{curl } u &= \omega, \\ \text{div } u &= 0. \end{aligned} \tag{1.3}$$

The function $H = P + \frac{1}{2}|u|^2$ is known as the Bernoulli function. The curl of the velocity field, $\omega = \text{curl } u$, is called the vorticity field. This is a measure of the local rotation of the fluid and it is of our interest to study the existence of solutions to the Euler equations where the vorticity field does not vanish identically. We shall call such a solution a *rotational* solution.

The starting point of this thesis is the paper [9] in which Lortz studies the equations governing magnetohydrostatic equilibrium. These equations take the form

$$\begin{aligned} j \times B &= \nabla p, \\ \text{curl } B &= j, \\ \text{div } B &= 0. \end{aligned} \tag{1.4}$$

Here B denotes the magnetic field, j the current density and p the hydrostatic pressure. Equation (1.4) precisely corresponds to the steady Euler equation (1.3). Indeed, we have the correspondence

$$\begin{aligned} B &\leftrightarrow u, \\ j &\leftrightarrow \omega, \\ p &\leftrightarrow -H. \end{aligned}$$

Magnetohydrodynamics, of which magnetohydrostatics is a part, studies the motion and behaviour of electrically conducting fluids, for example plasmas. This theory has many real world applications in physics and engineering, one example being the designing of fusion reactors in nuclear power plants. An example is the Tokamak construction, which is a toroidal shaped chamber confining a plasma which fills the torus. An electric current is run through the toroidal shaped chamber, inducing a sufficiently strong magnetic field keeping the plasma from escaping out of the chamber. The Tokamak reactors have been used in real world experiments involving fusion confinement, and have succesfully produced controlled release of deuterium-tritium thermonuclear fusion power. For more on this theory, and on its history and applications, one can consult the book [10].

Starting with an *irrotational* flow, that is curl u=0, Lortz constructs rotational solutions to equation (1.4) (and thus also to the Euler equation), with boundary condition (1.2), in toroidal domains which have a certain type of mirror symmetry with respect to a plane.

Our intention is to try to clarify some steps in the paper by Lortz, in particular to show rigorously that the solutions obtained are indeed rotational, under suitable assumptions. We shall also rewrite the proof using Banach's fixed point theorem. In particular, this approach allows us to show uniqueness of the obtained solution. This we do in section 2.

In section 3 we use similar ideas as in [9] to study the existence of rotational flow in periodic cylinder type domains. The domains we consider are described in cylindrical coordinates as

$$\Omega^\epsilon = \{(r, \theta, z) \mid 0 < r < d + \epsilon\eta(\theta, z)\ 0 \leq \theta < 2\pi, z \in \mathbb{R}\ \},$$

where $d > 0$ is a fixed positive number, and η a smooth function which is periodic in θ and z. $\epsilon > 0$ is a sufficiently small parameter so that $d + \epsilon\eta(\theta, z) > 0$ for all θ and z. Thus our domains can be seen as small perturbations of an ordinary circular cylinder of radius d.

One motivation for studying flow in domains of this type comes from the desire to understand three dimensional doubly periodic water waves. In usual Cartesian coordinates (x, y, z) one is interested in domains of type

$$\Omega^\eta = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < z < d + \eta(x, y)\},$$

where $d > 0$ is a fixed constant, and $\eta : \mathbb{R}^2 \to \mathbb{R}$ is some function which is periodic in x and y. The fluid occupying the domain Ω^η has the velocity field

u which obeys Euler's equation (1.1) with boundary condition (1.2) on the top and bottom boundaries of Ω^η, also called the kinematic boundary condition. One also impose the dynamic boundary condition on the top surface $\partial\Omega^\eta_{\text{top}} = \{z = d + \eta(x,y)\}$, given by

$$P = P_{\text{atm}} - 2\sigma K_M \quad \text{ on } \partial\Omega^\eta_{\text{top}}. \tag{1.5}$$

Here P_{atm} is the constant atmospheric pressure, $\sigma > 0$ the coefficient of surface tension and K_M the mean curvature of the surface $\partial\Omega^\eta_{\text{top}}$, given by

$$K_M = \frac{1}{2}\operatorname{div}\Big(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\Big).$$

In the most general case the function η is unknown. To solve the water wave problem one thus have to find the triple (u, P, η) solving problems (1.1), (1.2) and (1.5). Since the domain is specified by the function η, finding the domain Ω^η is part of solving the problem. Thus this problem is what is known as a free boundary problem, and is in general difficult to study.

We shall only consider periodic cylinder domains of the form Ω^ϵ for a fixed function η. Hence we do not work with a free boundary problem. In such a domain we consider the Euler equation (1.3) with boundary condition (1.2). There we show the existence of a unique smooth irrotational, divergence free vector field $u_0 = u_0(\epsilon)$, tangent to the boundary of Ω^ϵ, for each $\epsilon \geq 0$ sufficiently small. We can then use the ideas put forth by Lortz in [9] to show existence of solutions to problem (1.3) and (1.2) in Ω^ϵ, for ϵ sufficiently small, by perturbing the irrotational flow u_0 by a sufficiently small perturbation factor v to get the sought velocity field u via $u = u_0 + v$. In the special case where the function η takes the form $\eta = \cos(z)$ and $\eta = \cos(\theta)\cos(z)$, we can show the existence of rotational solutions for small $\epsilon > 0$. This could be a step in the direction of understanding the full doubly periodic water wave problem given by problems (1.1), (1.2) and (1.5) in Ω^η, and to construct rotational solutions to this problem.

The existence of rotational flow for the doubly periodic water wave problem has recently been considered in the paper [8]. There the authors consider the case of Beltrami flows, where the vorticity is proportional to the velocity.

2 Rotational flows in mirror symmetric, toroidal domains

2.1 Previous work

In this subsection we shall briefly describe the results obtained in the paper [9] by Lortz. Before we do this we shall need some preliminary discussion about the mathematical setting and notation.

We shall assume that Ω is a given bounded, open set in $\mathbb{R}^3$, with smooth boundary $\partial\Omega$. We shall further assume that the boundary $\partial\Omega$ is a surface of genus one, which we denote by T, described implicitly by a smooth function f as

$$T = f^{-1}(0),$$

so that

$$\nabla f \neq 0 \quad \text{on } T.$$

Finally we shall assume the following type of mirror symmetry. Let N denote a fixed unit vector in $\mathbb{R}^3$, and let Π be the plane described by

$$\Pi = \{x \cdot N = 0\},$$

with $x = (x_1, x_2, x_3)$ standard Cartesian coordinates. We let z be defined by $z := x \cdot N$. The function f then has the form $f = f(z, N \times x)$, and we require it to have the following symmetries (leaving out the argument $N \times x$)

$$\begin{aligned} f(-z) &= f(z), \\ N \cdot \nabla f(-z) &= -N \cdot \nabla f(z), \\ N \times \nabla f(-z) &= N \times \nabla f(z). \end{aligned} \tag{2.1}$$

As an example of such a domain we may take the interior of an ordinary circular torus, described implicitly in ordinary Cartesian coordinates as the zero set of the function

$$f(x_1, x_2, x_3) := x_3^2 + \left(\sqrt{x_1^2 + x_2^2} - (A + R)\right)^2 - R^2, \tag{2.2}$$

with $0 < A, R$ fixed numbers. As plane of symmetry one may take any rotation of the (x_2, x_3)-plane around the x_3-axis.

The corresponding symmetries imposed on the solution are given by

$$N \cdot u(-z) = N \cdot u(z), N \times u(-z) = -N \times u(z), \tag{2.3a}$$

$$N \cdot \omega(-z) = -N \cdot \omega(z), N \times \omega(-z) = N \times \omega(z), \tag{2.3b}$$

$$H(-z) = H(z). \tag{2.3c}$$

These symmetries are compatible with the Euler equation (1.3), and so with the equation (1.1). Indeed, that H has symmetry (2.3c) shows that the gradient ∇H has symmetry (2.3b). If u has symmetry (2.3a) one can check that $\omega = \text{curl } u$ has symmetry (2.3b). If u has symmetry (2.3a) and ω has symmetry (2.3b) we see that the cross product $u \times \omega$ has symmetry (2.3b). Hence the symmetries are compatible with the Euler equation.

The symmetry plane Π cuts the domain Ω into two disjoint, simply connected regions, connected by the surface $M := \Pi \cap \overline{\Omega}$, which in turn decomposes as the disjoint union of two surfaces $M := M_1 \cup M_2$. We can denote the two simply connected regions of Ω by Ω_L and Ω_R, with the L and R denoting that they lie on the left and right side of the symmetry plane Π, respectively. The boundary of Ω_R is the union $M_1 \cup M_2 \cup T_R$, with T_R being the part of T lying to the right of the symmetry plane Π.

To proceed further we will first need to find an irrotational field u_0, which is divergence free, that is div $u_0 = 0$, satisfying the boundary condition (1.2). We also want u_0 to be non-vanishing in Ω. This can be achieved as follows. We seek to write $u_0 = \nabla\phi$ for a harmonic function ϕ, whose normal derivative $n \cdot \nabla\phi$ vanishes on T. If ϕ is a harmonic function in the whole of Ω so that $\nabla\phi$ is tangential to T, we obtain that $u_0 = 0$. Hence one needs to introduce a discontinuity in ϕ so that $u_0 = \nabla\phi$ is continuous. We then consider the following mixed boundary value problem.

$$\begin{aligned} \Delta\phi &= 0, \quad \text{in } \Omega_R, \\ n \cdot \nabla\phi &= 0, \quad \text{on } T_R, \\ \phi &= 0, \quad \text{on } M_1, \\ \phi &= c, \quad \text{on } M_2. \end{aligned}$$

Here c is a given non-zero constant. This has a unique solution ϕ, see [6]. Thus we obtain $u_0 := \nabla\phi$ in Ω_R. If we reflect this solution in the symmetry plane Π, we get an irrotational field u_0 in the whole of Ω, which is divergence free, tangential to T and satisfies the symmetries (2.3a). Indeed, we reflect ϕ in the symmetry plane by setting

$$\phi(-z) := -\phi(z).$$

The derivatives of ϕ are then continuous in the whole of Ω. Further, the harmonicity of ϕ implies that they satisfy Laplace's equation in the weak sense. Classical elliptic regularity [6] then gives that these derivatives of ϕ are equal almost everywhere to strong solutions of Laplace's equation in Ω, and by continuity they are thus themselves strong solutions and hence smooth in Ω. Hence the corresponding gradient vector field u_0 is smooth in Ω, and further satisfies the symmetries (2.3a). Since T has genus one, it follows that, up to multiplicative constants, u_0 is the unique irrotational, divergence free vector field which is tangent to T; see for example [4]. We shall assume that u_0 is non-vanishing in Ω, as this is also an assumption made in [9]. We may however note that if we consider a flow in a circular torus described in cylindrical coordinates (r, θ, z), we can see directly that such a non-vanishing flow u_0 exists. Indeed, say that we consider the toroidal region Ω bounded by the torus (2.2), described in cylindrical coordinates, so that

$$\Omega = \{(r, \theta, z) \mid A < r < A + 2R,\ 0 \leq \theta < 2\pi,\ z^2 < R^2 - (r - (A + R))^2\} \quad (2.4)$$

Then we can consider the vector field

$$u_0(r, \theta, z) := \frac{1}{r} e_\theta,$$

with e_r, e_θ, e_z the standard basis for $\mathbb{R}^3$ expressed in cylindrical coordinates. This is obtained as the gradient of the (multi-valued) harmonic function

$$\phi(r, \theta, z) = \theta.$$

We now intend to briefly describe the method employed in the paper by Lortz. Having obtained u_0 as an irrotational, divergence free field in Ω which is tangential to the boundary and unique modulo constants, one can specify this constant by for example specifying the flux across the surface M_1, that is, specifying the number F defined by

$$F := \int_{M_1} u_0 \cdot \mathrm{d}S. \quad (2.5)$$

For this see for example the paper [4].

