THE HANGING ROPE: A CONVEX OPTIMIZATION PROBLEM IN THE CALCULUS OF VARIATIONS

ERIK STEEN

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LUND UNIVERSITY

Faculty of Engineering
Centre for Mathematical Sciences
Mathematics

CENTRUM SCIENTIARUM MATHEMATICARUM

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Abstract

We study the problem first introduced by Verma and Keller in 1984 of how to taper a heavy rope such that its elongation is minimized. The problem is stated as an optimization problem of a functional $J[w]$. Specifically we provide a proof of optimality for the solution using traditional convex optimization techniques. We also utilize the Legendre transformation when studying the Euler–Lagrange equation — this is nice because it sheds some light on the structure of the solution in a natural way. In the last section we consider a similar problem but where the functional J is a function of itself; $J = J[w, J]$. This problem is unfortunately not solved but might be subject to future research.

Acknowledgements

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Contents

1 Introduction

This is a thesis about how to taper a heavy rope that's hanging vertically (from the ceiling) so that its elongation is minimized. The rope is being stretched by a weight attached to its lower end as well as by its own weight. The initial (undeformed) length together with the total mass and density of the rope are known, while the parameter we can adjust is the cross section area. The problem is formulated and solved using the Calculus of Variations, and an optimality proof for the solution is given exploiting the convexity of the problem. This problem was originally stated in 1984 by Verma and Keller using a constant density and a linear stress-strain relation (Hooke's law) [6]. Later it was generalized to an arbitrary density and a non-linear stress-strain relation by Negrón-Marrero [2] and again revisited in 2018 by Overgaard [3]. The goals of this thesis are to

- Provide Negrón-Marreros solution with an optimality proof.
- Interpret the five assumptions made by Negrón-Marrero on the stress-strain relation. We will do so in relation to the mechanics of materials and to their relevance for the solution from a mathematical perspective.
- Give explicit solutions to interesting special cases of the model.

In the last section, we consider a different problem - "The Rotating Load" inspired by the original problem. Here, the "rope" is rotating around an axis, again with a weight attached to the outer end. The interesting case arises with large deformations since stretching the rope changes the forces acting on it, the centripetal force being proportional to the distance from the axis of rotation. This problem is unfortunately not solved; however, my hope is to be able to shed some light on some of the difficulties that have to be overcome, and present the results I've found.

2 Background

2.1 The Standard Problem

The Standard Problem in the Calculus of Variations can be described as follows: Given a function $L(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$, minimize the functional

$$J[y] = \int_a^b L(x, y(x), y'(x))\mathrm{d}x$$

over some admissible functions y, for example

$$y \in \{\mathcal{C}^1[a, b];\ y(a) = \alpha,\ y(b) = \beta\} = \mathcal{X}$$

for given α and β. The integrand $L(x_1, x_2, x_3)$ is called the Lagrangian, or the Lagrange function. We will assume it to have continuous partial derivatives.

To minimize $J[y]$, the approach used is to assume we've found the minimizing function $y_0(x)$, then add a variation $\epsilon v(x)$ to it. Here $v(x)$ is a variation to $y_0(x)$ while $\epsilon \in \mathbb{R}$ is used to scale this variation. We demand that $v(a) = v(b) = 0$ in order for $y(x) = y_0(x) + \epsilon v(x)$ to satisfy the boundary conditions, $y(a) = \alpha$ and $y(b) = \beta$.

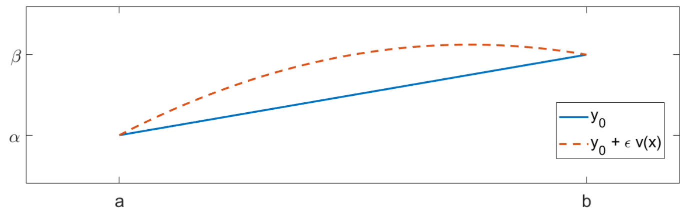

Figure 1: Illustration of a simple variation to y_0 (the straight line).

We now study the properties of the function

$$j(\epsilon) = J[y_0 + \epsilon v] = \int_a^b L(x, y_0(x) + \epsilon v(x), y_0'(x) + \epsilon v'(x))\mathrm{d}x.$$

Since y_0 by assumption is a minimizer of J we know that $j(\epsilon)$ has a minimum at $j(0) = J[y_0]$. Since it has a minimum at $\epsilon = 0$ we know that $j'(0) = 0$. Let's have a look at this term:

$$j'(\epsilon) = \frac{d}{d\epsilon}\int_a^b L(x, y_0 + \epsilon v, y_0' + \epsilon v')\mathrm{d}x =$$

$$= \int_a^b L_y(x, y_0 + \epsilon v, y_0' + \epsilon v')v + L_{y'}(x, y_0 + \epsilon v, y_0' + \epsilon v')v'\mathrm{d}x\ .$$

Here L_y and $L_{y'}$ denote the derivatives of the Lagrangian with respect to the second and third argument of the Lagrangian. It will also be convenient to use the notation $L_{y_0}(x) = L_y(x, y_0(x), y_0'(x))$ (and $L_{y_0'}(x) = L_{y'}(x, y_0(x), y_0'(x))$). That is the derivative of L with respect to the second (or third) variable, evaluated at $(x, y_0(x), y_0'(x))$. With $\epsilon = 0$ in the expression above we get

$$j'(0) = \int_a^b L_y(x, y_0, y_0')v + L_{y'}(x, y_0, y_0')v' \mathrm{d}x.$$

Now we want to integrate the first term by parts. By letting the primitive function of L_{y_0} be denoted by θ, such that $\theta' = L_{y_0}$, we then have

$$\begin{aligned} j'(0) &= \int_a^b \theta' v + L_{y_0'} v' \mathrm{d}x = \\ &= \Big[\theta v\Big]_a^b + \int_a^b (-\theta + L_{y_0'})v' \mathrm{d}x = 0. \end{aligned}$$

Since $v(a) = v(b) = 0 \implies [\theta v]_a^b = 0$ all that is left is

$$\int_a^b (-\theta + L_{y_0'})v' \mathrm{d}x = 0. \tag{1}$$

Since $v \in C_0^1[a, b]$, $L_{y_0'} \in C[a, b]$ and θ is clearly continuous (it's differentiable), we can use the following lemma.

Lemma 1 (du Bois–Reymond)**.** *If $N(x) \in C[a, b]$ and*

$$\int_a^b N(x)v'(x)\mathrm{d}x = 0$$

for all $v \in C_0^1[a, b]$ then

$$N(x) = \textit{constant}$$

We will postpone the proof of this lemma until the end of this section and instead return to the integral in equation (1). The lemma states that

$$-\theta + L_{y_0'} = c \tag{2}$$

for some constant c. Since θ is differentiable ($\theta' = L_{y_0}$) and $L_{y_0'} = c+\theta$ is the sum of a constant and a differentiable function we conclude that $L_{y_0'}$ is differentiable as well. We differentiate equation (2) and arrive at the Euler–Lagrange equation

$$-L_{y_0} + \frac{d}{dx} L_{y_0'} = 0 \tag{3}$$

where we used $\theta' = L_{y_0}$. This differential equation must be satisfied by any minimizer of the functional $J[y]$. However, also maximizers and "saddle points" of J will satisfy this ODE. To find the solution to the minimizing problem we'll need something more.

We'll end this section with a proof of du Bois-Reymonds lemma.

