Dynamical Borel–Cantelli Lemmas and Applications

Master's thesis

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Abstract

The classical Borel–Cantelli lemma is a beautiful discovery with wide applications in the mathematical field. The Borel–Cantelli lemmas in dynamical systems are particularly fascinating. Here, D. Kleinbock and G. Margulis [1] have given an important sufficient condition for the strongly Borel–Cantelli sequence, which is based on the work of W. M. Schmidt [2], [3].

This Master's thesis deals with an improvement of Kleinbock's and Margulis' theorem and obtains a weaker sufficient condition for the strongly Borel–Cantelli sequences. Several versions of the dynamical Borel–Cantelli lemmas will be deduced by extending another useful theorem by W. M. Schmidt [3], W. J. LeVeque [4], and W. Philipp [5].

Furthermore, some applications of our theorems will be discussed. Firstly, a characterization of the strongly Borel–Cantelli sequences in one-dimensional Gibbs–Markov systems will be established. This will improve the theorem of C. Gupta, M. Nicol, and W. Ott in [6]. Secondly, N. Haydn, M. Nicol, T. Persson, and S. Vaienti [7] proved the strong Borel–Cantelli property in sequences of balls in terms of a polynomial decay of correlations for Lipschitz observables. Our theorems will be applied to relax their inequality assumption. Finally, as a result of Y. Guivarch's and A. Raugi's findings [8], we know that the weakly mixing property could be characterized by Borel–Cantelli sequences that only contain a finite number of distinct sets, where each set has positive measure. This is a Borel–Cantelli result, although not strong. So a weakly β-mixing property will be introduced to imply the strong Borel–Cantelli property.

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Contents

1 Introduction

In 1909, the French mathematician Émile Borel [9] proved that if the sum of the probabilities of a collection of stochastic events is finite, then the probability that infinitely many of these events occur must be zero. Generally, the converse assertion will not be true. However, a couple of years later, the Italian Francesco Paolo Cantelli found that the converse assertion of Borel's result indeed holds if the events are independent. More specifically, Cantelli showed that if the sum of the probabilities of a collection of independent events is infinite, then the probability of the occurrence of infinitely many of these events must be one [10]. Nowadays, both results are together known as the classical Borel–Cantelli lemma.

The independence assumption in the classical Borel–Cantelli lemma is not necessary, and the lemma's application is sometimes also made more difficult by this assumption. Extending the classical Borel–Cantelli lemma to non-necessarily independent events has become an important topic in probability theory. There have been many attempts to weaken the independence assumption and various versions of the Borel–Cantelli lemma has been established. For instance, it is well known that the independence assumption could be relaxed by pairwise independence, as shown in Chapter 6 in [11].

The classical Borel–Cantelli lemma is a fundamental tool for many convergence theorems in probability theory. For example, the lemma is applied in the standard proof of the Law of Large Numbers, which states that the sample average of a large random sample is very close to the population average [11].

This Master's thesis will study the Borel–Cantelli lemmas in dynamical systems. Suppose that (X, μ, T) is a dynamical system and the transformation $T : X \to X$ preserves the measure μ. By applying the Borel–Cantelli lemma to the measure-preserving dynamical system (X, μ, T), we can directly obtain that if the n-th pre-image sets $T^{-n}(A_n)$ of the sets A_n in X are pairwise independent with respect to μ and the measures $\mu(A_n)$ are large enough such that $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, then the events $T^n(x) \in A_n$ occur infinitely many times for almost every x in X. In other words, the set of the points x such that $T^n(x) \in A_n$ holds for finitely many values n will have zero measure.

To be considered a Borel–Cantelli sequence, the measures $\mu(A_n)$ cannot approach zero too fast, since the classical Borel–Cantelli lemma implies that if

$\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then for almost every x in X, the events $T^n(x) \in A_n$ occur finitely often. On the other hand, the divergence condition $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ alone is not enough to guarantee that $T^n(x) \in A_n$ holds infinitely many times for almost every x in X. Through N. Chernov's and D. Kleinbock's result in [8], it is known that for any non-atomic measure-preserving dynamical system, we can always find a sequence $\{A_n\}$ of sets with the divergence condition $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ such that $T^n(x) \in A_n$ only holds for finitely many n for almost every x in X. Note that Chernov's and Kleinbock's result works well even for ergodic dynamical systems. In Example 1, we will give an ergodic dynamical system that contains a sequence $\{A_n\}$ with the divergence condition such that for all x in X, the event $T^n(x) \in A_n$ only holds for finitely many n.

It is also known that the pairwise independence condition for the events $T^{-n}(A_n)$ is not necessary. More specifically, it is seldom valid for measure-preserving transformations. To overcome this difficulty, we usually try to find conditions that are weaker than the pairwise independence condition. Plenty of research has been done in this direction. For example, D. Kleinbock and G. Margulis [1] defined

Definition 1. *A sequence $\{A_n\}$ is said to satisfy the condition $\triangle$ if there is a constant $C > 0$ such that*

$$\sum_{m,n=M}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \leq C \sum_{n=M}^{N} \mu(A_n)$$

holds for all $N \geq M \geq 1$.

They proved that if the sequence $\{A_n\}$ satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and the condition $\triangle$, then it is a strongly Borel–Cantelli sequence, i.e.

$$\frac{\sum_{n=1}^{m} \mathbf{1}_{T^{-n}(A_n)}(x)}{\sum_{n=1}^{m} \mu(A_n)} \longrightarrow 1 \quad \text{almost surely in } X \text{ as } m \to \infty.$$

Note that the numerator $\sum_{n=1}^{m} \mathbf{1}_{T^{-n}(A_n)}(x)$ corresponds to the number of integers $n \in [1, m]$ for which $T^n(x) \in A_n$. It is clear that the condition $\triangle$ holds if the sets $T^{-n}(A_n)$ are pairwise independent. In their proof, Kleinbock and Margulis use the asymptotic estimation

$$\sum_{n=1}^{m} \mathbf{1}_{T^{-n}(A_n)}(x) = \sum_{n=1}^{m} \mu(A_n) + O\left(\left(\sum_{n=1}^{m} \mu(A_n)\right)^{\frac{1}{2}} \log^{\frac{3}{2}+\varepsilon}\left(\sum_{n=1}^{m} \mu(A_n)\right)\right),$$

which holds almost everywhere for every given constant $\varepsilon > 0$. This asymptotic estimation is extremely useful and was first proven by W. M. Schmidt [2], [3]. Many researchers have applied the asymptotic estimation when trying to prove the strong Borel–Cantelli property. However, it is not necessary to apply a stronger asymptotic estimation in order to get the strong Borel–Cantelli property. Instead, let us introduce a weaker condition $\triangle_\alpha$.

Definition 2. *Let $0 \leq \alpha < 2$. A sequence $\{A_n\}$ is said to satisfy the condition $\triangle_\alpha$ if there is a constant $C > 0$ such that*

$$\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \leq C\left(\sum_{n=1}^{N} \mu(A_n)\right)^{\alpha}$$

holds for all large integers N.

The first main task in this thesis is to prove that the condition $\triangle$ in Kleinbock's and Margulis' theorem [1] could be replaced by the condition $\triangle_\alpha$ for some $1 \leq \alpha < 2$. The result is

Theorem 1. *If the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, and the condition $\triangle_\alpha$ holds for some $1 \leq \alpha < 2$, then $\{A_n\}$ is a strongly Borel–Cantelli sequence.*

There are two advantages of the condition $\triangle_\alpha$ over the condition $\triangle$ when studying the strong Borel–Cantelli property. First of all, it is more complicated to check the condition $\triangle$ than the condition $\triangle_\alpha$, since the condition $\triangle$ contains two free integers N, M, whereas the condition $\triangle_\alpha$ only has one free integer N. Secondly, the condition $\triangle_1$ is weaker than the condition $\triangle$, while the condition $\triangle_\alpha$ for $\alpha > 1$ is weaker than the condition $\triangle_1$. The condition $\triangle_1$ is actually a special case of the condition $\triangle$ at $M = 1$. Hence, Theorem 1 is an improvement of Kleinbock's and Margulis' Theorem 1.4 [1].

Furthermore, we will construct an example to show that for any $1 < \alpha < 2$, there exists a sequence in some ergodic dynamical system that satisfies the condition $\triangle_\alpha$ but not the condition $\triangle$ (see Example 2). There will be another example to show that the condition $\triangle_\alpha$ with $1 \le \alpha < 2$ in Theorem 1 cannot be replaced by the condition $\triangle_2$. We will also give slightly stronger conditions than the condition $\triangle_2$, which yields that $\sum_{n=1}^{m} \mathbf{1}_{T^{-n}(A_n)}(x) / \sum_{n=1}^{m} \mu(A_n) \longrightarrow 1$ in L_1 norm.

As an application of Theorem 1, several useful consequences will be deduced. It seems that one of the most popular results to prove a strongly Borel–Cantelli sequence should be the following theorem given by W. J .LeVeque [4], W. M. Schmidt [3] and W. Philipp [5].

Theorem D ([3], [4], [5]). *Suppose that the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, and there exists a positive series $\sum_{m=1}^{\infty} c_m < \infty$ such that*

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_{m-n}\,\mu(A_n)$$

holds for all $m > n$. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Theorem 1 gives the possibility to extend Theorem D. We will find that there are two consequences of Theorem 1 that strengthen Theorem D in different ways.

Corollary 1. *Suppose that the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, and there exist a positive sequence $\{c_m\}$ and constants $0 \le \beta < 1$ and $c > 0$ such that*

$$\sum_{m=1}^{n} c_m \le c \left(\sum_{m=1}^{n} \mu(A_m) \right)^{\beta}$$

holds for all sufficiently large integer n, and

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_{m-n}\,\mu(A_n)$$

holds for all $m > n$. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Obviously, the inequality assumption $\sum_{m=1}^{n} c_m \le c \left(\sum_{m=1}^{n} \mu(A_m) \right)^{\beta}$ in Corollary 1 is true if $\sum_{m=1}^{\infty} c_m < \infty$. So Corollary 1 improves Theorem D. Furthermore, in Corollary 1, we do not assume that the series $\sum_{m=1}^{\infty} c_m$ is convergent.

Corollary 2. *Suppose that the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, and there exists a positive series $\sum_{m=1}^{\infty} c_m < \infty$ and a constant $0 \le \beta < 1$ such that*

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_{m-n}^{1-\beta}\,\mu(A_m)^{\beta}\mu(A_n)$$

holds for all $m > n$. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Note that Corollary 2 for $\beta = 0$ corresponds to Theorem D.

Several applications of our theorems and corollaries will be presented. Firstly, C. Gupta, M. Nicol and W. Ott [6] applied the condition $\triangle$ to prove a strong Borel–Cantelli lemma for one-dimensional Gibbs–Markov systems under the additional condition $\mu(A_m) \le c\,\mu(A_n)$ for $m \ge n \ge 1$. As a direct consequence of our results, we get in Theorem 5 that the additional condition is superfluous. Hence, we will establish a characterization of the strong Borel–Cantelli property in one-dimensional Gibbs–Markov systems.

Secondly, through the inequality assumption $q > \frac{(2/\delta)+\gamma}{1-\gamma}$, N. Haydn, M. Nicol, T. Persson and S. Vaienti [7] proved a strong Borel–Cantelli lemma for sequences of the balls A_n with $\mu(A_n) \ge c_1\, n^{-\gamma}$ in a dynamical system satisfying the polynomial decay $p(k) \le c_2\, k^{-q}$ of correlations for Lipschitz observables. As an application of the results, we will show in Theorem 6 that their inequality assumption can be weakened by

$$q > \max\left\{\frac{\gamma}{\delta} + 1,\ \frac{(2/\delta)\gamma + \gamma}{1-\gamma}\right\},$$

which, if $\gamma \ge \frac{-3+\sqrt{13}}{2}$, is equivalent to the inequality

$$q > \frac{(2/\delta)\gamma + \gamma}{1-\gamma}.$$

Thirdly, it is well known that the ergodicity of a measure-preserving dynamical system could be characterized by both Borel–Cantelli and strongly Borel–Cantelli constant sequences. Y. Guivarch and A. Raugi [8] proved that the weakly mixing property could be characterized by Borel–Cantelli sequences that only contain a finite number of distinct sets, each with positive measure. This is a Borel–Cantelli result, but not a strong Borel–Cantelli result. Instead, we will introduce the weakly β-mixing property, which implies the strong Borel–Cantelli property in these sequences.