An integral curve, or streamline, of the velocity field u is a curve $\gamma : I \subset \mathbb{R} \to \Omega$, from an open interval I of the real line, defined by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \gamma(t) = u(\gamma(t)).$$

We shall have to assume that the streamlines of the vector field u_0 are closed, go from M_1 back to M_1, and are uniformly bounded in length by some positive number $L_0 > 0$. As $N \cdot u \neq 0$ on M_1 by symmetry (2.3a), we can assume $N \cdot u > 0$ on M_1 so that the streamlines have to go from M_1 into Ω_R. The assumption is then that they continue all the way to M_2, and then by reflection symmetry into Ω_L and back to M_1 in such a way that their lengths are uniformly bounded. See also lemma 2.1 and the paper [3] by Alber.

For a solution (u, H) to equation (1.3), we see that H is constant on streamlines of u and ω. As ω should be divergence free, one can make the ansatz

$$\omega = \nabla H \times \nabla \tau,$$

for some function τ. Then we see that

$$u \times \omega = u \times (\nabla H \times \nabla \tau) = (u \cdot \nabla \tau) \nabla H - (u \cdot \nabla H) \nabla \tau. \quad (2.6)$$

Then (1.3) leads to the equations

$$u \cdot \nabla H = 0,$$
$$u \cdot \nabla \tau = 1.$$

If we require τ to be zero on M_1, integrating the differential equation for τ along streamlines of u allows us to think of $\tau(x)$ as the time it takes to travel along the streamline starting on M_1 and passing through x. Lortz defines the function $q(x)$ as the time it takes to travel along streamlines starting at M_1 for one period. That is, $q(x)$ is obtained by integrating the equation $u \cdot \nabla \tau = 1$ over a closed streamline passing through the point x, for one period.

Now, thinking of τ as the time it takes to travel along streamlines means that if we follow a streamline for several revolutions around the toroidal domain Ω the values of τ increases with each revolution. Thus τ is a multi-valued function in this way. However, $\omega = \nabla H \times \nabla \tau$ of course needs to be well-defined single-valued in Ω. This forces us to put further restrictions on the function H. In particular, the part of $\nabla \tau$ which is orthogonal to ∇H must be single-valued. We claim that we must have that

$$\nabla H \times \nabla q = 0.$$

Indeed, we first show that this holds on M_1. Recall that N is normal to M_1 and $u \cdot N$ is the only non-zero component of u on M_1 by symmetry (2.3a). We write $\nabla H = \nabla_T H + \nabla_N H$, with $\nabla_T H$ tangential to M_1, and $\nabla_N H$ orthogonal to M_1. Since ∇H needs to be orthogonal to u by (1.3) we see that ∇H is tangential to M_1 on M_1, that is $\nabla H = \nabla_T H$ on M_1. We can similarly decompose $\nabla \tau = \nabla_T \tau + \nabla_N \tau$. Let $\Phi_t(x^0)$ denote the flow of u in Ω, starting at $x^0 \in M_1$ at time $t = 0$. Let $f|_{M_1^-}(x^0) = \lim_{t \searrow 0} f(\Phi_t(x^0))$ for a function or vector field f, and let $f|_{M_1^+}(x^0)$ be the value of f at $x^0 \in M_1$ after following $\Phi_t(x^0)$ around Ω for one time period. Thus we need to have that

$$\omega|_{M_1^+} - \omega|_{M_1^-} \equiv 0.$$

Now, the fact that $\tau|_{M_1^-} = 0$ implies that $\nabla_T \tau|_{M_1^-} = 0$. Then the fact that $\nabla H|_{M_1} = \nabla_T H|_{M_1}$ and the requirement that the part of $\nabla \tau$ orthogonal to ∇H is single-valued implies that $\nabla_N \tau|_{M_1^+} - \nabla_N \tau|_{M_1^-} = 0$. It follows that

$$\nabla H \times \nabla \tau|_{M_1^+} - \nabla H \times \nabla \tau|_{M_1^-} = \nabla_T H \times \nabla_T \tau|_{M_1^+}.$$

On the other hand, $\tau|_{M_1^+} = q|_{M_1^+}$ since $\tau|_{M_1^-} = 0$. Since q is constant on streamlines of u, symmetry (2.3a) implies that $\nabla q|_{M_1} = \nabla_T q|_{M_1}$. From this we see that

$$\nabla_T \tau|_{M_1^+} = \nabla_T q|_{M_1^+} = \nabla q|_{M_1^+}.$$

Finally, all this implies that we must have that

$$\nabla H \times \nabla q|_{M_1} = 0.$$

To see that we have $\nabla H \times \nabla q = 0$ in the whole of Ω we can argue as follows. First, for fixed t we let $D\Phi_t(x)$ denote the derivative of $\Phi_t(x)$ as a function of x. Recall that H and q are constant on streamlines of u. Thus $H(x) = H(\Phi_t(x))$ and $q(x) = q(\Phi_t(x))$ for all $x \in \Omega$ and times t. Differentiating these relations with respect to x, and using the facts that $\nabla H \times \nabla q|_{M_1} = 0$ and $D\Phi_t(x)$ is invertible for all t shows that $\nabla H \times \nabla q = 0$ holds identically in Ω.

Now, to obtain rotational solutions, Lortz starts with an irrotational flow and uses an iterative method. The fact that $\nabla H \times \nabla q = 0$ must hold in Ω suggests that H may be a function of q. One specifies a scalar function h in the Hölder space $C^{2,\alpha}$ (see the appendix) for a given $0 < \alpha < 1$. Further, one should assume that the derivative of h is non-vanishing, $h' \not\equiv 0$, and that its supremum norm is not too large, which can be achieved by multiplication by a small parameter $\beta > 0$.

The idea is then to let H be given by

$$H(x) := \beta h(q(x)).$$

With q and τ determined generated by a non-vanishing vector field u with symmetries (2.3a). If ∇q is not identically zero, then the vorticity field given by $\omega = \nabla H \times \nabla \tau$ is not identically zero. Thus one would hope that ω actually is the vorticity field of the vector field u which generated τ and q. In other words, with ω as above, one needs that u satisfies

$$\begin{aligned} \text{curl } u &= \omega, && \text{in } \Omega \\ \text{div } u &= 0, && \text{in } \Omega \\ u \cdot n &= 0, && \text{on } \partial\Omega \end{aligned} \tag{2.7}$$

as this would then produce a rotational solution (u, H) to the Euler equation (1.3). See also theorems 2.2 and 2.3.

That q actually is non-constant, so that ∇q is non-vanishing, is not obvious and is something we shall discuss more later on. However, in the case of a circular torus as (2.4) with the explicit irrotational flow in cylindrical coordinates, $u_0(r, \theta, z) = \frac{1}{r} e_\theta$, this is true. By continuity it holds also for small perturbations of such domains. We show another result of this type in section 3, working instead with a type of periodic cylinder as our domain. In the case of an ordinary circular, periodic cylinder in cylindrical coordinates (r, θ, z), of the form

$$\Omega^0 = \{(r, \theta, z) \mid 0 < r < d,\ 0 \le \theta < 2\pi,\ z \in \mathbb{R}\},$$

for a fixed $d > 0$ where we identify integer multiples of 2π in z, we have the explicit irrotational flow $u_0 = e_z$, modulo multiplicative constants. In this case the corresponding q is the time it takes to travel along a streamline starting on the surface $\{z = 0\}$ until reaching the surface $\{z = 2\pi\}$, and for $u_0 = e_z$ this is obviously constant. In this case it is not obvious what q looks like in small perturbations of the domain Ω^0. In section 3 we are able to show that q is indeed non-constant in small perturbations of the periodic cylinder, if we restrict the perturbation considerably. This is shown by doing explicit calculations.

The iteration method used by Lortz is described as follows. One starts with an irrotational field u_0, determined by its flux (2.5) across M_1. The functions τ_0 and q_0 are computed as described above. The function H_0 is defined by $H_0 := \beta h(q_0)$, and the vorticity $\omega_1 := \nabla H_0 \times \nabla \tau_0$. One then obtains a new field u_1 by solving the corresponding div-curl problem (2.7), and specifies it uniquely by imposing the condition that its flux (2.5) agrees with the flux F_0 of u_0. The algorithm then starts over with this new field u_1. We write it down as

$$\begin{aligned} u_m \to q_m \to H_m := \beta h(q_m), \\ u_m \cdot \nabla \tau_m = 1 \text{ in } \Omega,\ \tau_m = 0 \text{ on } M_1 \to \tau_m, \\ \tau_m, H_m \to \omega_{m+1} := \nabla H_m \times \nabla \tau_m, \end{aligned}$$

and

$$\begin{aligned} \text{curl } u_{m+1} = \omega_{m+1}, \quad \text{in } \Omega \\ \text{div } u_{m+1} = 0, \quad \text{in } \Omega \\ u_{m+1} \cdot n = 0, \quad \text{on } \partial\Omega \\ \int_{M_1} u_{m+1} \cdot \mathrm{d}S = F_0 \to u_{m+1}. \end{aligned}$$

The algorithm is taken for $m = 0, 1, 2, ...$, and is initiated with the irrotational field u_0, specified by the flux $F_0 \neq 0$.

Lortz shows that the fields u_m form a Cauchy sequence in the space $C^{1,\alpha}$, if the parameter β is not too large, and thus converge to some field u, with a vorticity determined by the uniform limit of the vorticities ω_m.

One problem here is that if q_0 is a constant function we only get back the irrotational field u_0 each time, as we then have that the vorticity ω_1 is zero. So to actually get a rotational solution one would have to verify that ∇q_0 is not identically zero.

2.2 A fixed point method

We shall now want to reformulate the proof by Lortz using Banach's fixed point theorem. In doing so we are very much inspired by Alber's paper [3]. We shall start by introducing appropriate Banach spaces. For integers $k = 1, 2, ...,$ and real numbers $0 < \alpha < 1$, we define the spaces

$$X_k := \{v \in C^{k,\alpha} \mid v \text{ has symmetry (2.2a)}\}.$$

Further, we introduce the subspace V of vector fields v in X_2, satisfying the additional properties

$$\begin{aligned} \operatorname{div} v = 0 &\qquad \text{in } \Omega, \\ v \cdot n = 0 &\qquad \text{on } \partial\Omega. \end{aligned}$$

For a positive number $\mu > 0$ we consider the closed ball V_μ in V, defined as

$$V_\mu := \{v \in V \mid \|v\|_{2,\alpha} \leq \mu\}.$$

We shall consider V_μ as a subset of X_1 and consider perturbations $u = u_0 + v$, of the irrotational field u_0, by fields v in V_μ. We then construct an operator B taking V_μ into V_μ, and use Banach's fixed point theorem to show that the operator B has a fixed point in V_μ. We shall assume, as in the paper by Lortz, that the irrotational field u_0 is non-vanishing in Ω. We further assume that all streamlines of u_0 are closed, going from M_1 back to M_1, and uniformly bounded in length by some number $L_0 > 0$. We set

$$\underline{u}_0 := \inf_{x \in \Omega} |u_0(x)|.$$

Now if μ is chosen so that $\mu < \underline{u}_0$, we find for a vector field $u = u_0 + v$, with $v \in V_\mu$, that

$$\underline{u} := \inf_{x \in \Omega} |u(x)| \geq \underline{u}_0 - \sup_{x \in \Omega} |v(x)| \geq \underline{u}_0 - \|v\|_{2,\alpha} > 0.$$

Thus any such perturbation is also non-vanishing in Ω. Arguments as in [3] (see section 3) shows that the streamlines of any such perturbation are closed and go from M_1 to M_1, and are uniformly bounded in length by some number L_μ.

Lemma 2.1
Let $\mu > 0$ be chosen so that $\mu < \underline{u}_0$. Under the assumption that all streamlines of u_0 are closed, are uniformly bounded in length and go from M_1 back to M_1 it follows that the streamlines of any perturbation $u = u_0 + v$ with $v \in V_\mu$ also has closed streamlines that go from M_1 to M_1. Furthermore, there is a constant $L_\mu > 0$, depending on μ, so that the length of any streamline of any such perturbation u is bounded by L_μ.

From now on we thus only consider $0 < \mu < \underline{u}_0$. Ultimately we want to prove the following theorem.

