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Portfolio Optimization using the Entropic Value-at-Risk

An Investor Preference Approach

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Abstract

It is very important for an investor to choose an accurate and effective risk measure when optimizing a portfolio of different assets. Recently, in addition to the standard risk measures such as variance or Value-at-Risk (VaR), more developed risk measures have emerged and one of them is the entropic Value-at-Risk (EVaR). This paper is testing the hypothesis stated by Ahmadi-Javid and Fallah-Tafti (2019) that entropic Value-at-Risk (EVaR) is the better risk measure to use in the portfolio optimization. To achieve this goal, the EVaR-optimized portfolio is compared to the mean-variance optimized portfolio (MV) for investors with different preferences. These preferences are exhibited through utility functions starting from the traditional utility functions such as the power and the exponential utility function and finishing with more complex functions such as the bilinear and S-shaped utility function. The conducted tests have shown that under different utility functions investors had different preferences for these two portfolios. EVaR optimized portfolio was mostly preferred by investors with the bilinear utility function when the kink has a negative value, which means that more risk averse investors were preferring this portfolio.

Keywords: Mean-variance framework, entropic value-at-risk, utility functions, risk preference, portfolio optimization

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Kevin Kühn & Polina Kuznetsova

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List of Abbreviations

CARA	Constant absolute risk aversion
CE	Certainty Equivalent
CRRA	Constant relative risk aversion
CVaR	Conditional Value-at-Risk
EVaR	Entropic Value-at-Risk
HARA	Hyperbolic absolute risk aversion
MV	Mean-variance approach
$u(w)$	Utility function
VaR	Value-at-Risk

1 Introduction

Due to the constantly arising financial distractions and crisis, the need in finding a perfect method of calculating the risk has significantly increased. Risk management means identifying and assessing the uncertainty of risk factors using appropriate and relevant methods. Alongside, with the growing importance of the correct risk measurement techniques, the necessity of the appropriate asset allocation, from which investors can gain profit, also increased. For many years, the Value-at-Risk has been a popular risk measure to use in forming an investment decision. Even today it is used by the Basel Committee on Banking Supervision as a risk measure to represent the estimation of the market risk in the Fundamental Review of the Trading Book, although in the future Value-at-Risk will be changed to conditional Value-at-Risk. It is an important part of portfolio management and asset allocation industries since it is easy to calculate and easy to interpret. In practice, the Value-at-Risk is a crucial measure of the extent to which a specific portfolio is exposed to the risk inherent in financial markets. Hence, it has also been used in portfolio optimization. Initially, portfolio optimization was introduced by Markowitz (1952) to achieve an optimal portfolio that has low risk and a high return. However, some researches argue about the efficiency of the mean-variance approach since the use of the standard deviation as an appropriate measure of risk implies that investors equally evaluate the probability of negative returns versus positive returns (Cid, Soler and Blanco, 2010).

According to Kahneman and Tversky (1979), investors tend to have loss aversion where they prefer to avoid losses rather than gain equivalent benefits. This theory, as well as advancing the mean-variance optimization technique, pushes the interest in creating new risk measures in order to include the loss aversion in the portfolio optimization approach. In the last few years, the prospect theory increased the demand for new risk measures in addition to VaR and CVaR. Recently, a new form of Value-at-Risk has emerged: The Entropic Value-at-Risk (EVaR). Ahmadi-Javid (2011) claims that this version has better properties that will enhance the portfolio optimization. Ahmadi-Javid and Fallah-Tafti (2019) researched further the topic on portfolio optimization using the EVaR.

Therefore, the overall purpose of this paper is to test the hypothesis stated by Ahmadi-Javid and Fallah-Tafti (2019), where they claim that the EVaR is a superior risk measure to use in the portfolio optimization or asset allocation. The general portfolio optimization approach by Markowitz (1952) is used as a starting point and further is compared to an optimization that incorporates the EVaR as the measure of the risk in portfolio. To be more specific, the aim is to investigate if the created portfolio optimization technique using EVaR is valid and outperforms the standard MV optimized portfolio technique for any kind risk preference of an individual investor. The different utility functions are used in order to exhibit how agent with different attitude towards risk will decide between the EVaR and the MV optimized portfolio. In the paper written by Ahmadi-Javid and Fallah-Tafti (2019), they have created EVaR optimized portfolios for several confidence levels and compared them based on the Sharpe ratio. Taking a different approach in this research paper, the EVaR portfolio is determined for one confidence level and is compared to a MV optimized portfolio in terms of the certainty equivalent. Hence, this research paper aims to widen the view on the readiness of investors with different preferences to pay for the EVaR optimized portfolio and connects the theory with the "real-world" application.

In this research paper, the EVaR optimized portfolio was only compared to the portfolio optimized by the standard mean-variance framework. Hence, the precise conclusion cannot be made whether portfolio with EVaR is the superior risk measure in comparison with other risk measures such as VaR and CVaR. In case of the MV-framework, the portfolio is optimized under the normal distribution assumption and hence, the proposed optimization technique does not account for higher moments such as skewness and kurtosis, whereas in case of the EVaR portfolio, it accounts for all moments. In regard to the data, the Dow Jones Industrial Average Index was chosen since the index is composed of the companies from different industries. The index only contains 30 US companies and therefore, Dow Jones Index does not suit the role of the index of total and global stock market activity.