2 Borel–Cantelli lemmas

2.1 The classical Borel–Cantelli lemma

Let $(X, \mathcal{F}, \mu)$ be a probability space, where $\mathcal{F}$ is a σ-algebra in the sample space X, and μ is a probability measure. Given a sequence of measurable sets $\{A_n\}$ in X, the indicator function of the subset A_n is defined as

$$\mathbf{1}_{A_n}(x) := \begin{cases} 1, & \text{if } x \in A_n \\ 0, & \text{if } x \notin A_n. \end{cases}$$

It is clear that $\sum_{n=1}^{\infty} \mathbf{1}_{A_n}(x) < \infty$ almost surely in X if and only if for almost every point x in X, there exist finitely many n such that $x \in A_n$. Similarly, $\sum_{n=1}^{\infty} \mathbf{1}_{A_n}(x) = \infty$ almost surely in X if and only if for almost every point x in X, there exist infinitely many n such that $x \in A_n$.

Lemma A (Borel–Cantelli lemma).
(i) *If* $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, *then almost every point* x *in* X *belongs to finitely many sets* A_n.
(ii) *If* $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ *and all* A_n *are independent, then almost every point* x *in* X *belongs to infinitely many sets* A_n.

It is well known that the condition for the independent sets A_n in the Borel–Cantelli lemma could be replaced by the pairwise independence of the sets A_n. That is, $\mu(A_n \cap A_m) = \mu(A_n)\,\mu(A_m)$ holds when $n \neq m$. The Borel–Cantelli lemma is an important technical result and has been found to be extremely useful when proving several limit results in probability theory. For instance, the Borel–Cantelli lemma is a key component when proving the strong Law of Large Numbers, which states that if $\{X_1,\ X_2,\ \ldots,\ X_n,\ \ldots\}$ is a sequence of independent and identically distributed random variables with the finite expected value $E(X_1)$, then the sample average

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \longrightarrow E(X_1) \quad \text{almost surely as} \quad n \to \infty.$$

The Borel–Cantelli lemma is also applied in the proof of another well known theorem, which states that if a sequence of functions $\{f_n(x)\}$ converges to $f(x)$ in measure in X, then there exists a subsequence $\{f_{n_k}(x)\}$ that converges to $f(x)$ almost everywhere in X. In fact, the convergence in measure of the sequence $\{f_n(x)\}$ implies that for each integer $k > 0$, there exists a sufficiently large n_k such that $\mu(A_k) \le 2^{-k}$, where $A_k = \left\{x; |f_{n_k}(x) - f(x)| \ge 2^{-k}\right\}$. Hence, $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and by the Borel–Cantelli lemma, we get that $\sum_{k=1}^{\infty} \mathbf{1}_{A_k}(x) < \infty$ almost everywhere in X. This implies that the subsequence $\{f_{n_k}(x)\}$ converges to $f(x)$ almost everywhere.

2.2 A dynamical Borel–Cantelli lemma

Let $T : X \to X$ be a transformation. Denote by $T^n(x)$ the n-th iterate of T at the point $x \in X$, where $T^{n+1} = T \circ T^n$ for $n = 1, 2, \ldots$. For a subset A in X, we define the n-th pre-image set

$$T^{-n}(A) = \{x : \ T^n(x) \in A\}.$$

For simplicity, the notation $T^{-n}A = T^{-n}(A)$ is used. Evidently, $x \in T^{-n}(A)$ if and only if $T^n(x) \in A$. The transformation $T : X \to X$ is called a μ-preserving transformation if it is measurable and

$$\mu\left(T^{-1}(A)\right) = \mu(A)$$

holds for all measurable sets $A \subset X$. If $T : X \to X$ is a transformation that preserves the probability measure μ in X, then (X, μ, T) is called a measure-preserving dynamical system. A measure-preserving dynamical system (X, μ, T) is called an ergodic dynamical system if whenever $T^{-1}(A) = A$ holds for some measurable subset A of X, then either $\mu(A) = 0$ or $\mu(A) = 1$.

A direct application of the Borel–Cantelli lemma on the measure-preserving dynamical system (X, μ, T) results in the following lemma.

Lemma B.

(i) *If* $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, *then for almost every point* x *in* X, *there exist at most finitely many* n *such that* $T^n(x) \in A_n$, *i.e.* $\sum_{n=1}^{\infty} \mathbf{1}_{T^{-n}A_n}(x) < \infty$ *holds almost everywhere in* X.

(ii) *If* $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ *and all* $T^{-n}(A_n)$ *are independent, then for almost every point* x *in* X, *there exist infinitely many* n *such that* $T^n(x) \in A_n$, *i.e.* $\sum_{n=1}^{\infty} \mathbf{1}_{T^{-n}A_n}(x) = \infty$ *holds almost everywhere in* X.

The additional condition for the independence in the assertion (ii) in Lemma B is not superfluous, as shown by Chernov and Kleinbock [8].

Theorem A ([8]). *If μ is a non-atomic measure (that is, any measurable set A in X with $\mu(A) > 0$ contains a measurable subset B of A such that $\mu(A) > \mu(B) > 0$), then for any μ-preserving transformation T of X, there exists a sequence $\{A_n\}$ of measurable subsets of X with $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ such that for almost every point $x \in X$, there exist at most finitely many n for which $T^n(x) \in A_n$.*

Theorem A works well for ergodic dynamical systems. Below, we will give a simple example to show that the independence in Lemma B is not superfluous. Let us construct a sequence of subsets $\{A_n\}$ with $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ in a measure-preserving dynamical system (X, μ, T) such that for each point $x \in X$, there exist at most finitely many n for which $T^n(x) \in A_n$.

Example 1. The ergodic measure-preserving dynamical system (X, μ, T) is studied, where μ is the product Lebesgue measure on the sample space $X = [0,1) \times [0,1)$, and T is the invertible μ-preserving Baker's transformation in X defined by

$$T(x,y) = \begin{cases} \left(\frac{1}{2}x, 2y\right), & \text{if } 0 \le y < \frac{1}{2} \\ \left(\frac{1}{2}x + \frac{1}{2}, 2y - 1\right), & \text{if } \frac{1}{2} \le y < 1. \end{cases}$$

Take the subsets

$$B_0 = \{0\} \times [0,1) \text{ and } B_k = \left[2^{-k}, 2^{-k+1}\right) \times [0,1) \text{ for } k = 1, 2, 3, \ldots$$

Hence, these sets are disjoint, $\mu(B_k) = \frac{1}{2^k}$ and $\cup_{k=0}^{\infty} B_k = X$. We denote by $\{D_n\}$ the following sequence

$$\underbrace{B_0}, \underbrace{B_1,\ B_1}, \underbrace{B_2,\ B_2,\ B_2,\ B_2}, \underbrace{B_3,\ B_3,\ B_3,\ B_3,\ B_3,\ B_3,\ B_3,\ B_3}, \ldots,$$

that is, $D_0 = B_0,\ D_1 = B_1,\ D_2 = B_1,\ D_3 = B_2,\ D_4 = B_2, \ldots$ and so on. Then $\sum_{n=0}^{\infty} \mu(D_n) = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \sum_{k=1}^{\infty} 1 = \infty$, and for every $(x,y) \in X$, there exist at most finitely many n such that $(x,y) \in D_n$. Let $A_n = T^n(D_n)$ for $n = 0, 1, \ldots$. Then $\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(D_n) = \infty$ and for all $(x,y) \in X$, we have that $T^n(x,y) \in A_n$ if and only if $(x,y) \in D_n$, which might happen at most finitely many times.

3 Borel–Cantelli lemmas in dynamical systems

3.1 Borel–Cantelli properties

Let

$$S_m(x) = \sum_{n=1}^{m} \mathbf{1}_{T^{-n}(A_n)}(x)$$

and

$$E_m = \int_X S_m(x)\, d\mu = \sum_{n=1}^{m} \mu(A_n)$$

for a sequence of measurable subsets $\{A_n\}$ in X. The norm $\|f\|$ of the function f in the space L_1 is given by

$$\|f\| = \int_X |f(x)|\, d\mu.$$

Definition 3.
(i) *A sequence of measurable subsets $\{A_n\}$ in X is called a Borel–Cantelli sequence relative to T if $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and for almost every point x in X, there exist infinitely many n such that $T^n(x) \in A_n$.*
(ii) *A sequence of measurable subsets $\{A_n\}$ in X is called an L_1 Borel–Cantelli sequence relative to T if $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and*

$$\left\| \frac{S_m(x)}{E_m} - 1 \right\| \longrightarrow 0 \qquad \text{as } m \to \infty.$$

(iii) *A sequence of measurable subsets $\{A_n\}$ in X is called a strongly Borel–Cantelli sequence relative to T if $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and*

$$\frac{S_m(x)}{E_m} \longrightarrow 1 \quad \text{almost surely in } X \text{ as } m \to \infty.$$

Note that $\{A_n\}$ is a Borel–Cantelli sequence if and only if $\{S_n(x)\}$ is an unbounded sequence for almost all x in X. Moreover, the following theorem shows that any strongly Borel–Cantelli sequence is an L_1 Borel–Cantelli sequence, and any L_1 Borel–Cantelli sequence is a Borel–Cantelli sequence.

Theorem B.
(i) *If $\{A_n\}$ is a strongly Borel–Cantelli sequence, then it is an L_1 Borel–Cantelli sequence as well.*
(ii) *If $\{A_n\}$ is an L_1 Borel–Cantelli sequence, then it is a Borel–Cantelli sequence as well.*

Proof. (i) Assume that $\{A_n\}$ is a strongly Borel–Cantelli sequence. Then $\frac{S_m(x)}{E_m} \to 1$ almost surely in X as $m \to \infty$. Since $\frac{S_m(x)}{E_m} \geq 0$ in X, we have

$$\left\| \frac{S_m(x)}{E_m} \right\| = \int_X \frac{S_m(x)}{E_m}\, d\mu = \frac{E_m}{E_m} = 1 = \|1\|.$$

By Scheffé's lemma (Lemma 5.4.3 in [12]), we get that $\| \frac{S_m(x)}{E_m} - 1 \| \to 0$ as $m \to \infty$. Hence, $\{A_n\}$ is an L_1 Borel–Cantelli sequence.
(ii) Assume that $\{A_n\}$ is an L_1 Borel–Cantelli sequence. Then for any $\varepsilon > 0$, we have

$$\mu \left\{ x \in X; \left| \frac{S_m(x)}{E_m} - 1 \right| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon} \left\| \frac{S_m(x)}{E_m} - 1 \right\| \longrightarrow 0 \quad \text{as } m \to \infty.$$

So $\frac{S_m(x)}{E_m}$ converges to 1 in measure in X. Thus, there exists a subsequence $\left\{ \frac{S_{m_k}(x)}{E_{m_k}} \right\}$ that converges to 1 almost everywhere in X. But $E_{m_k} \to \infty$ as $k \to \infty$. Hence, $S_{m_k}(x) \to \infty$ almost everywhere in X. Since $\{S_m(x)\}$ is an increasing sequence, we get that $S_m(x) \to \infty$ for almost all x in X. □

3.2 Sufficient conditions on strongly Borel–Cantelli sequences

D. Kleinbock and G. Margulis [1] proved that the independence of the sets $T^{-n}(A_n)$ in Lemma B could be weakened by the condition $\triangle$.

Theorem C ([1])**.** *If the sequence $\{A_n\}$ of subsets of X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and the condition $\triangle$ holds, then $\{A_n\}$ is a strongly Borel–Cantelli sequence.*

A more general condition was first obtained by W. M. Schmidt [2], [3], see also Lemma 10 in Chapter 1 in [13]. The following lemma is crucial to the proof of Theorem C.