Theorem 2.2 *Let u_0 be a non-vanishing, irrotational, divergence free vector field tangent to the boundary of Ω, satisfying symmetry (2.3a). Let $0 < \mu < \underline{u}_0$, and $0 < \alpha < 1$. Fix a scalar function $h \in C^{2,\alpha}(\Omega)$ whose derivative does not vanish identically. For each μ, there is a $\beta(\mu) > 0$ so that for each $0 < \beta < \beta(\mu)$, there is a vector field $v \in V_\mu$ for which the vector field $u = u_0 + v$ has symmetry (2.3a), and is a solution to Euler's equation (1.3) with boundary condition (1.2). The vorticity ω of u is given by $\omega = \nabla H \times \nabla \tau$, with τ solving problem*

(2.8), and the Bernoulli function H given by $H = \beta h(q)$. Here q is determined by u and defined as in section 2.1. The solution u is the only $C^{2,\alpha}$ solution with symmetry (2.3a) obeying the estimate $\|u - u_0\|_{2,\alpha} \leq \mu$. In particular, if q_0 generated by u_0 is not constant, we obtain for each sufficiently small μ a rotational solution to the Euler equation (1.3) with boundary condition (1.2).

The remainder of this section is devoted to proving this theorem. We shall begin by explaining in more detail how the operator B is defined, and why we are interested in finding fixed points of it.

We first construct for each $0 < \mu < \underline{u_0}$, an operator $B : V_\mu \to V$. Later we shall make sure that B actually maps into V_μ, and is a contraction. Given $v \in V_\mu$, we consider the vector field $u := u_0 + v$. The functions τ and q are constructed as explained above. Thus τ should solve the following transport problem

$$\begin{aligned} u \cdot \nabla\tau = 1 \qquad & \text{in } \Omega, \\ \tau = 0 \quad & \text{on } M_1, \end{aligned} \tag{2.8}$$

and have $C^{2,\alpha}$ regularity. To solve (2.8) we approximate u by a sequence of smooth vector fields, and solve (2.8) for these smooth vector fields. We can then obtain the solution of (2.8) for u as a limit of these. We use proposition B.1 in the appendix, which gives a sequence of smooth vector fields, u_j, converging to u in $C^{2,\lambda}$ for each $0 < \lambda < \alpha$, and a constant $C > 0$ for which the bound

$$\|u_j\|_{2,\alpha} \leq C \|u\|_{2,\alpha} \tag{2.9}$$

holds for all j. We can solve the problem corresponding to (2.8) for the vector fields u_j, namely

$$\begin{aligned} u_j \cdot \nabla\tau_j = 1 \qquad & \text{in } \Omega, \\ \tau_j = 0 \quad & \text{on } M_1. \end{aligned} \tag{2.10}$$

Let $\Phi_t^j(x)$ be the flow of the vector field u_j. As each vector field u_j is smooth, classical regularity results gives smoothness of the flow. The function τ_j solving problem (2.10) can be characterized via the equation

$$x = \Phi^j_{\tau_j(x)}(x^0), \tag{2.11}$$

with $x \in \Omega$ and $x^0 \in M_1$, which is seen by integrating problem (2.10) along streamlines of u_j. Thus τ_j is the inverse of the time component of the flow of u_j. More refined regularity results show that also τ_j is a smooth function. For more on regularity of the flow of vector fields see e.g. appendix A in [5] and chapter 6 in [14].

Lortz shows in the appendix of his paper that one can find a constant $\bar{C}$ so that the bound

$$\|\tau_j\|_{2,\alpha} \leq \bar{C}, \tag{2.12}$$

holds for all j. Thus (τ_j) is a bounded sequence in $C^{2,\alpha}$. For any $0 < \lambda < \alpha$ it holds that the space $C^{2,\alpha}(\Omega, \mathbb{R})$ is compactly embedded in $C^{2,\lambda}(\Omega, \mathbb{R})$; this is essentially a direct consequence of the Arzela-Ascoli theorem. Hence we may assume that $\tau_j \to \tau$ in $C^{2,\lambda}$. We claim further that $\tau \in C^{2,\alpha}$. Indeed, given a multi-index γ such that $|\gamma| = 2$, we have that $D^\gamma \tau_j \to D^\gamma \tau$ uniformly. As (2.12) gives the bound

$$|D^\gamma \tau_j(x) - D^\gamma \tau_j(y)| \leq \bar{C}|x-y|^\alpha,$$

for all $x, y \in \Omega$ and $j \geq 1$, it follows by taking limits that also the estimate

$$|D^\gamma \tau(x) - D^\gamma \tau(y)| \leq \bar{C}|x-y|^\alpha, \tag{2.13}$$

holds for all $x, y \in \Omega$. This shows $\tau \in C^{2,\alpha}$. Also τ solves problem (2.8) by taking limits in (2.10).

The streamlines of u starting on M_1 cover Ω, and go from M_1 to M_1. The function q is defined on M_1 by letting $q(x)$ be the value obtained by integrating the equation $u \cdot \nabla\tau = 1$ along the streamline of u starting at x on M_1 for one period. q is then defined for points in Ω by letting it be constant on streamlines for u. The $C^{2,\alpha}$ reqularity of u and τ shows that also q is $C^{2,\alpha}$. That q is constant on streamlines means that it satisfies the reflection formula

$$q(-z) = q(z).$$

Additionally ∇q satisfies the symmetry relation (2.3b).

We now fix a scalar function $h \in C^{2,\alpha}$ with non-vanishing derivative, $h'(x) \not\equiv 0$. Let $\beta > 0$ be a positive number and define the function

$$H(x) := \beta h(q(x)).$$

We further define the vector field

$$\omega := \nabla H \times \nabla \tau = \beta h'(q) \nabla q \times \nabla \tau.$$

As we saw before, ω is continuous in Ω. ω should have $C^{1,\alpha}$ regularity in Ω. Using that $\nabla\tau, \nabla q$ are $C^{1,\alpha}$ in Ω_L and Ω_R, and using the symmetry $(2.3b)$ of ω shows that ω is even C^1 in Ω. Applying estimate (2.16) below gives the $C^{1,\alpha}$ regularity of ω in Ω.

ω shall serve as the curl of the vector field $u = u_0 + v$, with v a fixed point of the operator B. Such a vector field u has symmetry (2.3a). Hence ω should be divergence free and satisfy the symmetry (2.3b). To see that ω is divergence free we can compute, since both H and τ are twice continuously differentiable, that

$$\text{div } \omega = \text{div } (\nabla H \times \nabla \tau) = (\text{curl } \nabla H) \cdot \nabla \tau - (\text{curl } \nabla \tau) \cdot \nabla H = 0.$$

To check that ω satisfy symmetry (2.3b) we note that since q is constant on streamlines, ∇q has symmetry (2.3b) and $\nabla\tau$ has symmetry (2.3a), the cross product $\nabla q \times \nabla \tau$ has the symmetry (2.3b), and so also ω has this symmetry.

We now want to solve the following div-curl problem

$$\begin{aligned} \text{curl } w &= \omega, \text{ in } \Omega, \\ \text{div } w &= 0, \text{ in } \Omega, \\ w \cdot n &= 0, \text{ on } \partial\Omega, \\ \int_{M_1} w \cdot \mathrm{d}S &= 0. \end{aligned} \tag{2.14}$$

The solution should be unique and have $C^{2,\alpha}$ regularity. That a solution must be unique follows by the theory in [4]. To actually solve this problem we can look at another problem, namely

$$\begin{aligned} -\Delta\xi &= \omega, \text{ in } \Omega, \\ \text{div } \xi &= 0, \text{ on } \partial\Omega, \\ \xi \times n &= 0, \text{ on } \partial\Omega, \end{aligned} \tag{2.15}$$

for the unknown vector field ξ. Here Δ stands for the vector Laplacian, that is the Laplacian acting on each component of the vector field ξ. If ξ is a solution to problem (2.15) of sufficient regularity, we get a solution to the first three equations in problem (2.14) by setting $w = \text{curl } \xi$. Indeed, we can first check that ξ is actually divergence free in the whole of Ω. This follows by the fact that div $\xi = 0$ on the boundary $\partial\Omega$, and the calculation

$$\Delta \text{div } \xi = \text{div } \Delta\xi = -\text{div } \omega = 0,$$

since ω was assumed divergence free. Then the first equation in problem (2.14) follows by the calculation

$$\text{curl } w = \text{curl curl } \xi = \nabla \text{div } \xi - \Delta\xi = \omega.$$

The second equation in (2.14) follows by the fact that div curl = 0. Finally, the third equation follows by using Stokes' theorem. Indeed, let γ be a simply connected smooth loop in $\partial\Omega$, forming the boundary of the surface $\text{int}(\gamma)$. Then as ξ is orthogonal to $\partial\Omega$ we find that

$$\int_{\text{int}(\gamma)} (w \cdot n)\mathrm{d}S = \int_{\text{int}(\gamma)} (\text{curl } \xi \cdot n)\mathrm{d}S = \int_\gamma \xi \cdot \mathrm{d}l = 0.$$

To solve problem (2.14) completely we then have to consider the integral constraint. This follows immediately by using Stokes' theorem again.

We thus turn to problem (2.15). We shall argue for the existence of a unique solution with correct regularity, but not provide all the details. Using proposition B.1 we may find a sequence of smooth vector fields ω_j in Ω, uniformly bounded in the $C^{1,\alpha}$ norm, converging to the vector field ω in $C^{1,\lambda}$ for each

$0 < \lambda < \alpha$. We can solve problem (2.15) with ω replaced by ω_j to find a unique solution ξ_j. Indeed, that such a solution exist and is unique follows by considerations in section 5.9 of [12]; it is even shown that the vector field ξ_j is smooth in Ω. Schauder type estimates [2], the convergence of the ω_j in $C^{1,\lambda}$ for all $0 < \lambda < \alpha$, and the fact that the ω_j are uniformly bounded in $C^{1,\alpha}$ gives a uniform bound in $C^{3,\alpha}$ on the vector fields ξ_j, and shows that they are actually Cauchy in $C^{3,\lambda}$ for all $0 < \lambda < \alpha$. These considerations imply the existence of a vector field $\xi \in C^{3,\alpha}$, so that $\xi_j \to \xi$ in $C^{3,\lambda}$ for all $0 < \lambda < \alpha$. The vector field ξ is then seen to satisfy problem (2.15). In this way we can obtain a particular solution to (2.15). However, [12] actually shows uniqueness of the solution in our case, taking into account the fact that we assumed that the boundary of Ω is smooth, and that it has genus 1. Hence we can uniquely solve (2.15) with a vector field ξ in $C^{3,\alpha}$. Thus the vector field $w := \operatorname{curl} \xi$ has $C^{2,\alpha}$ regularity, and as we argued above, uniquely solves problem (2.14).

Now we can define our operator B. Given $v \in V_\mu$ we set $u = u_0 + v$, compute the functions q and τ, and obtain the function $H = \beta h(q)$. Then the vector field w is obtained as the unique solution to the problem (2.14), with $\omega = \nabla H \times \nabla \tau$. That w solves problem (2.14), has $C^{2,\alpha}$ regularity and satisfies the symmetry (2.3a) shows that w lies in the space V. We define our operator B as

$$\begin{aligned} B : V_\mu &\to V, \\ v &\to B(v) := w. \end{aligned}$$

As the solution to problem (2.14) is unique, this is well-defined.

Now, we shall consider this operator as defined on subsets of the Banach space X_1, i.e. its continuity will be measured via the $\|\cdot\|_{1,\alpha}$ norm. We shall proceed to show that if the parameter β is chosen sufficiently small, then B takes V_μ into V_μ, and is a contraction on this subset in the $C^{1,\alpha}$ topology. If we show that V_μ is a closed subset of X_1 in the $C^{1,\alpha}$ topology we can use Banach's fixed point theorem to show that the operator B has a uniquely determined fixed point v in V_μ. The corresponding vector field $u = u_0 + v$ will then turn out to solve the Euler equation (1.1) with boundary condition (1.2) for a suitably defined pressure P. This is the content of the following theorem.

Theorem 2.3
Assume $\beta > 0$ is chosen so small that $B : V_\mu \to V_\mu$. Let $v \in V_\mu$ be a fixed point of the operator B, and let $u := u_0 + v$. Then u is the velocity field of a solution to the Euler equation (1.1), with boundary condition (1.2).