Proof. To prove du Bois-Raymonds lemma above let's consider a special variation

$$v(x) = \int_a^x N(\bar{x}) - \mu \, \mathrm{d}\bar{x}$$

where μ is the mean value of $N(x)$ on $[a, b]$. This $v(x)$ vanishes at $x = a$ and $x = b$, and is differentiable. By assumption

$$\int_a^b N(x)v'(x)\mathrm{d}x = 0$$

holds for all $v \in C_0^1[a, b]$, then it must hold for our constructed v, which implies

$$\int_a^b N(x)(N(x) - \mu)\mathrm{d}x = 0. \tag{4}$$

Also, clearly

$$\mu \int_a^b (N(x) - \mu)\mathrm{d}x = 0. \tag{5}$$

If we now subtract equation (5) from equation (4) we get

$$\int_a^b N(x)(N(x) - \mu)\mathrm{d}x - \mu \int_a^b N(x) - \mu \, \mathrm{d}x = 0 \iff$$

$$\int_a^b (N(x) - \mu)^2 \mathrm{d}x = 0$$

which shows that $N(x) = \mu = $ constant. □

2.2 Convex functions and functionals

In this section we go through the definition of convex functions and functionals, then provide a couple of theorems taken from [5] that will be used in the optimality proof for our problem. We start with a formal definition of a convex real valued function of one variable.

Definition 1. *Let φ be a function defined on an interval $D \subseteq \mathbb{R}$. The functiono φ is called convex on D if it satisfies*

$$(1-\lambda)\varphi(a) + \lambda\varphi(b) \geq \varphi((1-\lambda)a + \lambda b),\ \forall \lambda \in [0,1]$$

where $a, b \in D$. If we change the inequality to a strict inequality above and $\lambda \in (0,1)$, we say that φ is strictly convex.

The definition says that if you connect two points on the graph of φ with a straight line segment, then the line lies above the graph. Likewise, if we draw a tangent to φ it touches the graph from below as seen in Figure 2.

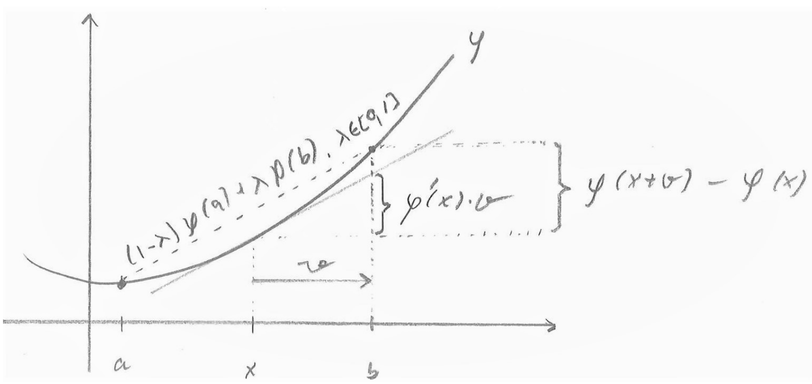

Figure 2: A convex function. The dashed line connecting $(a, \varphi(a))$ with $(b, \varphi(b))$ lies above the graph of φ on (a, b).

From the figure we conclude that if φ is differentiable at x then

$$\varphi(x+v) - \varphi(x) \geq \varphi'(x)v \tag{6}$$

Moving on to functions of n variables. Let $D \subseteq \mathbb{R}^n$ be a convex set. Then $\varphi : D \to \mathbb{R}$ is convex if and only if $t \mapsto \varphi(\mathbf{x}+t\mathbf{v})$ is convex for all $\mathbf{x} \in D$, $\mathbf{v} \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ such that $\mathbf{x} + t\mathbf{v} \in D$. Hence, if φ is convex it is convex in every direction $\mathbf{v}$ and we can write equation (6) as

$$\varphi(\mathbf{x}+\mathbf{v}) - \varphi(\mathbf{x}) \geq \nabla\varphi(\mathbf{x}) \cdot \mathbf{v} \tag{7}$$

where $\nabla\varphi(\mathbf{x}) \cdot \mathbf{v}$ is the directional derivative in the direction of $\mathbf{v}$, (scaled with the length of $\mathbf{v}$). For strictly convex φ equality in equation (7) is only obtained for $\mathbf{v} = \mathbf{0}$.

We would like to extend the concept of convexity to our functionals. As a stepping stone we introduce

Definition 2 (The Gâteaux Variations). *For $J : \mathcal{X} \to \mathbb{R}$, we define*

$$dJ(y; v) = \lim_{\epsilon \to 0} \frac{J[y + \epsilon v] - J[y]}{\epsilon}$$

where $y, y + v \in \mathcal{X}$.

This is the directional derivative of J, just as $\nabla\varphi(\mathbf{x}) \cdot v$ was the directional derivative of φ in equation (6). Also note that if

$$J[y] = \int_a^b L(x, y, y')\mathrm{d}x$$

and we apply the Gâteaux Variation, we get the first variation $j'(0)$, covered in section 2.1:

$$dJ(y; v) = \int_a^b L_y(x, y, y')v + L_{y'}(x, y, y')v'\mathrm{d}x.$$

This term equals zero for any y satisfying the Euler-Lagrange equation.

We define convexity of the functional J as

Definition 3. *The functional*

$$J : \mathcal{X} \mapsto \mathbb{R}$$

is said to be convex on $\mathcal{X}$ if

$$J[y + v] - J[y] \geq dJ(y; v)$$

for all y, $y+v \in \mathcal{X}$. For strict convexity equality is obtained if and only if $v = 0$.

We now have a definition for convex functionals. The reason we want this is that it makes it easy to show that solutions to minimization problems are global and unique. If we have a convex functional $J[y]$, then any y_0 making $dJ(y_0; v) = 0$ for all v is a global minimizer of J. If J is strictly convex then this y_0 is also unique. On the next page, Theorem 1 gives us a condition that guarantees J being strictly convex. After that, Theorem 2 states that any function y_0 satisfying $dJ(y_0; v) = 0$ for all v and J strictly convex is indeed the unique minimizer of J.

Theorem 1. *Let D be a convex domain in $\mathbb{R}^2$ and for given α, β, set*

$$\mathcal{X} = \{y \in \mathcal{C}^1[a,b];\ y(a) = \alpha,\ y(b) = \beta;\ (y(x), y'(x)) \in D\}$$

If $L(x, y, z) : [a,b] \times D \to \mathbb{R}$ satisfies

$$L(x, y+v, z+w) - L(x, y, z) \geq L_y(x, y, z)v + L_z(x, y, z)w$$

for all (y,z) and (y+v,z+w) $\in D$, with equality at (x,y,z) if and only if ***either*** *$v = 0$* ***or*** *$w = 0$, and L, L_y, L_z are continuous, then*

$$J[y] = \int_a^b L(x, y(x), y'(x))\mathrm{d}x$$

is strictly convex on $\mathcal{X}$.