Lemma C ([2], [3], [13])**.** *Let $\{f_k(x)\}$ be a sequence of non-negative measurable functions in X. Suppose that $\{a_k\}$ and $\{b_k\}$ are two sequences of real numbers satisfying*
(i) $\phi_n := \sum_{k=1}^{n} b_k \to \infty$ as $n \to \infty$*;*
(ii) *There is a constant $C_1 > 0$ such that*

$$0 \leq a_k \leq b_k \leq C_1 \quad \text{for } k = 1, 2, \ldots;$$

(iii) *There is a constant* $C_2 > 0$ *such that*

$$\int_X \left(\sum_{k=M}^{N} f_k(x) - \sum_{k=M}^{N} a_k\right)^2 d\mu \le C_2 \sum_{k=M}^{N} b_k \quad \text{for all integers } N \ge M > 0.$$

Then, for any constant $\varepsilon > 0$*, we have*

$$\sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} a_k + O\left(\phi_n^{\frac{1}{2}} \log^{\frac{3}{2}+\varepsilon} \phi_n\right) \text{ almost everywhere in } X.$$

Remark. The asymptotic estimation from the result in Lemma C means that for almost every $x \in X$, there exists a constant $c(x) > 0$ such that the inequality

$$\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n} a_k\right| \le c(x)\, \phi_n^{\frac{1}{2}} \log^{\frac{3}{2}+\varepsilon} \phi_n$$

holds for all $n = 1, 2, \ldots$, where the constant $c(x)$ might depend on the point x in X, see Remark 3.2 in [6].

It is not necessary to check the inequality assumption $\triangle$ for all integers $N \ge M > 0$ in Theorem C. Below, we will prove that it is enough to assume that the inequality assumption $\triangle$ only holds for all integers $N > 0$ and $M = 1$. Furthermore, in order to prove the strong Borel–Cantelli property, it will not be necessary to prove such an asymptotic estimation like in Lemma C. So a new direct proof of Theorem C will be given. Recall the condition $\triangle_\alpha$: There are constants $C > 0$ and $\alpha \ge 0$ such that

$$\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \le C\left(\sum_{n=1}^{N} \mu(A_n)\right)^{\alpha}$$

holds for all large integers N.

Note that we cannot adopt the stronger inequality

$$\sum_{m,n=1}^{N} \mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le C\left(\sum_{n=1}^{N} \mu(A_n)\right)^{\alpha}$$

instead of the condition $\triangle_\alpha$ for $1 \le \alpha < 2$. This is due to the fact that if it were true, then by Hölder's inequality, we would have

$$\left(\sum_{n=1}^{N} \mu(A_n)\right)^2 = \left(\int_X \sum_{n=1}^{N} \mathbf{1}_{T^{-n}(A_n)}(x)\, d\mu\right)^2$$

$$\le \int_X 1^2\, d\mu \int_X \left(\sum_{n=1}^{N} \mathbf{1}_{T^{-n}(A_n)}(x) \right)^2 d\mu$$

$$= \int_X \sum_{m,n=1}^{N} \mathbf{1}_{T^{-m}(A_m)}(x) \mathbf{1}_{T^{-n}(A_n)}(x)\, d\mu$$

$$= \sum_{m,n=1}^{N} \mu\left(T^{-m} A_m \cap T^{-n} A_n\right) \le C \left(\sum_{n=1}^{N} \mu(A_n) \right)^{\alpha}.$$

Hence,

$$\left(\sum_{n=1}^{N} \mu(A_n) \right)^{2-\alpha} \le C,$$

which is impossible because $2 - \alpha > 0$ and $\sum\limits_{n=1}^{\infty} \mu(A_n) = \infty$.

Evidently, the condition $\triangle$ implies the condition $\triangle_1$, while the condition $\triangle_1$ implies the condition $\triangle_\alpha$ for $\alpha \ge 1$. Now, let us improve Theorem C in terms of the condition $\triangle_\alpha$ for $1 \le \alpha < 2$.

Theorem 1. *If the sequence $\{A_n\}$ of sets in X satisfies $\sum\limits_{n=1}^{\infty} \mu(A_n) = \infty$ and the condition $\triangle_\alpha$ holds for some $1 \le \alpha < 2$, then $\{A_n\}$ is a strongly Borel–Cantelli sequence and moreover satisfies*

$$\left\| \left(\frac{S_n(x)}{E_n} - 1 \right)^2 \right\| \le \frac{C}{E_n^{2-\alpha}} \quad \text{for } n = 1, 2, \ldots$$

Proof. It follows from Hölder's inequality that for any integer $N > 0$, we have

$$\|S_N(x) - E_N\|^2 = \left\| \sum_{n=1}^{N} \left\{ \mathbf{1}_{T^{-n}(A_n)}(x) - \mu(A_n) \right\} \right\|^2$$

$$\le \int_X 1^2\, d\mu \int_X \left(\sum_{n=1}^{N} \left\{ \mathbf{1}_{T^{-n}(A_n)}(x) - \mu(A_n) \right\} \right)^2 d\mu$$

$$= \left\| (S_N(x) - E_N)^2 \right\| = \int_X \left(\sum_{n=1}^{N} \left\{ \mathbf{1}_{T^{-n}(A_n)}(x) - \mu(A_n) \right\} \right)^2 d\mu$$

$$= \int_X \sum_{m=1}^{N} \left\{ \mathbf{1}_{T^{-m}(A_m)}(x) - \mu(A_m) \right\} \sum_{n=1}^{N} \left\{ \mathbf{1}_{T^{-n}(A_n)}(x) - \mu(A_n) \right\} d\mu$$

$$= \sum_{m,n=1}^{N} \int_X \left\{ \mathbf{1}_{T^{-m}(A_m)}(x) - \mu(A_m) \right\} \left\{ \mathbf{1}_{T^{-n}(A_n)}(x) - \mu(A_n) \right\} d\mu$$

$$= \sum_{m,n=1}^{N} \int_X \mathbf{1}_{T^{-m}(A_m)}(x) \mathbf{1}_{T^{-n}(A_n)}(x) \, d\mu$$

$$- \sum_{m,n=1}^{N} \int_X \left\{ \mathbf{1}_{T^{-m}(A_m)}(x)\mu(A_n) + \mathbf{1}_{T^{-n}(A_n)}(x)\mu(A_m) - \mu(A_m)\mu(A_n)) \right\} d\mu$$

$$= \sum_{m,n=1}^{N} \left\{ \mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n) \right\}$$

$$\leq C \left(\sum_{n=1}^{N} \mu(A_n) \right)^{\alpha} = C E_N^{\alpha},$$

where the last inequality follows from the condition $\triangle_\alpha$. Thus, we get

$$\| \left(S_N(x) - E_N\right)^2 \| \leq C E_N^{\alpha},$$

which implies

$$\left\| \left(\frac{S_N(x)}{E_N} - 1 \right)^2 \right\| \leq \frac{C}{E_N^{2-\alpha}}$$

and

$$\|S_N(x) - E_N\| \leq C^{\frac{1}{2}} E_N^{\frac{\alpha}{2}}.$$

Hence, we have

$$\left\| \frac{S_N(x)}{E_N} - 1 \right\| \leq \frac{C^{\frac{1}{2}}}{E_N^{1-\frac{\alpha}{2}}} = \frac{C^{\frac{1}{2}}}{E_N^{\frac{2-\alpha}{2}}}.$$

From $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, it follows that we can take the smallest integer $N_1 > 0$ such that

$$E_{N_1} = \sum_{n=1}^{N_1} \mu(A_n) \geq 1^{\frac{4}{2-\alpha}}.$$

Since $\frac{4}{2-\alpha} > 2$ we get $(x+1)^{\frac{4}{2-\alpha}} > x^{\frac{4}{2-\alpha}} + 1$ for $x \geq 1$. Hence, there exists the smallest integer $N_2 > N_1$ such that

$$E_{N_2} = \sum_{n=1}^{N_2} \mu(A_n) \geq 2^{\frac{4}{2-\alpha}}.$$

By repeating this procedure, we can get a sequence of integers $0 < N_1 < \cdots < N_k < \cdots$ such that

$$E_{N_k} = \sum_{n=1}^{N_k} \mu(A_n) \geq k^{\frac{4}{2-\alpha}} > \sum_{n=1}^{N_k-1} \mu(A_n) = E_{N_k-1} \quad \text{for } k = 1, 2, \cdots$$

So when $\alpha < 2$, we get

$$\left\| \frac{S_{N_k}(x)}{E_{N_k}} - 1 \right\| \leq \frac{C^{\frac{1}{2}}}{E_{N_k}^{\frac{2-\alpha}{2}}} \leq \frac{C^{\frac{1}{2}}}{k^{\frac{4}{2-\alpha}\frac{2-\alpha}{2}}} = \frac{C^{\frac{1}{2}}}{k^2} \quad \text{for } k = 1, 2, \cdots$$

Hence, by using the triangle inequality for the norm $||\cdot||$, we get

$$\left\| \sum_{k=1}^{\infty} \left| \frac{S_{N_k}(x)}{E_{N_k}} - 1 \right| \right\| \leq \sum_{k=1}^{\infty} \left\| \frac{S_{N_k}(x)}{E_{N_k}} - 1 \right\| \leq \sum_{k=1}^{\infty} \frac{C^{\frac{1}{2}}}{k^2} < \infty,$$

which implies that the series $\sum_{k=1}^{\infty} \left| \frac{S_{N_k}(x)}{E_{N_k}} - 1 \right|$ is convergent almost everywhere in X. Hence, $\frac{S_{N_k}(x)}{E_{N_k}} \to 1$ almost surely as $k \to \infty$. Now, for any positive integer N, we take k such that $N_k \leq N < N_{k+1}$. Then

$$\frac{S_{N_k}(x)}{E_{N_k}} \frac{E_{N_k}}{E_{N_{k+1}}} \leq \frac{S_{N_k}(x)}{E_{N_k}} \frac{E_{N_k}}{E_N} \leq \frac{S_N(x)}{E_N} \leq \frac{S_{N_{k+1}}(x)}{E_{N_{k+1}}} \frac{E_{N_{k+1}}}{E_N} \leq \frac{S_{N_{k+1}}(x)}{E_{N_{k+1}}} \frac{E_{N_{k+1}}}{E_{N_k}}.$$

Hence, to show that $\frac{S_N(x)}{E_N} \to 1$ almost surely as $N \to \infty$, it is enough to show that $\frac{E_{N_{k+1}}}{E_{N_k}} \to 1$ as $k \to \infty$. For each k, we have

$$\begin{aligned} 1 \leq \frac{E_{N_{k+1}}}{E_{N_k}} &= \frac{\sum_{n=1}^{N_{k+1}} \mu(A_n)}{E_{N_k}} = \frac{\sum_{n=1}^{N_{k+1}-1} \mu(A_n) + \mu\left(A_{N_{k+1}}\right)}{E_{N_k}} \\ &= \frac{E_{N_{k+1}-1} + \mu\left(A_{N_{k+1}}\right)}{E_{N_k}} \leq \frac{(k+1)^{\frac{4}{2-\alpha}} + 1}{k^{\frac{4}{2-\alpha}}} \\ &= (1 + \frac{1}{k})^{\frac{4}{2-\alpha}} + \frac{1}{k^{\frac{4}{2-\alpha}}} \longrightarrow 1 \text{ as } k \to \infty, \end{aligned}$$

which implies that $\frac{E_{N_{k+1}}}{E_{N_k}} \to 1$ as $k \to \infty$. The proof is now complete. $\square$

The proof of Theorem 1 also implies the following theorem.

Theorem 2. *If the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and there exist constants $0 \le \beta < 1$ and $C > 0$ such that*

$$\|S_n(x) - E_n\| \le C\, E_n^{\beta} \quad \text{for all large} \quad n,$$

then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

For each $1 < \alpha < 2$, we construct a strongly Borel–Cantelli sequence $\{A_n\}$ of subsets in a measure-preserving dynamical system (X, μ, T) that satisfies the condition $\triangle_\alpha$ but not $\triangle$. In this sense, we can say that Theorem 1 is an improvement of Theorem C.

