Group Representations and a Study of M-groups

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Populärvetenskaplig sammanfattning

Herr O och Fru X spelar 15-spelet. Reglerna till 15-spelet är följande: Herr O börjar med att välja ett heltal mellan 1 och 9, Fru X fortsätter med att välja ett annat tal mellan 1 och 9, och så vidare. Vinnaren är den som först har samlat ihop tre tal vars summa är 15. Fru X känner att spelet är onödigt komplicerat och upptäcker att 15-spelet kan representeras med tre-i-rad. Betrakta följande parti (läsaren uppmanas att själv fylla i det motsvarande partiet i tre-i-rad):

Herr O	Fru X
5	6
3	7
2	8
9	1

$$\longleftrightarrow \qquad \begin{array}{c|c|c} 2 & 7 & 6 \\ \hline 9 & 5 & 1 \\ \hline 4 & 3 & 8 \end{array}$$

Denna uppsats handlar inte om hur man kan representera spel, utan hur man kan representera speciella matematiska strukturer som kallas *grupper*. Man kan inte vinna eller förlora i en grupp, men likt spel har en grupp vissa regler som måste följas. Till exempel om g_1 och g_2 är två element i en grupp så ska man kunna multiplicera ihop dem till ett element $g_1 g_2$ som också ska vara ett element i gruppen. Vi kommer till exempel att studera kvaterniongruppen Q_8 som består av elementen ± 1, $\pm i$, $\pm j$ och $\pm k$ med regeln

$$i^2 = j^2 = k^2 = ijk = -1.$$

Denna formel är inhuggen på Broom Bridge, Dublin, platsen där matematikern Sir William Rowan Hamilton promenerade vid tillfället då idén slog honom. Vi kommer använda Q_8 för att definiera mer komplicerade grupper med upp till 96 element. Vi kan då ta inspiration från Fru X och representera dessa komplicerade grupper i lättförståliga termer. Mer specifikt beskriver representationsteori bland annat hur grupper kan representeras som linjära avbildningar i vektorrum.

Herr O tycker sig ha hittat ytterliggare två representationer av 15-spelet i form av tre-i-rad (i den andra representationen är tanken att man fyller i två stycken tre-i-rad samtidigt):

$$\begin{array}{c|c|c} 6 & 1 & 8 \\ \hline 7 & 5 & 3 \\ \hline 2 & 9 & 4 \end{array} \quad \text{och} \quad \begin{array}{c|c|c|c|c|c} 2 & 7 & 6 & & & \\ \hline 9 & 5 & 1 & & & \\ \hline 4 & 9 & 8 & & & \\ \hline & & & 2 & 7 & 6 \\ \hline & & & 9 & 5 & 1 \\ \hline & & & 4 & 9 & 8 \end{array}.$$

Fru X är inte imponerad. Herr O har i viss mån tagit den redan givna representationen och först roterat den och sedan duplicerat den. Liknande situationer uppstår i representationsteori av grupper, där man rigoröst kan definiera vad det betyder att en representation är unik och att man inte kan bryta ned den i mindre representationer (som man kan göra i Herr O's andra förslag). En sådan representation kallas *irreducibel*. Dessutom kan man visa att givet en grupp, så finns det ett ändligt antal unika irreducibla representationer.

Givet en representation av en grupp så existerar det en *karaktär*. En karaktär är ett betydligt simplare objekt än en representation men som, förvånadsvärt nog, ändå ger mycket information om dess underliggande representation. Karaktärerna för en grupp kan sammanfattas i en *karaktärstabell*.

Ett syfte i denna uppsats är att använda metoder från representationsteori för att hitta de irreducibla representationerna av den förut nämnda gruppen med 96 element och bestämma dess karaktärstabell.

Abstract

In this thesis we introduce some basic concepts in representation theory such as Schur's Lemma, induced representations and M-groups. These concepts are then used in particular on a group G_{96} of order 96, in order to determine its irreducible representations and its character table. Furthermore, we show that G_{96} is an M-group and contains a subgroup which is not an M-group itself.

Introduction

This paper will introduce representation and character theory for a reader acquainted with basic group theory. Chapters 1 - 4 introduce the basic concepts of general representation and character theory for finite groups which are needed in later chapters. The results contained there can all be found in any book introducing representation theory. In particular, the author has based these chapters on [3] and [4]. The proof of The Second Orthogonality Relation (Theorem 3.2) can be found as an exercise in [4].

Chapters 5 - 6 are based on [2], although Chapter 5 mostly serves as a recap on the basic theory of the solvability of groups which can be found in most introductory texts on group theory, for example in [1]. Chapter 7 is based on an exercise in [2].

As far as structure goes, Chapter 1 - 6 contains theory that prepares for the more practical Chapter 7, where the main analysis takes place. In Chapter 1 - 4 elementary concepts of representations of finite groups (over $\mathbb{C}$), character theory and induced representations are introduced. In Chapter 5, as stated, the solvability of groups is discussed. In Chapter 6 the M-groups are defined and related to the concepts of Chapter 5. Chapter 7 is devoted to practical use of the theory from Chapter 1 - 4 and gives a relevant, and hopefully interesting, counterexample that is introduced in Chapter 6.

Contents

1 Group Representations

In this paper we let G denote a group and V a vector space. We restrict our attention to G being finite and V being finite-dimensional over the algebraically complete field $\mathbb{C}$, the complex numbers. We define the automorphism group $\mathrm{Aut}(V)$ to be the group under composition of all bijective linear transformations $f : V \to V$. Generally, let $\mathrm{Hom}(V, W)$ be the set of linear transformations between the two vector spaces V and W. The set of endomorphisms of V is $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

It is often convenient to fix a basis for V. Then we can associate every such automorphism by its invertable $n \times n$ matrix, where $n = \dim V$. Furthermore, $\mathrm{Aut}(V)$ is isomorphic as a vector space to $\mathrm{GL}(n, \mathbb{C})$, the general linear group of degree n over $\mathbb{C}$.

We now state the - for our purposes - most central definition.

Definition 1.1. A **representation** of a group G is an ordered pair (θ, V) where V is a vector space and θ is a homomorphism of groups

$$\theta : G \to \mathrm{Aut}(V).$$

Moreover, V is called a ***G*-module** and the **dimension** of θ is defined to be $\dim V$.

Remark. By abuse of notation, we will often say θ is a representation of G without explicitly stating its underlying G-module. Conversely, we will say that V is a G-module, with the assumption that a representation (θ, V) of G is defined such that V is a G-module with respect to this representation.

Example 1.1. *(Principal Representation)*

For any group G, the principal representation denotes the trivial homomorphism $\theta : G \to \mathrm{Aut}(\mathbb{C})$, given by $\theta_g = 1$ for all $g \in G$. Note that $\mathrm{Aut}(\mathbb{C}) \cong \mathbb{C}^{\times} = \mathbb{C} - 0$. The importance of this representation will be apparent later.

Example 1.2. *(Regular Representation)*

For any group G, let V be a vector space with $\dim V = |G|$. Choose a basis for V and index the basis vectors by the elements of G, that is $\{e_x\}_{x \in G}$. If we let G permute the basis vectors by the action of left multiplication we get the fruitful regular representation. Explicitly, define $\theta : G \to \mathrm{Aut}(V)$ by

$$\theta_g \sum_{x \in G} c_x e_x = \sum_{x \in G} c_x e_{gx},$$

where $\{c_x\}$ are complex coefficients.

An interesting feature of the regular G-module is that there exists a one-dimensional subspace that is invariant under its representation. In fact, consider the the line spanned by the vector $v = \sum_{x \in G} e_x$. Since for any $g \in G$ the map $\phi : x \mapsto gx$ is a bijection of G, we can change the summation index in

$$\theta_g v = \sum_{x \in G} e_{gx} = \sum_{y \in G} e_y = v.$$

Similarly $\theta_g \langle v \rangle = \langle v \rangle$ for any $g \in G$. It should be evident that $\langle v \rangle$ can, in itself, be regarded as a G-module and that it is in some way isomorphic to the principal representation. These notions will be discussed and made rigorous shortly.

Definition 1.2. Let (θ, V) be a representation of G and V_1 and subspace of V. If $\theta_g V_1 = V_1$ for all $g \in G$ then V_1 is called a **sub-*G*-module** of V.

The reader might find it apparent that V_1 is indeed a G-module, but we show this result for the sake of clarity.

Lemma 1.1. *Let (θ, V) be a representation of G and V_1 a sub-G-module of V. Then there exists a representation (ϑ, V_1) of G with V_1 as a G-module.*

Proof. For every g in G define $\vartheta_g = \theta_g|_{V_1}$. Seeing that V_1 is a sub-G-module, ϑ_g is a bijective linear map from V_1 to V_1. Since θ is a homomorphism of groups then so is $\vartheta : G \to \mathrm{Aut}(V_1)$. Thus (ϑ, V_1) is a representation of G. □

Remark. For showing that V_1 is a sub-G-module it is sufficient to check that $\theta_g v \in V_1$ for any $v \in V_1$ and $g \in G$, since an injective endomorphism of a finite vector space must also be bijective. Note that 0 and V are always sub-G-modules of any G-module V. The following definition now seems natural.

Definition 1.3. Let V be a G-module. The sub-G-modules 0 and V are called the **trivial** sub-G-modules of V and any other sub-G-modules are called **proper**. If V has no proper sub-G-modules then V is said to be **irreducible**.

Theorem 1.1. (Maschke's Theorem) *Let (θ, V) be a representation of G and V_1 a sub-G-module of V. Then there exists a complementary subspace V_2 of V_1 which is also a sub-G-module of V with respect to θ.*

Remark. This theorem allows us to decompose a G-module V into a direct sum of irreducible sub-G-modules $V_1, \dots, V_k$. In fact, if we suppose V is not irreducible, then Maschke's theorem gives us a decomposition $V = V_1 \oplus V_2$ of sub-G-modules V_1 and V_2. Since $\dim(V_1)$ and $\dim(V_2)$ are both strictly less than $\dim(V)$ and a one-dimensional G-module is always irreducible, the statement follows from induction.

Proof of Theorem 1.1. Let W be the orthogonal complement of V_1. Let $p : V \to V_1$ be the orthogonal projection onto V_1. We construct a new function $\hat{p} : V \to V$ by conjugating p with the elements of G and then taking the average:

$$\hat{p} = \frac{1}{|G|} \sum_{x \in G} \theta_x^{-1} p \theta_x.$$

Since V_1 is invariant under θ, we have for any $x \in G$ and $v \in V_1$

$$\theta_x^{-1}(p\theta_x v) = \theta_x^{-1}(\theta_x v) = v.$$

This implies that $\hat{p}(v) = v$ and $\hat{p}$ is a projection onto V_1 along some complementary subspace $\ker \hat{p} =: V_2$. We wish to show that V_2 is a sub-G-module of V. The key to this fact lies with that $\hat{p}$ commutes with θ_g for any $g \in G$:

$$\begin{aligned}
\theta_g^{-1} \hat{p} \theta_g &= \frac{1}{|G|} \sum_{x \in G} \theta_g^{-1} \theta_x^{-1} p \theta_x \theta_g \\
&= \frac{1}{|G|} \sum_{x \in G} \theta_{xg}^{-1} p \theta_{xg} \\
&= \frac{1}{|G|} \sum_{y \in G} \theta_y^{-1} p \theta_y = \hat{p} \\
\iff \hat{p}\theta_g &= \theta_g \hat{p}.
\end{aligned}$$

Take any $u \in V_2$, then $\theta_g u \in V_2 \iff \hat{p}\theta_g u = 0 \iff \theta_g \hat{p} u = 0$ which holds since u is in the kernel of $\hat{p}$. Therefore, V_2 is a sub-G-module and $V = V_1 \oplus V_2$. □

Now we introduce a concept describing when representations of a given group should be considered equivalent.

Definition 1.4. A ***G*-homomorphism** between two representations (θ, V) and (θ', V') of a group G is a linear map $f : V \to V'$ such that $f \circ \theta_g = \theta'_g \circ f$ holds for all g in G. If f is bijective, then it is called a ***G*-isomorphism** and we say that (θ, V) is ***G*-isomorphic** to (θ', V'). We denote this by $(\theta, V) \cong_G (\theta', V')$.

Remark. The linear map $f : V \to V'$ being a G-homomorphism is equivalent to the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\theta_g"'] & V' \arrow[d, "\theta'_g"] \\ V \arrow[r, "f"] & V' \end{tikzcd}$$

being commutative for all g in G.

Remark. By further extending the abuse of notation introduced earlier, we might simply write $\theta \cong_G \theta'$ or $V \cong_G V'$.

Remark. With a fixed basis for G-modules V and V', consider the matrices $A(g)$ and $B(g)$ corresponding to the representations θ_g and θ'_g respectively. If V is G-isomorphic to V' then it implies that there exists an invertible matrix T such that $B(g) = TA(g)T^{-1}$ holds for all g in G.

We are now prepared for a both theoretically and practically powerful result. It reveals a dichotomy of G-homomorphisms between irreducible G-modules.

Theorem 1.2. (Schur's lemma) *Let (θ, V) and (θ', V') be two irreducible representations of a group G and let $f : V \to V'$ be a G-homomorphism. Then f is either the trivial map $f = 0$, or a G-isomorphism and $(\theta, V) \cong_G (\theta', V')$. Furthermore, if $(\theta, V) = (\theta', V)$ then $f = \lambda 1_V$ for some λ in $\mathbb{C}$.*

Proof. Suppose that $f \neq 0$. We first show that the irreducibility of V and V' implies that $\ker f = \{0\}$ and $\operatorname{im} f = V'$.

$$\begin{aligned} f \neq 0 &\implies \{0\} \subset \ker f \subsetneq V. \\ v \in \ker f &\implies f\theta_g(v) = \theta'_g f(v) \\ &\implies \theta_g(v) \in \ker f. \end{aligned}$$

Similarly:

$$\begin{aligned} f \neq 0 &\implies \{0\} \subsetneq \operatorname{im} f \subset V'. \\ v' \in \operatorname{im} f &\implies v' = f(v), \text{ for some } v \in V \\ &\implies \theta'_g(v') = \theta'_g f(v) = f\theta_g(v) \\ &\implies \theta'_g(v') \in \operatorname{im} f. \end{aligned}$$

Hence both $\ker f$ and $\operatorname{im} f$ are sub-G-modules of irreducible V and V' respectively, which implies that $\ker f = \{0\}$ and $\operatorname{im} f = V'$. But this just says that f is bijective $\implies f$ is a G-isomorphism and $V \cong_G V'$.

Now we turn our attention to the special case $(\theta, V) = (\theta', V)$. In this case f is a endomorphism of V. Here we use the property that V is a vector space over the algebraically closed field $\mathbb{C}$. Hence f has an eigenvalue $\lambda \in \mathbb{C}$. Noting that $\lambda 1_V$ is also a G-homomorphism implies that the difference $(f - \lambda 1_V)$ is also a G-homomorphism.

The key property of this G-homomorphism is that it has a nonzero kernel, since an eigenvector corresponding to λ is an element of it. The irreducibility of V and a similar argument as above forces $\ker(f - \lambda I_V) = V$, which implies that $f = \lambda 1_V$. □

Two things are now worth mentioning.

First, suppose we have two representations (θ^1, V_1) and (θ^2, V_2) of a group G. Then we can naturally define another representation of G with G-module equal to the (external) direct sum $V_1 \oplus V_2$. Simply define $\theta : G \to \operatorname{Aut}(V_1 \oplus V_2)$ by

$$\theta_g(v_1, v_2) = (\theta_g^1 v_1, \theta_g^2 v_2),$$

and verify that θ is indeed a homomorphism of groups. Let us denote $\theta = \theta^1 \oplus \theta^2$.