Proof. If v is a fixed point of B, we have that $\operatorname{curl} u = \omega = \nabla H \times \nabla \tau$. Since q is constant on streamlines of u we find that

$$u \cdot \nabla q = 0,$$

and so we find that

$$u \cdot \nabla H = (\beta h'(q)) u \cdot \nabla q = 0.$$

Now we can compute

$$u \times \operatorname{curl} u = u \times \omega = u \times (\nabla H \times \nabla \tau) = (u \cdot \nabla \tau)\nabla H - (u \cdot \nabla H)\nabla \tau = \nabla H.$$

Since u is also divergence free and tangential to the boundary of Ω, we see that u is a solution to the Euler equation (1.3), with boundary condition (1.2). Defining the pressure via the formula

$$P := H - \frac{1}{2}|u|^2$$

then gives a solution (u, P) to (1.1). □

Hence it remains to show that V_μ is closed in X_1, and that the parameter β can be chosen so small that $B : V_\mu \to V_\mu$ is a contraction. We first show that V_μ is closed in X_1. Thus we let $(v_j)_{j=1}^{\infty}$ in V_μ converge to some vector field $v \in C^{1,\alpha}$. That the v_j have symmetry (2.3a) shows that $v \in X_1$. Now the fact that $\|v_j\|_{2,\alpha} \leq \mu$ for all j gives a subsequence $(v_{j_l})_{l=1}^{\infty}$ converging to a vector field $u \in C^2$ in the C^2 topology. We then see that actually $v = u \in C^2$, and $v_j \to v$ in C^2. Then the arguments giving (2.13) also gives that $v \in C^{2,\alpha}$, and we even have that $v \in V$. Pointwise for $x \neq y$ we have that

$$\begin{aligned} &\frac{\left(\sum_{i=1}^{3}\sum_{|\gamma|=2}|D^\gamma v^i(x) - D^\gamma v^i(y)|^2\right)^{1/2}}{|x-y|^\alpha} \\ &= \lim_{j\to\infty} \frac{\left(\sum_{i=1}^{3}\sum_{|\gamma|=2}|D^\gamma v_j^i(x) - D^\gamma v_j^i(y)|^2\right)^{1/2}}{|x-y|^\alpha}. \end{aligned}$$

With the notation of appendix A, we have for each j and $x \neq y$, the inequality

$$\frac{\left(\sum_{i=1}^{3}\sum_{|\gamma|=2}|D^\gamma v_j^i(x) - D^\gamma v_j^i(y)|^2\right)^{1/2}}{|x-y|^\alpha} \leq \mu - \sum_{l=0}^{2}\|v_j\|_l \,.$$

Letting $j \to \infty$ gives for each $x \neq y$ the inequality

$$\frac{\left(\sum_{i=1}^{3}\sum_{|\gamma|=2}|D^\gamma v^i(x) - D^\gamma v^i(y)|^2\right)^{1/2}}{|x-y|^\alpha} \leq \mu - \sum_{l=0}^{2}\|v\|_l$$

and so $[v]_{2,\alpha} \leq \mu - \sum_{l=0}^{2}\|v\|_l$, that is $\|v\|_{2,\alpha} \leq \mu$. Thus $v \in V_\mu$ and V_μ is closed in X_1.

We now turn to the problem of B mapping into V_μ. Given $v \in V_\mu$ we have the bound

$$\|B(v)\|_{2,\alpha} = \|w\|_{2,\alpha} \leq C^{(1)}\|\omega\|_{1,\alpha}\,,$$

with $C^{(1)}$ a positive constant only depending on Ω and α. Hence we shall turn to the norm $\|\omega\|_{1,\alpha}$. We can find a constant $C^{(2)} > 0$ for whihch we have the estimate

$$\|\omega\|_{1,\alpha} = \|(\beta h'(q))\nabla q \times \nabla \tau\|_{1,\alpha} \leq \beta C^{(2)} \|h\|_{2,\alpha} \|\nabla q\|_{1,\alpha} \|\nabla \tau\|_{1,\alpha} . \tag{2.16}$$

for all q, τ generated by vector fields $u = u_0 + v, v \in V_\mu$. Since h is a fixed function it suffices to estimate the norms of ∇q and $\nabla \tau$. These are considered in [9], and we obtain that there exists positive constants $C^{(3)}$ and $C^{(4)}$, depending on Ω, α, u_0 and μ, for which we have the bounds

$$\|\nabla q\|_{1,\alpha} \leq C^{(3)},$$
$$\|\nabla \tau\|_{1,\alpha} \leq C^{(4)},$$

compare also (2.12). Hence combining these estimates with (2.16) shows that

$$\|B(v)\|_{2,\alpha} \leq \beta C^{(5)}, \tag{2.17}$$

with $C^{(5)}$ being a positive constant depending on Ω, α, u_0, μ and h. To get that B maps V_μ into V_μ we thus have to choose $\beta \leq \frac{\mu}{C^{(5)}}$.

We now need to show that B is a contraction for a sufficiently small β. Recall that B should be a contraction when it is viewed as an operator defined on the closed subset V_μ of the Banach space X_1 equipped with its $C^{1,\alpha}$ topology. Hence, given $v_1, v_2 \in V_\mu$ we should estimate the difference

$$\|B(v_1) - B(v_2)\|_{1,\alpha} .$$

Analogously to our constant $C^{(1)}$ we can find a constant $C^{(6)} > 0$ so that

$$\|B(v_1) - B(v_2)\|_{1,\alpha} \leq C^{(6)} \|\omega_1 - \omega_2\|_{0,\alpha} , \tag{2.18}$$

with $\omega_j = \beta h'(q_j)\nabla q_j \times \nabla \tau_j$ being the corresponding vorticity derived from $u_j = u_0 + v_j$, $j = 1, 2$. We have that

$$\omega_1 - \omega_2 = \beta\Big(h'(q_1)\nabla q_1 \times \nabla \tau_1 - h'(q_2)\nabla q_2 \times \nabla \tau_2\Big).$$

Lortz ([9]) estimates the $C^{0,\alpha}$ norm of this difference $\omega_1 - \omega_2$ to eventually obtain a constant $C^{(7)}$ for which we have

$$\|\omega_1 - \omega_2\|_{0,\alpha} \leq \beta C^{(7)} \|v_1 - v_2\|_{1,\alpha} .$$

Combining this with (2.18) then implies that if we then choose

$$\beta < \min(\frac{\mu}{C^{(5)}}, \frac{1}{C^{(6)}C^{(7)}}),$$

we get an operator $B : V_\mu \to V_\mu$ which is a contraction, and hence has a unique fixed point in v in V_μ by Banach's fixed point theorem. According to theorem 2.3, this produces a solution $u = u_0 + v$ to the Euler equation (1.1) with boundary condition (1.2). Its vorticity is given by $\omega = \nabla H \times \nabla \tau$. τ solves the PDE (2.8) and $\nabla \tau$ is thus non-vanishing in Ω. $H = \beta h(q)$, and assuming h' is non-vanishing, u will be a rotational solution as long as q is not constant, i.e. as long as ∇q does not vanish identically. Banach's fixed point theorem also gives the uniqueness in that the obtained solution u is the only possible $C^{2,\alpha}$ solution with symmetry (2.3a) satisfying the estimate $\|u - u_0\|_{2,\alpha} \leq \mu$. The above considerations gives us theorem 2.2.

3 Flows in a periodic cylinder

We shall now consider flows in a cylindrical domain which have a certain periodicity condition. Let $0 < \alpha < 1$. Let (r, θ, z) be cylindrical coordinates in $\mathbb{R}^3$. To find rotational flow one can consider the problem

$$\begin{aligned} \text{curl } u &= \nabla H \times \nabla \tau && \text{in } \Omega^\epsilon, \\ \text{div } u &= 0 && \text{in } \Omega^\epsilon, \\ u \cdot n &= 0 && \text{on } \partial\Omega^\epsilon, \\ \int_{\{z=0\}} u \cdot \mathrm{d}S &= C. \end{aligned}$$

Here we have the domain

$$\Omega^\epsilon := \{(r, \theta, z) \mid 0 < r < d + \varepsilon\eta\},$$

where $d > 0$ is a fixed constant and $\eta = \eta(\theta, z)$ a fixed smooth function, which we take to be 2π periodic in θ and z. We shall further assume that the following symmetries hold

$$\begin{aligned} (u^1, u^2, u^3)(r, \theta, -z) &= (-u^1, -u^2, u^3)(r, \theta, z), \\ \eta(\theta, -z) &= \eta(\theta, z), \end{aligned}$$

with u^1, u^2, u^3 the components of the velocity field. The parameter $\epsilon^0 > 0$ is chosen sufficiently small, so that

$$d + \epsilon^0 \eta(\theta, z) > 0$$

for all θ and z, and we only consider $0 \leq \epsilon \leq \epsilon^0$. The vector field u should be 2π periodic in z. The functions H and τ are defined as in section 2. Here one can use the periodicity of η, and thus of the domain Ω^ϵ to identify multiples of 2π in z and obtain a toroidal domain. M_1 is then taken as the surface

$$M_1 := \{(r, \theta, z) \in \Omega^\epsilon \mid z = 0\} =: \{z = 0\}.$$

Then τ should solve the problem corresponding to (2.8), and q is defined on the surface $\{z = 0\}$ at the point $(r, \theta, 0)$ as the time it takes to travel along a streamline of u starting at $(r, \theta, 0)$ at time 0, until reaching the surface $\{z = 2\pi\}$. That is, $q(r, \theta, 0)$ is characterized by the equation

$$2\pi = \Phi_{z, q(r,\theta,0)}(r, \theta, 0),$$

with $\Phi_t(r, \theta, 0) = (\Phi_{r,t}(r, \theta, 0), \Phi_{\theta,t}(r, \theta, 0), \Phi_{z,t}(r, \theta, 0))$ the flow of u starting at $(r, \theta, 0)$ at time $t = 0$. q is then extended to Ω^ϵ as being constant on streamlines. Again, $H = \beta h(q)$ for a sufficiently small constant $\beta > 0$, and some $C^{2,\alpha}$ scalar function h. The fixed point method of section 2 applies to this case as well due to the periodicity of our domain, even though the domain itself is unbounded. Thus one looks at perturbations of an irrotational vector field and if the corresponding q is non-constant one obtain rotational solutions to Euler's equations. We do not intend to go into details of this, but shall instead turn our attention to the irrotational flow. We intend to show that the corresponding function q is non-constant for explicit choices of the function η, namely for $\eta = \cos(z)$ and $\eta = \cos(\theta)\cos(z)$. One can show continuous dependence of the flow map with respect to the vector field itself [5]. If we show that the irrotational flow has a non-constant q, it follows that for small enough perturbations also the perturbed vector field has non-constant q, and thus combining this with the fixed point method gives rotational flow. To show that q is non-constant for the irrotational field we shall first try to find this field, and thus we turn our attention to the problem

$$\begin{aligned} \text{curl } u &= 0 \qquad && \text{in } \Omega^\epsilon, \\ \text{div } u &= 0 && \text{in } \Omega^\epsilon, \\ u \cdot n &= 0 && \text{on } \partial\Omega^\epsilon, \\ \int_{\{z=0\}} u \cdot \mathrm{d}S &= \pi d^2. \end{aligned} \tag{3.1}$$

We shall consider flows u solving problem (3.1) which are 2π periodic in z. The fourth equation in problem (3.1) is the flux condition which uniquely specifies our irrotational flow. We have chosen it to be equal to πd^2 to simplify expressions that show up later on. One is of course free to choose any real constant in the flux constraint, which has the effect of multiplying the solution to (3.1) with a constant, but as long as it is non-zero the results we obtain in the following will not change.

Similar to section 2.2, let $\mu > 0$, and define the space of vector fields

$$V(\eta, \epsilon) := \{ v \in C^{2,\alpha}(\Omega^\epsilon) \mid v \text{ is } 2\pi \text{ periodic in } z, \text{div } v = 0, v \cdot n = 0 \ \},$$

and the subset

$$V_\mu(\eta, \epsilon) := \{ v \in V(\eta, \epsilon) \mid \ \|v\|_{2,\alpha} \leq \mu \},$$

with the norm taken in Ω^ϵ for $0 \leq z \leq 2\pi$.