Example 2. Let μ be the product Lebesgue measure on the sample space $X = [0,1) \times [0,1)$ and let T be the invertible μ-preserving Baker's transformation in X given in Example 1. For any $1 < \alpha < 2$, we define the subsets $A_n = T^n\left([(n+1)^{\alpha-2}, 1) \times [0,1)\right)$ in X for $n = 1, 2, \ldots$. Then $\{T^{-n}(A_n)\} = \{[(n+1)^{\alpha-2}, 1) \times [0,1)\}$ is an increasing sequence, i.e. $T^{-n}(A_n) \subset T^{-n-1}(A_{n+1})$ for $n = 1, 2, \ldots$, and $\mu(A_n) = \mu\left(T^{-n}(A_n)\right) = 1 - (n+1)^{\alpha-2}$. Hence, $\sum_{n=1}^{\infty} \mu(A_n) \ge \sum_{n=1}^{\infty} \mu(A_1) = \infty$, and for all $N \ge 1$, we have

$$\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\}$$

$$\begin{aligned}
&\le \sum_{m,n=1}^{N} \left\{\mu\left(T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} = \sum_{m,n=1}^{N} \left\{\mu\left(A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&= \sum_{n=1}^{N} \mu(A_n) \sum_{m=1}^{N} (1 - \mu(A_m)) = \sum_{n=1}^{N} \mu(A_n) \sum_{m=1}^{N} \frac{1}{(m+1)^{2-\alpha}} \\
&\le \sum_{n=1}^{N} \mu(A_n) \sum_{m=1}^{N} \int_m^{m+1} \frac{1}{x^{2-\alpha}}\, dx = \sum_{n=1}^{N} \mu(A_n) \int_1^{N+1} \frac{1}{x^{2-\alpha}}\, dx \\
&= \frac{(N+1)^{\alpha-1} - 1}{\alpha - 1} \sum_{n=1}^{N} \mu(A_n) \le \frac{(N+1)^{\alpha-1} - 1}{\alpha - 1} N \\
&\le \frac{(N+1)^{\alpha}}{\alpha - 1} \le \frac{2^{\alpha} N^{\alpha}}{\alpha - 1}.
\end{aligned}$$

On the other hand, the inequality $\mu(A_n) \ge \mu(A_1) = 1 - 2^{\alpha-2}$ holds for all $n = 1, 2, \ldots$. So we get

$$\left(\sum_{n=1}^{N} \mu(A_n)\right)^{\alpha} \ge \left(1 - 2^{\alpha-1}\right)^{\alpha} N^{\alpha},$$

and hence

$$\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} = \mathrm{O}\left(\left(\sum_{n=1}^{N}\mu(A_n)\right)^{\alpha}\right).$$

Thus, the sequence $\{A_n\}$ satisfies the condition $\triangle_\alpha$. By Theorem 1, we get that it is a strongly Borel–Cantelli sequence. Now, let us show that $\{A_n\}$ does not satisfy the condition $\triangle$. We have

$$\begin{aligned}
&\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&= 2\sum_{n=1}^{N}\sum_{m=1}^{n-1} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&\qquad + \sum_{n=1}^{N}\left\{\mu\left(A_n\right) - \mu(A_n)^2\right\} \\
&\geq 2\sum_{n=1}^{N}\sum_{m=1}^{n-1} \left\{\mu\left(A_m\right) - \mu(A_m)\mu(A_n)\right\} = 2\sum_{n=1}^{N}\left(1-\mu(A_n)\right)\sum_{m=1}^{n-1}\mu(A_m) \\
&\geq 2\mu(A_1)\sum_{n=1}^{N}\left(1-\mu(A_n)\right)(n-1) = 2\mu(A_1)\sum_{n=1}^{N}\frac{n-1}{(n+1)^{2-\alpha}},
\end{aligned}$$

which, by the inequality $2(n-1) \geq n+1$ when $n \geq 3$, is larger than

$$\begin{aligned}
&\mu(A_1)\sum_{n=3}^{N}\frac{1}{(n+1)^{1-\alpha}} \geq \mu(A_1)\sum_{n=3}^{N}\int_{n}^{n+1}\frac{1}{x^{1-\alpha}}\,dx \\
&= \mu(A_1)\int_{3}^{N+1}\frac{1}{x^{1-\alpha}}\,dx = \frac{\mu(A_1)\left((N+1)^{\alpha} - 3^{\alpha}\right)}{\alpha} \\
&\qquad = \frac{\mu(A_1)\left(\left(1+\frac{1}{N}\right)^{\alpha} - \left(\frac{3}{N}\right)^{\alpha}\right)}{\alpha}N^{\alpha},
\end{aligned}$$

where

$$\frac{\mu(A_1)\left(\left(1+\frac{1}{N}\right)^{\alpha} - \left(\frac{3}{N}\right)^{\alpha}\right)}{\alpha} \longrightarrow \frac{\mu(A_1)}{\alpha} > 0 \quad \text{as } N \to \infty.$$

So, by $1 < \alpha < 2$, we have shown that $\{A_n\}$ does not satisfy the condition $\triangle$.

The following theorem is very useful in the study of the strong Borel–Cantelli property and has been proven by W. J. LeVeque [4], W. M. Schmidt [3] and W. Philipp [5].

Theorem D ([3], [4], [5]). *Suppose that the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and that there exists a positive series $\sum_{m=1}^{\infty} c_m < \infty$ such that*

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \leq \mu(A_m)\mu(A_n) + c_{m-n}\,\mu(A_n)$$

holds for all $m > n$. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

As a simple application of Theorem 1, we will improve Theorem D and give the following two corollaries.

Corollary 1. *Suppose that the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and that there exist a positive sequence $\{c_m\}$ and constants $0 \leq \beta < 1$ and $c > 0$ such that*

$$\sum_{m=1}^{n} c_m \leq c\left(\sum_{m=1}^{n} \mu(A_m)\right)^{\beta}$$

holds for all sufficiently large integers n, and

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \leq \mu(A_m)\mu(A_n) + c_{m-n}\,\mu(A_n)$$

holds for all $m > n$. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

It is clear that the inequality assumption $\sum_{m=1}^{n} c_m \leq c\left(\sum_{m=1}^{n} \mu(A_m)\right)^{\beta}$ holds when $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and $\sum_{m=1}^{\infty} c_m < \infty$. Thus, Corollary 1 is an improvement of Theorem D.

Proof. By Theorem 1, it is enough to check the condition $\triangle_\alpha$ for some $1 \leq \alpha < 2$. For any large $N \geq 1$, we have

$$\begin{aligned}
&\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&= 2\sum_{n=1}^{N} \sum_{m=n+1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&\qquad + \sum_{n=1}^{N} \left\{\mu\left(A_n\right) - \mu(A_n)^2\right\}
\end{aligned}$$

$$\leq 2\sum_{n=1}^{N}\sum_{m=n+1}^{N} c_{m-n}\,\mu(A_n)+\sum_{n=1}^{N}\mu\left(A_n\right)$$

$$\leq \left(1+2\sum_{m=1}^{N}c_m\right)\sum_{n=1}^{N}\mu(A_n)\leq \sum_{n=1}^{N}\mu(A_n)+2c\left(\sum_{n=1}^{N}\mu(A_n)\right)^{1+\beta}$$

$$\leq (1+2c)\left(\sum_{n=1}^{N}\mu(A_n)\right)^{1+\beta}.$$

Hence, $\{A_n\}$ satisfies the condition $\triangle_{1+\beta}$ with $1<1+\beta<2$. □

Remark. It is fairly easy to see that the inequality assumption

$$\mu\left(T^{-m}A_m\cap T^{-n}A_n\right)\leq \mu(A_m)\mu(A_n)+c_{m-n}\,\mu(A_n)\quad\text{for all } m>n$$

in Corollary 1 could be replaced by one of the following inequalities

1. $\mu\left(T^{-m}A_m\cap T^{-n}A_n\right)\leq \mu(A_m)\mu(A_n)+c_{m-n}\,\mu(A_m)$ for all $m>n$

2. $\mu\left(T^{-m}A_m\cap T^{-n}A_n\right)\leq \mu(A_m)\mu(A_n)+c_m\,\mu(A_n)$ for all $m>n$

3. $\mu\left(T^{-m}A_m\cap T^{-n}A_n\right)\leq \mu(A_m)\mu(A_n)+c_n\,\mu(A_m)$ for all $m>n$

Corollary 2. *Suppose that the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty}\mu(A_n)=\infty$ and that there exists a positive series $\sum_{m=1}^{\infty}c_m<\infty$ and a constant $0\leq\beta<1$ such that*

$$\mu\left(T^{-m}A_m\cap T^{-n}A_n\right)\leq \mu(A_m)\mu(A_n)+c_{m-n}^{1-\beta}\,\mu(A_m)^{\beta}\mu(A_n)$$

holds for all $m>n$. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Note that Corollary 2 with $\beta=0$ corresponds to Theorem D. So Corollary 2 extends Theorem D.

Proof. By Theorem 1, it is enough to check the condition $\triangle_\alpha$ for some $1\leq\alpha<2$. If $\beta=0$, the proof of Corollary 1 gives that for any $N\geq 1$,

$$\sum_{m,n=1}^{N}\left\{\mu\left(T^{-m}A_m\cap T^{-n}A_n\right)-\mu(A_m)\mu(A_n)\right\}$$

$$\leq\left(1+2\sum_{m=1}^{N}c_m\right)\sum_{n=1}^{N}\mu(A_n)\leq\left(1+2\sum_{m=1}^{\infty}c_m\right)\sum_{n=1}^{N}\mu(A_n).$$

So $\{A_n\}$ satisfies the condition $\triangle_1$ and is therefore a strongly Borel–Cantelli sequence.

If $0 < \beta < 1$, then for any $N \geq 1$, we have

$$\begin{aligned}
&\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&= 2\sum_{n=1}^{N}\sum_{m=n+1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&\qquad + \sum_{n=1}^{N}\left\{\mu\left(A_n\right) - \mu(A_n)^2\right\} \\
&\leq 2\sum_{n=1}^{N}\sum_{m=n+1}^{N} c_{m-n}^{1-\beta}\,\mu(A_m)^{\beta}\mu(A_n) + \sum_{n=1}^{N}\mu\left(A_n\right) \\
&= 2\sum_{n=1}^{N}\mu(A_n)\sum_{m=n+1}^{N} c_{m-n}^{1-\beta}\,\mu(A_m)^{\beta} + \sum_{n=1}^{N}\mu\left(A_n\right).
\end{aligned}$$

By applying Hölder's inequality for $\frac{1}{p} + \frac{1}{q} = 1$, where $p = \frac{1}{\beta} > 1$ and $q = \frac{1}{1-\beta} > 1$, we get

$$\begin{aligned}
\sum_{m=n+1}^{N} c_{m-n}^{1-\beta}\,\mu(A_m)^{\beta} &\leq \left(\sum_{m=n+1}^{N}\left(c_{m-n}^{1-\beta}\right)^{\frac{1}{1-\beta}}\right)^{1-\beta}\left(\sum_{m=n+1}^{N}\left(\mu(A_m)^{\beta}\right)^{\frac{1}{\beta}}\right)^{\beta} \\
= \left(\sum_{m=n+1}^{N} c_{m-n}\right)^{1-\beta}\left(\sum_{m=n+1}^{N}\mu(A_m)\right)^{\beta} &\leq \left(\sum_{k=1}^{\infty} c_k\right)^{1-\beta}\left(\sum_{m=1}^{N}\mu(A_m)\right)^{\beta}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&\leq 2\left(\sum_{k=1}^{\infty} c_k\right)^{1-\beta}\left(\sum_{m=1}^{N}\mu(A_m)\right)^{1+\beta} + \sum_{n=1}^{N}\mu\left(A_n\right) \\
&\leq \left(1 + 2\left(\sum_{k=1}^{\infty} c_k\right)^{1-\beta}\right)\left(\sum_{m=1}^{N}\mu(A_m)\right)^{1+\beta}.
\end{aligned}$$

Thus, $\{A_n\}$ satisfies the condition $\triangle_{1+\beta}$ with $1 < 1+\beta < 2$ and is therefore a strongly Borel–Cantelli sequence. □

Remark. From the proof of Corollary 2, it turns out that the inequality assumption

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_{m-n}^{1-\beta}\,\mu(A_m)^\beta\mu(A_n) \quad \text{for all } m > n$$

in Corollary 2 can be replaced by one of the following inequalities

1. $\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_{m-n}^{1-\beta}\,\mu(A_m)\mu(A_n)^\beta \quad \text{for } m > n$

2. $\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_{m}^{1-\beta}\,\mu(A_m)^\beta\mu(A_n) \quad \text{for } m > n$

3. $\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_{m}^{1-\beta}\,\mu(A_m)\mu(A_n)^\beta \quad \text{for } m > n$

On the other hand, the inequality $\mu(A_m) \le \mu(A_m)^\beta$ holds for $0 < \beta \le 1$. This leads to the following corollary.