As it was mentioned above, the EVaR is a relatively new risk measure and it is not explored in detail yet. This means it is not widely used in forming an investment decision, and therefore it is important to research this topic further. Currently, investors use the VaR and CVaR for optimizing the portfolio under the mean-variance Markowitz (1952) framework, but there is a possibility that the portfolio can be optimized more efficiently using the EVaR. Also, due to a potential upcoming financial crisis caused by the coronavirus outbreak in 2020, risk management together with portfolio optimization become crucial topics in modern world society.

After conducting a specific number of tests, this master's thesis shows that an investor with different preferences is important in choosing an EVaR optimized portfolio. Investor with bilinear utility function would prefer this portfolio more than other types of investors. The agent with traditional utility functions such as power and exponential utility functions is rather indifferent between the EVaR and MV optimized portfolios. Whereas the investor with S-shaped utility function, or prospect theory utility function, would prefer EVaR optimized portfolio only at a specific inflection point.

As to the structure of the paper, the Introduction chapter provides valuable information about the aim of the paper and the relevance and importance of the topic. Further, in Chapter 2: Literature Review, the theoretical background is explained to lay out a better understanding of the ground of the thesis. Chapter 3 describes the methods used for optimizing the portfolio under the mean-variance framework and using the EVaR. Chapter 4 elaborates on the data used and provides a view on the data statistics. Chapter 5 provides the empirical results, and Chapter 6 represents the conclusion of the conducted research.

2 Literature Review

2.1 Portfolio Optimization

Portfolio optimization, or “optimal asset allocation” refers to the idea that an optimal portfolio is the portfolio which is mean-variance efficient meaning that the investor shall maximize the expected return on the portfolio while maintaining a level of risk the investor is already bearing (Rasmussen, 2003). While maintaining the balance, the investor should also think about the fact that every asset in the portfolio is unique and thus the weights of each asset in the portfolio shall vary. Diversification is one of the most important concepts in portfolio optimization theory, it is crucial for the investor to invest in the different asset types and classes to reduce the total risk of the portfolio (Rasmussen, 2003).

This section provides the theoretical background on the risk aversion and utility functions definitions. As well the main topics of the research are explained such as portfolio optimization under the mean-variance framework and minimization of the risk, where the risk measure is projected by EVaR.

2.2 Risk Aversion

Firstly, the concept was introduced by Bernoulli (1782 cited in Bell & Fishburn, 2000) in the 18th century. The main idea of his paper is that two people who encounter the same lottery can evaluate it differently due to the differences in their individual psychology. According to his studies, utility of wealth or *u(w)* is a measure of the intensity of a person’s preferences for wealth *w*, which should be estimated using the power of preference without reference to the risk or probability of the outcome (Bell & Fishburn, 2000). Arrow (1964) and Pratt (1965) were the first to notice the crucial role of the first and second derivative of the utility function and measures of risk aversion were firstly introduced in their work. According to Kahneman and Tversky (1979) it is said the agent is considered to be risk averse, when:

$$u \text{ is concave } (u''(x) > 0) \tag{1}$$

The concept, when the agent prefers the certain option over the risky opportunity with the same expected value, is then called risk aversion (Danthine and Donaldson, 2015). The utility function of the risk-averse agent is also assumed to be increasing (Ding, Chen and Zhang, 2009). The agent can also be risk-neutral, where the investor is indifferent between the certain and uncertain alternatives and risk-loving, where the agent would prefer the uncertain rather than the certain outcome (Danthine and Donaldson, 2015).

2.3 Certainty Equivalent

DeMiguel et al. (2009) underlined that there are three performance measures of the newly constructed portfolio: the Sharpe ratio, the certainty equivalent, and a measure of the portfolio turnover. In this paper the comparison between the portfolios is done by the certainty equivalents of the EVaR and mean-variance optimized portfolios under different utility functions.

The concept of certainty equivalence represents the guaranteed amount of money that a person would consider as desirable as a risky asset (Danthine and Donaldson, 2015). The certainty equivalent changes in accordance to the investors risk tolerance/aversion. The difference in certainty equivalents is explained as a certain return on investment corresponding to an increase in utility (for the utility function under consideration) (Hagströmer et al., 2008). For a risk-averse agent, CE is frequently less than the expected value. When comparing the certainty equivalent of the portfolio, the portfolio with bigger certainty equivalent shall be preferred (Ding, Chen and Zhang, 2009).

2.4 Utility functions

In his article, Tobin (1958) proposed that risk-averse agents can be divided in two types: diversifiers, whose utility functions concave upward and plungers, whose utility functions are either linear or convex upward. The assumption, under this theory, is that the investor’s preference is represented either by the quadratic utility function, or by the probability distribution of the returns, which is part of the two-parameter distribution class (Nigro and Glustoff, 1972).