Modifying the results in section 2 as explained above, we obtain the following theorem.

Theorem 3.1 *Let $0 < \alpha < 1$. Let $d > 0$ be a fixed constant and $\eta(\theta, z)$ a fixed smooth function, 2π periodic in θ and z. Let ϵ^0 be so that $d + \epsilon^0 \eta(\theta, z) > 0$ for all (θ, z), and define the domain Ω^ϵ, for $0 \leq \epsilon \leq \epsilon^0$. For each η there is an $\epsilon^0(\eta) \leq \epsilon^0$ for which the problem (3.1) has a unique smooth solution $u_0 = u_0(\epsilon)$, for each $0 \leq \epsilon \leq \epsilon^0(\eta)$. Fix $0 \leq \epsilon \leq \epsilon^0(\eta)$, and let $h_\epsilon \in C^{2,\alpha}(\Omega^\epsilon)$ be a fixed scalar function, 2π periodic in z, with derivative not identically zero. For each $\mu < \underline{u_0(\epsilon)}$ there is a $\beta(\mu) > 0$ and a vector field $v_\epsilon \in V_\mu(\eta, \epsilon)$ so that the vector field $\overline{u_\epsilon} = u_0(\epsilon) + v_\epsilon$ solves Euler's equation (1.3) with boundary condition (1.2)*

in Ω^ϵ. The vorticity ω_ϵ of u_ϵ is given by $\omega_\epsilon = \nabla H_\epsilon \times \nabla \tau_\epsilon$, with Bernoulli function $H_\epsilon = \beta h_\epsilon(q_\epsilon)$, and $\tau_\epsilon, q_\epsilon$ generated by u_ϵ.
For the choices $\eta = cos(z)$ and $\eta = cos(\theta)cos(z)$, the functions $q_0(\epsilon)$ generated by the $u_0(\epsilon)$ are non-constant for sufficiently small $\epsilon > 0$, and thus the u_ϵ are rotational solutions.

As mentioned earlier we shall not go into all details of this theorem. We shall however show that we can solve problem (3.1) for fixed η and sufficiently small $\epsilon \geq 0$, to find solutions $u_0(\epsilon)$. We shall also show that the functions $q_0(\epsilon)$ are non-constant for sufficiently small $\epsilon > 0$. From now on we drop the subscript 0 and dependence on ϵ of $u_0(\epsilon)$ and $q_0(\epsilon)$ to simplify notation.

We remark that if we consider the case of an ordinary circular torus as described in (2.4) we have the explicit irrotational field $u_0(r, \theta, z) = \frac{1}{r} e_\theta$. Here one can show that the corresponding q is non-constant. However, in the case of a circular periodic cylinder Ω^0 we can find the explicit irrotational field $u_0 = e_z$. In this case, q is constant. Thus the case we shall consider is topologically the same as that of a circular torus, but we do encounter a difference in that the q generated by the irrotational flow in Ω^0 is constant. In the case of the circular torus, one can use the fact that q corresponding to u_0 is non-constant to show that this holds true also for small perturbations of u_0, and of the domain, to obtain rotational solutions. In the case of the periodic cylinder Ω^0 we can not use any such arguments, and in Ω^ϵ we can not solve equation (3.1) explicitly. Thus we are forced to actually compute a Taylor expansion of q with respect to ϵ to show that q is not constant for sufficiently small $\epsilon > 0$. Here we have to actually specify the function η, and it turns out that the choices $\eta = \cos(z)$ and $\eta = \cos(\theta)\cos(z)$ are simple enough for us to perform the required calculations.

To solve problem (3.1) we shall first reformulate it. Ω^ϵ is a simply connected region in $\mathbb{R}^3$ and thus we may introduce a scalar potential φ for the vector field u, that is $u = \nabla\varphi$. However, φ will not be 2π periodic in z, as then the first three equations in problem (3.1) would force u to be identically zero, and thus not satisfy the integral constraint. However, we do have the condition that $\nabla\varphi$ should be 2π periodic in z. Problem (3.1) transforms to the problem

$$\begin{aligned} \Delta\varphi = 0 \qquad & \text{in } \Omega^\epsilon, \\ \nabla\varphi \cdot n = 0 \qquad & \text{on } \partial\Omega^\epsilon, \\ \int_{\{z=0\}} \partial_z \varphi(r, \theta, 0) r \mathrm{d}r \mathrm{d}\theta = \pi d^2. & \end{aligned} \tag{3.2}$$

The plan of the rest of this section is as follows. In section 3.2 we transform problem (3.2) for each ϵ to a problem in the domain Ω^0. We then use the implicit function theorem to find a unique solution for each ϵ sufficiently small. This gives solutions to problem (3.1). Sections 3.2-3.5 are devoted to showing that the functions q corresponding to the solutions of problem (3.1) are non-constant for sufficiently small $\epsilon > 0$ in the special cases of $\eta = \cos(z)$ and $\eta = \cos(\theta)\cos(z)$.

3.1 A flattening transform

Here we use the variables $(\bar{r}, \bar{\theta}, \bar{z})$ in the flat cylinder

$$\Omega^0 = \{(x, y, z) \mid \bar{r} < d\}.$$

Problem (3.2) is a boundary value problem where the boundary depends on the variable ϵ. To transform it to a problem in Ω^0 we shall introduce a flattening transformation. For this purpose we choose a function $\chi \in C^\infty(\mathbb{R})$ so that $\chi \geq 0,\ \chi' \geq 0$, and

$$\chi(s) = \begin{cases} 1 & s > \frac{3d}{4} \\ 0 & s < \frac{d}{4} \end{cases}.$$

We fix $\epsilon^0 < (2 \|\chi'\|_\infty \|\eta\|_\infty)^{-1}$ and consider $0 < \epsilon < \epsilon^0$. We can now introduce the flattening transformation $F_\epsilon : \Omega^0 \to \Omega^\epsilon$ via

$$F_\epsilon(\bar{r}, \bar{\theta}, \bar{z}) := (\bar{r} + \epsilon\chi(\bar{r})\eta(\bar{\theta}, \bar{z}), \bar{\theta}, \bar{z}) = (r, \theta, z).$$

We note that F_ϵ is the identity transformation for $\bar{r} < \frac{d}{4}$, and for $\epsilon = 0$ we see that F_0 is the identity transformation in Ω^0. Further, $1 + \epsilon\chi'(\bar{r})\eta(\bar{\theta}, \bar{z}) > 1 - \epsilon^0 \|\chi'\|_\infty \|\eta\|_\infty > \frac{1}{2}$. Thus F_ϵ is a diffeomorphism. In particular we find that $d + \epsilon\eta(\bar{\theta}, \bar{z}) > \frac{d}{4}$. Given $\varphi \in C^\infty(\Omega^\epsilon)$, we define the function $\bar{\varphi} \in C^\infty(\Omega^0)$ via

$$\bar{\varphi} := \varphi \circ F_\epsilon. \tag{3.3}$$

The function φ solves problem (3.2) if and only if $\bar{\varphi}$ solves the problem

$$\begin{aligned} \operatorname{div}(A_\epsilon \bar{\nabla} \bar{\varphi}) &= 0 \text{ in } \Omega^0, \\ A_\epsilon \bar{\nabla} \bar{\varphi} \cdot n &= 0 \text{ on } \partial\Omega^0, \\ I(\bar{\varphi}, \epsilon) &= \pi d^2, \end{aligned} \tag{3.4}$$

where $A_\epsilon = DF_\epsilon^{-1} DF_\epsilon^{-T} |\mathrm{det} DF_\epsilon|$, with everything in Cartesian coordinates, and

$$I(\bar{\varphi}, \epsilon) = \int_0^{2\pi} \int_0^d \partial_{\bar{z}} \bar{\varphi}(\bar{r}, \bar{\theta}, 0)(\bar{r} + \epsilon\chi(\bar{r})\eta(\bar{\theta}, 0))(1 + \epsilon\chi'(\bar{r})\eta(\theta, 0)) \mathrm{d}\bar{r} \mathrm{d}\bar{\theta},$$

where we use that $\partial_z \eta(\theta, 0) = 0$. We shall use the implicit function theorem to find a unique solution, $\bar{\varphi}_\epsilon$, to problem (3.4) for each ϵ, having sufficient regularity. Indeed, define the sets

$$\begin{aligned} \Omega_0^0 &:= \{(\bar{r}, \bar{\theta}, \bar{z}) \in \Omega^0 \mid 0 < \bar{z} < 2\pi\}, \\ S_0 &:= \{\bar{r} = d,\ 0 < \bar{\theta}, \bar{z} < 2\pi\}. \end{aligned}$$

We let $k \geq 2$ be an integer then define the spaces

$$\begin{aligned}
X &:= \{\bar{\varphi} \in C^{k,\alpha}(\Omega^0), \bar{\nabla}\bar{\varphi}\ 2\pi \text{ periodic in } \bar{z}\}/\mathbb{R},\\
Y_1 &:= \{f \in C^{k-2,\alpha}(\Omega^0), f\ 2\pi \text{ periodic in } \bar{z}\},\\
Y_2 &:= \{g \in C^{k-1,\alpha}(\partial\Omega^0), g\ 2\pi \text{ periodic in } \bar{z}\},\\
Y &:= \{(f,g) \in Y_1 \times Y_2 \mid \int_{\Omega_0^0} f \mathrm{d}\bar{V} = \int_{S_0} g \mathrm{d}\bar{S}\},\\
E &:= [0, \epsilon^0],
\end{aligned}$$

and equip these with the usual Hölder norms taken over the cell Ω_0^0. These are all Banach spaces. Indeed, it suffices to check that X is a Banach space, the rest is clear. So let $(\bar{\varphi}_j)_{j=1}^{\infty}$ be a Cauchy sequence in X. Then there are functions $\bar{\psi}_j$ in $C^{k,\alpha}$ which are 2π periodic in $\bar{z}$, and constants C_j, so that we have the representation $\bar{\varphi}_j = C_j\bar{z} + \bar{\psi}_j$. We have the explicit representation of the constants via $C_j = \frac{1}{2\pi}\int_0^{2\pi} \partial_{\bar{z}}\bar{\varphi}_j(\bar{r},\bar{\theta},\xi)\mathrm{d}\xi$. Since $k \geq 2$ we see that the constants also form a Cauchy sequence, and hence there is a constant C so that $C_j \to C$. From this we find by the triangle inequality that also $\bar{\psi}_j$ is a Cauchy sequence in $C^{k,\alpha}(\Omega_0^0)$. Hence there is a function $\bar{\psi}$ in $C^{k,\alpha}(\Omega_0^0)$ so that $\bar{\psi}_j \to \bar{\psi}$ in the $C^{k,\alpha}$ norm. The periodicity of the $\bar{\psi}_j$ shows that the function $\bar{\varphi} := C\bar{z} + \bar{\psi}$ exists in X, and is the limit of the $\bar{\varphi}_j$. It follows that X is a Banach space.