Corollary 3. *Suppose that the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty}\mu(A_n) = \infty$ and that there exists a positive series $\sum_{m=1}^{\infty} c_m < \infty$ and a constant $0 < \gamma \le 1$ such that*

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \le \mu(A_m)\mu(A_n) + c_m^{\gamma}\,\mu(A_m)\mu(A_n)$$

holds for all $m > n$. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Another consequence of Theorem 1 says that any $\{A_n\}$ with a very large measure $\mu(A_n)$ is a strongly Borel–Cantelli sequence.

Corollary 4. *If the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty}\mu(A_n) = \infty$ and there are constants $C > 0$ and $0 < \beta < 1$ such that*

$$\sum_{n=1}^{N}(1 - \mu(A_n)) \le C\left(\sum_{n=1}^{N}\mu(A_n)\right)^\beta$$

holds for all $N \ge 1$, then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Proof. For any $N \ge 1$, we have

$$\sum_{m,n=1}^{N}\left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\}$$

$$\le \sum_{m,n=1}^{N}\left\{\mu\left(T^{-m}A_m\right) - \mu(A_m)\mu(A_n)\right\} = \sum_{m,n=1}^{N}\left\{\mu\left(A_m\right) - \mu(A_m)\mu(A_n)\right\}$$

$$= \sum_{m=1}^{N} \mu(A_m) \sum_{n=1}^{N} (1 - \mu(A_n)) \le C \left(\sum_{n=1}^{N} \mu(A_n) \right)^{1+\beta}.$$

So the sequence $\{A_n\}$ satisfies the condition $\triangle_\alpha$ with $\alpha = 1 + \beta < 2$ and by Theorem 1, we get that $\{A_n\}$ is a strongly Borel–Cantelli sequence. □

The following example shows that Corollary 4 fails for $\beta = 1$.

Example 3. Assume that μ is the Lebesgue measure on $X = [0, 1]$ and T is the identity transformation in X. Let $A_n = [0, 1/2]$ for all $n = 1, 2, \ldots$. Clearly, $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, while $\{A_n\}$ is not a Borel–Cantelli sequence. Hence, it is not a strongly Borel–Cantelli sequence either. Moreover, for any $N \ge 1$, we have

$$\sum_{n=1}^{N} (1 - \mu(A_n)) = \frac{N}{2} = \sum_{n=1}^{N} \mu(A_n),$$

which implies that $\{A_n\}$ satisfies the inequality assumption in Corollary 4 for $\beta = 1$.

Example 3 also shows that the condition $\triangle_\alpha$ with $1 \le \alpha < 2$ in Theorem 1 cannot be improved by the condition $\triangle_2$. In fact, we have the following corollary of Theorem 1, which is slightly stronger than Corollary 4.

Corollary 5. *Let $1 \le \alpha < 2$ and $C > 0$. Suppose that the sequence $\{A_n\}$ with $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ satisfies one of the following conditions:*

(i) *The inequality*

$$\sum_{n=1}^{N} \mu(A_n) \sum_{m=n+1}^{N} (1 - \mu(A_m)) \le C \left(\sum_{n=1}^{N} \mu(A_n) \right)^{\alpha}$$

holds for all $N \ge 1$;

(ii) *The inequality*

$$\sum_{n=1}^{N} (1 - \mu(A_n)) \sum_{m=1}^{n-1} \mu(A_m) \le C \left(\sum_{n=1}^{N} \mu(A_n) \right)^{\alpha}$$

holds for all $N \ge 1$.

(iii) *The inequality*

$$\sum_{n=1}^{N} (1 - \mu(A_n)) \sum_{m=n+1}^{N} \mu(A_m) \le C \left(\sum_{n=1}^{N} \mu(A_n) \right)^{\alpha}$$

holds for all $N \geq 1$.

(iv) *The inequality*

$$\sum_{n=1}^{N} \mu(A_n) \sum_{m=1}^{n-1} (1-\mu(A_m)) \leq C \left(\sum_{n=1}^{N} \mu(A_n) \right)^{\alpha}$$

holds for all $N \geq 1$.

Then $\{A_n\}$ *is a strongly Borel–Cantelli sequence.*

Proof. There is only a need to prove (i) and (ii). The proofs of (iii) and (iv) are done in a similar way. Assume that assertion (i) is true. Then, for any $N \geq 1$, we write

$$\sum_{m,n=1}^{N} \left\{ \mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n) \right\}$$

$$= 2\sum_{n=1}^{N} \sum_{m=n+1}^{N} \left\{ \mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n) \right\} + \sum_{n=1}^{N} \left\{ \mu\left(A_n\right) - \mu(A_n)^2 \right\}$$

$$\leq 2\sum_{n=1}^{N} \sum_{m=n+1}^{N} \left\{ \mu\left(T^{-n}A_n\right) - \mu(A_m)\mu(A_n) \right\} + \sum_{n=1}^{N} \left\{ \mu\left(A_n\right) - \mu(A_n)^2 \right\}$$

$$\leq 2\sum_{n=1}^{N} \mu(A_n) \sum_{m=n+1}^{N} (1-\mu(A_m)) + \sum_{n=1}^{N} \mu\left(A_n\right)$$

$$\leq (1+2C) \left(\sum_{n=1}^{N} \mu(A_n) \right)^{\alpha}.$$

So by Theorem 1, we get that $\{A_n\}$ is a strongly Borel–Cantelli sequence.

If assertion (ii) is true, then we get

$$\sum_{m,n=1}^{N} \left\{ \mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n) \right\}$$

$$= 2\sum_{n=1}^{N} \sum_{m=1}^{n-1} \left\{ \mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n) \right\}$$

$$+\sum_{n=1}^{N}\left\{\mu\left(A_n\right)-\mu(A_n)^2\right\}$$

$$\leq 2\sum_{n=1}^{N}\left(1-\mu(A_n)\right)\sum_{m=1}^{n-1}\mu(A_m)+\sum_{n=1}^{N}\mu\left(A_n\right)$$

$$\leq (1+2C)\left(\sum_{n=1}^{N}\mu(A_n)\right)^{\alpha},$$

which, by Theorem 1, implies that $\{A_n\}$ is a strongly Borel–Cantelli sequence.

□

The proof shown above implies that the conditions (i) and (ii) in Corollary 5 are equivalent to the condition $\triangle_\alpha$ when the sequence $\{T^{-n}(A_n)\}$ is increasing, i.e. $T^{-1}(A_1) \subset T^{-2}(A_2) \subset T^{-3}(A_3) \subset \ldots$. Similarly, the conditions (iii) and (iv) in Corollary 5 are equivalent to the condition $\triangle_\alpha$ when the sequence $\{T^{-n}(A_n)\}$ is decreasing.

Finally, we will give another type of consequence of Theorem 1.

Corollary 6. *Let $0 \leq \nu < 1$ and $\beta < 1-2\nu$. Suppose that there are constants $c_1 > 0$ and $c_2 > 0$ such that both $\sum_{k=1}^{n}\mu(A_k) \geq c_1\, n^{1-\nu}$ and*

$$\sum_{m=1}^{n-1}\left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right)-\mu(A_m)\mu(A_n)\right\} \leq c_2\, n^{\beta}$$

hold for all large n. Then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Proof. From $\sum_{k=1}^{n}\mu(A_k) \geq c_1\, n^{1-\nu}$, it follows that $\sum_{k=1}^{\infty}\mu(A_k)=\infty$. For all large N, we have

$$\sum_{m,n=1}^{N}\left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right)-\mu(A_m)\mu(A_n)\right\}$$

$$=2\sum_{n=1}^{N}\sum_{m=1}^{n-1}\left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right)-\mu(A_m)\mu(A_n)\right\}$$

$$+\sum_{n=1}^{N}\left\{\mu\left(A_n\right)-\mu(A_n)^2\right\}$$

$$\le 2\sum_{n=1}^{N} c_2\, n^{\beta} + \sum_{n=1}^{N} \mu\left(A_n\right) \le 2c_2\, N^{\beta} \sum_{n=1}^{N} 1 + \sum_{n=1}^{N} \mu\left(A_n\right)$$

$$= 2c_2\, N^{\beta+1} + \sum_{n=1}^{N} \mu\left(A_n\right) \le 2c_2 \left(\frac{1}{c_1}\sum_{n=1}^{N} \mu\left(A_n\right)\right)^{\frac{\beta+1}{1-\nu}} + \sum_{n=1}^{N} \mu\left(A_n\right).$$

Since $\beta < 1 - 2\nu$, we have $\frac{\beta+1}{1-\nu} < 2$. Hence, the condition $\triangle_\alpha$ holds for the constant $\alpha = \frac{\beta+1}{1-\nu} < 2$. By Theorem 1, we know that $\{A_n\}$ is a strongly Borel–Cantelli sequence. □

3.3 Sufficient conditions on Borel–Cantelli properties with convergence in L_1

The condition $\triangle_2$ cannot guarantee the strongly Borel–Cantelli property. However, we have

Theorem 3. *If the sequence $\{A_n\}$ of sets in X satisfies $\sum\limits_{n=1}^{\infty} \mu(A_n) = \infty$ and*

$$\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\}$$

$$= o\left(\left(\sum_{n=1}^{N} \mu(A_n)\right)^2\right) \quad \text{as } N \to \infty,$$

where $N = 1, 2, \dots,$ then $\{A_n\}$ is an L_1 Borel–Cantelli sequence.

Proof. It follows from the proof of Theorem 1 that for each integer $N > 0$, we have

$$\|S_N(x) - E_N\|^2 \le \sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\}$$

$$= o\left(\left(\sum_{n=1}^{N} \mu(A_n)\right)^2\right) = o\left(E_N^2\right) \quad \text{as } N \to \infty.$$

Hence,

$$\left\|\frac{S_N(x)}{E_N} - 1\right\| \longrightarrow 0 \quad \text{as } N \to \infty.$$

Thus, $\{A_n\}$ is an L_1 Borel–Cantelli sequence. □

Similarly to the proof of Corollary 4, we can apply Theorem 3 to deduce the following corollary.

