A short book on fractions, symbols, and hyperbolics and what they have in common

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The study of Diophantine approximation goes back to the third century, and continues even today as an active area of research. Since first studied it has been found in many branches of mathematics. This can make it both more interesting and more difficult to learn about. This thesis aims to be an educative text to introduce the reader to the background of Diophantine approximation, as well as two very distinct case studies.

How do you find good approximations for numbers that cannot be expressed with only addition, multiplication, and division? That is to say, how do you best describe an irrational number using only rational numbers? This is the essence of Diophantine approximation. The basis of the subject is that for any irrational number it is possible to find a rational number that is arbitrarily close to it. You have probably even done something similar yourself at some point. For example, the constant pi can be approximated in this way by $\pi \approx 3.14 = 314/100$. One might make a game out of this to try and find the smallest numerator and denominator that is closer, and find that $\pi \approx 22/7$ is an even better approximation. In fact, 22/7 happens to be the best approximation until 333/106, in terms of the size of the denominators. This kind of thinking eventually leads to the theory of continued fractions, which through a chain of divisions and sums are able to encode any number as a sequence of whole number *coefficients*. This is an important statement, as it deeply ties together the concepts of Diophantine approximation and continued fractions. In particular, it is the appearance of the coefficients in surprising places that are the subject of exploration in this paper. We look at two specific cases. The first is through the world of symbolic sequences, and their interpretation as the dynamics of rotations on a circle. The other is the world of hyperbolic geometry, where we go from the very basics to the geodesic intersection on a hyperbolic tiling. What is important is that these seemingly unrelated topics both turn out to be rooted in the theory of Diophantine approximation. It is sometimes said that one of the last people to know all branches of mathematics of their time was Henri Poincaré, a mathematician who lived between 1854 and 1912. This stands as a reverent testament to the sheer breadth of human knowledge today, as what is known has expanded ever faster. It is in this spirit the paper tries to inspire thought on the importance of studying a broad range of mathematical topics. The fruits of this paper, then, is an introduction to the theory linking together very different topics that are beautiful in their own right, to a new generation of mathematicians.