Proof. L satisfies

$$L(x, y+v, z+w) - L(x, y, z) \geq L_y(x, y, z)v + L_z(x, y, z)w$$

with equality if and only if either $v = 0$ or $w = 0$. Then

$$L(x, y+v, y'+v') - L(x, y, y') \geq L_y(x, y, y')v + L_{y'}(x, y, y')v'$$

must hold for all (y, y'), $(y+v, y'+v') \in D$, with equality if and only if $v = 0$ or $v' = 0$. We integrate both sides and get

$$\int_a^b L(x, y+v, y'+v') - L(x, y, y')\mathrm{d}x \geq \int_a^b L_y(x, y, y')v + L_{y'}(x, y, y')v'\mathrm{d}x$$

$$\iff$$

$$J[y+v] - J[y] \geq dJ(y; v) \tag{8}$$

But if one of the functions v or v' equals zero at x then the product vv' must also be zero at x. We integrate vv' and get

$$\int vv'\mathrm{d}x = \frac{v^2}{2} = c$$

for some constant c. Since $v(0) = 0$ we conclude that $v = 0$ on the whole interval (and so is v'). We therefore have equality in equation (8) if and only if $v = 0$ and $v' = 0$, making J strictly convex. $\square$

Now to the uniqueness and optimality of the solution y_0.

Theorem 2. *If J is strictly convex on the convex set $\mathcal{X}$, then each $y_0 \in \mathcal{X}$ for which*

$$dJ(y_0; v) = 0, \ \forall y_0 + v \in \mathcal{X}$$

minimizes J on $\mathcal{X}$ uniquely.

Proof. J is strictly convex so

$$J[y+v] - J[y] \geq dJ(y;v), \ \ \forall y, y+v \in \mathcal{X}$$

with equality if and only if $v = 0$. Insert y_0 such that $dJ(y_0; v) = 0, \ \forall v; \ y_0 + v \in \mathcal{X}$ and get

$$J[y_0 + v] \geq J[y_0]$$

with equality if and only if $v = 0$. This proves uniqueness and optimality of y_0. □

2.3 The Legendre Transformation

This presentation of the Legendre transform is influenced by the article *Making Sense of the Legendre transform* [4]. My goal with this section is to give a brief explanation on how to calculate the transform and show some properties of it. While we've discussed the concept of convexity for many variables, we will only need Legendre transformations in one variable.

Computing the transform

The Legendre transform takes a strictly convex and differentiable function $\varphi(x)$ and maps it to another function $\varphi_*(s)$, containing the same information as φ, but now as a function of the new variable $s = \varphi'(x)$. The strict convexity of φ is needed for this relation to be bijective. This will make φ' strictly monotone and invertible. In other words, for any $s \in \text{Range}(\varphi'(x))$, we can find a unique x such that $s = \varphi'(x)$. We now proceed with the definition.

Definition 4 (The Legendre Transform)**.** *Let $\varphi_*(s)$ denote the Legendre transformation of the convex function $\varphi : D \to \mathbb{R}$. Then*

$$\varphi_*(s) := \sup_{x \in D}\{sx - \varphi(x)\}.$$

Since φ is strictly convex (and $-\varphi$ concave) the supremum is obtained when

$$\frac{d}{dx}(sx - \varphi(x)) = 0 \implies s = \varphi'(x).$$

To calculate the transform we start off by defining the new variable

$$s := \varphi'(x).$$

Now we want to find the inverse of $\varphi'(x)$, let us denote it by $x(s)$. This function takes the slope of φ as input argument and returns the x-value for which $\varphi'(x) = s$. The transformed function is given by

$$\varphi_*(s) = sx(s) - \varphi(x(s)) \tag{9}$$

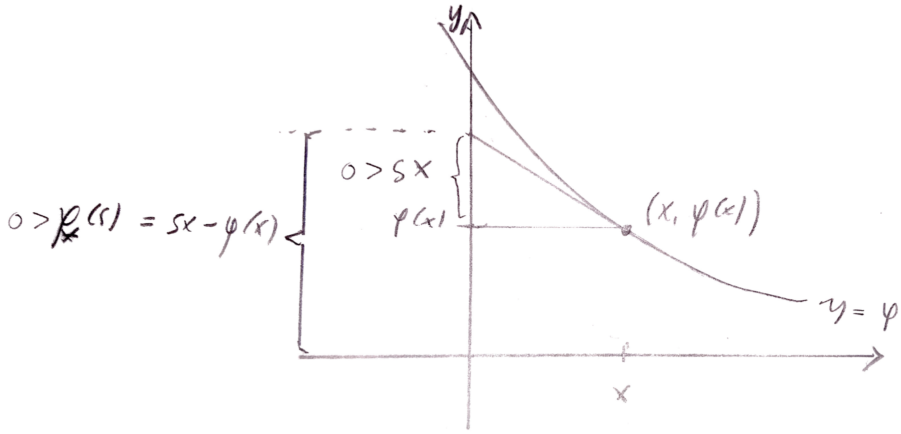

Figure 3: In the case illustrated in the figure $sx < 0$ since clearly $s = \varphi'(x) < 0$. Please note that the transformed function $\varphi_*(s) = sx - \varphi(x)$ is here also negative.

There is a geometrical interpretation of the transformation. Please consult Figure 3 above in what follows. We draw two lines from the point $(x, \varphi(x))$ to the y-axis, creating a triangle. The base of this triangle, parallel to the x-axis, obviously has length x. The hypotenuse should be parallel to the tangent of φ at x. With $s = \varphi'(x)$ we conclude that the height of the triangle is $\pm sx$ (depending on the sign of s). The transform $\varphi_*(s)$ is simply the difference between the height of the triangle sx and $\varphi(x)$.

The transform is its own inverse

If we perform the Legendre transform a second time we recover our original function. Let's do that. We define the new variable

$$y(s) = \frac{d}{ds}\varphi_*(s)$$

and invert the monotonic function $y(s)$ to $s(y)$. We now construct

$$\varphi_{**}(y) = ys(y) - \varphi_*(s(y))$$

If we now let φ_{**} and φ_* switch places and rename the variable y to x, since the name doesn't matter, we get

$$\varphi_*(s) = xs - \varphi_{**}(x)$$

Here we can identify φ_{**} with φ by comparing this expression with equation (9).

The variables s and x are called a conjugate pair and of course they are not independent of each other.

We now give an example

Example 1. *Let's transform* $\varphi(x) = \frac{1}{a}x^a$, $x \geq 0$, $a > 1$. *Let*

$$s = \varphi'(x) = x^{a-1} \iff x(s) = s^{\frac{1}{a-1}}$$

and

$$\varphi_*(s) = sx(s) - \varphi(x(s)) = s^{1+\frac{1}{a-1}} - \frac{1}{a}s^{\frac{a}{a-1}} = \frac{a-1}{a}s^{\frac{a}{a-1}}$$

We see that $\frac{1}{a}x^a$ transforms to $\frac{1}{b}s^b$ where $b = a/(a-1)$. The numbers a and b are also said to be conjugate to each other and they have the nice property

$$\frac{1}{a} + \frac{1}{b} = 1.$$

Scaling properties of the Legendre transformation

Here follows a couple of scaling rules for the Legendre transform. We are only considering functions on domains D that are scaling invariant. By this i mean that D satisfy: $x \in D \implies kx \in D$, $k > 0$.