Secondly we can, with the help of Maschke's theorem and the remark following it, go in the converse direction. Given any G-module V we can decompose it into irreducible sub-G-modules, $V = \bigoplus_{i=1}^m V_i$. Unfortunately, this decomposition is not unique. For a trivial example, consider the group $G = 1$ and G-module $\mathbb{C}^2$. Any two non coinciding lines will work as irreducible sub-G-modules and decompose $\mathbb{C}^2$.

Suppose we instead gather up all V_i that are G-isomorphic to each other. That is, let $V = \bigoplus_{i=1}^n W_i$, where W_1 is the direct sum of precisely those V_i which are G-isomorphic with V_1. Let V_j be the first sub-G-module not included in this direct sum. Now let W_2 be the direct sum of precisely those V_i which are G-isomorphic with V_j. Continue in this way until all V_i are included.

This is called the **canonical decomposition**. We will later show that it does not depend on the initial irreducible decomposition and is thus unique.

Some questions of interest: how can we systematically determine if a given representation is irreducible or not? And how many different (up to G-isomorphism) irreducible representations does there exist for a given group G? The answers to both of these questions will follow in the next chapter.

2 Character Theory

Definition 2.1. Let (θ, V) be a representation of a group G. The **character** of this representation is the function $\chi : G \to \mathbb{C}$ given by

$$\chi(g) = \operatorname{tr} \theta_g.$$

We say that a character is **irreducible** if its underlying representation is irreducible.

Remark. To clarify, fix a basis for V. Now $\chi(g)$ is well-defined and equals the trace of the matrix corresponding to θ_g. Note the independence of the choice of basis, implied from the well-known similarity invariance, $\operatorname{tr}(XY) = \operatorname{tr}(YX)$, and also from the fact that the trace of a matrix equals the sum of its eigenvalues.

It is remarkable just how much essential information of a representation we can retrieve from its character. We begin with some properties.

Lemma 2.1. *Let χ be the character of a representation (θ, V) of G and let $g \in G$.*

i. $\chi(1) = \dim(V)$,

ii. $\chi(g^{-1}) = \overline{\chi(g)}$,

iii. $\left|\chi(g)\right| \leq \chi(1)$,

iv. χ *is a* class function *on* G.

Proof.

i. $\chi(1) = \operatorname{tr} \theta_1 = \operatorname{tr} 1_V = \dim(V)$.

ii. Let $m = |G|$ and let λ be an eigenvalue of θ. Then λ^m is an eigenvalue of $(\theta_g)^m = \theta_{g^m} = \theta_1 = 1_V$. Hence λ is an mth root of unity $\implies \lambda^{-1} = \overline{\lambda}$. Now let $\{\lambda_k\}$ be the eigenvalues of θ_g. We evaluate:

$$\chi(g^{-1}) = \operatorname{tr} \theta_g^{-1} = \sum_k \lambda_k^{-1} = \sum_k \overline{\lambda_k} = \overline{\chi(g)}.$$

iii.

$$\left|\chi(g)\right| = \left|\sum_k \lambda_k\right| \leq \sum_k \left|\lambda_k\right| \underset{i.}{=} \dim V \underset{ii.}{=} \chi(1).$$

iv. We must show that χ is constant on each conjugacy class of G, that is for any $x \in G$ we have $\chi(xgx^{-1}) = \chi(g)$. This follows directly from the similarity invariance of the trace with the substitution $\tilde{g} = xg$ and $\tilde{x} = x^{-1}$.

□

Remark. Given two representations (θ^1, V_1), (θ^2, V_2) with characters χ_1, χ_2 respectively, let the corresponding matrices of θ_g^1 and θ_g^2 be A_1 and A_2 respectively (with respect to some choice of basis for V_1 and V_2). The corresponding matrix of $(\theta^1 \oplus \theta^2)_g$ is simply $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and has character $\chi_1 + \chi_2$.

Given a group G, let $\mathfrak{C}_{\text{class}}$ be the set of class functions and let $\mathrm{Irr}_G \subset \mathfrak{C}_{\text{class}}$ be the set of irreducible characters. We regard $\mathfrak{C}_{\text{class}}$ as a vector space over $\mathbb{C}$ by setting

$$(\alpha_1\varphi_1 + \alpha_2\varphi_2)(g) = \alpha_1\varphi_1(g) + \alpha_2\varphi_2(g), \quad \alpha_1, \alpha_2 \in \mathbb{C} \text{ and } \varphi_1, \varphi_2 \in \mathfrak{C}_{\text{class}}.$$

If $\mathrm{Cl}(g_1), \dots, \mathrm{Cl}(g_r)$ are the conjugacy classes of G, then $\mathfrak{C}_{\text{class}}$ has dimension r. Furthermore, we equip $\mathfrak{C}_{\text{class}}$ with an inner product

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi_1(g)\overline{\varphi_2(g)}.$$

Showing that $\langle \cdot, \cdot \rangle$ is in fact an inner product follows directly:

i. (Conjugate symmetry)

$$\begin{aligned}\langle \varphi_1, \varphi_2 \rangle &= \frac{1}{|G|} \sum_{g \in G} \varphi_1(g)\overline{\varphi_2(g)} \\ &= \frac{1}{|G|} \overline{\sum_{g \in G} \varphi_2(g)\overline{\varphi_1(g)}} = \overline{\langle \varphi_2, \varphi_1 \rangle},\end{aligned}$$

ii. (Linearity in the first argument)

$$\begin{aligned}\langle \alpha_1\varphi_1 + \alpha_2\varphi_2, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} (\alpha_1\varphi_1(g) + \alpha_2\varphi_2(g))\overline{\phi(g)} \\ &= \alpha_1\langle \varphi_1, \phi \rangle + \alpha_2\langle \varphi_2, \phi \rangle,\end{aligned}$$

iii. (Positive definite)

$$\langle \varphi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \left|\varphi(g)\right|^2 \geq 0$$

$$\text{with equality} \iff \varphi = 0.$$

The first main goal of this section is proving the following fundamental result, which will in turn answer both questions posed at the end of the last section.

Theorem 2.1. (The First Orthogonality Relation) *If $\chi, \chi' \in \mathrm{Irr}_G$, then*

$$\langle \chi, \chi' \rangle = \begin{cases} 1, & \text{if } \chi = \chi'. \\ 0, & \text{if } \chi \neq \chi'. \end{cases}$$

Remark. In other words, Irr_G is an orthonormal set in $\mathfrak{C}_{\text{class}}$. Furthermore, we have found an upper bound $\left|\mathrm{Irr}_G\right| \leq r$, where r is the number of conjugacy classes of G. In fact, Irr_G is also a basis for $\mathfrak{C}_{\text{class}}$ and $\left|\mathrm{Irr}_G\right| = r$. This result will be the second main goal of this section.

Remark. A small note on the equality of characters is needed.
Two characters χ, χ' coming from two representations $(\theta, V), (\theta', V')$ respectively (both of course representations of the same group G) are considered equal if they are equal *as functions* from G to $\mathbb{C}$. How equality of characters affect G-isomorphism between their respective representations, and vice versa, will be covered in this section.

In order to prove Theorem 2.1 the coming result is crucial.

Theorem 2.2. (The Schur Relations) *Let (θ, V) and (θ', V) be two irreducible representations of a group G. Fix a basis for V and V'. With respect to this basis, let $A(g) = (a_{rs}(g))$ and $A'(g) = (a'_{pq}(g))$ be the matrix representations of θ_g and θ'_g respectively, for any $g \in G$. Define a function S by*

$$S(p,q,r,s) = \sum_{g\in G} a'_{pq}(g^{-1})a_{rs}(g).$$

If $(\theta, V) \ncong_G (\theta', V')$, then $S = 0$.
On the other hand, if $(\theta, V) = (\theta', V')$, then

$$S(p,q,r,s) = \begin{cases} \frac{|G|}{\dim V}, & \text{if } p = s \text{ and } q = r. \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $n = \dim V$ and $n' = \dim V'$ and consider the linear transformation ϵ_{xy} from V to V' which is - with respect to the chosen basis - given by the $n' \times n$ matrix E_{xy} which has entry at (x,y) equal to 1 and zeros as the rest of the entries. Define another linear transformation $f_{xy} : V \to V'$ by

$$f_{xy} = \sum_{g\in G} \theta'_{g^{-1}} \epsilon_{xy} \theta_g.$$

First we show that f_{xy} is actually a G-homomorphism. Secondly, we show how f_{xy} and S relates to each other. Lastly, we use Schur's lemma.
We evaluate for any $h \in G$

$$f_{xy}\theta_h = \sum_{g\in G} \theta'_{g^{-1}} \epsilon_{xy} \theta_{gh} = \sum_{g\in G} \theta'_{hg^{-1}} \epsilon_{xy} \theta_g = \theta'_h f_{xy},$$

where in the second equality we used the bijection $g \mapsto gh$ of G. Hence f_{xy} is indeed a G-homomorphism. Note that the properties of ϵ_{xy} are never used here.
The corresponding matrix of f_{xy} equals

$$\sum_{g\in G} (a'_{pq}(g^{-1}))E_{xy}(a_{rs}(g)) = \left(\sum_{g\in G} a'_{px}(g^{-1})a_{ys}(g) \right)_{p,s} = (S(p,x,y,s))_{p,s}.$$

Now suppose $(\theta, V) \ncong_G (\theta', V')$. Schur's lemma (Theorem 1.2) forces $f_{xy} = 0$ for all $1 \le x \le n'$ and $1 \le y \le n$, which is just to say that $S = 0$.
Suppose instead our two representations are equal, then by Schur's lemma once again, $f_{xy} = \lambda 1_V$ for some $\lambda \in \mathbb{C}$. If $p \neq s$, then it follows directly that $S(p,q,r,s) = 0$. Suppose instead from now on that $p = s$. Evaluating the trace of f_{xy} in two ways yields

$$\begin{aligned} \lambda n = \operatorname{tr} \lambda 1_V = \operatorname{tr} f_{xy} &= \sum_{p=1}^{n} S(p,x,y,p) \\ &= \sum_{g\in G} \sum_{p=1}^{n} a_{px}(g^{-1})a_{yp}(g) \\ &= \sum_{g\in G} (a_{yx}(1)) = |G|\,(1_V)_{y,x}. \end{aligned}$$

Therefore,

$$x \neq y \implies \lambda = 0 \implies f_{xy} = 0,$$
$$x = y \implies \lambda = \frac{|G|}{n} \implies f_{xy} = \frac{|G|}{n} 1_V.$$

Lastly, evaluating $S(p,q,r,p)$ as the element at entry (p,p) of the matrix of f_{qr}, we get our desired conclusion. □

We are now in a position of proving a weaker version of the first orthogonality relation, as a corollary of the Schur relations.

Corollary 2.1. *Let χ and χ' be irreducible characters of representations (θ, V) and (θ', V') respectively. Then*

$$\langle \chi, \chi' \rangle = \begin{cases} 1, & \text{if } \chi = \chi'. \\ 0, & \text{if } (\theta, V) \ncong_G (\theta', V'). \end{cases}$$

Proof. If $\chi = \chi'$ then $\langle \chi, \chi' \rangle = \langle \chi, \chi \rangle$ shows us that we can without loss of generality suppose $(\theta, V) = (\theta', V')$. Keeping with the notation of the proof of the Schur relations gives us

$$\begin{aligned} |G| \langle \chi, \chi \rangle &= \sum_g \sum_{p,r} a_{pp}(g^{-1}) a_{rr}(g) \\ &= \sum_{p,r} S(p,p,r,r) = \sum_p S(p,p,p,p) \\ &= \sum_p \frac{|G|}{\dim V} = |G|. \end{aligned}$$

If $(\theta, V) \ncong_G (\theta', V')$ then similarly from the Schur relations:

$$|G| \langle \chi, \chi' \rangle = \sum_{p,r} S(p,p,r,r) = 0.$$

□

All that is left now for proving the first orthogonality relation is the following.

Lemma 2.2. *Let χ and χ' be irreducible characters of representations (θ, V) and (θ', V') respectively. Then*

$$\chi = \chi' \implies (\theta, V) \cong_G (\theta', V').$$

Proof. Striving for a contradiction, suppose that. $(\theta, V) \ncong_G (\theta', V')$. Keeping with the notation from the Schur relations,

$$\begin{aligned} 0 &= \sum_{p,r} S(p,p,r,r) = \sum_{p,r} \sum_{g \in G} a'_{pp}(g^{-1}) a_{rr}(g) \\ &= \sum_{g \in G} \left(\sum_p a'_{pp}(g^{-1}) \right) \left(\sum_r a_{rr}(g) \right) = \sum_{g \in G} \chi(g^{-1}) \chi(g) \\ &= \sum_{g \in G} \overline{\chi(g)} \chi(g) = \sum_g \left| \chi(g) \right|^2. \end{aligned}$$

But clearly $\chi(1) > 0$. $\lightning$ $\square$

The first orthogonality relation now follows directly.

Proof of Theorem 2.1. Lemma 2.2 gives that $(\theta, V) \ncong_G (\theta', V') \implies \chi \neq \chi'$ and the rest follows from Corollary 2.1. $\square$

We will now prove the general version of Lemma 2.2, which shows just how close of a connection the character of a representation has with the concept of G-isomorphism.

Theorem 2.3. *Let (θ, V) and (θ', V') be two representations of a group G, with character χ and χ' respectively. Then*

$$(\theta, V) \cong_G (\theta', V') \iff \chi = \chi'.$$

Proof. First suppose $(\theta, V) \cong_G (\theta', V')$. Take a G-isomorphism $f : V \to V'$ and simply deduce that the characters are equal:

$$\chi(g) = \operatorname{tr} \theta_g = \operatorname{tr} f^{-1} \theta'_g f = \operatorname{tr} \theta'_g = \chi'(g).$$

Now suppose that $\chi = \chi'$. Let $n = \max\{\dim V, \dim V'\}$. If $n = 1$ then χ and χ' are irreducible and the result is Lemma 2.2.
Suppose now $n > 1$. By Maschke's Theorem (Theorem 1.1) we can decompose V and V' into some direct sum of irreducible sub-G-modules

$$\begin{aligned} V &= V_1 \oplus \cdots \oplus V_m \\ V' &= V'_1 \oplus \cdots \oplus V'_r. \end{aligned}$$

Let V_i and V'_j have irreducible characters χ_i and χ'_j respectively. By the remark following Lemma 2.1 we have that

$$\begin{aligned} \chi &= \chi_1 + \cdots + \chi_m \\ \chi' &= \chi'_1 + \cdots + \chi'_r. \end{aligned}$$

With the help of the inner product, we find that $\langle \chi, \chi_1 \rangle = \langle \chi', \chi_1 \rangle \iff \sum_i \langle \chi_i, \chi_1 \rangle = \sum_j \langle \chi'_j, \chi_1 \rangle$. By Lemma 2.2, the number of sub-G-modules in these decompositions of V and V' that are G-isomorphic to V_1 is equal. Therefore we get

$$\begin{aligned} V &\cong_G \underbrace{V_1 \oplus \cdots \oplus V_1}_{\langle \chi, \chi_1 \rangle \text{ times}} \oplus \tilde{V} \\ V &\cong_G \overbrace{V_1 \oplus \cdots \oplus V_1} \oplus \tilde{V}' \end{aligned}$$

for some sub-G-modules $\tilde{V}$ and $\tilde{V}'$. Since $\langle \chi, \chi_1 \rangle \geq \langle \chi_1, \chi_1 \rangle = 1$ we have that $\max\{\dim \tilde{V}, \dim \tilde{V}'\} < n$. By induction over n we are done. $\square$

We now present, as a corollary, an easy method of deciding if a given representation is irreducible or not. This result is of great practical use.