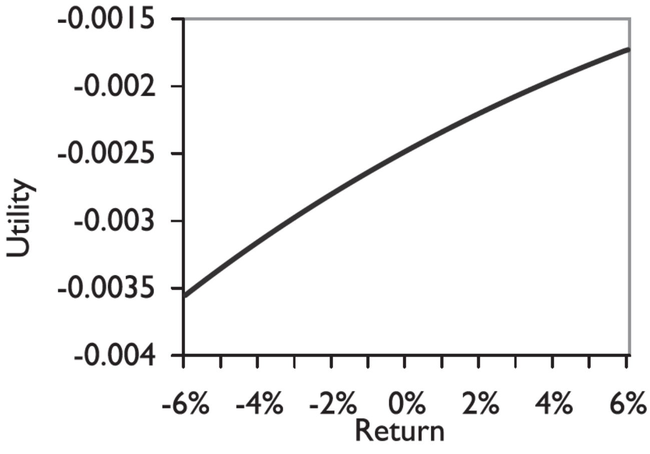
Since quadratic utility is not a realistic description of any investor's attitude to risk, investors’ preferences are frequently represented by more complex utility functions such as bilinear and S-shaped utility functions (Adler and Kritzman, 2007 and Sharpe, 2007). According to

Gourieroux and Monfort (2004), the utility functions are useful in the specification of the efficient portfolios because they can show the preferences of the individual investors and make it possible to compare market portfolio with the efficient batch of the individual portfolios in order to test equilibrium models. Until the beginning of the 21st century, the so-called plungers were never fully considered in research papers and only through researching the bilinear and S-shaped utility functions and hence, in this article, this type of risk-averse investors are taken into account. Additionally, in this article the standard power and exponential utility functions are considered. The utility functions themselves are further going to be represented in the Methodology chapter.

2.4.1 Exponential utility

Following the exponential utility function used in Hagströmer et al. (2008), we look at absolute risk aversion, which is represented in the function through the parameter A and hence, is part of the constant absolute risk aversion (CARA) class (Danthine and Donaldson, 2015). A graphic representation can be seen in Figure 1. Being part of this class refers to the property that the overall amount invested in risky assets is not affected if the level of wealth, or in this case portfolio return, changes (Danthine and Donaldson, 2015). This utility function is also part of the hyperbolic absolute risk aversion (HARA) class (Sharpe, 2007).

Figure 1: Exponential utility function

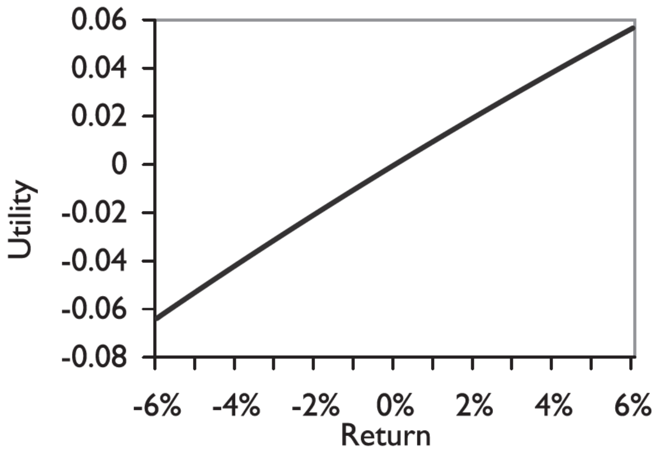


Source: Hagströmer et al. (2008)

2.4.2 Power utility functions

Danthine and Donaldson (2015) state that the power utility function belongs to the class of constant relative risk aversion (CRRA) and is a way to ensure the independence of risk aversion from wealth/portfolio returns. The function always implies a preference for upside deviations and therefore never slopes down (Adler & Kritzman, 2007). The applied power utility function is obtained from the paper by Hagströmer et al. (2008) and it reflects the degree of relative risk aversion in the parameter γ. For the case $\gamma = 1$ a log utility function is used (Hagströmer et al., 2008 and Danthine and Donaldson, 2015). Additionally, the optimal portfolio decision does not depend on wealth and the consumption decisions (Barucci, 2003). An example of the power utility function is shown in Figure 2.

Figure 2: Power utility function



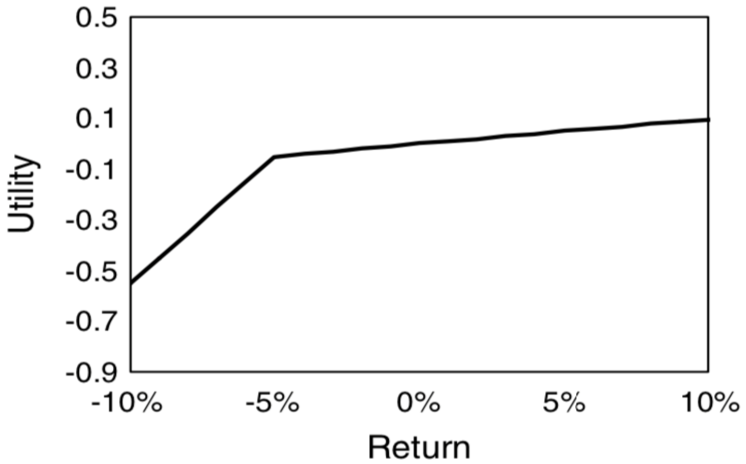
Source: Hagströmer et al. (2008)