Lemma 2. *(a) Let* $\varphi_*(s)$ *be the Legendre transform of* $\varphi(x)$ *defined on the positive real axis. For* $k > 0$ *we have*

$$\varphi(kx) \mapsto \varphi_*(\frac{s}{k}) \tag{10}$$

(b) Let $\varphi_*(s)$ *be the Legendre transform of* $\varphi(x)$ *defined on the positive real axis. For* $k > 0$ *we have*

$$k\varphi(x) \mapsto k\varphi_*(\frac{s}{k})$$

Proof. (a) We have

$$\varphi_*(s) = \sup_{x\in\mathbb{R}^+} \{sx - \varphi(x)\}$$

and let

$$g(x) = \varphi(kx),\ k > 0.$$

Then

$$\begin{aligned} g_*(s) &= \sup_{x\in\mathbb{R}^+} \{sx - g(x)\} \\ &= \sup_{x\in\mathbb{R}^+} \{sx - \varphi(kx)\} \\ &= \sup_{x\in\mathbb{R}^+} \{\frac{s}{k}kx - \varphi(kx)\} \\ &= \varphi_*(\frac{s}{k}) \end{aligned}$$

□

Proof. *(b)* We let
$$g(x) = k\varphi(x),\ k > 0 \tag{11}$$
and
$$g'(x) = k\varphi'(x). \tag{12}$$
The transform of φ is
$$\varphi_*(s) = sx(s) - \varphi(x(s)) \tag{13}$$
where $x(s)$ is the inverse of $\varphi'(x)$. Define
$$s = g'(x) = k\varphi'(x) \iff \frac{s}{k} = \varphi'(x).$$
We see that the inverse of $g'(x)$ is $x(s/k)$, where $x(\cdot)$ is the inverse of $\varphi'(\cdot)$. The transform of g is given by
$$\begin{aligned} g_*(s) &= sx(s/k) - g(x(s/k)) = \\ &= sx(s/k) - kf(x(s/k) = \\ &= k\big(\frac{s}{k}x(s/k) - \varphi(x(s/k))\big) = \\ &= k\varphi_*(s/k). \end{aligned} \tag{14}$$
$\square$

3 The hanging rope

Defining the problem

Consider an elastic rope hanging from a roof with a weight attached at the bottom. The rope is of a fixed (undeformed) length L and a fixed mass m. Further, the rope is under the influence of gravity giving rise to internal forces that cause the rope to stretch. We would like to design this rope by shaping it in such a way that its elongation is minimized. The only parameter that we can adjust is the cross section area $A(x)$. Lets set up a coordinate system aligning the x-axis and the rope with each other. We denote the deformation of the rope with $y(x)$ and let the lower end of the rope be fixed at $x = 0$ (at all times). What this means is that, in a deformed state the distance between the lower end of the rope and the "roof" will be $L + y(L)$.

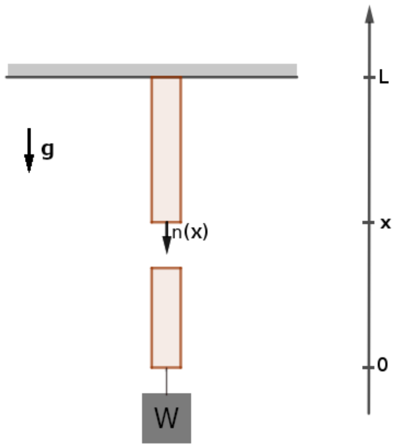

Figure 4: Free body diagram of the undeformed rope, with a cut at x. The bottom of the rope is fixed at $x = 0$

The internal force $n(x)$ that arises due to the attached weight W and the weight of the rope section that lies beneath x is given by

$$n(x) = W + g\int_0^x \rho(\bar{x})A(\bar{x})\mathrm{d}\bar{x} \tag{15}$$

where g is the gravitational constant, $\rho(x)$ is the density and $A(x)$ is the cross section area of the rope.

For elastic materials we have

$$n(x) = A(x)\hat{N}(y'(x)) \tag{16}$$

where $\hat{N} : (-1, \infty) \to \mathbb{R}$ is the stress as a function of the strain, $y'(x)$. Note that when the strain $y' = -1$ the rope elements are completely compressed. Since we're not considering compression at all we will limit the domain of $\hat{N}(\cdot)$ to the positive real axis. Negrón-Marerro makes five assumptions about $\hat{N}$ in his article [2], we'll make almost the the same ones but with a small modification to A3 (since we're not considering compression). The assumptions are:

A1 $\hat{N}(\cdot)$ is a strictly increasing differentiable function.

A2 $\hat{N}(\nu) \to \infty$ as $\nu \to \infty$.

A3 $\hat{N}(0) = 0$

It follows from A1 - A3 that $\hat{N} : \mathbb{R}^+ \to \mathbb{R}^+$ has a differentiable inverse $\hat{\nu} : \mathbb{R}^+ \to \mathbb{R}^+$. Further, we assume that this inverse satisfies the following conditions:

A4 $N \mapsto N^2 \hat{\nu}'(N)$ is strictly increasing on $\mathbb{R}^+$.

A5 $N^2 \hat{\nu}'(N) \to \infty$ as $N \to \infty$.

The stress is given by

$$\hat{N}(y'(x)) = A(x)^{-1} \big[W + g \int_0^x \rho(\bar{x}) A(\bar{x}) \mathrm{d}\bar{x} \big] \tag{17}$$

We apply the inverse function of $\hat{N}$ and get

$$y'(x) = \hat{\nu}\Big(A(x)^{-1} \big[W + g \int_0^x \rho(\bar{x}) A(\bar{x}) \mathrm{d}\bar{x} \big] \Big)$$

where $\hat{\nu}$ is the strain as a function of the stress. To get the total elongation we integrate and get

$$y(L) - y(0) = \int_0^L \hat{\nu}\Big(A(x)^{-1} \big[W + g \int_0^x \rho(\bar{x}) A(\bar{x}) \mathrm{d}x \big] \Big) \mathrm{d}x \tag{18}$$

where $y(0) = 0$.

We now want to state this as a problem of variational calculus. Let

$$\begin{aligned} B(x) &:= \frac{W}{g} + \int_0^x \rho(\bar{x}) A(\bar{x}) \mathrm{d}x \\ B'(x) &= \rho(x) A(x) \end{aligned} \tag{19}$$

Equation (18) now reads

$$y(L) = \int_0^L \hat{\nu}\Big(\frac{g \rho(x) B(x)}{B'(x)} \Big) \mathrm{d}x = \int_0^L \hat{\nu}\Big(\frac{g \rho(x)}{B'(x)/B(x)} \Big) \mathrm{d}x$$

We notice that we have a derivative in the denominator, let $w(x) = \ln(B(x)) \implies w'(x) = B'(x)/B(x)$ and construct the functional

$$J[w] = \int_0^L \hat{\nu}\Big(\frac{g\rho(x)}{w'(x)}\Big)\mathrm{d}x. \tag{20}$$

To get the boundary values of w we note that

$$w(x) = \ln\Big(\frac{W}{g} + \int_0^x \rho(\bar{x})A(\bar{x})\mathrm{d}x\Big)$$

from which it follows that

$$\begin{aligned} w(0) &= \ln\frac{W}{g} \\ w(L) &= \ln(\frac{W}{g} + M) \end{aligned}$$

where M is the total mass of the rope. We also note that

$$w'(x) = \frac{\rho(x)A(x)}{W/g + \int_0^x \rho(\bar{x})A(\bar{x})\mathrm{d}x}$$

and demand that $w'(x) > 0,\ \forall x \in [0,\ L]$, which is natural since we think of both density and as positive quantities. Furthermore, we expect both the area and density to be at least piecewise continuous. Let $\mathcal{D}^1[0, L]$ denote the set of continuous and piecewise continuously differentiable functions defined on the interval $[0, L]$. Then the set of admissible functions $\mathcal{X}$ is then

$$\mathcal{X} = \{w \in \mathcal{D}^1[0, L];\ w(0) = \ln(W/g),\ w(L) = \ln(W/g + M),\ w'(x) > 0\ \forall x\}$$