Corollary 2.2. *Let (θ, V) be a representation of a group G with character χ. Then*

$$(\theta, V) \textit{ is irreducible} \iff \|\chi\| = 1.$$

Proof. If (θ, V) is irreducible, then by definition $\chi \in \mathrm{Irr}_G$. By the first orthogonality relation, the norm of χ equals one.
Conversely, suppose $\|\chi\| = 1$. Decompose V as a direct sum of n irreducible sub-G-modules, $V = \bigoplus_{i=1}^n V_i$. If the character of V_i is χ_i, then $\chi = \sum_{i=1}^n \chi_i$ and

$$1 = \|\chi\|^2 = \sum_{i,j} \langle \chi_i, \chi_j \rangle \geq \sum_{i=1}^n \langle \chi_i, \chi_i \rangle = n.$$

In other words, V has no proper sub-G-module and (θ, V) is irreducible. □

Remark. Here we aim to clarify the inequality given in the proof, for the sceptic reader. We keep with the notation of the previous proof. The number of sub-G-modules in our decomposition G-isomorphic to V_1 equals $\langle \chi, \chi_1 \rangle$. Gathering these up into a direct sum yields W_1, and so forth (as was explained in the last paragraph of Chapter 1). Thus

$$V = \bigoplus_{j=1}^m W_j, \quad m \leq n.$$

After a possible renaming of the V_i's we can assume that

$$W_j \cong_G \bigoplus_{k=1}^{\langle \chi, \chi_j \rangle} V_j =: \langle \chi, \chi_j \rangle V_j.$$

And so we arrive at

$$V \cong_G \bigoplus_{j=1}^m \langle \chi, \chi_j \rangle V_j,$$

$$\chi = \sum_{j=1}^m \langle \chi, \chi_j \rangle \chi_j.$$

Furthermore,

$$\|\chi\|^2 = \sum_{j=1}^m \langle \chi, \chi_j \rangle^2 \geq \sum_{j=1}^m \langle \chi, \chi_j \rangle = n.$$

Example 2.1. *(Characters of the Principal and Regular Representation)* The principal representation $(\theta^1, \mathbb{C})$ of any group G given by $\theta^1_g = 1$ has the trivial character $\chi_1 = 1$.

Consider now the regular representation (θ, V) of G. Recall that the basis of V is indexed by the elements of G. With respect to this basis, the matrix corresponding to θ_g is a permutation matrix. Evaluating the regular character is easily done:

$$\begin{aligned} \chi(g) &= \text{number of 1's in the diagonal of } \theta_g \\ &= \left|\{ h \in G \mid gh = h \}\right| = \begin{cases} |G|, & \text{if } g = 1. \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now let $\{\chi_i\}_{i=1}^r$ be the the set of irreducible characters coming from the set (up to G-isomorphism) of irreducible G-modules $\{V_i\}_{i=1}^r$, respectively. Then

$$\langle \chi, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi_i(g^{-1}) = \frac{1}{|G|} \chi(1) \chi_i(1) = \chi_i(1),$$

and so

$$V \cong_G \bigoplus_{i=1}^r \chi_i(1) V_i = \bigoplus_{i=1}^r \dim(V_i) V_i,$$
$$\chi = \sum_{i=1}^r \chi_i(1) \chi_i = \sum_{i=1}^r \dim(V_i) \chi_i.$$

Remark. Informally, this shows that, for a given group G, the regular representation contains every (up to G-isomorphism) irreducible G-module and that the regular character contains every irreducible character. This result seems reasonable for two reasons.
Firstly, the dimension of the regular representation, $|G|$, is quite large, so the complex structure is expected. Secondly, the structure of the regular representation is, in some sense, exactly equal to the structure of its underlying group. The only difference is that the binary operation now occurs in the more rich setting of a vector space. It should then come as no surprise that every irreducible representation - that captures some aspects of the structure of G - is found in the regular representation.

We have seen that the span of the orthonormal set $\mathrm{Irr}_G = \{\chi_i\}_{i=1}^r$ contains the subspace of $\mathfrak{C}_{\text{class}}$ consisting of the characters. Shortly we will show that actually $\langle \mathrm{Irr}_G \rangle = \mathfrak{C}_{\text{class}}$. We will end this section by proving the uniqueness of the canonical decomposition. Surprisingly, both these results share a lemma.

Lemma 2.3. *Let (θ, V) be an irreducible representation of a group G, with character χ. Let φ be a class function of G. Define a map $f : V \to V$ by*

$$f = \sum_{g \in G} \varphi(g) \theta_{g^{-1}}.$$

Then f is a homothety $f = \lambda 1_V$ with $\lambda = \frac{|G|}{\dim V} \langle \varphi, \chi \rangle$.

Proof. Evidently we must show that f is a G-homomorphism and then use Schur's lemma. The fact that f is linear follows from θ_g being linear. Now we evaluate for any $h \in G$

$$f \theta_h = \sum_{g \in G} \varphi(g) \theta_{g^{-1}h} = \theta_h \sum_{g \in G} \varphi(g) \theta_{h^{-1}g^{-1}h}$$
$$= \theta_h \sum_{g \in G} \varphi(hgh^{-1}) \theta_{g^{-1}} = \theta_h f.$$

In the third equality we used the bijection $g \mapsto hgh^{-1}$ of G and in the last equality we used φ being a class function. By Schur's Lemma $\implies f = \lambda 1_V$ for some $\lambda \in \mathbb{C}$. We compute λ by the trace of f

$$\dim(V) \lambda = \operatorname{tr} f = \sum_{g \in G} \varphi(g) \chi(g^{-1})$$
$$= |G| \langle \varphi, \chi \rangle.$$

□

Theorem 2.4. *The orthonormal set* Irr_G *forms a basis for the vector space* $\mathfrak{C}_{\text{class}}$ *of class functions of* G.

Proof. The first orthogonality relation (Theorem 2.1) says that $\operatorname{Irr}_G = \{\chi_i\}_{i=1}^r$ is a linearly independent and orthonormal set. Let φ be any class function and suppose that $\langle \varphi, \chi_i \rangle = 0$ for any $i \in \{1, \ldots, r\}$, in other words $\varphi \in \langle \operatorname{Irr}_G \rangle^\perp$. If we can show that $\varphi = 0$, then $\mathfrak{C}_{\text{class}} = \langle \operatorname{Irr}_G \rangle \oplus \langle \operatorname{Irr}_G \rangle^\perp = \langle \operatorname{Irr}_G \rangle$ and we are done.

Let (θ, V) be the regular representation of G and $\{(\theta^j, V_j)\}_{j=1}^m$ be an irreducible decomposition of V. Define $\varphi_0 : V \to V$ by

$$\varphi_0 = \sum_{g \in G} \varphi(g) \theta_{g^{-1}}.$$

We want to show that $\varphi_0 = 0$ and for that end we expand

$$\begin{aligned}
\varphi_0 &= \sum_{g \in G} \varphi(g) \bigoplus_{j=1}^m \theta^j_{g^{-1}} \\
&= \bigoplus_{j=1}^m \sum_{g \in G} \varphi(g) \theta^j_{g^{-1}} \\
&= \bigoplus_{j=1}^m 0 = 0.
\end{aligned}$$

The third equality is Lemma 2.3 used m times. In particular for the basis $\{e_g\}_{g \in G}$ of V we have

$$0 = \varphi_0(e_1) = \sum_{g \in G} \varphi(g^{-1}) \theta_g e_1 = \sum_{g \in G} \varphi(g^{-1}) e_g$$

$$\implies \varphi(g^{-1}) = 0, \text{ for any } g \in G \implies \varphi = 0.$$

□

Theorem 2.5. *Let* V *be a* G*-module and let* $\{V_j'\}_{j=1}^m$ *be an irreducible decomposition of sub-*G*-modules. Let* $\{V_i\}_{i=1}^r$ *be the distinct (up to* G*-isomorphism) irreducible* G*-modules, with respective characters* $\{\chi_i\}_{i=1}^r$*. Define* W_i *as the direct sum of those* V_j' *which are* G*-isomorphic to* V_i*. Then the decomposition*

$$V = W_1 \oplus \cdots \oplus W_r$$

does not *depend on the original decomposition.*

Remark. The decomposition $V = W_1 \oplus \cdots \oplus W_r$ is precisely the **canonical decomposition**. Note that some W_i may be zero.

Remark. It is interesting that when we apply a character as the class function in Lemma 2.3 we naturally get this proof.

Proof. Let θ be the representation of the G-module V and let the character of V_i be χ_i for $i = 1, \ldots, r$. Fix such an i, and define $f_i : V \to V$ by

$$f_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \chi_i(g) \theta_{g^{-1}}.$$

The function above is defined without any mention of the original decomposition. When we restrict f_i with the V_i's we can use Lemma 2.3

$$f_i|_{V_k} = \begin{cases} 1, & \text{if } k = i. \\ 0, & \text{if } k \neq i. \end{cases}$$
$$\Longrightarrow f_i|_{W_k} = \begin{cases} 1, & \text{if } k = i. \\ 0, & \text{if } k \neq i. \end{cases}$$

and so f_i is the projection of V onto W_i. Since f_i is independent from the original decomposition, then so is the decomposition $V = W_1 \oplus \cdots \oplus W_r$. $\square$

3 Selected Results on Irreducible Characters

This chapter will be covering what impact the inherent structures of a group G has on its irreducible representations and characters. All the following results will come to use later. Chapter 2 gave us the following.

Corollary 3.1. *Let* $\{\mathrm{Cl}(g_i)\}_{i=1}^r$ *be the conjugacy classes of a group* G. *Then* $r = \left|\mathrm{Irr}_G\right|$.

Proof. The set indicator functions $\{\mathbb{1}_{\mathrm{Cl}(g_i)}\}_{i=1}^r$ is a basis for the vector space $\mathfrak{C}_{\mathrm{class}}$. The result now follows from Theorem 2.4. □

Theorem 3.1. (The Degree Equation) *Let* G *be a group, then*

$$|G| = \sum_{\chi \in \mathrm{Irr}_G} \chi(1)^2.$$

Proof. Recall from Example 2.1 that the regular character χ' of G is given by

$$\chi' = \sum_{\chi \in \mathrm{Irr}_G} \chi(1)\chi.$$

Evaluating at $g = 1$ gives

$$|G| = \chi'(1) = \sum_{\chi \in \mathrm{Irr}_G} \chi(1)^2.$$

□

Theorem 3.2. (The Second Orthogonality Relation) *Let* $\mathrm{Cl}(h)$ *be a conjugacy class of a group* G. *Then*

$$\sum_{\chi \in \mathrm{Irr}_G} \overline{\chi(h)} \cdot \chi = \frac{|G|}{\left|\mathrm{Cl}(h)\right|} \mathbb{1}_{\mathrm{Cl}(h)},$$

where $\mathbb{1}_{\mathrm{Cl}(h)}$ *is the indicator function of* $\mathrm{Cl}(h)$.

Remark. Recall from Lagrange's Theorem that $\frac{|G|}{|\mathrm{Cl}(h)|} = \left|\mathrm{C}_G(h)\right|$ where $\mathrm{C}_G(h)$ is the centralizer of h in G.

Proof. The indicator function $\mathbb{1}_{\mathrm{Cl}(h)}$ is a class function of G. By Theorem 2.4 we can express $\mathbb{1}_{\mathrm{Cl}(h)}$ in the basis of irreducible characters as

$$\mathbb{1}_{\mathrm{Cl}(h)} = \sum_{\chi \in \mathrm{Irr}_{G G}} a_\chi \chi,$$

with coefficients

$$\begin{aligned} a_\chi = \langle \mathbb{1}_{\mathrm{Cl}(h)}, \chi \rangle &= \frac{1}{|G|} \sum_{g \in G} \mathbb{1}_{\mathrm{Cl}(h)}(g) \overline{\chi(g)} \\ &= \frac{\left|\mathrm{Cl}(h)\right|}{|G|} \cdot \overline{\chi(h)}. \end{aligned}$$

The result now follows directly. □

Let us now define an important subset of Irr_G.

Definition 3.1. Let G be a group. Define the **linear characters** of G as the set

$$\mathrm{Lin}_G = \{\chi \in \mathrm{Irr}_G \mid \dim \chi = 1\}.$$

Remark. If a representation (θ, V) has a linear character χ we can identify θ by its character in the following sense:

$$\begin{aligned} \theta : G &\to \mathrm{Aut}\, V \\ & \quad\ \ \wr\| \\ \chi : G &\longrightarrow \mathbb{C}^\times \end{aligned}$$

Theorem 3.3. *Let G be a group. Then G is abelian if and only if* $\mathrm{Lin}_G = \mathrm{Irr}_G$.

Proof.

$$\begin{aligned} G \text{ is abelian} &\iff |G| = \text{the number of conjugacy classes of } G \\ &\iff |G| = \left|\mathrm{Irr}_G\right| \\ &\iff \left|\mathrm{Lin}_G\right| = \left|\mathrm{Irr}_G\right|. \end{aligned}$$

The second and third equivalences follow from Corollary 3.1 and Theorem 3.1, respectively. □

Definition 3.2. Let χ be a character of a group G. Define the **kernel** of χ as

$$\ker \chi = \{g \in G \mid \chi(g) = \chi(1)\}.$$

Definition 3.3. Let (θ, V) be a representation of a group G. The representation is called **faithful** if $\ker \theta = 1$. In other words, if θ is an embedding.

We are interested in faithful representations. By the following theorem, this property is entirely captured by characters.

Theorem 3.4. *Let (θ, V) be a representation of a group G with character χ. Then*

$$\ker \theta = \ker \chi.$$

Proof. If $g \in \ker \theta \implies \theta_g = 1_V \implies g \in \ker \chi$. For the harder implication, now let $g \in \ker \chi$. Let the dimension of V equal n. Restricting our representation to $(\theta|_{\langle g \rangle}, V)$ we get a representation of the abelian subgroup $\langle g \rangle$. Consider a decomposition into irreducible representations

$$\theta|_{\langle g \rangle} = \bigoplus_{i=1}^{n} \theta^i.$$

By Theorem 3.3, θ^i is one dimensional and has a character $\chi_i(g)$ (identified with θ^i_g) equal to an mth root of unity, where $m = |G|$. We arrive at

$$n = \chi(1) = \chi(g) = \chi_1(g) + \cdots + \chi_n(g)$$

which implies that $\chi_i(g) = 1$ and $\theta^i(g) = 1_{\mathbb{C}} \implies \theta(g) = \theta|_{\langle g \rangle}(g) = 1_V$. □

We will now discuss how the characters of a quotient group G/N preserve or provide characters of the original group G.

Theorem 3.5. *Let G be a group with a normal subgroup N. Then there exists a bijective correspondence between the set of irreducible representations of G/N and the set of irreducible representations of G which contain N in their kernel.*

Remark. By our abuse of notation, Theorem 2.3 and Theorem 3.4, the bijective correspondence given above gives rise to another bijective correspondence between $\mathrm{Irr}_{G/N}$ and $\{\chi \in \mathrm{Irr}_G \mid N \subset \ker \chi\}$.

Proof. Given an irreducible representation (θ, V) of G/N, define a representation $(\tilde{\theta}, V)$ of G by $\tilde{\theta}_g = \theta_{gN}$. This makes $\tilde{\theta}$ into a homomorphism. In fact,

$$\tilde{\theta}_{gh} = \theta_{ghN} = \theta_{gNhN} = \theta_{gN}\theta_{hN} = \tilde{\theta}_g\tilde{\theta}_h.$$

An invariant sub-G-module of V with respect to $\tilde{\theta}$ is easily seen to be invariant with respect to θ which implies that $\tilde{\theta}$ is irreducible. If $x \in N$ then

$$\tilde{\theta}_x = \theta_{xN} = \theta_N = \tilde{\theta}_1$$

and $x \in \ker \tilde{\theta}$. Likewise, suppose we have another irreducible representation (θ', V') of G/N which gives an irreducible representation $(\tilde{\theta}', V')$ of G by the above mapping. It is easy to see that a G-isomorphism between $\tilde{\theta}$ and $\tilde{\theta}'$ is also a G-isomorphism between θ and θ'. Thus this mapping of representations is injective up to G-isomorphism.
Conversely, now let (θ, V) be an irreducible representation of G such that $\ker \theta \supset N$. If $gN = hN \implies g^{-1}h \in N \implies \theta_{g^{-1}h} = \theta_1$ and so $\theta_g = \theta_h$. This means we can define a representation $(\underset{\sim}{\theta}, V)$ of G/N by $\underset{\sim}{\theta}_{gN} = \theta_g$. Similarly one checks that $\underset{\sim}{\theta}$ is irreducible and that this mapping of representations is injective up to G-isomorphism. Also, both of these maps are bijective, since $\tilde{\underset{\sim}{\theta}} = \theta$ (which holds in both orders of applying the tildes). $\square$

Remark. The representation $\tilde{\theta}$ will be called the **lifted representation** of θ. The character $\tilde{\chi}$ of $\tilde{\theta}$ will likewise be called the **lifted character** of χ, where χ is the character of θ.