2.4.3 Bilinear utility functions

To follow the prospect theory introduced by Kahneman and Tversky (1979), the equations are supposed to preserve the bilinear form that emphasize the expected utility theory. According to prospect theory, investors feel losses more severely than gains (Danthine and Donaldson, 2015). In Figure 3, it is visible that at a reference point the function hit its steepest point and it is where the value function suddenly changes. The location of the reference point, and the method of coding and editing the choice problems emerge as critical factors in the analysis of decisions. In the case of this research, the bilinear utility function proposed by Adler and Kritzman (2007) is used, where the reference point is called kink or critical return level x. The parameter P

exhibits the penalty for the returns lower than the kink. At $P = 1$ the utility function shows neither risk aversion nor loss aversion since there is no kink. The investors, whose behavior is described by this utility function, are worried about penetrating the threshold. It is said the function simply describes the view of the investors on the risk (Adler & Kritzman. 2007). The bilinear function does not indicate risk aversion since the function is linear and the marginal utility is not declining in returns (Hagströmer et al., 2008).

Figure 3: Kinked (bilinear) utility function

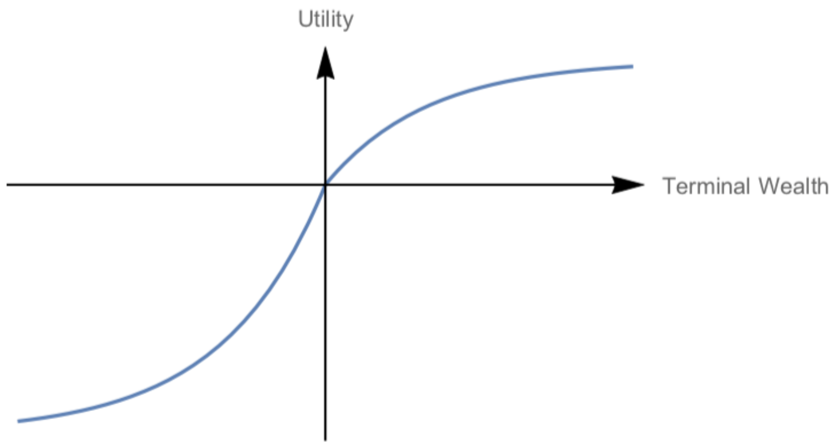


Source: Adler, M. & Kritzman, T. (2007)

2.4.4 S-shaped utility function

Kahneman and Tversky (1979) were the pioneers to propose the idea that the investor's utility function might be S-shaped. Figure 4 visually represents the value function proposed by Kahneman and Tversky (1979). The parameters included in the function are represented by γ_1 and γ_2 as the upside and downside shape parameters (respectively), and A and B as the upside and downside magnitude parameters. When the returns are located below the critical value or inflection point z, the investor is considered to be risk-loving, whereas when it is above the inflection point, the investor is risk averse. The higher the ratio $\frac{A}{B}$ gets, the less risk is chosen for the portfolio. The $\gamma_1$and γ_2 parameters do not have a big influence on the allocation but primarily determine the bend of the S-shape (Hagströmer et al., 2008).

Figure 4: S-shaped utility function.



Source: Armstrong,J. & Brigo, D. (2017)

2.5 Markowitz: Mean-Variance portfolio optimization review

The foundation of any investment decision process is the desire to receive excess returns over the risk-free rate on the investment. The theory developed by Harry Markowitz (1952) forms a ground for the whole asset management industry and is still used widely nowadays. It is considered as one of the first efforts to determine the relationship between the expected return and risk. The general rule is: "the investor does (or should) consider expected return a desirable thing and a variance of the return an undesirable thing" (Markowitz, 1952). To clarify, the investor should maximize the expected return on investment, for a given risk (variance) level, or minimize the risk level while maintaining a given return level.

After the application of the mean-variance portfolio theory, the resulting portfolios given a risk level are almost all diversified. In addition, there are several other assumptions stated by Markowitz (1952):

1) The market is efficient, and all the investors have symmetric information about the market conditions, therefore all investors are considered to be equal.
2) All investors are risk averse and invest only in the risky assets
3) The returns are normally distributed
4) If the investor is willing to reduce the risk (or volatility), the new investment should be added.

5) The investor believes that the more profit he receives from the investment, the higher the risk of the investment and the other way around.
6) The theory implies diversification for a vast range of μ_i and $\sigma_{i,j}$, and the investors do not believe that the adequacy of diversification depends solely on the number or the types of different securities held.
7) Assets in the portfolio are not perfectly correlated.