Corollary 7. *If the sequence* $\{A_n\}$ *of sets in* X *satisfies* $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ *and*

$$\sum_{n=1}^{N} (1 - \mu(A_n)) = o\left(\sum_{n=1}^{N} \mu(A_n)\right) \quad \text{as } N \to \infty,$$

then $\{A_n\}$ *is an* L_1 *Borel–Cantelli sequence.*

Another useful consequence of Theorem 3 is

Corollary 8. *Suppose that the sequence* $\{A_n\}$ *of sets in* X *satisfies* $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ *and that there exists a positive sequence* $\{c_n\}$ *with* $\lim_{n\to\infty} c_n = 0$ *such that*

$$\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) \leq \mu(A_m)\mu(A_n) + c_m\, \mu(A_m)\mu(A_n)$$

holds for all integers $m > n$. *Then* $\{A_n\}$ *is an* L_1 *Borel–Cantelli sequence.*

Proof. From the assumption, we have

$$\begin{aligned}
&\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&= 2\sum_{n=1}^{N} \sum_{m=n+1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} \\
&\qquad + \sum_{n=1}^{N} \left\{\mu\left(A_n\right) - \mu(A_n)^2\right\} \\
&\leq 2\sum_{n=1}^{N} \mu(A_n) \sum_{m=1}^{N} c_m\, \mu(A_m) + \sum_{n=1}^{N} \mu\left(A_n\right).
\end{aligned}$$

From $\lim_{n\to\infty} c_n = 0$, it follows that for each $\varepsilon > 0$, we can choose $M > 0$ such that $c_m < \varepsilon$ for all $m > M$. So for $N > M$, we get

$$\sum_{m=1}^{N} c_m\, \mu(A_m) \leq \sum_{m=1}^{M} c_m\, \mu(A_m) + \varepsilon \sum_{m=M+1}^{N} \mu(A_m),$$

which, by $\sum_{m=1}^{\infty} \mu(A_m) = \infty$, implies that

$$\sum_{m=1}^{N} c_m \, \mu(A_m) = o\left(\sum_{m=1}^{N} \mu(A_m)\right) \quad \text{as } N \to \infty.$$

Thus,

$$\sum_{m,n=1}^{N} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\} = o\left(\left(\sum_{n=1}^{N} \mu(A_n)\right)^2\right)$$

as $N \to \infty$. Hence, Theorem 3 gives that $\{A_n\}$ is an L_1 Borel–Cantelli sequence. □

Theorem 4. *If the sequence $\{A_n\}$ of sets in X satisfies $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and there exist integers $n_1 < n_2 < \cdots < n_k < \ldots$ such that*

$$\sum_{m,n=1}^{n_k} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\}$$

$$= o\left(\left(\sum_{n=1}^{n_k} \mu(A_n)\right)^2\right) \quad \text{as } k \to \infty,$$

then $\{A_n\}$ is a Borel–Cantelli sequence.

Proof. From the proof of Theorem 1, we get

$$||S_{n_k}(x) - E_{n_k}||^2 \le \sum_{m,n=1}^{n_k} \left\{\mu\left(T^{-m}A_m \cap T^{-n}A_n\right) - \mu(A_m)\mu(A_n)\right\}$$

$$= o\left(\left(\sum_{n=1}^{n_k} \mu(A_n)\right)^2\right) = o\left(E_{n_k}^2\right) \quad \text{as } k \to \infty.$$

Hence, we have

$$\left\|\frac{S_{n_k}(x)}{E_{n_k}} - 1\right\| \longrightarrow 0 \quad \text{as } k \to \infty,$$

which implies that $\frac{S_{n_k}(x)}{E_{n_k}}$ converges to 1 in measure. So there exists a subsequence $\left\{\frac{S_{n_{k_l}}(x)}{E_{n_{k_l}}}\right\}$ of $\left\{\frac{S_{n_k}(x)}{E_{n_k}}\right\}$ such that $\frac{S_{n_{k_l}}(x)}{E_{n_{k_l}}}$ converges to 1 almost surely in X. Therefore, by $E_{n_{k_l}} \to \infty$, we get that $S_{n_{k_l}}(x) \to \infty$ almost surely in X. Finally, since $S_n(x) = \sum_{m=1}^{n} \mathbf{1}_{T^{-m}(A_m)}(x)$ increases as n increases, we can see that $S_n(x) \to \infty$ almost surely in X. This is the same as saying that $\{A_n\}$ is a Borel–Cantelli sequence. □

4 Applications of dynamical Borel–Cantelli lemmas

In this section, we will apply our results to improve some of the already known theorems.

4.1 One-dimensional Gibbs–Markov systems

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space. Also, let $\mathcal{P}$ be a countable partition of the compact interval X into subintervals such that $\mu(\alpha) > 0$ for all $\alpha \in \mathcal{P}$. For each $n \in \mathbb{N}$, we define

$$\mathcal{P}_n = \left\{ \bigcap_{i=0}^{n-1} T^{-i}(\alpha_i);\ \alpha_i \in \mathcal{P} \text{ for } 0 \le i \le n-1 \right\}.$$

A μ-preserving map $T : X \to X$ is said to be a Gibbs–Markov map if the following conditions are satisfied [6]:

(1) We have $\mathcal{B} = \sigma\left(\{T^{-i}(\alpha);\ \alpha \in \mathcal{P}, i \in \mathbb{Z}^+\}\right) \pmod{\mu}$, where $\sigma(A)$ denotes the σ-algebra that is generated by the set A.

(2) For all $\alpha, \beta \in \mathcal{P}$, if $\mu\left(T(\alpha) \cap \beta\right) > 0$, then $\beta \subset T(\alpha) \pmod{\mu}$.

(3) For all $\alpha \in \mathcal{P}$, the restriction map $T\,|_\alpha$ is invertible.

(4) For all $\alpha \in \mathcal{P}$, $T(\alpha) = X \pmod{\mu}$.

(5) There exists $c_1 > 0$ and $0 < \gamma_1 < 1$ such that $\mu(\alpha) \le c_1 \gamma_1^n$ for all $n \in \mathbb{N}$ and $\alpha \in \mathcal{P}_n$.

(6) There exists $c_2 > 0$ and $0 < \gamma_2 < 1$ such that for all $n \in \mathbb{N}$ and $\alpha \in \mathcal{P}_n$, we have that

$$\left| \log\left(\frac{J_T(x)}{J_T(y)} \right) \right| \le c_2 \gamma_2^n$$

holds for all $x, y \in \alpha$, where $J_T = \frac{d(\mu \circ T)}{d\mu}$.

Assume that $T : X \to X$ is a measure-preserving Gibbs–Markov map in X. So we have the one-dimensional Gibbs–Markov system $(X, \mathcal{B}, \mu, T, \mathcal{P})$. C. Gupta, M. Nicol and W. Ott [6] obtained a sufficient condition for strongly Borel–Cantelli sequences in one-dimensional Gibbs–Markov systems.

Theorem E ([6]). *Let $(X, \mathcal{B}, \mu, T, \mathcal{P})$ be a one-dimensional Gibbs–Markov system. Also, let $\{A_n\}$ be a sequence of intervals in X for which there exists a constant $C > 0$ such that $\mu(A_j) \le C\,\mu(A_i)$ holds for all $j \ge i \ge 1$. If $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, then $\{A_n\}$ is a strongly Borel–Cantelli sequence.*

C. Gupta, M. Nicol and W. Ott applied the condition $\triangle$ to prove Theorem E. As an application of our theorems, we will prove that the inequality assumption $\mu(A_j) \le C\,\mu(A_i)$ for $j \ge i \ge 1$ in Theorem E is superfluous. So we will establish a characterization of the strongly Borel–Cantelli sequences in one-dimensional Gibbs–Markov systems.

Theorem 5. *Let $(X, \mathcal{B}, \mu, T, \mathcal{P})$ be a one-dimensional Gibbs–Markov system and let $\{A_n\}$ be a sequence of intervals in X. Then $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ if and only if the sequence $\{A_n\}$ is a strongly Borel–Cantelli sequence.*

Proof. Assume that $\{A_n\}$ is a strongly Borel–Cantelli sequence. Then $\{A_n\}$ is a dynamical Borel–Cantelli sequence as well. However, from Lemma B, it follows that the convergence condition $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ implies that $\{A_n\}$ is not a dynamical Borel–Cantelli sequence. Thus, we must have $\sum_{n=1}^{\infty} \mu(A_n) = \infty$.

On the other hand, assume that $\sum_{n=1}^{\infty} \mu(A_n) = \infty$. In the proof of Theorem 1 in reference [6], C. Gupta, M. Nicol and W. Ott discussed four different cases of the indexes i and j to get the estimation

$$\mu\left(T^{-j}A_j \cap T^{-i}A_i\right) - \mu(A_j)\mu(A_i) \le C_1 \tau^{\frac{j-i}{2}} \mu(A_j) \quad \text{for all } j > i \ge 1,$$

where the constants $C_1 > 0$ and $0 < \tau < 1$. So by Corollary 1 and the following remark after the proof of Corollary 1, we know that the sequence $\{A_n\}$ is a strongly Borel–Cantelli sequence. $\square$

4.2 A dynamical Borel–Cantelli lemma on sequences of balls

Suppose that (X, μ, T) is a measure-preserving dynamical system, where X is a metric space with the probability measure μ. We will use the notation $E(f) := \int_X f d\mu$ for any integrable function f. Following paper [7], the conditions (A) and (B) are defined below:

(A) There exist positive constants $p(k)$ with $k = 0, 1, 2, \ldots$ such that the inequality

$$\left|E\left(\phi\,\psi \circ T^k\right) - E(\phi)E(\psi)\right| \le p(k)||\phi||_{Lip}||\psi||_{Lip}$$

holds for all Lipschitz functions $\phi,\ \psi$ in X.

(B) There exist constants $r_0 > 0$ and $0 < \delta < 1$ such that for all points $p \in X$ and all $0 < \varepsilon < r \le r_0$,

$$\mu\left\{x : r < d(x,p) < r + \varepsilon\right\} \le \varepsilon^{\delta}.$$

N. Haydn, M. Nicol, T. Persson and S. Vaienti [7] gave a sufficient condition for the strong Borel–Cantelli property in sequences of balls in X. They proved

Theorem F ([7])**.** *Let c_1 and c_2 be positive constants. Assume that (X, μ, T) satisfies the conditions* (A) *and* (B) *with $p(k) \le c_1\, k^{-q}$, and assume that the balls A_k satisfy $c_2/k^{\gamma} \le \mu(A_k)$ for some constant $0 < \gamma < 1$. If*

$$q > \frac{(2/\delta) + \gamma}{1 - \gamma},$$

then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Let us apply Theorem 2 with a modification of their proof to show that the inequality assumption $q > \frac{(2/\delta)+\gamma}{1-\gamma}$ in Theorem F could be weakened. The result is

Theorem 6. *Let c_1 and c_2 be positive constants. Assume that (X, μ, T) satisfies the conditions* (A) *and* (B) *with $p(k) \le c_1\, k^{-q}$, and assume that the balls A_k satisfy $c_2/k^{\gamma} \le \mu(A_k)$ for some constant $0 < \gamma < 1$. If*

$$q > \max\left\{\frac{\gamma}{\delta} + 1,\ \frac{(2/\delta)\gamma + \gamma}{1 - \gamma}\right\},$$

then $\{A_n\}$ is a strongly Borel–Cantelli sequence.

Note that the maximum in Theorem 6 is smaller than $\frac{(2/\delta)+\gamma}{1-\gamma}$ in Theorem F, since we have

$$\frac{(2/\delta) + \gamma}{1 - \gamma} > \frac{2}{\delta(1 - \gamma)} > \frac{1}{\delta} + \frac{1}{\delta} > \frac{\gamma}{\delta} + 1.$$

Proof. From $\sum_{k=1}^{n} \mu(A_k) \ge \sum_{k=1}^{n} c_2/k^{\gamma} \ge c_0\, n^{1-\gamma}$, it follows that $\sum_{k=1}^{\infty} \mu(A_k) = \infty$. By Theorem 2, it is enough to prove that there exist constants $0 \le \beta < 1$ and $C > 0$ such that $||S_n(x) - E_n|| \le C\, E_n^{\beta}$ holds for all large n.