Recall that G' denotes the derived subgroup of G. We can completely describe those irreducible characters that contain G' in its kernel.

Lemma 3.1. *Let (θ, V) be an irreducible representation of a group G with character χ. Then*

$$G' \subset \ker \chi \iff \chi \in \mathrm{Lin}_G.$$

Proof. If $G' \subset \ker \chi$, by Theorem 3.5

$$\theta_1 = \underset{\sim}{\theta}_{G'} = 1_V.$$

But $\underset{\sim}{\theta}$ is an irreducible representation of the abelian quotient group G/G' and by Theorem 3.3

$$\underset{\sim}{\chi} \in \mathrm{Lin}_{G/G'} \implies \dim V = 1 \implies \chi \in \mathrm{Lin}_G.$$

Suppose now that $\chi \in \mathrm{Lin}_G$ and let $x \in G'$. By the identification of linear characters we have that

$$\chi(x) = \chi(1) \iff \theta_x = \theta_1.$$

Since

$$\theta_{g^{-1}h^{-1}gh} = \theta_1$$

holds for all $g, h \in G$ we have that $\theta_x = \theta_1$ and the proof done. $\square$

Remark. Note the special case when G is abelian. Then $G' = 1$ and every irreducible character contains G' in its kernel. The conclusion of Theorem 3.3 follows.

The following results show the connection between characters of a group and the normal subgroups.

Lemma 3.2. *Let G be a group. Then*

$$1 = \bigcap_{\chi \in \mathrm{Irr}_G} \ker \chi.$$

Theorem 3.6. *Let G be group with a normal subgroup N. Then*

$$N = \bigcap_{\substack{\chi \in \mathrm{Irr}_G \\ N \subset \ker \chi}} \ker \chi.$$

Corollary 3.2. *Let G be a group. Then*

$$G' = \bigcap_{\chi \in \mathrm{Lin}_G} \ker \chi.$$

Remark. Lemma 3.2 is needed for Theorem 3.6 which implies Corollary 3.2. Naturally, we prove them in the opposite order.

Proof of Corollary 3.2. Direct use of Theorem 3.6 with the identification of Lemma 3.1 yields the conclusion. □

Proof of Theorem 3.6. Using the identification of characters described in the remark following Theorem 3.5 we have that

$$\bigcap_{\substack{\chi \in \mathrm{Irr}_G \\ N \subset \ker \chi}} \ker \chi = \bigcap_{\chi \in \mathrm{Irr}_{G/N}} \ker \tilde{\chi}.$$

Using the fact that $\ker \tilde{\chi}/N \cong \ker \chi$ we get that

$$\bigcap_{\chi \in \mathrm{Irr}_{G/N}} \ker \chi = \bigcup_g \{gN\} \iff \bigcap_{\chi \in \mathrm{Irr}_{G/N}} \ker \tilde{\chi} = \bigcup_g gN.$$

Finally, by Lemma 3.2 we have that $\bigcap_{\chi \in \mathrm{Irr}_{G/N}} \ker \chi = \{N\}$ from which the result follows. □

Proof of Lemma 3.2. The trick is, once again, to consider the regular representation (θ, V). Let χ be the regular character and $\{\chi_i\}_{i=1}^r$ the irreducible characters of G. By Example 2.1 $\chi = \sum_{i=1}^r \chi_i(1)\chi_i$. The evaluation of the regular character shows that θ is faithful and $\ker \chi = 1$. Now suppose that $g \in \bigcap_{i=1}^r \ker \chi_i$. Then

$$\begin{aligned} \chi(g) &= \sum_{i=1}^r \chi_i(1)\chi_i(g) \\ &= \sum_{i=1}^r \chi(1)^2 = |G| = \chi(1) \\ &\implies g \in \ker \chi. \end{aligned}$$

The third equality follows from The Degree Equation (Theorem 3.1). □

4 Induced Representations

Given a representation (θ, V) of a group G and a subgroup H of G we consider the restriction $\vartheta = \theta|_H$. Intuitively, the less complicated subgroup H may not need the full complexities of the G-module V (considered on its own right as an H-module) and so may have a proper sub-H-module W of V. Note that this might even happen even if V is irreducible when regarded as a G-module. Our hope is that some structure of the representations of G can be retrieved from the representations of its subgroups.

Let T be a set of left-coset representatives of H in G and define for any $t \in T$, W_t to equal $\theta_t W$. This definition is actually independent on the choice of representatives. In fact, let $s \in tH \implies s = th$ for some $h \in H$, then

$$\theta_s W = \theta_t \theta_h W = \theta_t \vartheta_h W = \theta_t W = W_t.$$

Therefore, we can well-define $W_{tH} = W_t$ and arrive at the following definition.

Definition 4.1. If H is a subgroup of a group G with representations (ϑ, W) and (θ, V) respectively, such that $\vartheta = \theta|_H$ and W is a sub-H-module of V. The representation (θ, V) is said to be **induced by** (ϑ, W) if

$$V = \bigoplus_{tH \in G/H} W_t.$$

In this case, we will also say that (ϑ, W) **induces** (θ, V).

Remark. Note that G acts on $\{W_t\}_{t\in T}$ by permuting $\theta_g W_t = \theta_{gt} W = W_{gt}$ and the action is transitive.

Remark. Let us examine how θ relates to ϑ in a situation like this. First put $v = \sum_{i=1}^n v_i$ where $v_i \in W_{t_i}$ and $T = \{t_i\}_{i=1}^n$ is a set of representations of H in G. Further write $v_i = \theta_{t_i} w_i$ for some $w_i \in W$. For each i there is a j and an $h_i \in H$ such that $gt_i = t_j h$. Putting this all together yields

$$\begin{aligned} \theta_g v = \theta_g \sum_{i=1}^n \theta_{t_i} w_i = \sum_{i=1}^n \theta_{gt_i} w_i &= \sum_{i=1}^n \theta_{t_j h_i} w_i \\ &= \sum_{i=1}^n \theta_{t_j} \vartheta_{h_i} w_i. \end{aligned}$$

Lemma 4.1. *Let H be a subgroup of G. If two representations (ϑ^1, W_1) and (ϑ^2, W_2) of H induces the representations (θ^1, V_1) and (θ^2, V_2) respectively, then the representation $(\vartheta^1 \oplus \vartheta^2, W_1 \oplus W_2)$ of H induces the representation $(\theta^1 \oplus \theta^2, V_1 \oplus V_2)$ of G.*

Proof. Let T be a set of left-coset representatives of H in G. It follows that $(\theta^1 \oplus \theta^2)|_{W_1 \oplus W_2} = \vartheta^1 \oplus \vartheta^2$ and

$$\begin{aligned} V_1 \oplus V_2 &= \left(\bigoplus_{t\in T} \theta_t^1 W_1 \right) \oplus \left(\bigoplus_{t\in T} \theta_t^2 W_2 \right) \\ &= \bigoplus_{t\in T} (\theta_t^1 W_1 \oplus \theta_t^2 W_2) \\ &= \bigoplus_{t\in T} (\theta^1 \oplus \theta^2)_t (W_1 \oplus W_2) = \bigoplus_{t\in T} (W_1 \oplus W_2)_t. \end{aligned}$$

$\square$

We now show that the regular representation of a group can always be induced from any of its subgroups.

Lemma 4.2. *Let (θ, V) be the regular representation of G with H any subgroup of G and ϑ the restriction of θ to H. Then there exists a representation (ϑ, W) of H that induces (θ, V). Furthermore, (ϑ, W) is the regular representation of H.*

Proof. Let T be a set of left-coset representatives of H in G. Naturally define $W = \langle e_h \rangle_{h \in H}$ which is trivially invariant under ϑ and a sub-H-module of V. In fact, W is the regular H-module. A simple verification shows that

$$\bigoplus_{t \in T} W_t = \bigoplus_{t \in T} \theta_t W = \bigoplus_{t \in T} \langle e_{th} \rangle_{h \in H} = V.$$

□

We now evaluate the character of an induced representation.

Theorem 4.1. *Let H be a subgroup of a group G with T a set of left-coset representatives of H in G. Suppose that the representation (ϑ, W) of H, with character φ, induces the representation (θ, V) of G with character χ. Then*

$$\chi(g) = \sum_{\substack{t \in T \\ t^{-1} g t \in H}} \varphi(t^{-1} g t).$$

Proof. Let $\{e_j\}_{j=1}^{n}$ be a basis for W. Extend this to a basis $\{f_{tj}\}$ for V where $f_{tj} = \theta_t e_j$ for $t \in T$ and $j = 1, \ldots, n$. In the basis $\{e_i\}$ let the matrix of φ_h equal $(a_{ij}(h))$. For a given f_{tj} we evaluate

$$\begin{aligned} \theta_g f_{tj} &= \theta_{gt} e_j = \theta_{\tilde{t}h} e_j = \theta_{\tilde{t}} \varphi_h e_j \\ &= \theta_{\tilde{t}} \sum_{k=1}^{n} a_{kj} e_k = \sum_{k=1}^{n} a_{kj} f_{\tilde{t}k} \end{aligned}$$

where $\tilde{t} \in T$ and $h \in H$ is uniquely defined by

$$gt = \tilde{t}h.$$

We see here that if $t \neq \tilde{t}$ then we get no contribution to the character χ. In fact, θ_g permutes $\theta_g W_t \neq W_t$. In the case that $t = \tilde{t} \iff t^{-1} g t \in H$ we get a contribution a_{jj}. Putting this together yields

$$\chi(g) = \sum_{\substack{t \in T \\ t^{-1} g t \in H}} \sum_{i=1}^{n} a_{ii} = \sum_{\substack{t \in T \\ t^{-1} g t \in H}} \varphi(h) = \sum_{\substack{t \in T \\ t^{-1} g t \in H}} \varphi(t^{-1} g t).$$

□

With the intention of generalizing this result for class functions on H we make the following definition.

Definition 4.2. Let H be a subgroup of a group G with T a set of left-coset representatives of H in G. Let ψ be a class function on H and define a function $\psi^o : G \to \mathbb{C}$ by $\psi^o(h) = \psi(h)$ for $h \in H$ and zero otherwise. The function $\psi^G : G \to \mathbb{C}$ defined by

$$\psi^G(g) = \sum_{t \in T} \psi^o(t^{-1} g t)$$

is called the **induction map** of ψ to G.

Remark. By Theorem 4.1, if (ϑ, W) is a representation of a subgroup H, with character φ, which induces a representation (θ, V) of G with character χ, then the induction map behaves as expected, namely $\varphi^G = \chi$.

Remark. If $t^{-1}gt \in H$ then the mapping $t^{-1}gt \mapsto h^{-1}t^{-1}gth$ is a bijection of H and ψ, being a class function, gives $\psi(t^{-1}gt) = \psi(h^{-1}t^{-1}gth)$. Also, $t^{-1}gt \in H \iff h^{-1}t^{-1}gth \in H$. Putting all this together yields a different way of writing the induction map of ψ:

$$\psi^G(g) = \frac{1}{|H|}\sum_{h\in H}\sum_{t\in T}\psi^o((th)^{-1}gth).$$

Furthermore, since $G = \cup_{t\in T} tH$ we get the often useful form

$$\psi^G(g) = \frac{1}{|H|}\sum_{x\in G}\psi^o(x^{-1}gx).$$

Lemma 4.3. *If H is a subgroup of G with a class function ψ on H, the the induction map ψ^G is a class function on G.*

Proof. For any $x \in G$ we have that

$$\begin{aligned}\psi^G(x^{-1}gx) &= \sum_{t\in T}\psi^o((xt)^{-1}gxt)\\ &= \sum_{t\in T'}\psi(t^{-1}gt) = \psi^G(g),\end{aligned}$$

where $T' = \{xt \mid t \in T\}$ is another set of left-coset representatives of H in G. □

The following result will be used in later chapters.

Lemma 4.4. (Transitivity of Induction Map) *If H and K are subgroups of G with H contained in K. Then if ψ is a class function on H we have that*

$$(\psi^K)^G = \psi^G.$$

Proof. Note that by Lemma 4.3, ψ^K is a class function on K and $(\psi^K)^G$ is well defined. If $g \notin K$ then $x^{-1}gx \notin K \implies x^{-1}gx \notin H$ for any $x \in K$. Hence

$$(\psi^K)^o(g) = \frac{1}{|H|}\sum_{x\in K}\psi^o(x^{-1}gx)$$

holds for all $g \in G$. Now we evaluate

$$\begin{aligned}(\psi^K)^G(g) &= \frac{1}{|K|}\sum_{y\in G}(\psi^K)^o(y^{-1}gy) = \frac{1}{|K|}\sum_{y\in G}\frac{1}{|H|}\sum_{x\in K}\psi^o((xy)^{-1}gxy)\\ &= \frac{1}{|H|}\sum_{y\in G}\psi^o(y^{-1}gxy) = \psi^G(g)\end{aligned}$$

where we in the third equality used the bijection $xy \mapsto y$ of G, for any $x \in K$. □

At this point in time, we do not know that if ψ is a character of a representation of a subgroup H of G, then its induction map ψ^G is necessarily a character of G. If we knew that there existed a representation of G, induced by the original representation of H, then Theorem 4.1 would answer this question in a positive way: yes, ψ^G is indeed a character of G.

The proof of existence and uniqueness (up to G-isomorphism) of the induced representation will end this chapter. Amazingly enough, one could show that ψ^G is a character given that ψ is a character without the proof of existence. The following result (which is also of great practical use) is the key point of this argument.

Theorem 4.2. (Frobenius Reciprocity Theorem) *Let H be a subgroup of G and let ψ and φ be class functions on H and G respectively. Then*

$$\langle \psi, \varphi|_H \rangle = \langle \psi^G, \varphi \rangle.$$

Remark. To clarify, the inner product on the left hand side is on the space of class functions of H, while the right hand side is on the space of class functions of G.

Theorem 4.3. *If H is a subgroup of G and ψ a character of some representation of H. Then the induction map ψ^G is a character of some representation of G.*

Proof of Theorem 4.3. From Lemma 4.3 we know that ψ^G is a class function of G, and so

$$\psi^G = \sum_{\chi \in \mathrm{Irr}_G} a_\chi \chi$$

where $a_\chi = \langle \psi^G, \chi \rangle$. Now ψ^G is a character if and only if a_χ is a non-negative integer for every $\chi \in \mathrm{Irr}_G$. By Frobenius reciprocity, $\langle \psi^G, \chi \rangle = \langle \psi, \chi|_H \rangle$ equals a non-negative integer, since ψ is a character of H. □

Proof of Frobenius Reciprocity Theorem. Simply evaluating the right hand side yields

$$\begin{aligned}
|G||H|\langle \psi^G, \varphi \rangle &= |H| \sum_{g \in G} \psi^G(g)\overline{\varphi(g)} \\
&= \sum_{g \in G} \sum_{x \in G} \psi^o(x^{-1}gx)\overline{\varphi(g)} \\
&= \sum_{g \in G} \sum_{x \in G} \psi^o(g)\overline{\varphi(xgx^{-1})} \\
&= \sum_{g \in G} \sum_{x \in G} \psi^o(g)\overline{\varphi(g)} \\
&= |G| \sum_{h \in H} \psi(h)\overline{\varphi(h)} = |G||H|\langle \psi, \varphi|_H \rangle.
\end{aligned}$$

□

Let us now prove the uniqueness and existence of the induced representation. We begin with showing uniqueness, which follows directly.