However, the mean-variance model is criticized that the assumptions are not realistic. In the article by Rice (2017), the author disapproves of the approach and argues that it does not account for the risks that drive the current market. The mean-variance approach assumes the normal distribution of the returns in the portfolio, but financial data usually has fat-tails and hence, is not normally distributed (Stoyanov, S., Rachev, S., Racheva-Iotova, B. and Fabozzi, F.). If investors have the quadratic utility, the mean-variance approach assumes that they are indifferent to other features of the distribution (Adler & Kritzman, 2007).

According to Rasmussen (2003), the portfolio return can be calculated as the weighted sum of the returns on the assets which are composing the portfolio itself. The portfolio return r_p consisting of N assets with portfolio weights w_n and returns r_n can be written as:

$$r_p = \sum_{n=1}^{N} w_n \, r_n \tag{2}$$

Or in a matrix notation:

$$r_p = w^T \, r_n = \begin{bmatrix} w_1 \\ \vdots \\ w_n \\ \vdots \\ w_N \end{bmatrix}^T \begin{bmatrix} r_1 \\ \vdots \\ r_n \\ \vdots \\ r_N \end{bmatrix} \tag{3}$$

The risk of a portfolio, determined by its volatility, highly depends on the exact nature and magnitude of the covariance or correlation between asset returns (Rasmussen, 2003 p.77). It is also possible to reduce the risk level of the total portfolio by choosing suitable assets and their weights, but only if assets in the portfolio are correlated. In other words, when dealing with risk in the portfolio, it is necessary to understand the way in which the assets in the portfolio are related to each other.

As it was mentioned above, the risk of the portfolio is quantified by its volatility or standard deviation. Since this study uses an initial portfolio of N assets, the formula for the variance of N assets is defined as follows:

$$Var_p = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j Cov(r_i, r_j) \quad (4)$$

where the portfolio volatility using the matrix, notation is:

$$\sigma_p = \sqrt{Var_p} = \sqrt{w^T \Sigma w} \quad (5)$$

The formula originates from Markowitz' (1952) article, but for this paper, the equation by Rasmussen (2003) was used. This calculation of risk adds more complexity to the modern portfolio theory approach since it demands the determination of asset-by-asset correlations and the individual volatility of assets. Under the mean-variance framework, Markowitz (1952) has stated that the efficient frontier can be constructed by combining the portfolios with the maximum possible expected return for a given level of risk.

Some researchers still argue whether the optimization approach itself is better than the naïve diversification rule where the share of 1/N of wealth is evenly distributed to each of the N assets available for investment (DeMiguel et al., 2009). Since it does not require any complex mathematical models and does not demand the approximation of the distribution moments; this approach is still used by asset management professionals (DeMiguel et al., 2009 and Benartzi & Thaler, 2001). According to the findings by DeMiguel et al. (2009), the equally-weighted portfolio surprisingly outperforms mean-variance optimization approach. Nevertheless, in this research, the naïve approach is used as the initial portfolio which then is optimized using the EVaR and the MV-framework.

Further, in this paper the focus is on the minimum-variance portfolio where the assets weights generate the lowest possible risk (Rasmussen, 2003). In this case, the mean-variance investor either does not take the expected returns into the account or, equivalently, limits the expected return so that it is the same for all assets (DeMiguel et al., 2009). No additional limitations on the level of the expected return are used.

2.6 Risk measure

2.6.1 VaR (Value-at-Risk)

There is an effective way to quantify risk in every single market. However, each method is closely related to its specific market and cannot be applied directly to other markets. In general, VaR and other risk measures are considered as efforts to create a single measure that sums up the total risk in the portfolio. This risk measure can first be traced to Baumol (1963) but became famous only after JPMorgan created the RiskMetrics in 1996 (Guégan and Hassani, 2019). The goal of the VaR is to provide a probability-based boundary for possible losses over a specified holding period and confidence level (James, 2003). In order to calculate VaR, it is necessary to choose two parameters: time horizon (or holding period) and the confidence level. The later one indicates the probability that we will get a result no worse than the VaR, and can take any value between 0 to 1. In regard to the time horizon, it is the period of time during which we measure profit or loss on our portfolio. Whenever the confidence level is increasing, the VaR estimate changes too (Dowd, 2005).

The concept of VaR is about assessing the potential change in portfolio value within a particular time frame at a particular level of certainty (Rasmussen, 2003). An advantage of the VaR calculation is, that it is easy to understand. However, some scientists argue that VaR is an inadequate measure of risk because some of the VaR assumptions are unrealistic. In addition, VaR is not a coherent risk measure (does not fulfil subadditivity) and therefore, does not always contribute to diversification (Artzner et al., 1999). Further, the tail risk together with illiquidity is not captured by the VaR approach (Guégan and Hassani, 2019).