Let $0 < \varepsilon_1 < 1-\gamma$. By using linear interpolation, we can choose a Lipschitz function $\tilde{f}_k(x)$ in X such that $\tilde{f}_k(x) = 1$ if $x \in A_k$, $\tilde{f}_k(x) = 0$ if $d(A_k, x) \geq (c_2/k^{\gamma+\varepsilon_1})^{1/\delta}$, $0 \leq \tilde{f}_k(x) \leq 1$ and $\|\tilde{f}_k\|_{Lip} \leq (k^{\gamma+\varepsilon_1}/c_2)^{1/\delta}$. Write $\tilde{f}_k(x) = \mathbf{1}_{A_k}(x) + \tilde{h}_k(x)$, $f_k(x) = \tilde{f}_k\left(T^k(x)\right)$ and $h_k(x) = \tilde{h}_k\left(T^k(x)\right)$. Then $f_k(x) = \mathbf{1}_{T^{-k}A_k}(x) + h_k(x)$ and $0 \leq h_k(x) \leq 1$. By the triangle inequality, we get

$$||S_n(x) - E_n|| = \int_X \left|\sum_{k=1}^n \mathbf{1}_{T^{-k}A_k}(x) - \sum_{k=1}^n \mu(A_k)\right| d\mu$$

$$\leq \int_X \left|\sum_{k=1}^n \mathbf{1}_{T^{-k}A_k}(x) - \sum_{k=1}^n f_k(x)\right| d\mu + \int_X \left|\sum_{k=1}^n f_k(x) - \sum_{k=1}^n E(f_k)\right| d\mu$$

$$+ \int_X \left|\sum_{k=1}^n E(f_k) - \sum_{k=1}^n \mu(A_k)\right| d\mu := I + II + III.$$

Since T is μ-preserving, the condition (B) gives

$$\mu(A_k) \leq E(\tilde{f}_k) = E(f_k) = \mu(A_k) + E\left(h_k(x)\right) \leq \mu(A_k) + \frac{c_2}{k^{\gamma+\varepsilon_1}}.$$

Hence,

$$\sum_{k=1}^n \mu(A_k) \leq \sum_{k=1}^n E(f_k) \leq \sum_{k=1}^n \mu(A_k) + c_3 n^{1-\gamma-\varepsilon_1},$$

which implies

$$III = \left|\sum_{k=1}^n E(f_k) - \sum_{k=1}^n \mu(A_k)\right| \leq c_3 n^{1-\gamma-\varepsilon_1} \leq c_4 \left(\sum_{k=1}^n \mu(A_k)\right)^{\frac{1-\gamma-\varepsilon_1}{1-\gamma}}.$$

From $f_k(x) = \mathbf{1}_{T^{-k}A_k}(x) + h_k(x)$ and the condition (B), we get

$$I = \int_X \left|\sum_{k=1}^n \mathbf{1}_{T^{-k}A_k}(x) - \sum_{k=1}^n f_k(x)\right| d\mu = \int_X \left|\sum_{k=1}^n h_k(x)\right| d\mu$$

$$= \sum_{k=1}^n \int_X h_k(x) d\mu = \sum_{k=1}^n \int_X \tilde{h}_k\left(T^k(x)\right) d\mu$$

$$\leq \sum_{k=1}^n \frac{c_2}{k^{\gamma+\varepsilon_1}} \leq c_3 n^{1-\gamma-\varepsilon_1} \leq c_4 \left(\sum_{k=1}^n \mu(A_k)\right)^{\frac{1-\gamma-\varepsilon_1}{1-\gamma}}.$$

It follows from Hölder's inequality that

$$II = \int_X \left| \sum_{k=1}^n f_k(x) - \sum_{k=1}^n E(f_k) \right| d\mu$$

$$\leq \left(\int_X 1^2 \, d\mu \right)^{1/2} \left(\int_X \left(\sum_{k=1}^n f_k(x) - \sum_{k=1}^n E(f_k) \right)^2 d\mu \right)^{1/2}$$

$$= \left(\int_X \left(\sum_{k=1}^n f_k(x) - \sum_{k=1}^n E(f_k) \right)^2 d\mu \right)^{1/2}$$

$$= \left(\sum_{i,j=1}^n \{ E(f_i f_j) - E(f_i)E(f_j) \} \right)^{1/2}$$

$$= \left(2 \sum_{i=1}^n \sum_{j=i+1}^n \{ E(f_i f_j) - E(f_i)E(f_j) \} + \sum_{i=1}^n \left\{ E\left(f_i^2\right) - E(f_i)^2 \right\} \right)^{1/2},$$

which, by $0 \leq f_k \leq 1$, is less than

$$\left(2 \sum_{i=1}^n \sum_{j=i+1}^n |E(f_i f_j) - E(f_i)E(f_j)| + \sum_{i=1}^n E(f_i) \right)^{1/2}$$

$$\leq 2^{1/2} \left(\sum_{i=1}^n \sum_{j=i+1}^n |E(f_i f_j) - E(f_i)E(f_j)| \right)^{1/2} + \left(\sum_{i=1}^n E(f_i) \right)^{1/2}.$$

So we only need to prove that there exists a constant $\alpha < 2$ such that

$$\sum_{i=1}^n \sum_{j=i+1}^n |E(f_i f_j) - E(f_i)E(f_j)| \leq c_5 \left(\sum_{i=1}^n E(f_i) \right)^{\alpha}.$$

Let $\varepsilon_2 > 0$. Take the constant $\sigma = 1 - \gamma - \varepsilon_2 > 0$. Write

$$\sum_{i=1}^n \sum_{j=i+1}^n |E(f_i f_j) - E(f_i)E(f_j)| = \sum_{i=1}^n \sum_{j=i+1}^{n \wedge (i+[i^\sigma])} |E(f_i f_j) - E(f_i)E(f_j)|$$

$$+ \sum_{i=1}^n \sum_{j=i+[i^\sigma]+1}^n |E(f_i f_j) - E(f_i)E(f_j)| := II_1 + II_2,$$

where we use the notation $a \wedge b = \min(a, b)$ and $[i^\sigma]$ denotes the integer part of the number i^σ. By $0 \le f_i, f_j \le 1$ we get

$$|E(f_i f_j) - E(f_i)E(f_j)| \le |E(f_i f_j)| + |E(f_i)E(f_j)| \le 2\,|E(f_i)|\,,$$

and hence

$$II_1 \le 2\sum_{i=1}^{n}\sum_{j=i+1}^{n\wedge(i+[i^\sigma])} E(f_i) \le 2\sum_{i=1}^{n}\sum_{j=i+1}^{n\wedge(i+[i^\sigma])}\left(\mu(A_i) + \frac{c_2}{i^{\gamma+\varepsilon_1}}\right)$$

$$\le 2\sum_{i=1}^{n}\left(i^\sigma\mu(A_i) + \frac{c_2 i^\sigma}{i^{\gamma+\varepsilon_1}}\right) \le 2n^\sigma\sum_{i=1}^{n}\mu(A_i) + c_6 n^{1+\sigma-\gamma-\varepsilon_1}$$

$$\le c_7\left(\sum_{i=1}^{n}\mu(A_i)\right)^{\frac{\sigma}{1-\gamma}+1} + c_8\left(\sum_{i=1}^{n}\mu(A_i)\right)^{\frac{1+\sigma-\gamma-\varepsilon_1}{1-\gamma}}$$

$$\le c_9\left(\sum_{i=1}^{n}\mu(A_i)\right)^{\frac{2-2\gamma-\varepsilon_2}{1-\gamma}} = c_9\left(\sum_{i=1}^{n}\mu(A_i)\right)^{2-\frac{\varepsilon_2}{1-\gamma}}.$$

On the other hand, by the condition (A), we get

$$II_2 = \sum_{i=1}^{n}\sum_{j=i+[i^\sigma]+1}^{n}|E(f_i f_j) - E(f_i)E(f_j)|$$

$$= \sum_{i=1}^{n}\sum_{j=i+[i^\sigma]+1}^{n}\left|\int_X \tilde{f}_i\left(T^i(x)\right)\tilde{f}_j\left(T^j(x)\right)d\mu - E(f_i)E(f_j)\right|$$

$$= \sum_{i=1}^{n}\sum_{j=i+[i^\sigma]+1}^{n}\left|\int_X \tilde{f}_i(x)\tilde{f}_j\left(T^{j-i}(x)\right)d\mu - E(f_i)E(f_j)\right|$$

$$\le \sum_{i=1}^{n}\sum_{j=i+[i^\sigma]+1}^{n}\|\tilde{f}_i\|_{Lip}\|\tilde{f}_j\|_{Lip}\,p(j-i)$$

$$\le c_{10}\sum_{i=1}^{n}\sum_{j=i+[i^\sigma]+1}^{n}\frac{i^{\frac{\gamma+\varepsilon_1}{\delta}}j^{\frac{\gamma+\varepsilon_1}{\delta}}}{(j-i)^q} \le c_{10}\sum_{i=1}^{n}i^{\frac{\gamma+\varepsilon_1}{\delta}}\sum_{k=1}^{\infty}\frac{(i+[i^\sigma]+k)^{\frac{\gamma+\varepsilon_1}{\delta}}}{([i^\sigma]+k)^q}$$

$$\le c_{10}\sum_{i=1}^{n}i^{\frac{\gamma+\varepsilon_1}{\delta}}\sum_{k=1}^{\infty}2^{\frac{\gamma+\varepsilon_1}{\delta}}\frac{(i+k)^{\frac{\gamma+\varepsilon_1}{\delta}}}{([i^\sigma]+k)^q}$$

$$= c_{11} \sum_{i=1}^{n} i^{\frac{\gamma+\varepsilon_1}{\delta}} \sum_{k=1}^{\infty} ([i^\sigma]+k)^{\frac{\gamma+\varepsilon_1}{\delta}-q} \left(\frac{i+k}{[i^\sigma]+k} \right)^{\frac{\gamma+\varepsilon_1}{\delta}},$$

where

$$\left(\frac{i+k}{[i^\sigma]+k} \right)^{\frac{\gamma+\varepsilon_1}{\delta}} = \left(1 + \frac{i-[i^\sigma]}{[i^\sigma]+k} \right)^{\frac{\gamma+\varepsilon_1}{\delta}} \le \left(1 + \frac{i-[i^\sigma]}{[i^\sigma]} \right)^{\frac{\gamma+\varepsilon_1}{\delta}}$$

$$= \left(\frac{i}{[i^\sigma]} \right)^{\frac{\gamma+\varepsilon_1}{\delta}} \le \left(\frac{2\,i}{i^\sigma} \right)^{\frac{\gamma+\varepsilon_1}{\delta}} = 2^{\frac{\gamma+\varepsilon_1}{\delta}} i^{(1-\sigma)\frac{\gamma+\varepsilon_1}{\delta}}.$$

Thus, we get

$$II_2 \le c_{12} \sum_{i=1}^{n} i^{\frac{\gamma+\varepsilon_1}{\delta}+(1-\sigma)\frac{\gamma+\varepsilon_1}{\delta}} \sum_{k=1}^{\infty} ([i^\sigma]+k)^{\frac{\gamma+\varepsilon_1}{\delta}-q}$$

$$\le c_{13} \sum_{i=1}^{n} i^{\frac{2\gamma-\sigma\gamma+2\varepsilon_1-\sigma\varepsilon_1}{\delta}} \sum_{k=1}^{\infty} (i^\sigma+k)^{\frac{\gamma+\varepsilon_1}{\delta}-q},$$

which, if $\frac{\gamma+\varepsilon_1}{\delta} - q < -1$, is less than

$$c_{14} \sum_{i=1}^{n} i^{\frac{2\gamma-\sigma\gamma+2\varepsilon_1-\sigma\varepsilon_1}{\delta}} i^{\sigma\left(\frac{\gamma+\varepsilon_1}{\delta}-q+1\right)} = c_{14} \sum_{i=1}^{n} i^{\frac{2\gamma}{\delta}-\sigma q+\sigma+\frac{2\varepsilon_1}{\delta}}$$

$$\le c_{15} n^{\frac{2\gamma}{\delta}-\sigma q+\sigma+1+\frac{2\varepsilon_1}{\delta}} = c_{15} n^{\frac{2\gamma}{\delta}+1+(1-\gamma)-(1-\gamma)q-\varepsilon_2-\varepsilon_2 q+\frac{2\varepsilon_1}{\delta}}$$

$$\le c_{16} \left(\sum_{k=1}^{n} \mu(A_k) \right)^{\frac{\frac{2\gamma}{\delta}+1}{1-\gamma}+1-q+\frac{-\varepsilon_2-\varepsilon_2 q+\frac{2\varepsilon_1}{\delta}}{1-\gamma}}.$$

By the arbitrariness of ε_1 and ε_2, we have proven Theorem 6 as long as we assume that $q > \frac{\gamma}{\delta} + 1$ and $\frac{\frac{2\gamma}{\delta}+1}{1-\gamma} + 1 - q < 2$, i.e.

$$q > \max \left\{ \frac{\gamma}{\delta} + 1,\ \frac{(2/\delta)\gamma + \gamma}{1-\gamma} \right\}.$$

□

Note that if $\gamma \ge \frac{-3+\sqrt{13}}{2} = 0.30278\ldots$, then

$$\max \left\{ \frac{\gamma}{\delta} + 1,\ \frac{(2/\delta)\gamma + \gamma}{1-\gamma} \right\} = \frac{(2/\delta)\gamma + \gamma}{1-\gamma},$$

since we have

$$\frac{\gamma}{\delta} + 1 \le \frac{(2/\delta)\gamma+\gamma}{1-\gamma} \Longleftrightarrow \frac{\gamma^2}{\delta} + \frac{\gamma}{\delta} + 2\gamma - 1 \ge 0$$

$$\Longleftarrow \gamma^2 + 3\gamma - 1 \ge 0,$$

which is true when $\gamma \ge \frac{-3+\sqrt{13}}{2}$.