Theorem 4.4. (Uniqueness) *Let H be a subgroup of G with representation (ϑ, W) of H and representations (θ, V) and (θ', V') of G. If both (θ, V) and (θ', V') are induced by (ϑ, W), then (θ, V) is G-isomorphic to (θ', V').*

Proof. Let ψ, χ and χ' be the characters of (ϑ, W), (θ, V) and (θ', V') respectively. By Theorem 4.1, $\chi = \psi^G = \chi'$, and so by Theorem 2.3, $(\theta, V) \cong_G (\theta', V')$. □

For showing the existence we need the following Lemmas.

Lemma 4.5. *Let H, with representation (ϑ, W), be a subgroup of G, with representation (θ, V) that is induced by the representation of H. Suppose that W' is a sub-H-module with respect to (ϑ, W), then the representation with H-module W' induces a representation with G-module V' such that V' is a sub-G-module with respect to (θ, V).*

Proof. Let T be a set of left-coset representations of H in G. Define $V' = \sum_{t \in T} \theta_t W'$. Since $V = \oplus_{t \in T} \theta_t W$, the sum of V' is also direct:

$$V' = \bigoplus_{t \in T} \theta_t W'.$$

Furthermore, we see that V' is invariant with respect to θ:

$$\theta_g V' = \bigoplus_{t \in T} \theta_{gt} W' = \bigoplus_{t \in T'} \theta_t W' = V',$$

with $T' = gT$ being another set of left-coset representations. The restriction to H of the representation with G-module V' is clearly the representation with H-module W', since $\theta|_H = \vartheta$. □

Lemma 4.6. *Let H be a subgroup of G. Suppose that a representation (ϑ, W) of H induces a representation (θ, V) of G and that $(\vartheta', W') \cong_H (\vartheta, W)$, then there exists a representation (θ', V') of G that is induced by (ϑ', W').*

Proof. Let T be a set of left-coset representatives of H in G. Let $W'_t = W'$ and the external direct sum $V' = \oplus_{t \in T} W'_t$ of copies of W' (formally, we define $W'_t = t \times W'$ and identify each W'_t with W'). Let $f : W \to W'$ be an H-isomorphism. Extend $f^* : V \to V'$ by

$$f^*(v_t) = f^*(\theta_t w) = f(w) \in W'_t$$

for any $v_t \in V'_t$ and $v_t = \theta_t w$ for some $w \in W$. It is evident that f^* is a linear bijection. Define $\theta'_g : V \to V$ by

$$\theta'_g = f^* \theta_g (f^*)^{-1}.$$

It follows directly from the definition that θ' is a homomorphism and its restriction to H equals ϑ'. All we need to check now is that

$$\begin{aligned}
V' &= \bigoplus_{t \in T} W'_t \\
\iff (f^*)^{-1} V' &= (f^*)^{-1} \bigoplus_{t \in T} \theta'_t W' \\
\iff V &= \bigoplus_{t \in T} (f^*)^{-1} \theta'_t W' \\
&= \bigoplus_{t \in T} \theta_t (f^*)^{-1} W' \\
&= \bigoplus_{t \in T} \theta_t W = \bigoplus_{t \in T} W_t
\end{aligned}$$

which holds by hypothesis. □

The desired Theorem of existence now follows easily.

Theorem 4.5. (Existence) *Let H be a subgroup of G and (ϑ, W) a representation of H, then there exists a representation (θ, V) of G that is induced by (ϑ, W).*

Proof. We can by Lemma 4.1 assume without loss of generality that (ϑ, W) is irreducible. Let (ϑ', W') be a representation in the canonical decomposition of the regular representation of H, such that (ϑ', W') is H-isomorphic to (ϑ, W). By Lemma 4.2, the regular representation of H induces the regular representation of G, which by Lemma 4.5 gives an induced representation of (ϑ', W'). By Lemma 4.6, this gives an induced representation of (ϑ, W). $\square$

Remark. With existence and uniqueness proven, we can from this point speak of *the* induced representation of (ϑ, W), when (ϑ, W) is a representation of a subgroup H of G.

We end this section with explicitly stating a method of how to use these results in practice, for finding the irreducible characters for a given group G.

Formula.

i. Find a subgroup H of G with left-coset representatives T.

ii. If H is normal, then compute the irreducible characters $\{\chi_i\}$ of G/H and lift them up to respective irreducible characters $\{\tilde{\chi}_i\}$ of G

iii. Compute the irreducible character $\{\psi_i\}$ of H and their respective induction map characters $\{\psi_i^G\}$ of G.

iv. Given an induction character ψ^G, compute $\langle \psi^G, \psi^G \rangle$ to see how far off it is from being irreducible. If the norm equals one, then ψ^G is irreducible.

v. Use any known irreducible character χ of G and compute with Frobenius reciprocity Theorem (for easier computations), $\langle \psi^G, \chi \rangle = \langle \psi, \chi|_H \rangle = n$. If $n > 0$ then continue from step iv. with the character $(\psi^G - n\chi)$ until it hopefully produces a new irreducible character of G.

5 A Note on Solvable Groups

Let us for the moment step back from the theory of representations and recap some facts from elementary group theory. Specifically, let us define what a solvable, supersolvable and a nilpotent group is. In the next chapter, the results in this chapter will be used to show the connection between M-groups and solvable groups.

Definition 5.1. A group G is called **solvable** if there exists a sequence, called a **normal series**, of proper subgroups $\{G_i\}_{i=1}^n$ of G such that

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

and the quotient group G_i/G_{i-1} is abelian for all $i = 1, \ldots, n$.

Lemma 5.1. *If N is a normal subgroup of G and suppose that N and the quotient group G/N are both solvable, then G is solvable.*

Proof. By hypothesis, there exists normal series

$$1 = N_0 \triangleleft \cdots \triangleleft N_m = N$$
$$N/N = H_0/N \triangleleft \cdots \triangleleft H_k/N = G/K$$

with subgroups H_i of G all containing N (by the Correspondence Theorem). The Third Isomorphism Theorem gives that

$$\frac{H_{i+1}/N}{H_i/N} \cong H_{i+1}/H_i$$

and we have a normal series

$$1 = N_0 \triangleleft \cdots \triangleleft N_m \triangleleft H_1 \triangleleft \cdots \triangleleft H_k = G.$$

□

Lemma 5.2. *If G is solvable, then $G' \neq G$.*

Proof. In a normal series of G, we have in particular that G/G_{n-1} is abelian. But G' is the smallest subgroup of G such that its quotient with G is abelian.

$$\implies G' \subset G_{n-1} \subsetneq G.$$

□

Remark. By induction, it follows readily that $G^{(r)} = 1$ for some $r > 0$.

Definition 5.2. A group G is called **supersolvable** if there exists a sequence, called a **supernormal series** of proper subgroups $\{N_i\}_{i=1}^n$ of G such that

$$1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_n = G$$

with $N_i \triangleleft G$ and the quotient group N_i/N_{i-1} is cyclic for all $i = 1, \ldots, n$.

Lemma 5.3. *A supersolvable group is solvable.*

Proof. It follows directly from the definitions. □

Definition 5.3. A group G is called **nilpotent** if it equals the direct product of its Sylow subgroups.

Lemma 5.4. *A nilpotent group is supersolvable.*

Proof. Let G be a nilpotent group with Sylow subgroups $\{P_i\}_{i=0}^{n}$, so that

$$G = P_0 \cdots P_n.$$

Clearly, each P_i is normal in G. Define $N_i = \prod_{j=0}^{i} P_j$ for $i = 0, \dots, n$ where we for $i = 0$ regard the product as empty and $N_0 = 1$. A product of normal subgroups is also normal, therefore

$$1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_n = G$$

is a normal series of G. Furthermore,

$$|N_i/N_{i-1}| = |P_i| = p_i^{n_i}$$

for some prime p_i and $n_i > 0$. If $n_i = 1$ then the quotient group is cyclic. Otherwise, if we can find a supernormal series

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r = P_i$$

then by the correspondence theorem applied to each P_i we have a supernormal series for G.
We show this by induction over $n = n_i$. As the p-group $P = P_i$ has a nontrivial center $Z(P)$ which contains an element g of order p, where $|P| = p^n$. Set $H_0 = \langle g \rangle$ which is normal in P (since $g \in Z(P)$). Now, P/H_0 has order p^{n-1}. By induction and the correspondence theorem, we can find a supernormal series

$$1 = H_0/H_0 \triangleleft H_1/H_0 \triangleleft \cdots \triangleleft H_r/H_0 = P/H_0$$

which we can lift up to get a supernormal series of P,

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r = P.$$

□

Lemma 5.5. *If G is a group such that $G' \subset Z(G)$, then G is nilpotent.*

Proof. Let $|G| = p_1^{m_1} \cdots p_n^{m_n}$ be the prime decomposition of the order of G. Suppose for each p_i that a Sylow p_i-subgroup P_i of order $p_i^{n_i}$ is actually the only Sylow p_i-subgroup in G. Then two different Sylow subgroups have trivial intersection and the product of all Sylow subgroups of G, $P_1 \cdots P_n$, is a subgroup with order

$$|P_1 \cdots P_n| = |P_1| \cdots |P_n| = |G| \implies P_1 \cdots P_n = G$$

and G is nilpotent.
Thus all that is left to show is that if P is a Sylow p-subgroup then it is the unique Sylow p-subgroup of G. From the Sylow theorems, this is implied if P is normal in G, since all Sylow p-subgroup are conjugate with each other. The normalizer $\mathrm{N}_G(P) = N$ is the largest subgroup of G that contains P as a normal subgroup. Our claim is that $N = G$.
By the Sylow theorems, P is the unique Sylow p-subgroup of N. Now if $g \in \mathrm{N}_G(N)$ so that $gNg^{-1} = N$ then $gPg^{-1} = P$ and $g \in N$. Therefore the normalizer of N in G is N itself.
If $g \in Z(G)$ then of course $gNg^{-1} = N \implies g \in \mathrm{N}_G(N) = N$ and by the hypothesis we have the following

$$G' \subset Z(G) \subset N \subset G.$$

The quotient group N/G' is a subgroup of the abelian G/G' which implies that in fact

$$N/G' \triangleleft G/G' \implies N \triangleleft G.$$

Thus, the normalizer of N is G, but $G = \mathrm{N}_G(N) = N$. $\square$

6 *M*-groups

In this section we will define what an M-group is. It is defined using the induced representation, but is actually closely related to the theory of solvable groups. Indeed, we will here show that the set of M-groups contains all supersolvable (and nilpotent) groups and is contained in the set of solvable groups.

It follows almost directly from the definitions that subgroups of supersolvable and solvable groups are supersolvable and solvable, respectively. Interestingly, this is not a feature that is shared with the M-groups. That is, there exists a subgroup of an M-group that is not itself an M-group. This example will be the main topic in the following chapter.

Now for the main definition and some important consequences.

Definition 6.1. Let G be a group with a representation (θ, V). The representation (θ, V) is called **monomial** if θ is a direct sum of induced representations,

$$\theta = \bigoplus \psi_i^G,$$

where each ψ_i is a one-dimensional representation of some subgroup H_i of G. Furthermore, a group G is called an ***M*-group** if every irreducible representation of G is monomial.

Remark. A one-dimensional representation is trivially monomial. Hence by Theorem 3.3.3, abelian groups are M-groups.

A monomial representation has a simple interpretation in its matrix form, when a suitable basis is defined. Each induced ψ_i^G has exactly one non-zero entry in each column and row, which means that the direct sum shares this property. Such a matrix is called a *monomial matrix*. A special case of monomial matrices are the permutation matrices, where each such non-zero entry equals 1. Given an monomial matrix (and representation) we can set all such non-zero entries to 1 and get a permutation matrix (and a so called *permutation representation*). We will later on in this chapter return to the permutation matrix and use some of its properties for further understanding of the monomial representation.

Definition 6.2. Let $\mathcal{M}$ be the set of groups such that if $G \in M$ then

i. All subgroups and homomorphic images of G are in $\mathcal{M}$

ii. If G is not abelian then there exists an abelian normal subgroup A of G such that $A \subsetneq Z(G)$.

Our first goal of this chapter is showing that $\mathcal{M}$ is actually a subset of the set of M-groups.

We start with a technical Lemma, which the reader might find obvious. In a way it connects the lifted representation introduced in Chapter 3 with the induced representation.

Lemma 6.1. *Suppose that A is a normal subgroup of some group G with a representation (θ, V) of G, such that $A \subset \ker\theta$. Suppose further that the representation $(\underset{\sim}{\theta}, V)$ of the quotient group G/A is monomial, then the representation (θ, V) is also monomial.*

Proof. Let K denote the quotient group G/A and let T be a set of left-coset representatives of A in G. By hypothesis, $\underset{\sim}{\theta} = \bigoplus \psi_i^K$ for some one-dimensional representations ψ_i of some subgroups of K. By the correspondence theorem, let the subgroup of ψ_i be H_i/A where H_i is some subgroup G containing A. Let i be fixed, and denote for simplicity $\psi = \psi_i$ and $H = H_i$.

We have that

$$\begin{aligned}\psi^K(gA) &= \frac{1}{|G:A|}\sum_{x\in K}\psi^o(x^{-1}(gA)x)\\ &= \frac{|A|}{|G|}\sum_{t\in T}\psi^o(t^{-1}gtA)\\ &= \frac{1}{|G|}\sum_{x\in G}\psi^o(x^{-1}gxA).\end{aligned}$$

Now lifting it up yields

$$\widetilde{(\psi^K)}(g) = \frac{1}{|G|}\sum_{x\in G}\widetilde{\psi^o}(x^{-1}gx) = \tilde{\psi}^G(g)$$

and so the result follows, since $\tilde{\psi_i}$ is a one-dimensional representation of subgroup H_i of G and

$$\theta = \bigoplus \widetilde{\psi_i^K} = \bigoplus \tilde{\psi_i}^G.$$

□

Lemma 6.2. *Suppose that A is a normal subgroup of some group G with an irreducible representation (θ, V) of G. Consider the canonical decomposition of V regarded as an A-module*

$$V = W_1 \oplus \cdots \oplus W_m$$

then the following holds:

i. G acts transitively on $\{W_i\}$ by the action $g \mapsto \theta_g W_i$.

ii. If H is defined as the subset of G that fixes W_1, then H is a subgroup of G and W_1 is an irreducible H-module with respect to some representation (ψ, W_1) of H. Furthermore, $\psi^G = \theta$.

Proof.

i. Let V_i be an A-module representative of the A-ismorphism class of irreducible A-modules which is included in the direct sum of W_i. Meaning W_i is A-isomorphic to a direct sum of a number of V_i's. Consider the sub-A-module

$$U_i = \bigoplus\{\theta_x V_1 \mid x \in G,\ \theta_x V_1 \cong_A V_i\}.$$

Clearly $U_i \subset W_i$. Since V is irreducible as a G-module, we have that $V = \bigoplus U_i$, which implies that $U_i = W_i$.