2.6.2 Coherent Risk measures

According to Artzner et al. (1999) a risk measure fulfils the property of coherence when it satisfies all of the following axioms:

1. Translation invariance: It describes that increasing the amount of it should decrease the risk measure by the same amount which means that:

   $for\ all\ X \in G\ and\ for\ all\ real\ numbers\ \alpha\ \ =>\ \ \rho(X + \alpha \cdot r) = \rho(X) - \alpha$

2. Subadditivity: It describes that the merged risk of two assets should be smaller than the sum of the individual risks that:

   $for\ all\ X_1 and\ X_2 \in G\ \ =>\ \ \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$

3. Positive homogeneity: It describes that while scaling a portfolio, the risk measure is scaling proportionally with the portfolio, so that:

   $for\ all\ \lambda \geq 0\ and\ all\ X \in G\ \ =>\ \ \rho(\lambda X) = \lambda\rho(X)$

4. Monotonicity: It describes that if the loss of one portfolio is smaller that of another portfolio, the risk measure of this portfolio is smaller too, so that:

   $for\ all\ X\ and\ Y \in G\ with\ X \leq Y\ \ =>\ \ \rho(X) \leq \rho(Y)$

If the risk measure does not fulfil all axioms, then it is not a coherent risk measure (Artzner et al., 1999).

2.6.3 Conditional Value-at-Risk (Expected Shortfall)

Since the VaR is seen as unstable, suffers to be not coherent and further does not provide any information beyond the VaR (Rockafellar & Uryasev, 2002) another risk measure is needed. Hence, Rockafellar and Uryasev (2002) suggest the use of the Conditional Value-at-Risk (CVaR) also known as Expected Shortfall (Hull, 2015). They claim that the CVaR has superior properties such as coherence and further considers "fat-tails" and high losses with small probabilities in the calculation.

2.6.4 Entropic Value-at-Risk

The Entropic Value-at-Risk is classified as a coherent risk measure and is described as the tightest upper bound for both VaR and CVaR with the same confidence levels obtained from the Chernoff inequality for the VaR (Ahmadi-Javid, 2012). Chernoff bound, or inequality, was introduced by Herman Chernoff (1952) and it defines the tail of the distribution (Nelson, 1995). Since EVaR originated from the Chernoff bound, it accounts for all moments of the distribution by using the moment-generating function $M_X(z)$ in the equation (Nelson, 1995 and Ahmadi-Javid, 2012). Chernoff bound for any constant a and for a random variable X is as following:

$$\Pr(X \geq a) \leq e^{-za} M_X(z), \ \ \forall z > 0 \tag{6}$$

Further, by solving the equation with respect to a for $\alpha \in (0,1]$:

$$e^{-za} M_X(z) = \alpha \tag{7}$$

where α: Risk level

The following result is obtained:

$$a_X(\alpha, z) \coloneqq z^{-1} \ln(\frac{M_X(z)}{\alpha}) \tag{8}$$

where $\Pr(X \geq a_X(\alpha, z) \leq \alpha)$

By Amir Ahmadi-Javid (2012), EVaR is considered to be a more attractive risk measure in comparison to its prominent competitors VaR and CVaR as it is strongly monotone over its domain and strictly monotone over its sub-domain. One of the most important advantages of EVaR is that it can solve the wide class of stochastic optimization problems since it can be efficiently computed in certain cases where CVaR cannot. At the same confidence levels, EVaR is considered to be a more risk-averse measure than CVaR. As a matter of fact, each coherent risk measure has its dual representation, and the dual representation of EVaR is related to the relative entropy or Kullback-Leiber divergence from where the new risk measure takes its name (Ahmadi-Javid, 2012).

EVaR also belongs to the convex risk measures class and hence, is differentiable (Ahmadi-Javid & Fallah-Tafti, 2019). According to the paper by Fischer, Moser and Pfeuffer (2018), the EVaR is law invariant, but it is not elicitable. Even though it is said that it cannot be backtested due to the non-elicitability property, Acerbi and Skezely (2015) have proven that it is not true and the risk measure does not necessarily need to be elicitable to be backtested. Therefore, backtesting of EVaR represents a potential area for future research. In addition, new concepts in regard to EVaR are constantly introduced: for example, cumulative entropic Value-at-Risk was described in the article by Assa et al. (2016) and belongs to the collective risk models class.

The EVaR mathematical representation, as the best upper bound, of X ∈ $\mathbf{L}_{M+}$ with confidence level $1 - \alpha$ and under the condition of $z > 0$ is:

$$EVaR_{1-\alpha}(X) \coloneqq \inf_{z>0}\{z^{-1}\ln(\frac{M_x(z)}{\alpha})\}, \tag{9}$$

where $\mathbf{L}_{M+}$ is the set of all Borel measurable functions, whose moment-generating function $M_x(z) = E(z^{zX})$ exists for all $z \in \mathbb{R}$ and $\mathbf{L}_{M+}$.

Random variable $X \in \mathbf{L}_{M+}$ represents the losses of a portfolio

The risk measure is also proved to be coherent for all (0,1] (Ahmadi - Javid, 2012). Since the tightest bound may be obtained as a limiting value, "inf" (or infinum) was applied instead of using "min" (Nelson, 1995). The condition of $z > 0$ indicates that it is an upper bound (Chernoff, 1952). Hence, $\text{EVaR}_{1-\alpha}(X)$ is proven to be the tightest upper bound of VaR and CVaR (Ahmadi - Javid, 2012).