We define the map $\Psi : X \times E \to Y \times \mathbb{R}$ by

$$\Psi(\bar{\varphi},\epsilon) = \Big((\mathrm{div}(A_\epsilon\bar{\nabla}\bar{\varphi}), A_\epsilon\bar{\nabla}\bar{\varphi}\cdot n|_{\bar{r}=d}, I(\bar{\varphi},\epsilon) - \pi d^2\Big).$$

We note that a solution to (3.4) for $\epsilon = 0$ is the function

$$\bar{\varphi}_0 = \bar{z},$$

so that $\Psi(\bar{\varphi}_0, 0) = (0,0,0)$. Let $D_1\Psi(\bar{\varphi},\epsilon)$ be the Fréchet derivative of Ψ with respect to the first variable evaluated at the point $(\bar{\varphi},\epsilon) \in X \times E$. Then for $\bar{\varphi} \in X$

$$D_1\Psi(\bar{\varphi}_0,0)\bar{\varphi} = \begin{pmatrix} \Big(\bar{\Delta}\bar{\varphi}(\bar{r},\bar{\theta},\bar{z}), \partial_{\bar{r}}\bar{\varphi}(d,\bar{\theta},\bar{z})\Big) \\ \int_0^{2\pi}\int_0^d \partial_{\bar{z}}\bar{\varphi}(\bar{r},\bar{\theta},0)\bar{r}\mathrm{d}\bar{r}\mathrm{d}\bar{\theta} \end{pmatrix}$$

with $\bar{\Delta}$ the Laplacian in the $(\bar{r},\bar{\theta},\bar{z})$ coordinates. To use the implicit function theorem we need to show that $D_1\Psi(\bar{\varphi},0) : X \to Y \times \mathbb{R}$ is bijective. Thus, pick $\bar{\varphi} \in X$ so that $D_1\Psi(\bar{\varphi}_0,0)\bar{\varphi} = 0$, i.e. $\bar{\varphi}$ solves

$$\begin{aligned}
\bar{\Delta}\bar{\varphi} &= 0 \quad \text{in } \Omega^0,\\
\partial_{\bar{r}}\bar{\varphi} &= 0 \quad \text{on } \partial\Omega^0,\\
\int_0^{2\pi}\int_0^d \partial_{\bar{z}}\bar{\varphi}(\bar{r},\bar{\theta},0)\bar{r}\mathrm{d}\bar{r}\mathrm{d}\bar{\theta} &= 0.
\end{aligned} \tag{3.5}$$

As before, $\bar{\nabla}\bar{\varphi}$ is 2π periodic in $\bar{z}$, so $\bar{\varphi}$ can be written $\bar{\varphi} = C\bar{z} + \bar{\psi}$ with $\bar{\psi}$ 2π periodic in $\bar{z}$. Then $\bar{\psi}$ solves the first two equations in (3.5) so $\bar{\psi} = D$ =constant.

Thus $\bar{\varphi} = C\bar{z} + D$ and the integral in (3.5) shows that $\bar{\varphi} = D$. So modulo constant we have injectivity. We move to surjectivity. If $((f, g), c) \in Y \times \mathbb{R}$ we thus seek $\bar{\varphi} \in X$ so that $\bar{\varphi}$ solves

$$\begin{aligned} \bar{\Delta}\bar{\varphi} &= f \qquad \text{in } \ \Omega^0, \\ \partial_{\bar{r}}\bar{\varphi} &= g \quad \text{ on } \ \partial\Omega^0, \\ \int_0^{2\pi}\int_0^d \partial_{\bar{z}}\bar{\varphi}(\bar{r}, \bar{\theta}, 0)\bar{r}\mathrm{d}\bar{r}\mathrm{d}\bar{\theta} &= c, \end{aligned} \tag{3.6}$$

Again, $\bar{\varphi} = C\bar{z} + \bar{\psi}$ with $\bar{\psi} \in C^{k,\alpha}$ 2π periodic in $\bar{z}$ solving the first two problems in (3.6). That such a solution $\bar{\psi}$ exists and is unique modulo constants can be shown via classical theory; see e.g. [6]. Here the integral condition in the definition of the space Y is a necessary and sufficient condition for the existence of a solution, after identifying multiples of 2π in $\bar{z}$ in our domain. After this the constant C is determined uniquely by the integral condition in (3.6). So modulo constant we solve problem (3.6) uniquely in $C^{k,\alpha}$, and hence uniquely in X. Then $D_1\Psi(\bar{\varphi}_0, 0)$ is invertible and we can use the implicit function theorem (see appendix C) to find a unique solution curve $\epsilon \mapsto \bar{\varphi}(\bar{r}, \bar{\theta}, \bar{z}, \epsilon) \in X$ to the equation

$$\Psi(\bar{\varphi}, \epsilon) = 0,$$

for $0 \leq \epsilon \leq \epsilon^0$ sufficiently small. Clearly Ψ is smooth in ϵ and $D_1\Psi$ is linear in $\bar{\varphi}$. It follows that the solution $\bar{\varphi}(\bar{r}, \bar{\theta}, \bar{z}, \epsilon) =: \bar{\varphi}_\epsilon$ depends smoothly on ϵ, as a map from E into X. As the above arguments work for each integer $k \geq 2$ the solution can be taken to be as regular as we want. In particular, we choose k so large that the calculations we perform in the remaining part of the thesis are justified.

In any $C^{k,\alpha}$ norm we have the Taylor expansion

$$\bar{\varphi}_\epsilon = \bar{\varphi}_0 + \epsilon\bar{\varphi}_1 + \epsilon^2\bar{\varphi}_2 + \mathcal{O}(\epsilon^3).$$

Let $G_\epsilon = F_\epsilon^{-1}$. Then we can also expand $\varphi_\epsilon = \bar{\varphi}_\epsilon \circ G_\epsilon$. We can extend F_ϵ to a global diffeomorphism, and using the extension theorem 4 of chapter 6 in [11], any $C^{k,\alpha}$ function in $\bar{\Omega}^0$ to a $C^{k,\alpha}$ function in $\mathbb{R}^3$, for any k. We then obtain a Taylor expansion of the form

$$\varphi_\epsilon = \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2 + \mathcal{O}(\epsilon^3), \tag{3.7}$$

in the $C^{k,\alpha}(\mathbb{R}^3)$ norm. Since $\Delta\varphi_\epsilon = 0$ in Ω^ϵ for all ϵ, it follows that $\Delta\varphi_j = 0$ in Ω^0 for all j. By continuity it holds up the boundary.

We shall now find boundary and integral conditions for each φ_j in Ω^0 by expanding the corresponding conditions of problem (3.2) in ϵ. We use variables (r, θ, z) from now on. Since $n = (1, -\epsilon\frac{1}{r}\eta_\theta, -\epsilon\eta_z)$, the boundary condition in (3.2) has the explicit form

$$\partial_r\varphi_\epsilon = \epsilon\Big(\frac{1}{r^2}\eta_\theta\partial_\theta\varphi_\epsilon + \eta_z\partial_z\varphi_\epsilon\Big), \tag{3.8}$$

at $r = d + \epsilon\eta(\theta, z)$. We have the following expansion, holding uniformly in θ and z

$$\begin{aligned}
\partial_r\varphi_\epsilon(d + \epsilon\eta(\theta, z), \theta, z) &= \partial_r\varphi_0(d + \epsilon\eta(\theta, z), \theta, z) + \epsilon\partial_r\varphi_1(d + \epsilon\eta(\theta, z), \theta, z) \\
&\quad + \epsilon^2\partial_r\varphi_2(d + \epsilon\eta(\theta, z), \theta, z) + \mathcal{O}(\epsilon^3) \\
&= \partial_r\varphi_0(d, \theta, z) \\
&\quad + \epsilon\Big(\eta(\theta, z)\partial_r^2\varphi_0(d, \theta, z) + \partial_r\varphi_1(d, \theta, z)\Big) \\
&\quad + \epsilon^2\Big(\frac{1}{2}\eta(\theta, z)^2\partial_r^3\varphi_0(d, \theta, z) + \eta(\theta, z)\partial_r^2\varphi_1(d, \theta, z) + \partial_r\varphi_2(d, \theta, z)\Big) \\
&\quad + \mathcal{O}(\epsilon^3).
\end{aligned}$$

A similar expansion on the right hand side of (3.8) leads to the equations

$$\begin{aligned}
\partial_r\varphi_0 &= 0, \\
\partial_r\varphi_1 &= \frac{1}{d^2}\eta_\theta\partial_\theta\varphi_0 + \eta_z\partial_z\varphi_0 - \eta\partial_r^2\varphi_0, \\
\partial_r\varphi_2 &= -\frac{2}{d^3}\eta\eta_\theta\partial_\theta\varphi_0 + \frac{1}{d^2}\eta\eta_\theta\partial_\theta\partial_r\varphi_0 + \eta\eta_z\partial_z\partial_r\varphi_0 \\
&\quad - \frac{1}{2}\eta^2\partial_r^3\varphi_0 + \frac{1}{d^2}\eta_\theta\partial_\theta\varphi_1 + \eta_z\partial_z\varphi_1 - \eta\partial_r^2\varphi_1,
\end{aligned} \tag{3.9}$$

for $r = d$. Expanding the integral condition in (3.2) leads to the equations

$$\begin{aligned}
\int_0^{2\pi}\int_0^d \partial_z\varphi_0(r, \theta, 0)r\mathrm{d}r\mathrm{d}\theta &= \pi d^2, \\
\int_0^{2\pi}\int_0^d \partial_z\varphi_1(r, \theta, 0)r\mathrm{d}r\mathrm{d}\theta &= -d\int_0^{2\pi}\eta(\theta, 0)\partial_z\varphi_0(d, \theta, 0)\mathrm{d}\theta, \\
\int_0^{2\pi}\int_0^d \partial_z\varphi_2(r, \theta, 0)r\mathrm{d}r\mathrm{d}\theta &= -d\int_0^{2\pi}\eta(\theta, 0)\partial_z\varphi_1(d, \theta, 0)\mathrm{d}\theta \\
&\quad - d\int_0^{2\pi}\eta^2(\theta, 0)\partial_z\partial_r\varphi_0(d, \theta, 0)\mathrm{d}\theta \\
&\quad - \int_0^{2\pi}\eta^2(\theta, 0)\partial_z\varphi_0(d, \theta, 0)\mathrm{d}\theta.
\end{aligned} \tag{3.10}$$

We note that φ_0 satisfies the problem

$$\begin{aligned}
\Delta\varphi_0 &= 0 \text{ in } \Omega^0, \\
\partial_r\varphi_0 &= 0 \text{ on } \partial\Omega^0, \\
\int_0^{2\pi}\int_0^d \partial_z\varphi_0(r, \theta, 0)r\mathrm{d}r\mathrm{d}\theta &= \pi d^2,
\end{aligned}$$

with the condition that φ_0 is 2π periodic in z. Hence $\varphi_0 = z$ modulo additive constants.

Using the relations (3.9) and (3.10), we shall compute an explicit expression for φ_ϵ up to second order in ϵ in the special cases when the boundary profile η has the form

$$\eta(\theta, z) = \cos(z).$$

and

$$\eta(\theta, z) = \cos(\theta)\cos(z).$$

These choices are simple enough for us to perform the required calculations, and subsequently show that function q_ϵ corresponding to the vector field u_ϵ is non-constant for small positive ϵ. This occupies the remaining part of this thesis.

3.2 u in the case of $\eta(\theta, z) = \cos(z)$

We start by finding φ_1. Since $\varphi_0 = z$, we find the problem for φ_1 in Ω^0 to be

$$\begin{aligned} \Delta\varphi_1 &= 0 \text{ in } \Omega^0, \\ \partial_r\varphi_1 &= -\sin(z) \text{ on } \partial\Omega^0, \\ \int_0^{2\pi}\int_0^d \partial_z\varphi_1(r,\theta,0)r\mathrm{d}r\mathrm{d}\theta &= -2\pi d. \end{aligned}$$

Since $\nabla\varphi_1$ is 2π periodic in z, the solution is unique modulo additive constants. We find that

$$\varphi_1(r,\theta,z) = -\frac{I_0(r)}{I_1(d)}\sin(z). \tag{3.11}$$

Here $I_\nu(t)$ denotes a modified Bessel function of the first kind. For more on these, see e.g. [1]. We have used the following identity for derivatives,

$$I_\nu^{(k)}(r) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}I_{\nu-k+2j}(r),\ \nu \geq 0, k \in \mathbb{N},$$

which is found as relation 9.6.29 in [1]. It holds that $I_{-1}(r) = I_1(r)$, so in particular we see that $I_0'(r) = I_1(r)$. The integral condition can be checked by using the relation

$$\int_0^x r^\nu I_{\nu-1}(r)\mathrm{d}r = x^\nu I_\nu(x),\ \nu > 0, x \in \mathbb{R},$$

which can be found as relation 11.3.25 in [1]. These identities will be used without reference throughout the remaining parts of this thesis.