Now we show if $\theta_x V_1 \cong_A \theta_y V_1$ for some x and y in G, then so is $\theta_{gx} V_1 \cong_A \theta_{gy} V_1$ for any g in G. If $f : \theta_x V_1 \to \theta_y V_1$ is an A-isomorphism then $\theta_g f \theta_{g^{-1}} : \theta_{gx} V_1 \to \theta_{gy} V_1$ is also an A-isomorphism. In fact, if $a \in A$ then for any $v_1 \in V_1$

$$\begin{aligned}\theta_g f \theta_{g^{-1}}(\theta_a \theta_{gx} v_1) &= \theta_g f \theta_{g^{-1}ag}\theta_x v_1 = \theta_g \theta_{g^{-1}ag} f \theta_x v_1\\ &= \theta_a \theta_g f \theta_x v_1 = \theta_a(\theta_g f \theta_{g^{-1}})(\theta_{gx} v_1)\end{aligned}$$

where the third equality uses the normality of A. So if for some i, we have $\theta_x V_1 \cong_A \theta_y V_1 \cong_A V_i$ then $\theta_{gx} V_1 \cong_A \theta_{gy} V_1 \cong_A V_j$ for some j. Which implies that $\theta_g W_i \subset W_j$. If we can show equality, then the irreducibility of G would imply that the action is transitive.

If $\theta_g W_i \subset W_j$ then $\dim W_i \leq \dim W_j$ and $W_i \subset \theta_{g^{-1}} W_j \subset W_k$ for some k, by the same argument as above. Repeating this process for the finite $\{W_i\}$ eventually yields $\dim W_j \leq \dim W_i$ and so $\theta_g W_i = W_j$.

ii. If h_1 and h_2 are elements of H, then $\theta_{h_1 h_2} W_1 = \theta_{h_1} W_1 = W_1$ so $h_1 h_2 \in H$ and H is a subgroup of G. By i. we can choose a set $T = \{t_i\}$ such that $\theta_{t_i} W_1 = W_i$. Now

$$t_i H = t_j H \iff \theta_{t_i t_j^{-1}} W_1 = W_1 \iff t_i = t_j$$

and hence T is a set of left-coset representatives of H in G. If W_1 were not irreducible as an H-module, then the induced representation of this proper sub-H-module would yield a proper representation of G, which contradicts the irreducibility of G. Hence, W_1 is an irreducible H-module with respect to the representation $\psi = \theta|_H$ and

$$\bigoplus_{t_i \in T} \theta_{t_i} W_1 = \bigoplus_i W_i = V \implies \psi^G = \theta.$$

□

Now we are ready for the first main theorem of this chapter.

Theorem 6.1. *If $G \in \mathcal{M}$ then G is an M-group.*

Proof. We proceed by proving with induction over the order of G.

If $|G| = 1$, then G is trivially an M-group.

Suppose now that the statement holds for all groups of order less than $|G|$. Let (θ, V) be an irreducible representation of G. If θ is not faithful, then the quotient group $G/\ker\theta$ is of smaller order than G, and by our induction hypothesis the representation $(\underset{\sim}{\theta}, V)$ is monomial. Lemma 6.1 implies that (θ, V) is also monomial and we are done. Suppose from now on that θ is faithful.

If G is abelian, then (θ, V) is again monomial, so we suppose further that G is not abelian. By the definition of $\mathcal{M}$, there exists an abelian subgroup A of G that is not contained in the center of G. If $\dim V = n$ and V is regarded as an A-module, then following the notation as in Lemma 6.2 and its proof, we can decompose

$$V = \bigoplus_{i=1}^{m} W_i = \bigoplus_{i=1}^{n} V_i,$$

with the first decomposition being the canonical decomposition and the second an irreducible decomposition into one-dimensional sub-A-modules. We claim that $m > 1$ which is equivalent to that not all $\{V_i\}_{i=1}^n$ are A-isomorphic. Suppose, in hope of a contradiction, that they are all A-isomorphic.

Set $\phi_i : V_1 \to V_i$ to be an A-isomorphism. Let $V_i = \mathbb{C} v_i$ for some $v_i \in V$ and take $a \in A$. Set

$$\theta_a v_1 = \lambda v_1,$$
$$\phi_i v_1 = \gamma_i v_i$$

for some $\lambda, \gamma_i \in \mathbb{C} - 0$. But now we get

$$\begin{aligned}\theta_a v_i &= \theta_a \gamma_i^{-1} \gamma_i v_i = \theta_a \gamma_i^{-1} \phi_i v_1 \\ &= \gamma_i^{-1} \theta_a \phi_i v_1 = \gamma_i^{-1} \phi_i \theta_a v_1 \\ &= \gamma_i^{-1} \phi_1 \lambda v_1 = \gamma_i^{-1} \lambda \gamma_i v_i = \lambda v_i,\end{aligned}$$

and so for any $v \in V$ we have $\theta_a v = \lambda v$. This implies that for all $x \in G$ we have that

$$\theta_{x^{-1}ax} = \theta_{x^{-1}} \lambda \theta_x = \lambda = \theta_a.$$

Recall that θ was assumed to be faithful, and so $x^{-1}ax = a$. Since this holds for all x in G, $a \in Z(G) \implies A \subset Z(G)$, which is the sought contradiction. Now with $m > 1$, the subgroup H defined in Lemma 6.2 is a proper subgroup of G and W_1 is an irreducible H-module with representation ψ such that $\psi^G = \theta$. Also, since H is a proper subgroup, our induction hypothesis gives that ψ is monomial, and

$$\begin{aligned}\psi &= \bigoplus \psi_i^H \\ \implies \theta = \psi^G &= (\bigoplus \psi_i^H)^G \\ &= \bigoplus (\psi_i^H)^G = \bigoplus \psi_i^G.\end{aligned}$$

where the second to last equality follows from Lemma 4.4.1 and the last equality is the transitivity of the induction map (Lemma 4.4.4). Hence, the irreducible representation (θ, V) is monomial and G is an M-group. □

We can now prove our first inclusion of M-groups into the theory of solvable groups.

Corollary 6.1. *If a group G is supersolvable, then G is an M-group.*

Proof. We show that the set of all supersolvable groups G is included in $\mathcal{M}$ and so the result follows from Theorem 6.1. Clearly, subgroups and homomorphic images of a supersolvable group is also supersolvable. It remains to show that if G is not abelian, then there exists an abelian normal subgroup A that is not contained in the center of G. From the supernormal series

$$1 = N_0 \triangleleft \cdots \triangleleft N_n = G$$

we can choose some $G_i \subset Z(G)$ and $G_{i+1} \subsetneq Z(G)$. Indeed, the center is properly contained in G. The quotient group G_{i+1}/G_i is of prime order, hence a cyclic group. It is now a standard result that $A = G_{i+1}$ is indeed an abelian group. For the doubting reader, the following argument is included.

Since $G_i \subset Z(G) \implies G_i \subset Z(G_{i+1})$ we take $x \in G_{i+1} - G_i$ and show that $x \in Z(G_{i+1})$. The coset xG_i generates G_{i+1}/G_i and so for each $y \in G_{i+1}$ we can write $y = x^k g$ for some $g \in G_i$ and

$$yx = x^k gx = x^{k+1} g = xy.$$

Therefore $Z(G_{i+1}) = G_{i+1}$ and G_{i+1} is abelian. □

From Lemma 5.4 and Lemma 5.5, if the derived normal subgroup G' is contained in the center of G, then G is an M-group. Note that in this case, both G' and the quotient G/G' are both abelian. This property can be generalised.

Theorem 6.2. *If G has a normal subgroup N such that both N and G/N are abelian, then G is an M-group.*

We need the following Lemma.

Lemma 6.3. *If G has a normal subgroup N such that N is solvable, G/N is supersolvable and all Sylow subgroups of N are abelian, then G is an M-group.*

Proof of Theorem 6.2. For an abelian group the center equals the group itself (also the derived subgroup is trivial), so by Lemma 5.5 the group is nilpotent. Therefore, both N and G/N are nilpotent, supersolvable and solvable, with subgroups that are abelian, in particular the Sylow subgroups. By Lemma 6.3 we are done. $\square$

In the following proof, we are rewarded for the results proven in Chapter 5.

Proof of Lemma 6.3. Again we show that the set of all groups that satisfy the hypothesis is a subset of $\mathcal{M}$. Properties such as a subgroup begin normal and solvable, quotient groups being supersolvable and Sylow subgroups being abelian, are purely group theoretical properties, and are preserved through homomorphisms. Furthermore, they are preserved by taking subgroups. Showing that G satisfies the second condition of the definition of $\mathcal{M}$ remains. Therefore, suppose that G is not abelian. If N itself is abelian, then if $N \subset Z(G)$ we can proceed exactly as in the proof of Corollary 6.1, with 1 replaced with N, using the supernormal series given from G/N. In that case we can find some normal abelian subgroup A that is not contained in the center of G. If $N \subsetneq Z(G)$ then $A = N$ will do.

If N is not abelian, let A be the maximal normal subgroup of G contained in N. We claim that $A \subsetneq Z(G)$, for otherwise we contradict the maximality of A. In fact, let $A \subset Z(G) \implies A \subset Z(N)$ and consider the quotient group N/A. Let the subgroup H/A (by the correspondence theorem), such that $A/A \subset H/A \subset N/A$, and H/A is the minimal nontrivial normal subgroup of G/A. Our goal is to show that H is also abelian, which contradicts the maximality of A.

Since G/A is solvable, Lemma 5.2 implies that the derived

$$\begin{aligned} &(H/A)' \neq H/A \\ \implies &(H/A)' = A/A, \end{aligned}$$

by the minimality of H, which gives that H/A is abelian. But this can only happen if the derived subgroup H' is contained in $A \subset Z(H)$. By Lemma 5.5, H is nilpotent and can be written as a direct product of its Sylow subgroups contained in N, which are by hypothesis abelian. Therefore, H is abelian and we have our contradiction. Thus, in all cases we can find some abelian normal A of G satisfying the second property of $\mathcal{M}$. $\square$

The third and last main result of this chapter will be showing that every M-group is solvable. First we need the notion of a permutation representation and a basic fact concerning such representations. We begin with a definition.

Definition 6.3. Let G be a group with a representation (θ, V). We call this representation a **permutation representation of degree *n*** if $\dim V = n$ and θ_g is associated with a permutation matrix for every $g \in G$, in a suitable basis for V.

Remark. The regular representation of a group G is a permutation representation of degree $|G|$.

Remark. It is evident that a permutation representation of degree n describes a group homomorphism from G into the symmetric group on n letters. Conversely, every such homomorphism gives a natural permutation representation after identifying $\{1,\ldots,n\}$ with the basis $\{e_1,\ldots,e_n\}$ of a vector space of dimension n. The character of such a natural permutation representation will in the following Lemma be described as a *permutation character* of G.

Lemma 6.4. *Let G be a permutation group on $\Omega = \{1,\ldots,n\}$ with permutation character χ. Let t equal the number of orbits of the G-set Ω. Then $\langle \chi, \chi_1 \rangle = t$, where χ_1 is the principal character of G.*

Proof. The Lemma follows directly from Burnside's Lemma, since $\chi(g)$ equals the number of fixed points of g with respect to the action of G on Ω. Indeed,

$$\langle \chi, \chi_1 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) = t.$$

□

The following Lemma and its proof will be used in the upcoming Theorem.

Lemma 6.5. *If a group G has a unique minimal normal subgroup N, then there exists a faithful irreducible representation of G.*

Proof. Suppose that $N \subset \ker\theta$ for all irreducible representations θ of G, then by Theorem 3.3.6, $N = 1$, contradicting the non-triviality of N. Therefore, there exists some irreducible θ such that for some $x \in N - 1$, we have $\theta_x \neq 1$. Then the following holds:

$$\begin{aligned} & N \subsetneq \ker\theta \\ \implies & N \cap \ker\theta \neq N \\ \implies & N \cap \ker\theta = 1 \\ \implies & \ker\theta = 1 \end{aligned}$$

where the second implication follows from the minimality of N and the third implication follows from the uniqueness of N. Hence, a faithful irreducible representation of G must exist. □

Theorem 6.3. *If G is an M-group, then G is solvable.*

Proof. We proceed with induction over $|G|$. The case $|G| = 1$ is trivial. Now suppose the theorem holds for all groups of order less than $|G| > 1$. First we can suppose without loss of generality that G has a unique minimal normal subgroup. If G had two distinct minimal normal subgroups N_1 and N_2, then G/N_1 and G/N_2 would be M-groups with order strictly less than the order of G. By our induction hypothesis, the quotient groups are solvable and so the direct product $G/N_1 \times G/N_2$ is solvable. The homomorphism $\phi : G \to G/N_1 \times G/N_2$ given by

$$\phi(g) = (gN_1,\ gN_2)$$

has kernel $N_1 \cap N_2$, which is a normal subgroup strictly contained in N_1 and N_2. By the minimality of N_1 and N_2 we must have $\ker\phi = 1$ and G is isomorphic to a solvable group, and hence solvable. Therefore, we now assume that G has a unique minimal normal subgroup N. By Lemma 6.5, there exists some faithful irreducible representation (θ, V) of G of minimal degree. In a suitable basis, the monomial θ has an associated monomial matrix. Change all non-zero entries to 1 and arrive at a permutation representation (θ^p, V) of degree $\dim V$. Let $K = \ker\theta^p$. We first claim that K is abelian. Indeed, $K = \{g \in G \mid \theta_g \text{ is diagonal}\}$ is isomorphic to the abelian matrix group $\{\theta_g \mid \theta_g \text{ is diagonal}\}$ with isomorphism map $g \mapsto \theta_g$. Secondly, we claim that K is non-trivial.

Suppose that K is trivial $\iff$ θ^p is faithful, in hope of a contradiction. By Lemma 6.4, θ^p contains at least one copy of the principal representation. Furthermore, it is reducible, for if θ^p were to equal the principal representation then θ^p would not be faithful. Therefore, θ^p can be decomposed into a direct sum of irreducible components

$$\theta^p = \bigoplus \varphi^i.$$

Take $x \in N - 1$, then there must exist some φ^i such that $\varphi^i_x \neq 1$. In fact, otherwise

$$\theta^p_x = \bigoplus \varphi^i_x = 1$$

which contradicts θ^p being faithful. Following the proof of Lemma 6.5 with θ replaced by φ^i we get that $\ker\varphi^i = 1$ and φ^i being a faithful irreducible representation of G, with $\dim\varphi^i < \dim\theta^p = \dim\theta$ contradicting the minimality of θ.

To summarise, K is abelian and the quotient G/K is an M-group of order less than G. By our induction hypothesis, G/K is solvable. But K is trivially solvable, so by Lemma 5.1, G itself must be solvable.

$\square$

7 Character Tables of Groups of Increasing Size

In this chapter we will work primarily with four different groups, which we will denote as Q_8, G_{24}, P_{32} and G_{96} (with their index denoting their order). We will find the so called ***character tables*** of these groups, which will help us find the irreducible representations. In particular, the faithful irreducible representations are of particular interest. Here we will get a chance of using the theory of the previous chapters. The last group of order 96 will also give an example of an M-group that contains a subgroup that is not an M-group itself.

7.1 A group of order 8

We begin with the group Q_8 with generators c, d and relations

$$c^4 = d^4 = 1, \quad c^2 = d^2, \quad d^{-1}cd = c^{-1}.$$

After identifying

$$\begin{aligned} c &\leftrightarrow i, \\ d &\leftrightarrow j, \\ cd &\leftrightarrow k, \end{aligned}$$

we notice that Q_8 is in fact the quaternion group of order 8, and we can easily verify that $Z(Q_8) = Q_8' = \langle c^2 \rangle \leftrightarrow \{\pm 1\}$. We begin by finding the character table of Q_8, but in order to do so we must first explain what a character table is.

If a group has conjugacy classes $\{\mathrm{Cl}(g_i)\}_{i=1}^r$ and irreducible characters $\{\chi_i\}_{i=1}^r$ we let the character table be a table of the following form:

Character Table

	$\mathrm{Cl}(g_1)$	$\mathrm{Cl}(g_2)$	$\dots$	$\mathrm{Cl}(g_r)$
χ_1	x_{11}	x_{12}	$\dots$	x_{1r}
χ_2	x_{21}	x_{22}	$\dots$	x_{2r}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	x_{r1}	x_{r2}	$\dots$	x_{rr}

where $x_{ij} = \chi_i(g_j)$. We will always order the irreducible characters in increasing order with χ_1 being the principal character (hence $x_{1j} = 1$ for all j). We will also set $\mathrm{Cl}(g_1) = \mathrm{Cl}(1) = 1$ (hence $x_{i1} = \dim \chi_i$ for all i). For extra readability, we will also write out $\mathrm{Cl}(g)_m^n$ where $m = \left|\mathrm{Cl}(g)\right|$ and $n = |G|/m$ (hence n equals the order of the centralizer of g in G). The second most trivial character table is for the group $\mathbb{Z}_2 = \{1, \beta\}$.