Under the normality assumption $X \sim N(\mu, \sigma^2)$, EVaR can be defined in terms of the mean and variance and it is equal to the mean-standard-deviation risk measure for different values of λ.

$$MV_{\lambda}(X) \coloneqq E(X) + \lambda STD(X), \lambda > 0, \tag{10}$$

$$EVaR_{1-\alpha}(X) = \mu + \sqrt{-2\ln\alpha}\,\sigma, \tag{11}$$

In the portfolio optimization approach of Ahmadi-Javid and Fallah-Tafti (2019), they have introduced the primal-dual interior point algorithm to solve the optimization problem with EVaR. Since this article is written under the Markowitz framework (1952), it is assumed that the returns are normally distributed.

3 Methodology

3.1 Core Assumptions

For the analysis, the normality approach is used and hence, the normal distribution is assumed, which allows for the above definitions of the Entropic Value-at-Risk. This assumption relies on the central limit theorem in statistics, which states that the distribution of the mean tends to the normal distribution when the number of observations tends to infinity (Brooks, 2014).

The FRTB by the Basel Committee on Banking Supervision (2019) suggests that in the calculation of the Conditional Value-at-Risk (Expected Shortfall) should be calculated by using an α of 2.5% which then results in a confidence level of 97.5%. Hence, since the Entropic Value-at-Risk is a coherent risk measure as well, the same approach has been used.

For the different utility functions, the parameters vary between the values stated in Table 1 which we obtained from Hagströmer et al. (2008):

Table 1: Parameters for the utility functions

Utility function	*Parameter values*
Exponential	A: $0.5 \leq A \leq 6$
Power	γ: $1 \leq \gamma \leq 5$
Bilinear	x: $-0.04 \leq x \leq 0.005$
	P: $1 \leq P \leq 10$
S-shaped	z: $-0.05 \leq z \leq 0.00$
	γ_1: $0.05 \leq \gamma_1 \leq 0.5$
	γ_2: $0.5 \leq \gamma_2 \leq 0.95$
	A: $1.5 \leq A \leq 2.9$
	B: $1.5 \leq B \leq 0.1$

3.2 Optimization

In the applied optimization technique, it is assumed that the obtained weights can only be positive which translates into the restriction that the investor does not allow for short selling. Further, the restriction that the sum of the weights shall be equal to one is determined, which means that the investor always invests the full amount of money and does not hold a cash position in the portfolio.

In the first step, the EVaR portfolio is determined. For the optimization problem under a sample-based setting, a convex objection function is used. To result in this function, the z in the definition function above is replaced with the parameter t^{-1}, which convexifies the optimization problem. The proof of the convexity of the function was achieved by Ahmadi-Javid and Fallah-Tafti (2019) and can be found in their paper. Hence the objective function for this optimization is as follows:

$$\min_{w} t \ln\left(\sum_{m=1}^{N} p_m e^{t^{-1}(r_m w')}\right) - t \ln(\alpha) \tag{12}$$

$$\text{s.t. } sum(w) = 1$$
$$0.001 \leq w_i \leq 1$$

where: w': Asset weights transposed (nx1 vector)

r_m: Return matrix (nxm vector)

$r_m w'$: Portfolio return at point m

p_m: Probability

α: Risk level

Minimizing the EVaR of the portfolio results in the optimal weights which are then applied to calculate the expected portfolio return.

$$r_{EVaR} = r_p = \bar{r}_i w_{EVaR} \tag{13}$$

where: $\bar{r}_i$: Average return for each asset

w_{EVaR}: Weights obtained from the EVaR optimization

In step 2, the mean-variance portfolio is determined. In this optimization, the variance is minimized while the restriction, where the expected return in the MV optimizations should be equal to the expected return calculated in step 1, is applied.

$$\min_{w}(w\Omega w') \tag{14}$$
$$\text{s.t. } sum(w) = 1$$
$$0.001 \leq w_i \leq 1$$
$$r_p = w\bar{r}_i{}'$$

Where: w: Asset weights (1xn Vector)
w': Asset weights transposed (nx1 Vector)
Ω: Variance-Covariance Matrix
$\bar{r}_i{}'$: Average returns for each asset transposed
r_p: Expected portfolio return

The obtained weights are summarized in Table A 1 in the Appendix. Using these weights, two portfolios are obtained on which the four chosen utility functions are applied. The different utility functions determined by Hagströmer et al. (2008), are shown in Table 2.