We can now state our variational problem P as

$$P: \quad \min_{w\in\mathcal{X}} J[w] \tag{21}$$

3.1 Solution

Based on $\hat{\nu}$, we define the new function

$$h(\xi) := \hat{\nu}(\frac{1}{\xi}). \tag{22}$$

If our Lagrange function is $L(x, w, w')$, then

$$L(x, w, w') = h(\frac{w'}{g\rho})$$

which is independent of w. The function h also has the following nice property:

Lemma 3. *The assumption*

$$\textbf{\textit{A4.}}\ N \mapsto N^2\hat{\nu}'(N) \textit{is strictly increasing on } N \in [0, \infty)$$

is equivalent with the statement

$$h(\xi) = \hat{\nu}(\frac{1}{\xi}) \textit{ is strictly convex.}$$

Proof. Let us denote the function in A4 by

$$g(N) = N^2\hat{\nu}'(N)$$

We have

$$h'(\xi) = -\frac{1}{\xi^2}\hat{\nu}'(1/\xi) = -g\big(1/\xi\big)$$

We let

$$N = \frac{1}{\xi}$$

and note that

$$\begin{cases} N \text{ is decreasing with increasing } \xi \\ g \text{ is strictly increasing} \end{cases} \implies g(1/\xi) \text{ is strictly decreasing with increasing } \xi$$

$$\iff -g(1/\xi) = h'(\xi) \text{ is strictly increasing with } \xi$$

which means that $h(\xi)$ is strictly convex.

$\square$

Since h is strictly convex we conclude from Theorem 1 that

$$J[w] = \int_0^L h(\frac{w'}{\rho g})\mathrm{d}x$$

is strictly convex. Theorem 2 then states that any admissible solution to the Euler–Lagrange equation is the unique minimizer of the problem.

The Euler–Lagrange equation is given by

$$L_w(x,w,w') - \frac{d}{dx}L_{w'}(x,w,w') = 0 \iff \frac{d}{dx}\Big(\frac{1}{g\rho(x)}h'(\frac{w'(x)}{g\rho(x)})\Big) = 0.$$

We integrate and rearrange to get

$$h'(\frac{w'(x)}{g\rho(x)}) = c_1\rho(x)$$

for some constant c_1. We want to invert this expression and the inverse of $h'(\xi)$ is given by $\frac{d}{ds}h_*(s) = h'_*(s)$, where $h_*(s)$ is the Legendre transform of $h(\xi)$.

$$\frac{w'(x)}{g\rho(x)} = h'_*(c_1\rho(x)) \iff w'(x) = g\rho(x)h'_*(c_1\rho(x)) \implies$$

$$w(x) = \int_0^x g\rho(\bar{x})h'_*(c_1\rho(\bar{x}))\mathrm{d}x + c_2.$$

We have

$$w(0) = \ln(W/g) \implies c_2 = \ln(W/g)$$

and

$$w(L) = \ln(W/g+M) \implies$$

$$\ln(W/g+M) = \int_0^L g\rho(x)h'_*(c_1\rho(x))\mathrm{d}x + \ln(W/g) \iff$$

$$\ln(1+Mg/W) = \int_0^L g\rho(x)h'_*(c_1\rho(x))\mathrm{d}x. \qquad (23)$$

This equation can be solved for c_1 which can be seen quite easily when thinking about "what $h'_*(s)$ does" from the perspective of the Legendre transform. This function takes the slope of $h(\xi)$ (that is $s = h'(\xi)$), as its input argument and returns the ξ for which $h'(\xi) = s$. Not only do we know that this ξ is unique due to the strict convexity of h, we also know that the range of $h'_*(s)$ equals the domain of $h(\xi)$ which is the positive real line. We saw in Lemma 3 that $h' < 0$ for all ξ.What this means is that there exists a $c_1 < 0$ (and $\rho > 0$) such that $h'_*(c_1\rho) = c_2$ for any $c_2 \in \mathbb{R}^+$. Also note that $\ln(1+Mg/W) > 0$. By using the mean value theorem equation (23) can be solved for c_1.

Optimal Area

We include the expression for the optimal area for convenience sake. We have

$$B(x) = e^{w(x)} = \frac{W}{g}\exp\Big\{\int_0^x g\rho(\bar{x})h'_*(c_1\rho(\bar{x}))\mathrm{d}\bar{x}\Big\}$$

and the optimal area is given by

$$A(x) = B'(x)/\rho(x) = Wh'_*(c_1\rho(x))\exp\Big\{\int_0^x g\rho(\bar{x})h'_*(c_1\rho(\bar{x}))\mathrm{d}\bar{x}\Big\}$$

3.2 Special Cases

Homogeneous pairs

We solve the problem for stress-strain relations of the form

$$\hat{\nu}(N) = A_1 N^p, \quad A_1 > 0, \quad p > 0$$

using the Legendre transform. This relation satisfies all our assumptions A1 – A5.

We saw in the previous section that

$$B(x) = \frac{W}{g} \exp\Big\{ \int_0^x g\rho(\bar{x}) h_*'(c_1\rho(\bar{x}))\mathrm{d}\bar{x} \Big\}$$

where c_1 satisfies equation (23). All we have to do is to find h_*'. Based on $\hat{\nu}$, we define

$$h(\xi) := \hat{\nu}(\frac{1}{\xi}) = A_1 \xi^{-p}, \ \xi > 0$$

We now compute the Legendre transform of $h(\xi)$ with $\rho = g = 1$, then use the scaling rule of the Legendre transform to get our sought function. Let

$$s := h'(\xi) = -pA_1\xi^{-(p+1)}, \ s < 0 \iff$$

$$\xi(s) = \Big(\frac{(-s)}{pA_1}\Big)^{-\frac{1}{p+1}}$$

and

$$\begin{aligned} h_*(s) &= s\xi(s) - h(\xi(s)) = \\ &= -(-s)\Big(\frac{(-s)}{pA_1}\Big)^{-\frac{1}{p+1}} - A_1\Big(\frac{(-s)}{pA_1}\Big)^{(-\frac{1}{p+1})(-p)} = \\ &= \ldots = \\ &= -(-s)^{\frac{p}{p+1}} A_1^{\frac{1}{p+1}} (p^{-\frac{p}{p+1}} + p^{\frac{1}{p+1}}) = \\ &= -(-s)^{\frac{p}{p+1}} A_1^{\frac{1}{p+1}} p^{-\frac{p}{p+1}} (1+p) \end{aligned}$$

And the derivative is

$$h_*'(s) = (-s)^{-\frac{1}{p+1}} A_1^{\frac{1}{p+1}} p^{\frac{1}{p+1}}, \ s < 0$$

We can now express the solution by using this h_*' in the expression for B above. Please note that there is no need for A_1 and p to be constant on $[0, L]$. Recently it's been more and more common to use functionally graded materials, or FMGs, that are materials with variable properties such as elasticity and/or density [1]. For example, in such material it could be of interest to optimize over these properties instead of the area.