Using (3.11), we find the equation for φ_2 to be

$$\begin{aligned} \Delta\varphi_2 &= 0 \text{ in } \Omega^0, \\ \partial_r\varphi_2 &= \frac{1}{4I_1(d)}\Big(3I_0(d) + I_2(d)\Big)\sin(2z) \text{ on } \partial\Omega^0, \\ \int_0^{2\pi}\int_0^d \partial_z\varphi_2(r,\theta,0)r\mathrm{d}r\mathrm{d}\theta &= A, \end{aligned}$$

for some constant A. Here the solution is again unique modulo constants, and we find that

$$\varphi_2(r,\theta,z) = Cz + \frac{(3I_0(d)+I_2(d))}{8I_1(d)I_1(2d)} I_0(2r)\sin(2z), \tag{3.12}$$

where C is chosen to make the integral condition hold. The exact expression for this constant is not important for our purposes. If $u(r,\theta,z,\epsilon) = (u^1,u^2,u^3)(r,\theta,z,\epsilon)$ is the vector field corresponding to the potential φ, we have the expansion

$$u(r,\theta,z,\epsilon) = u_0(r,\theta,z) + \epsilon u_1(r,\theta,z) + \epsilon^2 u_2(r,\theta,z) + \mathcal{O}(\epsilon^3), \tag{3.13}$$

with $u_j = \nabla\varphi_j = (\partial_r\varphi_j, \frac{1}{r}\partial_\theta\varphi_j, \partial_z\varphi_j)$. Hence we can determine u up to second order in ϵ by determining its components u^i, $i=1,2,3$. By (3.11) and (3.12) we find that

$$\begin{aligned}
u^1(r,\theta,z,\epsilon) &= -\epsilon\frac{I_1(r)}{I_1(d)}\sin(z) + \frac{(3I_0(d)+I_2(d))}{4I_1(d)I_1(2d)} I_1(2r)\sin(2z) + \mathcal{O}(\epsilon^3),\\
u^2(r,\theta,z,\epsilon) &= \mathcal{O}(\epsilon^3),\\
u^3(r,\theta,z,\epsilon) &= 1 - \epsilon\frac{I_0(r)}{I_1(d)}\cos(z)\\
&\qquad + \epsilon^2\Big(C + \frac{(3I_0(d)+I_2(d))}{4I_1(d)I_1(2d)} I_0(2r)\cos(2z)\Big) + \mathcal{O}(\epsilon^3),
\end{aligned} \tag{3.14}$$

3.3 q in the case of $\eta(\theta,z) = \cos(z)$

We define the function $q = q(r,\theta,\epsilon)$ as the time it takes to travel along a streamline of the vector field $u(r,\theta,z,\epsilon)$ for one period, starting on the surface $\{z=0\}$. q is then extended by being constant on streamlines of u. In this case $\eta = \cos(z)$ so the period is 2π, and q is defined as follows. Let $\Phi_t(r,\theta,0,\epsilon)$ be the flow of u starting at the point $(r,\theta,0)$. Then $q(r,\theta,\epsilon)$ is obtained via

$$2\pi = \Phi_{z,q(r,\theta,\epsilon)}(r,\theta,0,\epsilon). \tag{3.15}$$

Here $\Phi_t = (\Phi_{r,t}, \Phi_{\theta,t}, \Phi_{z,t})$. Taylor expanding the right hand side of (3.15) with respect to ϵ we find an expression of form

$$2\pi = 2\pi + \epsilon a_1 + \frac{\epsilon^2}{2}a_2 + \mathcal{O}(\epsilon^3).$$

Thus $a_j = 0$ for $j \geq 1$. We find that

$$a_1 = \partial_\epsilon\big[\Phi_{z,q(r,\theta,\epsilon)}(r,\theta,0,\epsilon)\big]\big|_{\epsilon=0}.$$

Clearly $\Phi_t(r,\theta,0,0) = (r,\theta,t)$ and $q(r,\theta,0) = 2\pi$, so $a_1 = 0$ is equivalent to

$$\partial_\epsilon q(r,\theta,0) = -(\partial_\epsilon\Phi)_{z,2\pi}(r,\theta,0,0).$$

We have

$$\Phi_{z,t}(r,\theta,0,\epsilon) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\Phi_{z,s}(r,\theta,0,\epsilon)\mathrm{d}s = \int_0^t u^3(\Phi_s(r,\theta,0,\epsilon),\epsilon)\mathrm{d}s. \tag{3.16}$$

We then find by (3.11), (3.12) and (3.14) and the expansion of u in terms of ϵ, that

$$(\partial_\epsilon\Phi)_{z,2\pi}(r,\theta,0,0) = \int_0^{2\pi} \partial_z\varphi_1(r,\theta,s)\mathrm{d}s = 0.$$

Thus

$$\partial_\epsilon q(r,\theta,0) = 0. \tag{3.17}$$

Then one has that

$$0 = a_2 = \partial_\epsilon^2\big[\Phi_{z,q(r,\theta,\epsilon)}(r,\theta,0,\epsilon)\big]\big|_{\epsilon=0}$$

Using (3.17) this means

$$\partial_\epsilon^2 q(r,\theta,0) = -(\partial_\epsilon^2\Phi)_{z,2\pi}(r,\theta,0,0).$$

We find that

$$\begin{aligned}(\partial_\epsilon^2\Phi)_{z,2\pi}(r,\theta,0,0) = 2\int_0^{2\pi} &((\partial_\epsilon\Phi)_s(r,\theta,0,0)\cdot(\partial_r,\partial_\theta,\partial_z)u_1^3)(\Phi_s(r,\theta,0,0))\mathrm{d}s\\ &+ 2\int_0^{2\pi} u_2^3(\Phi_s(r,\theta,0,0))\mathrm{d}s.\end{aligned}$$

The expression for φ_2 in (3.12) shows that the integral of $u_2^3 = \partial_z\varphi_2$ is equal to $4\pi C$. Thus we should turn to the other integral. As in (3.16) we can compute $(\partial_\epsilon\Phi)_{r,s}$ and $(\partial_\epsilon\Phi)_{z,s}$. As $u_1^3 = \partial_z\varphi_1$ is independent of θ by (3.11), the θ term is not important. We find that

$$(\partial_\epsilon\Phi)_{r,s}(r,\theta,0,0) = \frac{I_1(r)}{I_1(d)}(\cos(s)-1),$$

and

$$(\partial_\epsilon\Phi)_{z,s}(r,\theta,0,0) = -\frac{I_0(r)}{I_1(d)}\sin(s).$$

Straightforward computations then show that

$$\int_0^{2\pi}((\partial_\epsilon\Phi)_s(r,\theta,0,0)\cdot(\partial_r,\partial_\theta,\partial_z)u_1^3)(\Phi_s(r,\theta,0,0))\mathrm{d}s = -\frac{\pi}{I_1(d)^2}\Big(I_0(r)^2+I_1(r)^2\Big).$$

Then

$$\partial_\epsilon^2 q(r,\theta,0) = \frac{2\pi}{I_1(d)^2}\Big(I_0(r)^2+I_1(r)^2\Big) - 4\pi C =: f(r). \tag{3.18}$$

We note that f is a non-trivial function of r. Now Taylor's formula gives

$$q(r,\theta,\epsilon) = 2\pi + \frac{\epsilon^2}{2}f(r) + \int_0^\epsilon \frac{(\epsilon-t)^2}{2}\partial_\epsilon^3 q(r,\theta,t)\mathrm{d}t. \tag{3.19}$$

Thus q is non-constant for small positive ϵ.

3.4 u in the case of $\eta(\theta, z) = \cos(\theta)\cos(z)$

Here the problem for φ_1 reads

$$\begin{aligned}\Delta\varphi_1 &= 0 \text{ in } \Omega^0,\\ \partial_r\varphi_1 &= -\cos(\theta)\sin(z) \text{ on } \partial\Omega^0,\\ \int_0^{2\pi}\int_0^d \partial_z\varphi_1(r,\theta,0) r\mathrm{d}r\mathrm{d}\theta &= 0.\end{aligned}$$

Here we find the solution

$$\varphi_1(r,\theta,z) = -\frac{I_1(r)}{I_1'(d)}\cos(\theta)\sin(z). \tag{3.20}$$

Using this expression for φ_1, we find the problem for φ_2 as

$$\begin{aligned}\Delta\varphi_2 &= 0 \text{ in } \Omega^0,\\ \partial_r\varphi_2 &= A_1\sin(2z) + A_2\cos(2\theta)\sin(2z) \text{ on } \partial\Omega^0,\\ \int_0^{2\pi}\int_0^d \partial_z\varphi_2(r,\theta,0) r\mathrm{d}r\mathrm{d}\theta &= A,\end{aligned}$$

with the constants A_1, A_2 given by

$$\begin{aligned}A_1 &= \frac{1}{4dI_1'(d)}\Big(dI_1''(d) + (d - \frac{1}{d})I_1(d)\Big),\\ A_2 &= \frac{1}{4dI_1'(d)}\Big(dI_1''(d) + (d + \frac{1}{d})I_1(d)\Big),\end{aligned}$$

and A some unimportant constant. From this we can see that φ_2 is given by

$$\varphi_2(r,\theta,z) = Cz + B_1I_0(2r)\sin(2z) + B_2I_2(2r)\cos(2\theta)\sin(2z), \tag{3.21}$$

with the constants B_1, B_2 given by

$$\begin{aligned}B_1 &= \frac{1}{2I_1(2d)}A_1,\\ B_2 &= \frac{1}{2I_2'(2d)}A_2.\end{aligned} \tag{3.22}$$

The constant C is chosen so that the integral condition holds. We can now find the components $u^j, j = 1, 2, 3$ of u as

$$
\begin{aligned}
u^1(r,\theta,z,\epsilon) &= -\epsilon \frac{I_1'(r)}{I_1'(d)} \cos(\theta)\sin(z) \\
&\quad + \epsilon^2 \Big(2B_1 I_1(2r) + 2B_2 I_2'(2r)\cos(2\theta) \Big) \sin(2z) \\
&\quad + \mathcal{O}(\epsilon^3) \\
u^2(r,\theta,z,\epsilon) &= \epsilon \frac{1}{r} \frac{I_1(r)}{I_1'(d)} \sin(\theta)\sin(z) \\
&\quad - 2\epsilon^2 \frac{1}{r} B_2 I_2(2r)\sin(2\theta)\sin(2z) + \mathcal{O}(\epsilon^3), \\
u^3(r,\theta,z,\epsilon) &= 1 - \epsilon \frac{I_1(r)}{I_1'(d)} \cos(\theta)\cos(z) \\
&\quad + \epsilon^2 \Big(C + 2\Big(B_1 I_0(2r) + B_2 I_2(2r)\cos(2\theta) \Big) \cos(2z) \Big) \\
&\quad + \mathcal{O}(\epsilon^3),
\end{aligned} \tag{3.23}
$$

3.5 q in the case of $\eta(\theta, z) = \cos(\theta)\cos(z)$

We let $\Phi_t(r,\theta,0,\epsilon)$ and $q(r,\theta,\epsilon)$ be defined in the same way as in section 3.3. Again one has that

$$
\begin{aligned}
\Phi_t(r,\theta,0,0) &= (r,\theta,t), \\
q(r,\theta,0) &= (r,\theta,2\pi).
\end{aligned}
$$

As in section 3.3 we find that

$$\partial_\epsilon q(r,\theta,0) = 0,$$

so we have to move to the second order derivative. As in section 3.3 we have that

$$\partial_\epsilon^2 q(r,\theta,0) = -(\partial_\epsilon^2 \Phi)_{z,2\pi}(r,\theta,0,0),$$

and

$$
\begin{aligned}
(\partial_\epsilon^2 \Phi)_{z,2\pi}(r,\theta,0,0) = 2\int_0^{2\pi} &((\partial_\epsilon \Phi)_s(r,\theta,0,0)\cdot(\partial_r,\partial_\theta,\partial_z)u_1^3)(\Phi_s(r,\theta,0,0))\mathrm{d}s \\
&+ 2\int_0^{2\pi} u_2^3(\Phi_s(r,\theta,0,0))\mathrm{d}s.
\end{aligned}
$$

Using $u_2^3 = \partial_z \varphi_2$ and (3.21) we find that

$$2\int_0^{2\pi} u_2^3(\Phi_s(r,\theta,0,0))\mathrm{d}s = 4\pi C. \tag{3.24}$$

We find that

$$
\begin{aligned}
(\partial_\epsilon \Phi)_{r,s}(r,\theta,0,0) &= \frac{I_1'(r)}{I_1'(d)}\cos(\theta)(\cos(s)-1),\\
(\partial_\epsilon \Phi)_{\theta,s}(r,\theta,0,0) &= -\frac{I_1(r)}{rI_1'(d)}\sin(\theta)(\cos(s)-1),\\
(\partial_\epsilon \Phi)_{z,s}(r,\theta,0,0) &= -\frac{I_1(r)}{I_1'(d)}\cos(\theta)\sin(s).
\end{aligned}
$$

Using $u_1^3 = \partial_z \varphi_1$ and (3.20) we obtain after straightforward calculations that

$$
\begin{aligned}
&2\int_0^{2\pi}((\partial_\epsilon \Phi)_s(r,\theta,0,0)\cdot(\partial_r,\partial_\theta,\partial_z)u_1^3)(\Phi_s(r,\theta,0,0))\mathrm{d}s\\
&= -\frac{2\pi}{(I_1'(d))^2}\Big((I_1^2(r)+(I_1'(r))^2)\cos^2(\theta)+\frac{1}{r}I_1^2(r)\sin^2(\theta)\Big) := -g(r,\theta)
\end{aligned}
\tag{3.25}
$$

It follows from (3.25) and (3.24) that

$$
\partial_\epsilon^2 q(r,\theta,0) = g(r,\theta) - 4\pi C, \tag{3.26}
$$

so that again $q(r,\theta,\epsilon)$ is non-constant in r and θ for small ϵ.