Character Table of $\mathbb{Z}_2$

	$\mathrm{Cl}(1)_1^2$	$\mathrm{Cl}(\beta)_1^2$
χ_1	1	1
χ_2	1	−1

Now we return to our group Q_8 and construct the character table. Notice that we are able to do this without knowing any of the irreducible representations of Q_8. First we find the five conjugacy classes of Q_8 and so we know the dimensions of the character table.

Unfinished Character Table of Q_8

	$\mathrm{Cl}(1)_1^8$	$\mathrm{Cl}(c^2)_1^8$	$\mathrm{Cl}(c)_2^4$	$\mathrm{Cl}(d)_2^4$	$\mathrm{Cl}(cd)_2^4$
χ_1	1	1	1	1	1
χ_2	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}
χ_3	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}
χ_4	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}
χ_5	x_{51}	x_{52}	x_{53}	x_{54}	x_{55}

We will move in a snake-like fashion and decide the whole table in five parts, using five different relations. The Degree Equation (Theorem 3.1) gives a condition on the first column

$$\sum_{i=1}^{5} x_{i1}^2 = 8$$

and so $x_{21} = x_{31} = x_{41} = 1$ and $x_{51} = 2$. With all the linear characters found we can use Corollary 3.2 and get

$$\langle c^2 \rangle = \bigcap_{\chi \in \mathrm{Lin}_G} \ker \chi \implies x_{22} = x_{32} = x_{42} = 1.$$

The last unknown of the second column is given by the Second Orthogonality Relation (Theorem 3.2)

$$4 + 2x_{52} = 0 \implies x_{52} = -2.$$

The last row is now given directly from the First Orthogonality Relation (Theorem 2.1)

$$\|\chi_5\|^2 = \frac{4 + 4 + 2|x_{53}|^2 + 2|x_{54}|^2 + 2|x_{55}|^2}{8} = 1 \implies x_{53} = x_{54} = x_{55} = 0.$$

For the last nine entries, consider the three normal subgroups

$$N_2 = \{1,\ c^2,\ c,\ c^3\},$$
$$N_3 = \{1,\ c^2,\ d,\ d^3\},$$
$$N_4 = \{1,\ c^2,\ cd,\ cd^3\},$$

that all yield an isomorphic quotient group. Indeed,

$$G/N_2 \cong G/N_3 \cong G/N_4 \cong \mathbb{Z}_2.$$

Using Theorem 3.5, we can lift the non-principal irreducible representation of $\mathbb{Z}_2$ for G/N_2, G/N_3 and G/N_4, which yields the characters χ_2, χ_3 and χ_4 respectively. The character table of Q_8 is now complete.

Character Table of Q_8

	$\mathrm{Cl}(1)_1^8$	$\mathrm{Cl}(c^2)_1^8$	$\mathrm{Cl}(c)_2^4$	$\mathrm{Cl}(d)_2^4$	$\mathrm{Cl}(cd)_2^4$
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	1	−1	1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

We now seek the irreducible representation of degree 2, which is faithful by the character table (and Theorem 3.4). This representation is probably known to the reader. It can be found directly with the use of the induced representation.

We know from Lemma 5.5 that Q_8 is an M-group, hence χ_5 is monomial. Take $H = \{1,\ c,\ c^2,\ c^3\} \cong \mathbb{Z}_4$ and $T = \{1,\ d\}$ as a set of left-coset representatives of H in Q_8. Let ψ be a representation of H and consider the induced representation ψ^G. It is our hope that the character of ψ^G is χ_5. Take the irreducible representation $\psi_{c^k} = i^k$ and compute relations $c1 = 1c$ and $cd = dc^3$, which yields

$$\psi_c^G = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Similarly $d1 = d1$ and $dd = 1c^2$, gives

$$\psi_d^G = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies \psi_{cd}^G = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

One immediately sees that the character of ψ^G is χ_5 and ψ^G is the irreducible representation of degree 2, of our first group Q_8.

7.2 A group of order 24

Before defining a group G_{24} of order 24 we will do some preparations. By Theorem 3.3, the character table of any cyclic group, such as $\mathbb{Z}_3 = \{1,\ \beta,\ \beta^2\}$, is trivial to construct. Consider the following one.

Character Table of $\mathbb{Z}_3$

	$\mathrm{Cl}(1)_1^3$	$\mathrm{Cl}(\beta)_1^3$	$\mathrm{Cl}(\beta^2)_1^3$
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

$\omega = e^{2\pi i/3}$

Define an automorphism $\beta : Q_8 \to Q_8$ by

$$\begin{gathered} 1 \mapsto 1, \\ c^2 \mapsto c^2, \\ c \mapsto d \mapsto cd \mapsto c, \\ c^3 \mapsto d^3 \mapsto cd^3 \mapsto c^3. \end{gathered}$$

In our identification of Q_8, we recognize β as the renaming $i \mapsto j \mapsto k$. From the definition above, $\beta^3 = 1$ and β is of order 3. Identify $\{1,\ \beta,\ \beta^2\}$ with $\mathbb{Z}_3$ and define our group of order 24 as the outer semidirect product

$$G_{24} = \mathbb{Z}_3 \ltimes_\beta Q_8,$$

with Q_8 isomorphic to a normal subgroup of G_{24}. With this definition at hand, we can write out a set of generators and their relations of G_{24}. Explicitly, G_{24} has generators c, d and β with relations

$$\begin{gathered} c^4 = d^4 = 1, \quad c^2 = d^2, \quad c^d = c^{-1}, \\ \beta^3 = 1, \quad c^\beta = d, \quad d^\beta = cd. \end{gathered}$$

Now $Q_8 \cong \langle c, d\rangle$ and $\langle c, d\rangle \triangleleft G_{24}$. Furthermore, $Q_8' \subset G_{24}' \subset Q_8$ and $[cd,\ \beta] = c,\ [c,\ \beta] = d$ implies that $G_{24}' = Q_8$. The center of G_{24} is also easily computed. If $g \in Z(G_{24})$ with $g = \beta^{-i}x$ for some $x \in Q_8$ we must have

$$x = \beta^i\beta^{-i}x = \beta^{-i}x\beta^i = x^{\beta^i}$$

which can only happen when $i = 0 \implies x \in Z(Q_8) = \langle c^2\rangle$. We verify that c^2 is in the center of G_{24} by computing for any $g = \beta^i x \in G_{24}$

$$g^{-1}c^2g = (\beta^i x)^{-1}c^2\beta^i x = x^{-1}c^2x = c^2.$$

Therefore $Z(G_{24}) = \langle c^2\rangle$. The quotient group $G_{24}/Z(G_{24})$ of order 12 is isomophic to A_4, the alternating group of four letters, by the isomorphism φ given by the mappings (we abuse notation of the left coset $g\langle c^2\rangle = \pm g$)

$$\begin{gathered} \pm 1 \mapsto (),\ \pm c \mapsto (13)(24),\ \pm d \mapsto (14)(23),\ \pm cd = (12)(34) \\ \pm\beta \mapsto (123),\ \pm\beta c \mapsto (243),\ \pm\beta d \mapsto (142),\ \pm\beta cd = (234) \\ \pm\beta^2 \mapsto (132),\ \pm\beta^2 cd \mapsto (234),\ \pm\beta^2 c \mapsto (124),\ \pm\beta^2 d = (243). \end{gathered}$$

Notice that the first row is the Klein four-group H_4 which is normal in A_4. With this in mind, we compute the character table of A_4, which will give us some information of G_{24}. First, A_4 has five conjugacy classes and $A_4/H_4 \cong \mathbb{Z}_3$, which gives three irreducible characters and representations. By the Degree Equation, the fifth irreducible representation χ_5 must be of degree 3. This representation is easily found by the power of the induced representation. Take the subgroup H_4 of A_4 with left-coset representatives $T = \{(), (123), (132)\}$. A non-principal representation ψ of H_4 is for example given by χ_2 in its character table (which is easily produced by three lifted characters from the irreducible non-principal character of $\mathbb{Z}_2$).

Character Table of H_4

	$\mathrm{Cl}(())_1^4$	$\mathrm{Cl}((12)(34))_1^4$	$\mathrm{Cl}((13)(24))_1^4$	$\mathrm{Cl}((14)(23))_1^4$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Since A_4 is generated by the three-cycles (123) and (124) it is enough to determine ψ^G for these values. We compute

$$
\begin{aligned}
(123)() &= (123)(), \ (124)() = (132)(13)(24), \\
(123)(123) &= (132)(), \ (124)(123) = ()(14)(23), \\
(123)(132) &= ()(), \ (124)(132) = (123)(12)(34),
\end{aligned}
$$

which yields the induced representation (which to our knowledge might not be irreducible!)

$$
\psi^G_{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \psi^G_{(124)} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.
$$

The character of ψ^G is now known for three of the four conjugacy classes of A_4. We take the representative $(14)(23)$ out of the last conjugacy class (namely H_4) and compute

$$
\psi^G_{(14)(23)} = \psi^G_{(124)}\psi^G_{(123)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
$$

Now we can use the First Orthogonality Relation and verify that the character of ψ^G is indeed irreducible and equal to χ_4 of A_4.

Character Table of A_4

	$\mathrm{Cl}(1)^{12}_1$	$\mathrm{Cl}((12)(34))^4_3$	$\mathrm{Cl}(123)^4_3$	$\mathrm{Cl}(124)^4_3$
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

$\omega = e^{2\pi i/3}$

Now we turn our attention to computing the character table of G_{24}. First we see that G_{24} has seven conjugacy classes. We list them here for the sake of completion:

$$\mathrm{Cl}(1) = \{1\}, \quad \mathrm{Cl}(c^2) = \{c^2\},$$
$$\mathrm{Cl}(c) = \{c,\ c^3,\ d,\ d^3,\ cd,\ cd^3\},$$
$$\mathrm{Cl}(\beta) = \{\beta,\ \beta c,\ \beta d,\ \beta cd\}, \quad \mathrm{Cl}(\beta c^2) = \{\beta c^2,\ \beta c^3,\ \beta d^3,\ \beta cd^3\},$$
$$\mathrm{Cl}(\beta^2) = \{\beta^2,\ \beta^2 c^3,\ \beta^2 d^3,\ \beta^2 cd^3\}, \quad \mathrm{Cl}(\beta^2 c^2) = \{\beta^2 c^2,\ \beta^2 c,\ \beta^2 d,\ \beta^2 cd\}.$$

With four irreducible characters lifted from A_4, the Degree Equation gives that the remaining three degrees n_1, n_2 and n_3 must satisfy

$$n_1^2 + n_2^2 + n_3^2 = 24 - 12 = 12 \implies n_1 = n_2 = n_3 = 2$$

and so

Unfinished Character Table of G_{24}

	$\mathrm{Cl}(1)^{24}_1$	$\mathrm{Cl}(c^2)^{24}_1$	$\mathrm{Cl}(c)^4_6$	$\mathrm{Cl}(\beta)^6_4$	$\mathrm{Cl}(\beta c^2)^6_4$	$\mathrm{Cl}(\beta^2)^6_4$	$\mathrm{Cl}(\beta^2 c^2)^6_4$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω	ω^2	ω^2
χ_3	1	1	1	ω^2	ω^2	ω	ω
χ_4	2	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}
χ_5	2	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}
χ_6	2	x_{62}	x_{63}	x_{64}	x_{65}	x_{66}	x_{67}
χ_7	3	3	-1	0	0	0	0

$\omega = e^{2\pi i/3}$

We use the Second Orthogonality Relation twice in order to completely determine the second and third column. Firstly, applying the relation on column one and two yields

$$3 + 2(x_{42} + x_{52} + x_{62}) + 9 = 0 \implies x_{42} + x_{52} + x_{62} = -6.$$

Since $c^2 \in Z(G_{24})$, we must have by Lemma 2.1 that x_{42}, x_{52} and x_{62} are all real numbers and that $x_{42} = x_{52} = x_{62} = -2$. Secondly, applying the relation on column three yields (note that $\mathrm{Cl}(c) = Q_8 - \langle c^2 \rangle$)

$$3 + x_{43}^2 + x_{53}^2 + x_{63}^2 + 1 = 4 \implies x_{43} = x_{53} = x_{63} = 0.$$

Consider the similarities between row four, five and six in the character table for G_{24} and row five in the character table of Q_8. Let θ be the irreducible character corresponding to χ_4 of G_{24}, then $\theta|_{Q_8}$ could potentially equal the irreducible representation of degree 2 of Q_8. We make this ansatz. From the relations $\theta_{\beta c} = \theta_{d\beta}$ and $\theta_{\beta d} = \theta_{cd\beta}$ the matrix corresponding to θ_β must be of the form

$$\theta_\beta = a \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}.$$

With the additional constraint that $\theta_{\beta^3} = \theta_1$ we get the three distinct solutions

$$a = \frac{1}{\sqrt{2}} \exp\left(\frac{(3+8k)\pi i}{12} \right),$$

which correspond to the three distinct (up to G-isomorphism) two-dimensional representations of G_{24}. In fact, $k = 0$ gives χ_4, $k = 1$ gives χ_5 and $k = 2$ gives χ_6. We have now found all irreducible representations of G_{24} and have the full character table.

Character Table of G_{24}

	$\mathrm{Cl}(1)_1^{24}$	$\mathrm{Cl}(c^2)_1^{24}$	$\mathrm{Cl}(c)_6^4$	$\mathrm{Cl}(\beta)_4^6$	$\mathrm{Cl}(\beta c^2)_4^6$	$\mathrm{Cl}(\beta^2)_4^6$	$\mathrm{Cl}(\beta^2 c^2)_4^6$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω	ω^2	ω^2
χ_3	1	1	1	ω^2	ω^2	ω	ω
χ_4	2	-2	0	-1	1	-1	1
χ_5	2	-2	0	$-\omega$	ω	$-\omega^2$	ω^2
χ_6	2	-2	0	$-\omega^2$	ω^2	$-\omega$	ω
χ_7	3	3	-1	0	0	0	0

$\omega = e^{2\pi i/3}$

Note that the irreducible representations corresponding to χ_4, χ_5 and χ_6 are faithful. Furthermore, these two-dimensional representations are not monomial. This follows from the fact that G_{24} does not have any subgroup of index 2 (a necessary condition). The argument for this fact is short.

Suppose that H is a subgroup of G_{24} of index 2. If $Z(G_{24}) = \langle c^2 \rangle$ is contained in H, then the quotient $H/\langle c^2 \rangle$ is a subgroup of $G_{24}/\langle c^2 \rangle \cong A_4$ of index 2. But A_4 has no such subgroup. Hence we are left with the possibility that the center of G_{24} is not contained in H. This implies that $Q_8 \cap H = 1 \implies Q_8 \cap c^2 H = \{c^2\}$ which is a clear contradiction, since $H \cup c^2 H = G_{24}$.

We can now deduce that G_{24} is solvable (by Lemma 5.1 with the normal subgroup being Q_8) but is not an M-group.