Hooke's Law, constant density

We now implement the previous solution for Hookes Law of elasticity. We want to minimize

$$J[w] = \int_0^L \frac{1}{E} \frac{g\rho}{w'(x)} \mathrm{d}x$$

subject to

$$w \in \mathcal{X} = \{w \in C[0, L];\ w(0) = \ln(W/g),\ w(L) = \ln(W/g + M)\}$$

where E is Young's module of elasticity, M is the total mass of the rope, g is the gravitational force per mass, ρ is the constant density and W is the weight at the bottom of the rope. In the solution above we identify $A_1 = 1/E$ and $p = 1$. That means $h'_*(s)$ is given by

$$h'_*(s) = (-s)^{-1/2} E^{-1/2}$$

and

$$B(x) = \frac{W}{g} \exp\Big\{ \int_0^x g\rho h'_*(c_1\rho)\mathrm{d}\bar{x} \Big\}$$

Note that

$$g\rho h'_*(c_1\rho) = c_2 = \text{ constant}$$

so $B(x) = \frac{W}{g} e^{c_2 x}$. To find c_2 we solve equation (23) and get

$$\ln(1 + Mg/W) = \int_0^L g\rho h'_*(c_1\rho)\mathrm{d}x = \int_0^L c_2 \mathrm{d}x \iff$$

$$c_2 = \frac{\ln(1 + Mg/W)}{L}$$

The area is given by $A(x) = B'(x)/\rho$

$$A(x) = \frac{W}{\rho L g} \ln(1 + Mg/W)(1 + Mg/W)^{x/L}$$

which is the same result that Verma and Keller got in their original paper [6].

4 The Rotating Load

We now consider a problem of a rotating rod with a weight attached in its outer end. The rod has a natural length L, mass M and is fastened perpendicular onto a rotating cylinder with radius $r > 0$. The distance between the center of the cylinder and the outer edge of the rod is thus $R_0 = L + r$ when the rod is undeformed, as seen in Figure 5.

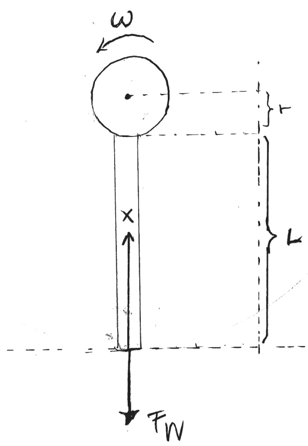

Figure 5: The rod is fastened on a rotating cylinder with radius r, rotating with an angular velocity of ω. The rod is perpendicular to the axis of rotation. We set up the coordinates such that the outer edge of the undeformed rod is located at $x = 0$ while the rod is fastened to the cylinder at $x = L$.

We will denote, as in the previous section, the displacement by $y(x)$; however we also introduce $u(x) = x + y(x)$. This function takes positions x in the undeformed rod and maps them to their deformed locations. We will have slightly different boundary conditions on $y(x)$ compared with the Hanging Rope. This time we let $x = L$ correspond to the point of contact between the cylinder and the rod at all times. This means that $y(L) = 0$ and $y(0)$ will be a negative number. We let $x = 0$ correspond to the outer edge of the rod in its undeformed state, but this time we set $y(L) = 0$. That is, the displacement is zero where the rod is fastened to the cylinder. The distance between a point in the rod and the point of rotation is given by $R - u(x)$. Please note that these boundary conditions make $y(x) < 0$ for $x \in [0, L)$ when the rod is stretched. Specifically, $y(0)$ will be a negative number with the magnitude of the total elongation of the rod. The reason I have chosen this coordinate system with these boundary conditions is that it makes it straightforward to express the distance between an element of the rod and the axis of rotation [1].

[1]This distance would have been $R - x - y(L - x)$ if we would have used the boundary

The centripetal force on a particle in circular motion is given by

$$F = \omega^2 x m$$

where ω is the angular velocity, x the radius of the motion and m the mass of the paricle. To find the normal force of the rod let us make a free body diagram. We begin with slicing the rod up in N discrete elements, see Figure 6.

The bottom element is pulled outwards by the force from the weight, F_W. It's also accelerating with acceleration a_1, so the force inwards must be $F_W + F_1$, where $F_1 = a_1 m_1$. The next element is pulled outward by this force $F_W + F_1$ and is accelerated inwards by a_2. The force inwards must therefore be $F_W + F_1 + F_2$, where $F_2 = a_2 m_2$. Proceeding in this fashion we see that element i is exposed to a normal force

$$n_i = F_W + \sum_{j=1}^{i-1} F_j$$

which is stretching it and a force $F_i = a_i m_i$ accelerating it inward. The $F_i : s$ are given by

$$F_i = \omega^2 (R - u_i) m_i$$

where ω is the angular velocity, $R - u_i$ the distance to the axis of rotation. The mass is given by $m_i = \rho_i A_i \Delta x_i$ where ρ_i, A_i, Δx_i are the density, cross section area and length of element i respectively. Putting it all together we get

$$n_i = F_W + \sum_{j=1}^{i-1} \omega^2 (R - u_i) \rho_i A_i \Delta x_i$$

and moving to the continuous case we get

$$n(x) = F_W + \int_0^x \omega^2 (R - u(\bar{x})) \rho(\bar{x}) A(\bar{x}) \mathrm{d}\bar{x}$$

For elastic materials the stress is given by

$$\hat{N}(y'(x)) = \frac{n(x)}{A(x)} = \frac{1}{A(x)} \Big(F_W + \int_0^x \omega^2 (R - u(\bar{x})) \rho(\bar{x}) A(\bar{x}) \mathrm{d}\bar{x} \Big).$$

We will make the same assumptions A1 - A5 on the stress-strain relation as we did in the first problem, see page 16. As before we denote the strain function by $\hat{\nu}(N)$. The strain is then given by

$$y'(x) = \hat{\nu} \Big[\frac{1}{A(x)} \Big(F_W + \int_0^x \omega^2 (R - u(\bar{x})) \rho(\bar{x}) A(\bar{x}) \mathrm{d}\bar{x} \Big) \Big] \tag{24}$$

The problem here is that $u(x) = x + y(x)$ depends on $A(x)$, which makes the Lagrange function hard to handle. We will try to illustrate this in the following section.

conditions $y(0) = 0$ and I didn't want to work with the function $y(L - x)$ when examining the differential equations arising from this problem.