4.3 Ergodic dynamical systems

Following Chapter 6 in [12], there are several notions of mixing for measure-preserving transformations in X.

Definition 4. *Let (X, μ, T) be a measure-preserving dynamical system.*
(i) *T is called a mixing if, for all measurable sets A and B in X, we have*

$$\lim_{n\to\infty} \mu\left(T^{-n}(A) \cap B\right) = \mu(A)\mu(B).$$

(ii) *T is called a weakly mixing if, for all measurable sets A and B in X, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\mu\left(T^{-k}(A) \cap B\right) - \mu(A)\mu(B)\right| = 0.$$

(iii) *T is called ergodic if, for all measurable sets A and B in X, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left\{\mu\left(T^{-k}(A) \cap B\right) - \mu(A)\mu(B)\right\} = 0,$$

that is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu\left(T^{-k}(A) \cap B\right) = \mu(A)\mu(B).$$

Evidently, weakly mixing implies ergodicity. From Chapter 6.2 in [12], we get that mixing implies weakly mixing as well. Generally, the converse implications do not hold.

The ergodic and weakly mixing properties of measure-preserving dynamical systems could be characterized by means of Borel–Cantelli and strongly Borel–Cantelli sequences, as shown in the following two Theorems.

Theorem G ([8])**.** *Let (X, μ, T) be a measure-preserving dynamical system. Then the following statements are equivalent.*

(a) *T is ergodic;*
(b) *Each constant sequence $\{A\}$ with $\mu(A) > 0$ is a Borel–Cantelli sequence;*
(c) *Each constant sequence $\{A\}$ with $\mu(A) > 0$ is a strongly Borel-Cantelli sequence.*

Theorem H ([8])**.** *Let (X, μ, T) be a measure-preserving dynamical system. Then the following statements are equivalent.*

(a) *T is weakly mixing;*

(b) *Each sequence $\{A_n\}$ that only contains finitely many distinct sets, each with positive measure, is a Borel–Cantelli sequence;*
(c) *Each sequence $\{A_n\}$ that only contains finitely many distinct sets, each with positive measure, is a Borel–Cantelli sequence in the L^2 metric, i.e.*

$$\left\| \left(\frac{S_n(x)}{E_n} - 1 \right)^2 \right\| \longrightarrow 0 \quad \text{as } n \to \infty.$$

According to paper [8], the equivalence to the assertions (a) and (b) in Theorem H was first proven by Y. Guivarch and A. Raugi. The equivalence to the assertions (a) and (c) was given by N. Chernov and D. Kleinbock. However, Theorem H does not discuss the strong Borel–Cantelli property of the sequence $\{A_n\}$. For this purpose, let us introduce a new type of weakly mixing transformations.

Definition 5. *Let $0 < \beta \leq 1$. A μ-preserving transformation T is called a weakly β-mixing with respect to a collection of measurable sets in X if, for any two sets A and B in the collection, there exists a constant $C > 0$ such that*

$$\sum_{k=0}^{n-1} \left| \mu\left(T^{-k}(A) \cap B\right) - \mu(A)\mu(B) \right| \leq C n^{\beta} \quad \text{for } n = 1, 2, \ldots,$$

that is,

$$\sum_{k=0}^{n-1} \left| \mu\left(T^{-k}(A) \cap B\right) - \mu(A)\mu(B) \right| = O\left(n^{\beta}\right).$$

Now, we have

Theorem 7. *Let $0 < \beta < 1$ and (X, μ, T) be a measure-preserving dynamical system. If the map T is weakly β-mixing with respect to the sequence $\{A_n\}$, which only contains finitely many distinct sets, each with positive measure, then $\{A_n\}$ is a strongly Borel–Cantelli sequence and satisfies*

$$\left\| \left(\frac{S_n(x)}{E_n} - 1 \right)^2 \right\| = O\left(\frac{1}{n^{1-\beta}} \right).$$

Proof. We have

$$\|\left(S_n(x) - E_n\right)^2\| = \sum_{k,m=1}^{n} \left\{ \mu\left(T^{-k}A_k \cap T^{-m}A_m\right) - \mu(A_k)\mu(A_m) \right\}$$

$$= 2\sum_{m=1}^{n}\sum_{k=m+1}^{n}\left\{\mu\left(T^{-k}A_k\cap T^{-m}A_m\right)-\mu(A_k)\mu(A_m)\right\}$$

$$+\sum_{m=1}^{n}\left\{\mu\left(A_m\right)-\mu(A_m)^2\right\}$$

$$= 2\sum_{m=1}^{n}\sum_{k=m+1}^{n}\left\{\mu\left(T^{-m}\left(T^{-(k-m)}A_k\right)\cap T^{-m}A_m\right)-\mu(A_k)\mu(A_m)\right\}$$

$$+\sum_{m=1}^{n}\left\{\mu\left(A_m\right)-\mu(A_m)^2\right\},$$

which, by the equality $T^{-m}(A)\cap T^{-m}(B)=T^{-m}(A\cap B)$ for all sets $A,\ B$, is equal to

$$2\sum_{m=1}^{n}\sum_{k=m+1}^{n}\left\{\mu\left(T^{-(k-m)}A_k\cap A_m\right)-\mu(A_k)\mu(A_m)\right\}$$

$$+\sum_{m=1}^{n}\left\{\mu\left(A_m\right)-\mu(A_m)^2\right\}$$

$$\leq 2\sum_{m=1}^{n}\sum_{k=m+1}^{n}\left|\mu\left(T^{-(k-m)}\left(A_k\right)\cap A_m\right)-\mu(A_k)\mu(A_m)\right|+E_n.$$

The sequence $\{A_k\}$ only contains a finite number of distinct sets (each with positive measure), say, $D_1, D_2, \ldots, D_L$. So there exist two positive constants c_1 and c_2 such that $c_1\leq\mu(A_k)\leq c_2$ for all $k=1,2,\cdots$. Hence,

$$c_1 n\leq\sum_{k=1}^{n}\mu(A_k)\leq c_2 n,\quad \text{i.e.}\quad c_1 n\leq E_n\leq c_2 n\quad \text{for } n=1,2,\cdots.$$

Thus, we get

$$\|\left(S_n(x)-E_n\right)^2\|$$

$$\leq 2\sum_{m=1}^{n}\sum_{l,w=1}^{L}\sum_{s=1}^{n-m}\left|\mu\left(T^{-s}(D_l)\cap D_w\right)-\mu(D_l)\mu(D_w)\right|+E_n,$$

where $E_n=O(n)=O\left(n^{1+\beta}\right)$. On the other hand, by the weakly β-mixing property of T with respect to $\{A_n\}$, we get

$$\sum_{m=1}^{n}\sum_{l,w=1}^{L}\sum_{s=1}^{n-m}\left|\mu\left(T^{-s}(D_l)\cap D_w\right)-\mu(D_l)\mu(D_w)\right|$$

$$= O\left(\sum_{m=1}^{n}\sum_{l,w=1}^{L}(n-m)^{\beta}\right) = O\left(\sum_{m=1}^{n}(n-m)^{\beta}\right)$$

$$= O\left(\sum_{k=1}^{n-1}k^{\beta}\right) = O\left(\int_{1}^{n}x^{\beta}\,dx\right) = O\left(n^{1+\beta}\right).$$

Thus, we have

$$\left|\left|(S_n(x)-E_n)^2\right|\right| = O\left(n^{1+\beta}\right),$$

which implies

$$\left\|\left(\frac{S_n(x)}{E_n}-1\right)^2\right\| = O\left(\frac{n^{1+\beta}}{E_n^2}\right) = O\left(\frac{1}{n^{1-\beta}}\right).$$

It follows from $c_1 n \le E_n \le c_2 n$ that $\sum_{n=1}^{\infty}\mu(A_n) = \lim_{n\to\infty}E_n = \infty$ and

$$\sum_{k,m=1}^{n}\left\{\mu\left(T^{-k}A_k \cap T^{-m}A_m\right) - \mu(A_k)\mu(A_m)\right\} = \|\,(S_n(x)-E_n)^2\,\|$$

$$= O\left(n^{1+\beta}\right) = O\left(E_n^{1+\beta}\right).$$

So $\{A_n\}$ satisfies the condition $\triangle_{1+\beta}$ with $1 < 1+\beta < 2$. By Theorem 1, we get that $\{A_n\}$ is a strongly Borel–Cantelli sequence. □

5 Unsolved problems and future work

In this Master's thesis, we have investigated several versions of the sufficient condition for the strong Borel–Cantelli property. We have also discussed some of their applications. Hopefully, these findings can be applied to further improve some of the already known results.

This thesis has mainly dealt with the strong Borel–Cantelli property. Consequently, it becomes natural to think about the sufficient condition for the Borel–Cantelli property itself.

We have proven that the weakly β-mixing with $0 < \beta < 1$ implies the strong Borel–Cantelli property in sequences of sets that only consist of finitely many distinct sets, where each set has positive measure. However, the converse assertion is still unknown, so for a future work, it would be rather interesting to research the characteristics of the strong Borel–Cantelli property in those sequences by means of some kind of mixing property.

References

[1] Dmitry Kleinbock and Grigory Margulis. Logarithm laws for flows on homogeneous spaces. *Inv. Math.*, 138:451–494, (1999).

[2] Wolfgang M. Schmidt. A metrical theorem in of Diophantine approximation. *Canad. J. Math.*, 12:619–631, (1960).

[3] Wolfgang M. Schmidt. Metrical theorems on fractional parts of sequences. *Transactions AMS*, 110:493–518, (1964).

[4] William Judson Leveque. On the frequency of small fractional parts in certain real sequences III. *Journal reine angew. Math.*, 202:215–220, (1959).

[5] Walter Philipp. Some metrical theorems in number theory. *Pacific J. Math.*, 20:109–127, (1967).

[6] Chinmaya Gupta, Matthew Nicol, and William Ott. A Borel–Cantelli lemma for non-uniformly expanding dynamical systems. *Nonlinearity*, 23(8):1991–2008, (2010).

[7] Nicolai Haydn, Matthew Nicol, Tomas Persson, and Sandro Vaienti. A note on Borel–Cantelli lemmas for non-uniformly hyperbolic dynamical systems. *Ergod. Th. Dynam. Sys.*, 33:475–498, (2013).

[8] Nikolai Chernov and Dmitry Kleinbock. Dynamical Borel–Cantelli lemmas for Gibbs measures. *Israel J. Math.*, 122:1–27, (2001).

[9] Emile Borel. Les probabilités dénombrables et leurs applications arithmétiques. *Rend. Circ. Mat. Palermo*, 27:247–271, (1909).

[10] Francesco Paolo Cantelli. Sulla probabilità come limite della frequenza. *Atti delta Reale Accademia Nationale dei Lincei*, 26:39–45, (1917).

[11] Daval Khoshnevisan. *Probability (Graduate Studies in Mathematics)*, volume 80. American Mathematical Society, (2007).

[12] César Ernesto Silva. *Invitation to Ergodic Theory.* American Mathematical Society, (2008).

[13] Vladimir Gennadievich. *Metric theory of Diophantine approximations.* J. Wiley Sons, New York–Toronto–London, (1979).