7.3 A group of order 32

Consider a group P_{32} of order 32 with generators a, b, c, d and relations

$$a^4 = b^4 = c^4 = d^4 = 1, \quad a^2 = b^2 = c^2 = d^2,$$
$$a^b = a^{-1}, \quad c^d = c^{-1},$$
$$ac = ca, \quad ad = da, \quad bc = cb, \quad bd = db.$$

We should think of P_{32} being the product of two copies of Q_8 (called a *central product*). Indeed, let $Q_1 = \langle a,\ b\rangle$ and $Q_2 = \langle c,\ d\rangle$, then it is clear that

$$Q_1 \cong Q_2 \cong Q_8.$$

Furthermore, the third line in the relations above show that all elements of Q_1 commute with all elements of Q_2 and vice versa. Since $Z(Q_1) = Z(Q_2) = \langle c^2\rangle \implies Z(P_{32}) = \langle c^2\rangle$. There are a total number of 17 conjugacy classes of P_{32}. Write $g \in P_{32}$ as $g = g_1 g_2$ for $g_1 \in Q_1$ and $g_2 \in Q_2$. Note that this composition of g is not unique. In any case,

$$g^{-1}xg = g_1^{-1}x_1g_1g_2^{-1}x_2g_2$$

holds for any $x = x_1x_2 \in P_{32}$, again with $x_1 \in Q_1$ and $x_2 \in Q_2$. In this way we can consider the conjugacy classes of P_{32} as direct products of the conjugacy classes of Q_1 with the conjugacy classes of Q_2.

All but one irreducible representations of P_{32} are given by the quotient group

$$P_{32}/\langle c^2\rangle = \langle a\langle c^2\rangle,\ b\langle c^2\rangle,\ c\langle c^2\rangle,\ d\langle c^2\rangle\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^4.$$

Indeed such an isomorphism exists by mapping the general element $a^{n_1}b^{n_2}c^{n_3}d^{n_4}\langle c^2\rangle \in P_{32}/\langle c^2\rangle$ for $n_i \in \{0,\ 1\}$ to $(n_1, n_2, n_3, n_4) \in \mathbb{Z}_2^4$. Since $\mathbb{Z}_2^4$ is an abelian group of order 16, we can lift 16 one-dimensional irreducible representations from the elementary abelian group $\mathbb{Z}_2^4$ up to P_{32}.

As the reader might have expected from the character tables of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and H_4, an abelian group G is isomorphic to its group of linear characters $\mathrm{Lin}_G = \mathrm{Irr}_G$ (with group operation $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$). This fact is not hard to prove and should be intuitively clear from these examples. We trust for the moment this unproven hunch and find four irreducible linear characters $\chi_2, \ldots, \chi_5$ of $\mathbb{Z}_2^4$ which hopefully should generate all 16 irreducible characters of $\mathbb{Z}_2^4$, in the same way that $(1,0,0,0), \ldots, (0,0,0,1)$ generate $\mathbb{Z}_2^4$. Identify $\mathbb{Z}_2$ by the obvious four quotient groups of $\mathbb{Z}_2^4$, lift up the non-principal representation to get $\chi_2, \ldots, \chi_5$ and then define

$$\chi_1 = \chi_2^2$$
$$\chi_6 = \chi_2\chi_3, \quad \chi_7 = \chi_2\chi_4, \quad \chi_8 = \chi_2\chi_5,$$
$$\chi_9 = \chi_3\chi_4, \quad \chi_{10} = \chi_3\chi_5, \quad \chi_{11} = \chi_4\chi_5,$$
$$\chi_{12} = \chi_2\chi_3\chi_4, \quad \chi_{13} = \chi_2\chi_3\chi_5,$$
$$\chi_{14} = \chi_2\chi_4\chi_5, \quad \chi_{15} = \chi_3\chi_4\chi_5$$
$$\chi_{16} = \chi_2\chi_3\chi_4\chi_5$$

It is easy to check that all $\chi_1, \ldots, \chi_{16}$ are irreducible characters of $\mathbb{Z}_2^4$ which we now lift up to P_{32}. By the Degree Equation, the last irreducible character of P_{32}, χ_{17}, is of degree 4 and is fully known by applying the Second Orthogonality Relation a total of 16 times with respect to the first column.

Character Table of P_{32}

	$\mathrm{Cl}(1)_1^{32}$	$\mathrm{Cl}(c^2)_1^{32}$	$\mathrm{Cl}(a)_2^{16}$	$\mathrm{Cl}(b)_2^{16}$	$\mathrm{Cl}(ab)_2^{16}$	$\mathrm{Cl}(c)_2^{16}$	$\mathrm{Cl}(d)_2^{16}$	$\mathrm{Cl}(cd)_2^{16}$	$\mathrm{Cl}(ac)_2^{16}$	$\mathrm{Cl}(ad)_2^{16}$	$\mathrm{Cl}(acd)_2^{16}$	$\mathrm{Cl}(bc)_2^{16}$	$\mathrm{Cl}(bd)_2^{16}$	$\mathrm{Cl}(bcd)_2^{16}$	$\mathrm{Cl}(abc)_2^{16}$	$\mathrm{Cl}(abd)_2^{16}$	$\mathrm{Cl}(abcd)_2^{16}$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	1	−1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1
χ_3	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_4	1	1	1	1	1	−1	1	−1	−1	1	−1	−1	1	−1	−1	1	−1
χ_5	1	1	1	1	1	1	−1	−1	1	−1	−1	1	−1	−1	1	−1	−1
χ_6	1	1	−1	−1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1
χ_7	1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1
χ_8	1	1	−1	1	−1	1	−1	−1	−1	1	1	1	−1	−1	−1	1	1
χ_9	1	1	1	−1	−1	−1	1	−1	−1	1	−1	1	−1	1	1	−1	1
χ_{10}	1	1	1	−1	−1	1	−1	−1	1	−1	−1	−1	1	1	−1	1	1
χ_{11}	1	1	1	1	1	−1	−1	1	−1	−1	1	−1	−1	1	−1	−1	1
χ_{12}	1	1	−1	−1	1	−1	1	−1	1	−1	1	1	−1	1	−1	1	−1
χ_{13}	1	1	−1	−1	1	1	−1	−1	−1	1	1	−1	1	1	1	−1	−1
χ_{14}	1	1	−1	1	−1	−1	−1	1	1	1	−1	−1	−1	1	1	1	−1
χ_{15}	1	1	1	−1	−1	−1	−1	1	−1	−1	1	1	1	−1	1	1	−1
χ_{16}	1	1	−1	−1	1	−1	−1	1	1	1	−1	1	1	−1	−1	−1	1
χ_{17}	4	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

We see that the representation coming from χ_{17} is the only irreducible faithful representation of P_{32}. In order to determine this representation we consider the subgroup $H = \langle b,\ c,\ d\rangle$ of index 2 in P_{32}, with left-coset representatives $T = \{1,\ a\}$. Consider further the unique two-dimensional irreducible representation ψ of $\langle c,\ d\rangle \triangleleft H$. Assume θ is a two-dimensional irreducible representation of H (which we can show exists with methods used before), that extends ψ to H, that is $\theta|_{Q_2} = \psi$. Then $\theta_b\psi_g = \psi_g\theta_b$ for all $g \in Q_2$ and $\theta_b = \omega 1$ for some $\omega \in \mathbb{C}$, by Schur's Lemma (Theorem 1.2). Furthermore, $\theta_{b^4} = 1$ and so in hope of interesting consequences choose $\omega = i$ (in fact, choosing $\omega = -1$ gives a reducible induced representation later on). Now

$$\theta_b = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad \theta_c = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \theta_d = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \theta_{cd} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},$$
$$\theta_{bc} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \theta_{bd} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \theta_{bcd} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

from which we can verify that θ is in fact irreducible which we lift up to P_{32} to some four-dimensional representation $\theta^{P_{32}}$. Once again, for the sake of completion, we list some of the matrices of $\theta^{P_{32}}$, which we for simplicity will rename as $\theta = \theta^{P_{32}}$.

$$\theta_a = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \theta_b = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad \theta_c = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad \theta_d = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\theta_{ab} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad \theta_{ac} = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \quad \theta_{ad} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$
$$\theta_{bc} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \theta_{bd} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \quad \theta_{cd} = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix},$$
$$\theta_{abc} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \theta_{abd} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \theta_{acd} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$
$$\theta_{bcd} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \theta_{abcd} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

One can now verify that θ is irreducible and equals the unique faithful irreducible representation of P_{32}. Thus, the character table of P_{32} is complete and all irreducible representations of P_{32} are known.

7.4 A group of order 96

It is time to move on to the largest group we will take under consideration in this paper, a group G_{96} of order 96. It will be created from P_{32} much the same way that G_{24} was created from Q_8. To this end, let β^1 and β^2 be the before mentioned automorphisms of $Q_1,\ Q_2 \triangleleft P_{32}$ respectively. We will extend these automorphisms to an automorphism α of P_{32} of order 3. In other words, let

$$a^\alpha = b, \quad b^\alpha = ab, \quad c^\alpha = d, \quad d^\alpha = cd.$$

Like in Section 7.2 we identify $\{1,\ \alpha,\ \alpha^2\}$ with $\mathbb{Z}_3$ and construct

$$G_{96} = \mathbb{Z}_3 \ltimes_\alpha P_{32}.$$

Since $G_{96}/P_{32} \cong \mathbb{Z}_3$ we can immediately lift up the two linear characters χ_2 and χ_3 to G_{96} from the non-principal characters of $\mathbb{Z}_3$. The next step is inducing the linear characters of P_{32} with left-coset representatives $T = \{1,\ \alpha,\ \alpha^2\}$. Most of these one-dimensional representations of P_{32} induce G_{96}-isomorphic representations of G_{96}. For a more aesthetically pleasing character table, we lift up χ_4, χ_2, χ_7, χ_3 and χ_{12} from P_{32} and receive induced characters χ_4, χ_5, χ_6, χ_7 and χ_8 of G_{96}. We check that all these are irreducible, again with the First Orthogonality Relation. These representations are in the opinion of the author, not very interesting. Consider for example the irreducible representation θ induced from the representation of P_{32} given by χ_2:

$$\theta_a = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \theta_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$
$$\theta_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \theta_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\theta_\alpha = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

If this representation seems familiar, it seems so because it is. Indeed, $\ker\theta = \langle c,\ d\rangle$ and it can be verified that $G_{96}/\ker\theta \cong A_4$ with for example $a\ker\theta$ having order 2 and $\alpha\ker\theta$ having order 3. Similar results hold for the rest of these irreducible three-dimensional representations of G_{96}.

The conjugacy classes of G_{96} are: $\mathrm{Cl}(1), \mathrm{Cl}(c^2)$ of order 1 and make up the center of G_{96}, $\mathrm{Cl}(a)$, $\mathrm{Cl}(c)$, $\mathrm{Cl}(ac)$, $\mathrm{Cl}(ad)$, $\mathrm{Cl}(acd)$ of order 6 which we can identify with P_{32} and $\mathrm{Cl}(\alpha)$, $\mathrm{Cl}(\alpha c^2)$, $\mathrm{Cl}(\alpha^2)$, $\mathrm{Cl}(\alpha^2 c^2)$ of order 16. At first, the Degree Equation really gave us no information, since 96 can be written as a sum of 11 squares in a number of ways. Although now, with eight known irreducible representations of G_{96}, we can deduce that the remaining three irreducible representations must be of degree 4.

We wish to find a one-dimensional representation, which induced will give such an irreducible representation of degree 4 of G_{96}. To that end, consider $A = \langle ac,\ bd\rangle$. Notice that both ac and bd are of order 2 and A is of order 4 $\implies A \cong H_4$. Furthermore, α fixes A:

$$(ac)^\alpha = bd, \quad (bd)^\alpha = abcd, \quad (abcd)^\alpha = ac.$$

Define a subgroup of G_{96} by $H = \langle \alpha,\ c^2,\ ac,\ bd\rangle$ with left-coset representatives $T = \{1,\ c,\ d,\ cd\}$ of H in G_{96}. Since α fixes A we can decompose $H = \langle\alpha\rangle\langle c^2\rangle\langle ac,\ bd\rangle$ and the order of H is 24. It turns out that three one-dimensional representations of H induce the remaining three irreducible representations of degree 4 of G_{96}. In fact, consider the following partial character table of H.

Partial Character Table of H

	$\mathrm{Cl}(1)_1^{24}$	$\mathrm{Cl}(ac)_3^8$	$\mathrm{Cl}(\alpha)_4^6$	$\mathrm{Cl}(\alpha^2)_4^6$	$\mathrm{Cl}(c^2)_1^{24}$	$\mathrm{Cl}(ac^3)_3^8$	$\mathrm{Cl}(\alpha c^2)_4^6$	$\mathrm{Cl}(\alpha^2 c^2)_4^6$
ψ_1	1	1	1	1	-1	-1	-1	-1
ψ_2	1	1	ω	ω^2	-1	-1	$-\omega$	$-\omega^2$
ψ_3	1	1	ω^2	ω	-1	-1	$-\omega^2$	$-\omega$

$\omega = e^{2\pi i/3}$

From the computations

$$\begin{aligned}
&a1 = c(ac^3), \quad ac = cd(bd^3), \quad ad = cd(ac), \quad acd = d(ac^3),\\
&b1 = d(bd^3), \quad bc = cd(bd^3), \quad bd = 1(bd), \quad bcd = c(bd),\\
&c1 = c(1), \quad cc = 1(c^2), \quad cd = cd(1), \quad ccd = d(c^2),\\
&d1 = d(1), \quad dc = cd(c^2), \quad dd = 1(c^2), \quad dcd = c(1),\\
&\alpha 1 = 1(\alpha), \quad \alpha c = cd(\alpha), \quad \alpha d = c(\alpha), \quad \alpha cd = d(\alpha),
\end{aligned}$$

we obtain the induced representation $\theta = \psi_1^{G_{96}}$ with the following matrices coming from the generators of G_{96}:

$$\theta_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \theta_b = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\theta_c = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \theta_d = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\theta_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \theta_{c^2} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Inducing the representations ψ_2 and ψ_3 up to G_{96} differ only from θ in θ_α by a homothety $\omega 1$ and $\omega^2 1$ respectively. It is now straightforward to check that these representations of degree 4 are all irreducible. Moreover, they are' faithful. All 11 irreducible representations of G_{96} are now known and we can fill in the character table.

Character Table of G_{96}

	$\mathrm{Cl}(1)_1^{96}$	$\mathrm{Cl}(c^2)_1^{96}$	$\mathrm{Cl}(a)_6^{16}$	$\mathrm{Cl}(c)_6^{16}$	$\mathrm{Cl}(ac)_6^{16}$	$\mathrm{Cl}(ad)_6^{16}$	$\mathrm{Cl}(acd)_6^{16}$	$\mathrm{Cl}(\alpha)_{16}^{6}$	$\mathrm{Cl}(\alpha c^2)_{16}^{6}$	$\mathrm{Cl}(\alpha^2)_{16}^{6}$	$\mathrm{Cl}(\alpha^2 c^2)_{16}^{6}$
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	ω	ω	ω^2	ω^2
χ_3	1	1	1	1	1	1	1	ω^2	ω^2	ω	ω
χ_4	3	3	3	-1	-1	-1	-1	0	0	0	0
χ_5	3	3	-1	3	-1	-1	-1	0	0	0	0
χ_6	3	3	-1	-1	3	-1	-1	0	0	0	0
χ_7	3	3	-1	-1	-1	3	-1	0	0	0	0
χ_8	3	3	-1	-1	-1	-1	3	0	0	0	0
χ_9	4	-4	0	0	0	0	0	1	-1	1	-1
χ_{10}	4	-4	0	0	0	0	0	ω	$-\omega$	ω^2	$-\omega^2$
χ_{11}	4	-4	0	0	0	0	0	ω^2	$-\omega^2$	ω	$-\omega$

$\omega = e^{2\pi i/3}$

Note that we have shown that all irreducible representations of G_{96} are monomial and so G_{96} is an M-group. Clearly, we can identify the subgroup $\langle \alpha,\ c,\ d \rangle$ with G_{24}, which is, by Section 7.2, not an M-group. As promised in Chapter 6, we have here produced an example of a subgroup of an M-group which is not an M-group itself.

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