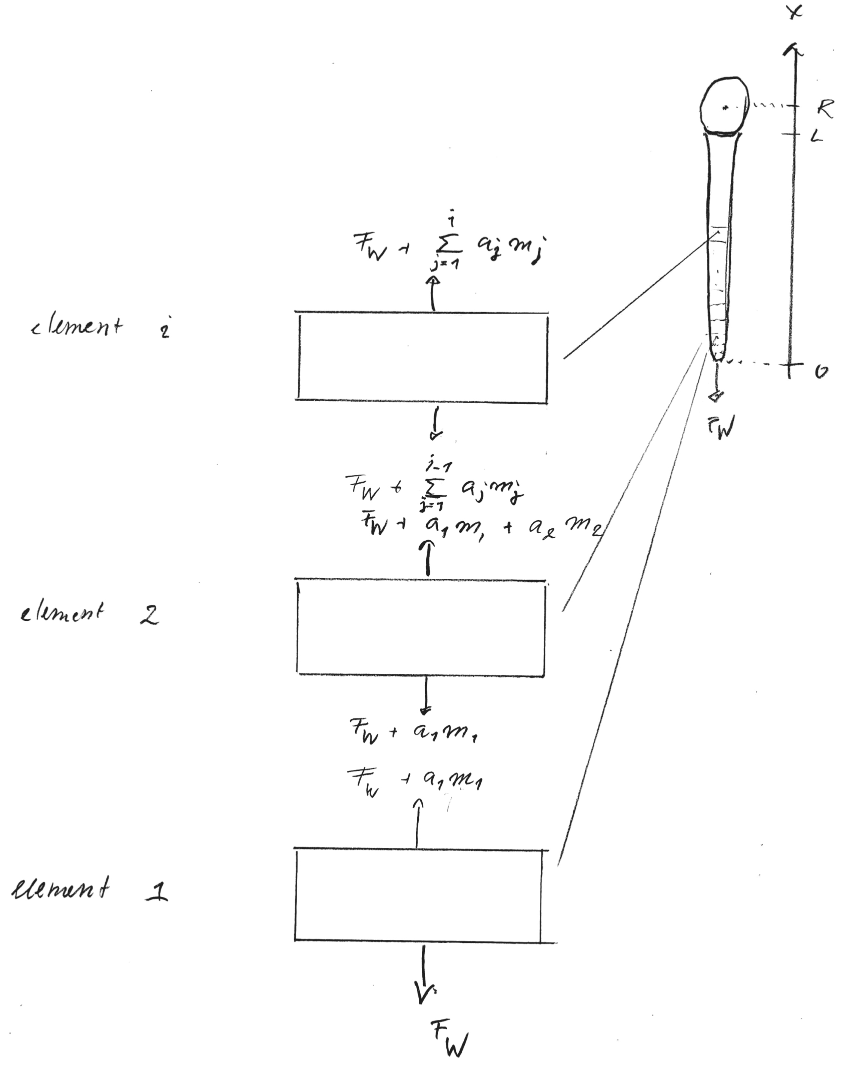

Figure 6: Free body diagram of the elements of the undeformed rod.

Large Deformations

We return to the strain given in equation (24), but this time writing $u(x) = x + y(x)$:

$$y'(x) = \hat{\nu}\Big[\frac{1}{A(x)}\Big(F_W + \int_0^x \omega^2(R - \bar{x} - y(\bar{x}))\rho(\bar{x})A(\bar{x})\mathrm{d}\bar{x}\Big)\Big].$$

As we can see, we have the function y present in both the left and right hand side, differentiated on the left side and in an integral on the right. Let's define the B-function again

$$\begin{aligned} B(x) &:= F_W + \int_0^x \omega^2(R - u(\bar{x}))\rho(\bar{x})A(\bar{x})\mathrm{d}\bar{x} \\ B'(x) &= \omega^2(R - u(x))\rho(x)A(x) \end{aligned}$$

with

$$\begin{aligned} B(0) &= F_W \\ \int_0^L \frac{B'(x)}{\omega^2(R - u(x))}\mathrm{d}x &= M \end{aligned}$$

Remember that $u(x) = x + y(x)$, where $y(x) = \int_0^x L(\bar{x}, B(\bar{x}), B'(\bar{x}))\mathrm{d}\bar{x}$. We can now see the recursive feature of the Lagrangian

$$L(x, B(x), B'(x)) = \hat{\nu}\Big(\frac{\omega^2(R - x - \int_0^x L(\bar{x}, B(\bar{x}), B'(\bar{x}))\mathrm{d}\bar{\bar{x}})\rho(x)B(x)}{B'(x)}\Big)$$

If we add a variation $\epsilon v(x)$ to $B(x)$ and $B'(x)$, then differentiate with respect to ϵ, we would start an infinite regress.

Small Deformations

We consider the case of small deformations where $y(x) \ll x$. We simply approximate $u(x) = x + y(x)$ with x. Again we introduce the help function

$$B(x) := F_W + \int_0^x \omega^2(R - \bar{x})\rho(\bar{x})A(\bar{x})\mathrm{d}\bar{x}$$
$$B'(x) = \omega^2(R - x)\rho(x)A(x)$$

and can now write

$$y'(x) = \hat{\nu}\Big[\frac{\omega^2(R - x)\rho(x)B(x)}{B'(x)}\Big] = \hat{\nu}\Big[\frac{\omega^2(R - x)\rho(x)}{B'(x)/B(x)}\Big]$$

Note that

$$\begin{aligned} B(0) &= F_W \\ \int_0^L B'(x)/(\omega^2(R - x))\mathrm{d}x &= M \end{aligned} \tag{25}$$

where M is the total mass of the rod. Again we introduce

$$w(x) := \ln B(x)$$
$$w'(x) = B'(x)/B(x)$$

and define our Lagrangian

$$L(x, w(x), w'(x)) := \hat{\nu}\Big[\frac{\omega^2(R - x)\rho(x)}{w'(x)}\Big]$$

We can now state the optimization problem on standard form

$$\min_{w \in \mathcal{X}} \int_0^L L(x, w(x), w'(x))\mathrm{d}x \tag{26}$$

where $\mathcal{X}$ is the set of all admissible functions, meaning they satisfy equation (25), and $w'(x) > 0$ on $x \in [0, L]$. We still assume A1 - A5, stated in the first problem. It follows that L is convex in w' and that any admissible function satisfying the Euler-Lagrange equation is the unique solution for this problem. We would like to solve this for Hooks law, $\hat{\nu}(N(x)) = \frac{N(x)}{E(x)}$, where $E(x)$ is Youngs modulus of elasticity.

Solution

Define

$$h(\xi) := \frac{1}{\xi}$$

Our Lagrangian can now be written as

$$L(x, w(x), w'(x)) = h(\frac{w'(x)}{k(x)})$$

where $k(x) = \omega^2(R-x)\rho(x)/E(x)$. We've seen that the Legendre transform of h is given by

$$h_*(s) = -2(-s)^{1/2},\ s < 0$$

and

$$h_*'(s) = -(-s)^{-1/2}$$

The Euler-Lagrange equation is given by

$$L_w - \frac{d}{dx}L_{w'} = 0 \iff \frac{d}{dx}\Big(\frac{1}{k(x)}h'(\frac{w'(x)}{k(x)})\Big) = 0$$

Integrating and rearranging we get

$$h'(\frac{w'(x)}{k(x)}) = -ck(x)$$

for some constant $c > 0$. The inverse of h' is given by $h_*'(s) = -(-s)^{-1/2}$. We apply this and get

$$\frac{w'(x)}{k(x)} = -\Big(ck(x)\Big)^{-1/2} \iff w'(x) = c_1 k(x)^{1/2}$$

for some constant $c_1 = -c^{-1/2} < 0$.

$$w(x) = c_1 \int_0^x k(x)^{1/2} + c_2$$

We have $B(x) = e^{w(x)}$, and

$$B(x) = \exp\Big\{c_1 \int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x} + c_2\Big\}$$

We use the conditions from equation (25), the first one stating $B(0) = e^{c_2} = F_W \implies c_2 = \ln F_W$. The second condition implies

$$\int_0^L B'(x)/(\omega^2(R-x))\mathrm{d}x = M \iff$$

$$\iff \int_0^L \frac{c_1 F_W}{\omega^2}\sqrt{\frac{\rho(x)}{(R-x)E(x)}}\exp\Big\{c_1 \int_0^x \sqrt{(R-\bar{x})\rho(\bar{x})/E(\bar{x})}\mathrm{d}\bar{x}\Big\}\mathrm{d}x = M$$

which needs to be solved numerically. The optimal area is given by

$$A(x) = \frac{B'(x)}{\omega^2(R-x)\rho(x)} = \frac{B'(x)}{k(x)E(x)} =$$

$$= F_W k(x)^{-1/2}E(x)^{-1}\exp\Big\{c_1 \int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x}\Big\}$$