.

A Hölder spaces

Let $0 < \alpha \leq 1$. Let d, k, n be a non-negative integers. Let Ω be an open subset of $\mathbb{R}^d$. For a vector field $u = (u^1, ..., u^n) : \Omega \to \mathbb{R}^n$ with bounded and continuous derivatives up to order k, we define for integers $0 \leq j \leq k$, the semi-norms

$$\|u\|_j := \sup_{x\in\Omega}\Big(\sum_{i=1}^{n}\sum_{|\gamma|=j}|D^\gamma u^i(x)|^2\Big)^{1/2}.$$

Here $\gamma = (\gamma_1, ..., \gamma_d)$ is a multi-index of non-negative integers, and $|\gamma| := \gamma_1 + ... + \gamma_d$ its size. $D^\gamma := \partial_{\gamma_1} \cdots \partial_{\gamma_d}$, with $\partial_i := \partial_{x_i} = \frac{\partial}{\partial_{x_i}}$, is interpreted as a differential operator.

We define additional semi-norms, depending on α and k, via

$$[u]_{k,\alpha} := \sup_{x,y\in\Omega, x\neq y}\frac{\Big(\sum_{i=1}^{n}\sum_{|\gamma|=k}|D^\gamma u^i(x) - D^\gamma u^i(y)|^2\Big)^{1/2}}{|x-y|^\alpha}.$$

Here $|x|$ denotes the Euclidean norm for points $x \in \mathbb{R}^d$.

We denote by

$$C^{k,\alpha}(\Omega, \mathbb{R}^n)$$

the collection of vector fields with bounded and continuous derivatives up to order k, for which the norm

$$\|u\|_{k,\alpha} := \sum_{j=0}^{k}\|u\|_j + [u]_{k,\alpha}$$

is finite. To simplify notation we also write $C^{k,\alpha}$ instead of $C^{k,\alpha}(\Omega, \mathbb{R}^n)$ when there is no risk of confusion to what we mean. Each $C^{k,\alpha}(\Omega, \mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{k,\alpha}$ is a Banach space.

B Approximation

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary. Let $0 < \alpha < 1$ be a real number and $k \geq 2$ an integer. We shall consider the following proposition.

Proposition B.1. Let $u : \Omega \to \mathbb{R}^3$ be a vector field of type $C^{k,\alpha}$. Then there is a sequence of smooth vector fields u_j converging to u in $C^{k,\lambda}$ for all $0 < \lambda < \alpha$. Further, the u_j may be chosen so that

$$\|u_j\|_{k,\alpha} \leq C\,\|u\|_{k,\alpha}\,,$$

for all j. Here $C > 0$ is a constant, independent of j.

Proof. We prove it without loss of generality under the assumption that $u : \Omega \to \mathbb{R}$ is a function of type $C^{k,\alpha}$, as this simplifies the exposition considerably. We shall first extend u to be defined on the whole of $\mathbb{R}^3$, with compact support. This can be achieved by using the extension theorem 4 of chapter 6 in [11], giving a continuous linear extension operator

$$E : C^{k,\alpha}(\Omega, \mathbb{R}) \to C^{k,\alpha}(\mathbb{R}^3, \mathbb{R}),$$

and multiplication with a cut-off function. Hence we can assume that $u \in C^{k,\alpha}(\mathbb{R}^3, \mathbb{R})$ with compact support. Now we choose a smooth function $0 \leq \chi \leq 1$ with compact support, so that χ is identically equal to 1 in a neighborhood of 0 with

$$\int_{\mathbb{R}^3} \chi(x) \mathrm{d}x = 1.$$

Choosing a sequence $\epsilon_j > 0$, $j = 1, 2, 3, ...$, so that $\lim_{j\to\infty} \epsilon_j = 0$, we set

$$\chi_j(x) := \frac{1}{\epsilon_j^3} \chi(\frac{x}{\epsilon_j}),$$

and then define

$$u_j(x) := u * \chi_j(x),$$

where $*$ denotes convolution. The functions u_j are smooth with compact support, and it is easy to see that $D^\gamma u_j$ converge to $D^\gamma u$ uniformly for each multi-index γ such that $|\gamma| \leq k$. It thus suffices to check that

$$[u_j - u]_{k,\lambda} \to 0,$$

for each $0 < \lambda < \alpha$. So choose a multi-index γ with $|\gamma| = k$. Let $x \neq y$. We shall show that

$$\frac{|D^\gamma u(x) - D^\gamma u_j(x) - (D^\gamma u(y) - D^\gamma u_j(y))|}{|x - y|^\lambda},$$

goes to zero uniformly in x, y as $j \to \infty$. This will suffice. We first let $0 < \delta < 1$ be some fixed real number. Suppose now that $|x - y| \leq \delta$. We find that

$$\begin{aligned}
&\frac{|D^\gamma u(x) - D^\gamma u_j(x) - (D^\gamma u(y) - D^\gamma u_j(y))|}{|x - y|^\lambda} \\
&\leq \int \frac{|\chi_j(z)|}{|x - y|^\lambda} |D^\gamma u(x) - D^\gamma u(x - z) - (D^\gamma u(y) - D^\gamma u(y - z))| \mathrm{d}z \\
&= \int \frac{|\chi_j(z)|}{|x - y|^{\lambda - \alpha}} \frac{|D^\gamma u(x) - D^\gamma u(x - z) - (D^\gamma u(y) - D^\gamma u(y - z))|}{|x - y|^\alpha} \mathrm{d}z \\
&\leq \int \frac{|\chi_j(z)|}{|x - y|^{\lambda - \alpha}} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x - y|^\alpha} \mathrm{d}z \\
&+ \int \frac{|\chi_j(z)|}{|x - y|^{\lambda - \alpha}} \frac{|D^\gamma u(x - z) - D^\gamma u(y - z)|}{|x - y|^\alpha} \mathrm{d}z.
\end{aligned}$$

Using that $x - y = (x - z) - (y - z)$ we get the estimate

$$\begin{aligned}
&\frac{|D^\gamma u(x) - D^\gamma u_j(x) - (D^\gamma u(y) - D^\gamma u_j(y))|}{|x-y|^\lambda} \\
&\leq 2\,\|u\|_{k,\alpha}\,|x-y|^{\alpha-\lambda}\int \chi_j(z)\mathrm{d}z \\
&\leq 2\,\|u\|_{k,\alpha}\,\delta^{\alpha-\lambda}.
\end{aligned} \tag{1}$$

We then turn to the case that $|x - y| \geq \delta$. Then one has that $\frac{1}{|x-y|^\lambda} \leq \frac{1}{\delta^\lambda}$. Let $\mu > 0$ be given. By the compact support of χ and uniform continuity of $D^\gamma u$ we can choose a $j_0 = j_0(\mu)$ so that for all $j \geq j_0$ we get that

$$\begin{aligned}
&\frac{|D^\gamma u(x) - D^\gamma u_j(x) - (D^\gamma u(y) - D^\gamma u_j(y))|}{|x-y|^\lambda} \\
&\leq \int \frac{|\chi_j(z)|}{|x-y|^\lambda}|D^\gamma u(x) - D^\gamma u(x-z) - (D^\gamma u(y) - D^\gamma u(y-z))|\mathrm{d}z \\
&\leq 2\mu\delta^{-\lambda}\int \chi_j(z)\mathrm{d}z \\
&= 2\mu\delta^{-\lambda}
\end{aligned} \tag{2}$$

From this we obtain that

$$\begin{aligned}
&[u - u_j]_{k,\lambda} \\
&= \sup_{x\neq y}\frac{|D^\gamma u(x) - D^\gamma u_j(x) - (D^\gamma u(y) - D^\gamma u_j(y))|}{|x-y|^\lambda} \\
&\leq \sup_{\substack{x\neq y\\ |x-y|\leq\delta}}\frac{|D^\gamma u(x) - D^\gamma u_j(x) - (D^\gamma u(y) - D^\gamma u_j(y))|}{|x-y|^\lambda} \\
&+ \sup_{\substack{x\neq y\\ |x-y|\geq\delta}}\frac{|D^\gamma u(x) - D^\gamma u_j(x) - (D^\gamma u(y) - D^\gamma u_j(y))|}{|x-y|^\lambda} \\
&\leq 2\,\|u\|_{k,\alpha}\,\delta^{\alpha-\lambda} + 2\mu\delta^{-\lambda}.
\end{aligned} \tag{3}$$

From this we get convergence. Indeed, given $\epsilon > 0$ we choose $\delta > 0$ so small that $2\,\|u\|_{k,\alpha}\,\delta^{\alpha-\lambda} \leq \frac{\epsilon}{2}$. Note that we used that $\alpha - \lambda > 0$ here. Since $\mu > 0$ was arbitrary we choose μ so small that $2\mu\delta^{-\lambda} \leq \frac{\epsilon}{2}$. With these choices of δ and μ we find for all $j \geq j_0(\mu)$ that $[u - u_j]_{k,\lambda} \leq \epsilon$. These considerations gives the convergence of the u_j to u in $C^{k,\lambda}$. Since $0 < \lambda < \alpha$ was arbitrary, this proves one part of the proposition.

It is straightforward to check that the estimate

$$\|u_j\|_{k,\alpha} \leq \|u\|_{k,\alpha}\,,$$

hold for each j. Since the extension operator E is continuous, the rest of the proposition follows. $\square$

C The implicit function theorem

In this section we simply state the implicit function theorem. We refer to chapter 16 of [13] for a more detailed discussion of this theorem.

Let V, W, X, Y, Z be Banach spaces. Let $k \geq 1$ be an integer, and let $C^k(V, W)$ denote the k times Fréchet differentiable maps $\Psi : V \to W$. If $\Psi \in C^k(X \times Y, Z)$ and $(x_0, y_0) \in X \times Y$, we let $D_1\Psi(x_0, y_0)$ be the Fréchet derivative of Ψ with respect to the variable $x \in X$ evaluated at the point (x_0, y_0).

Implicit function theorem Let $\Psi \in C^k(X \times Y, Z)$. Let $(x_0, y_0) \in X \times Y$ be points so that $\Psi(x_0, y_0) = 0$, and $D_1\Psi(x_0, y_0)$ is invertible. Then there exist open sets $x_0 \in U_{x_0} \subset X$ and $y_0 \in U_{y_0} \subset Y$, and a unique map $g \in C^k(U_{y_0}, U_{x_0})$ so that $\Psi(g(y), y) = 0$ for all $y \in U_{y_0}$. Furthermore, $\Psi(x, y) = 0$ for $(x, y) \in U_{x_0} \times U_{y_0}$ if and only if $x = g(y)$.

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Master's Theses in Mathematical Sciences 2020:E7
ISSN 1404-6342

LUNFMA-3114-2020

Mathematics
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund, Sweden

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