Let's calculate the strain, given by equation (24), but with $u(x) \approx x$. Note that $\omega^2(R-x)\rho(x) = E(x)k(x)$.

$$y'(x) = \frac{1}{E(x)A(x)}\Big[F_W + \int_0^x E(\bar{x})k(\bar{x})A(\bar{x})\mathrm{d}x\Big] =$$

$$= F_W^{-1}k(x)^{1/2}\exp\Big\{-c_1\int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x}\Big\}\Big[F_W + \int_0^x F_W k(\bar{x})^{1/2}\exp\Big\{c_1\int_0^{\bar{x}} k(\bar{\bar{x}})^{1/2}\mathrm{d}\bar{\bar{x}}\Big\}\mathrm{d}\bar{x}\Big] =$$

(intgrate last term, cancel F_W)

$$= k^{1/2}\exp\Big\{-c_1\int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x}\Big\}\Big[1 + \frac{1}{c_1}\exp\Big\{c_1\int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x}\Big\}\Big] =$$

$$= k(x)^{1/2}\Big(\exp\Big\{-c_1\int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x}\Big\} + \frac{1}{c_1}\Big) = y'(x)$$

which integrate nicely to

$$y(x) = -\frac{1}{c_1}\exp\Big\{-c_1\int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x}\Big\} + \frac{1}{c_1}\int_0^x k(\bar{x})^{1/2}\mathrm{d}\bar{x} + c_2$$

where c_2 satisfies the boundary condition $y(L) = 0$.

I've provided a plot of $A(x)$ for three different values of ω. The other parameters were set to $\rho = 1$, $m = 1$, $\omega \in \{1,\ 10,\ 100\}$, $F_W = 10\omega$, $L = 10$, $R = L + r = 10{,}5$, $E = 1$.

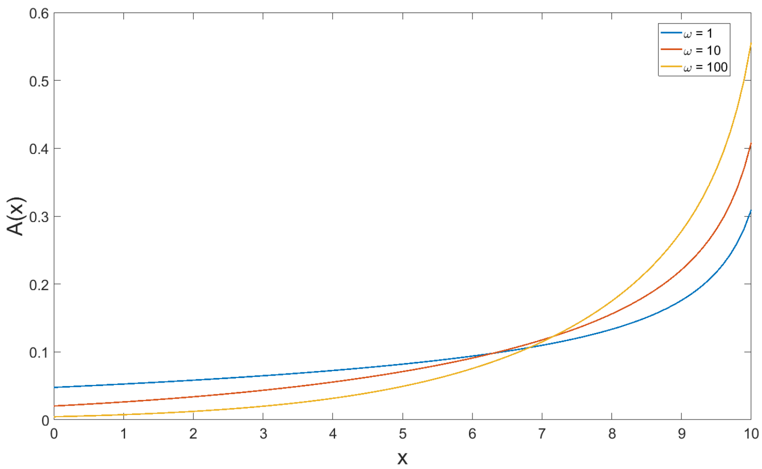

Figure 7: Solutions for three different values of ω.

Future work

What if we instead solve the problem for

$$y'(x) = \hat{\nu}\Big[\frac{1}{A(x)}\big(F_W + \int_0^x a(\bar{x})\rho(\bar{x})A(\bar{x})\mathrm{d}\bar{x}\big)\Big] \tag{27}$$

where $a(x) > 0$ is an arbitrary acceleration? This problem is equivalent with the hanging rope so we have in fact already solved it. However, we have new boundary conditions this time around in need of some special attention. Let us introduce the function $k(x) = a(x)\rho(x) > 0$ and write

$$\begin{aligned} B(x) &= F_W + \int_0^x k(\bar{x})A(\bar{x})\mathrm{d}\bar{x} \\ B'(x) &= k(x)A(x) \end{aligned}$$

where the boundary conditions are given by

$$B(0) = F_W \tag{28}$$

$$\int_0^L B'(x)/a(x)\mathrm{d}x = M \tag{29}$$

where F_W is some positive number that might depend on the solution and M is the total mass of the rope. We now have

$$y'(x) = \hat{\nu}\Big[\frac{k(x)B(x)}{B'(x)}\Big]$$

Again, we introduce the help function

$$\begin{aligned} w(x) &= \ln B(x) \\ w'(x) &= B'(x)/B(x) \end{aligned}$$

and construct the Lagrangian

$$L(x, w, w') = \hat{\nu}\Big[\frac{k(x)}{w'(x)}\Big]$$

The Lagrangian is convex in w' and so any solution to the Euler-Lagrange equation is a unique solution to the minimization problem

$$\min_{w\in\mathcal{X}} \int_0^L L(x, w, w')\mathrm{d}x$$

where $\mathcal{X}$ is the set of admissible functions.

If we follow the solution on page 18 we see that the optimal area is given by

$$A(x) = F_W h'_*(c_1 k(x)) \exp\Big\{ \int_0^x k(\bar{x}) h'_*(c_1 k(\bar{x}) \mathrm{d}\bar{x} \Big\}$$

where h_* is the Legendre transform of $h(\xi) = \hat{\nu}(1/\xi)$ and c_1 is some integration constant. We now have

$$y'(x) = \hat{\nu}\Big[\frac{1}{A(x)} (F_W + \int_0^x k(\bar{x}) A(\bar{x}) \mathrm{d}\bar{x}) \Big] \tag{30}$$

where

$$k(x)A(x) = F_W k(x) h'_*(c_1 k(x)) \exp\Big\{ \int_0^x k(\bar{x}) h'_*(c_1 k(\bar{x})) \mathrm{d}\bar{x} \Big\}.$$

This function can be easily integrated to

$$\int_0^x k(\bar{x}) A(\bar{x}) \mathrm{d}\bar{x} = F_W \exp\Big\{ \int_0^x k(\bar{x}) h'_*(c_1 k(\bar{x})) \mathrm{d}\bar{x} \Big\} + c_2.$$

We can now write equation (30) as

$$y'(x) = \hat{\nu}\Big[\frac{F_W (1 + \exp\Big\{ \int_0^x k(\bar{x}) h'_*(c_1 k(\bar{x})) \mathrm{d}\bar{x} \Big\}) + c_2}{A(x)} \Big]$$

Simplifying this we get (with a new value for c_2)

$$y'(x) = \hat{\nu}\Big[\frac{1}{h'_*(c_1 k(x))} \big(1 + c_2 \exp\Big\{ - \int_0^x k(\bar{x}) h'_*(c_1 k(\bar{x}) \mathrm{d}\bar{x} \Big\}\big) \Big]$$

If we now let $k(x) = \omega^2 (R - x - y(x)) \rho(x)$ we get a third order ordinary differential equation for $y(x)$. This will give rise to three more integration constants giving a total of 5 unknowns. We have given two boundary conditions in equations (28) and (29), furthermore we have an additional boundary condition $y(L) = 0$. This leaves us wanting for two more equations, unfortunately this problem has not yet been solved.

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Mathematics
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund, Sweden

http://www.maths.lth.se/