Hence, values for each $u(r_p)$ are obtained which are then used to calculate the certainty equivalent for each parameter and each portfolio. Finally, the difference of the certainty equivalents between the portfolios is determined for each parameter as follows:

$$\Delta CE = r_{CE}^{EVaR} - r_{CE}^{MV} \tag{15}$$

Table 2: Utility functions and Certainty Equivalent

Utility function		*Certainty Equivalent*
Exponential		
$u(r_p) = -\exp\,[-A(1 + r_p)]$		$r_{CE} = -\frac{1}{A}\ln(-\bar{u}) - 1$
Power		
$u(r_p) = \frac{(1+r_p)^{1-\gamma}-1}{1-\gamma}$	for $\gamma > 0$	$r_{CE} = [1 + (1-\gamma)\bar{u}]^{1/(1-\gamma)} - 1$
$u(r_p) = \ln\,(1 + r_p)$	for $\gamma = 1$	$r_{CE} = \exp(\bar{u}) - 1$
Bilinear		
$u(r_p) = \ln(1 + r_p)$	for $r_p \geq x$	$r_{CE} = \exp(\bar{u}) - 1$
$u(r_p) = P(r_p - x) + \ln(1 + x)$	for $r_p < x, P > 0$	(see notes)
S-shaped		
$u(r_p) = -A(z - r_p)^{\gamma_1}$	for $r_p \leq z$	$r_{CE} = z - \left(\frac{\bar{u}}{-A}\right)^{1/\gamma_1}$
$u(r_p) = +B(r_p - z)^{\gamma_2}$	for $r_p > z$	$r_{CE} = z + \left(\frac{\bar{u}}{+B}\right)^{1/\gamma_2}$

Notes: To calculate the CE under the bilinear utility, there is no explicit solution for the case when $\bar{u} < u(x)$. Hence, the CE is determined by a standard search algorithm (Hagströmer et al., 2008)

4 Data

For the analysis prices of 28 assets of the Dow Jones 30 Industrial are obtained from Compustat via the Wharton Research Data Services (2020). The index consists of the 30 so called blue-chip companies which are considered to be the leaders in their respective industries (Bloomberg, 2020). The requirement for the determination of the initial portfolio in this analysis is that the individual asset has monthly prices over the ten-year period from 01/2010-12/2019. Further, rebalancing and/or exchange of assets over the ten-year period is not considered in this research. Hence the input of our calculations are 120 prices resulting in 119 monthly returns for each asset. The prices are all denoted in US Dollar ($). The chosen companies and their determined weights are listed in Table A 1 in the annex.

The performance of the initial portfolio with equally-weighted assets can be seen as the green line in Figure 5. One can observe that the portfolio has an upward trend with a strong decrease from the 53rd to the 54th observation which is the start of a decreasing trend until the 74th observation. This is followed by another upwards trend until the end of the observed period. An average monthly return of 0.76% and a standard deviation of 3.52% has been calculated over the ten-year period (see Table 3).

Figure 5: Portfolio performance

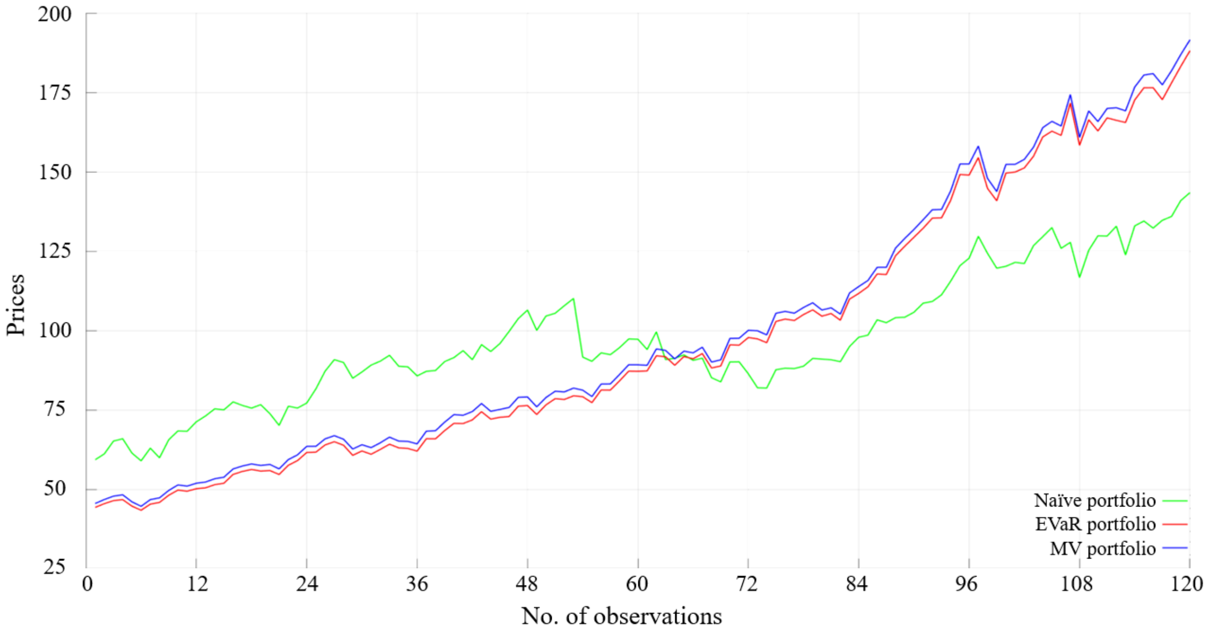


Table 3: Data properties

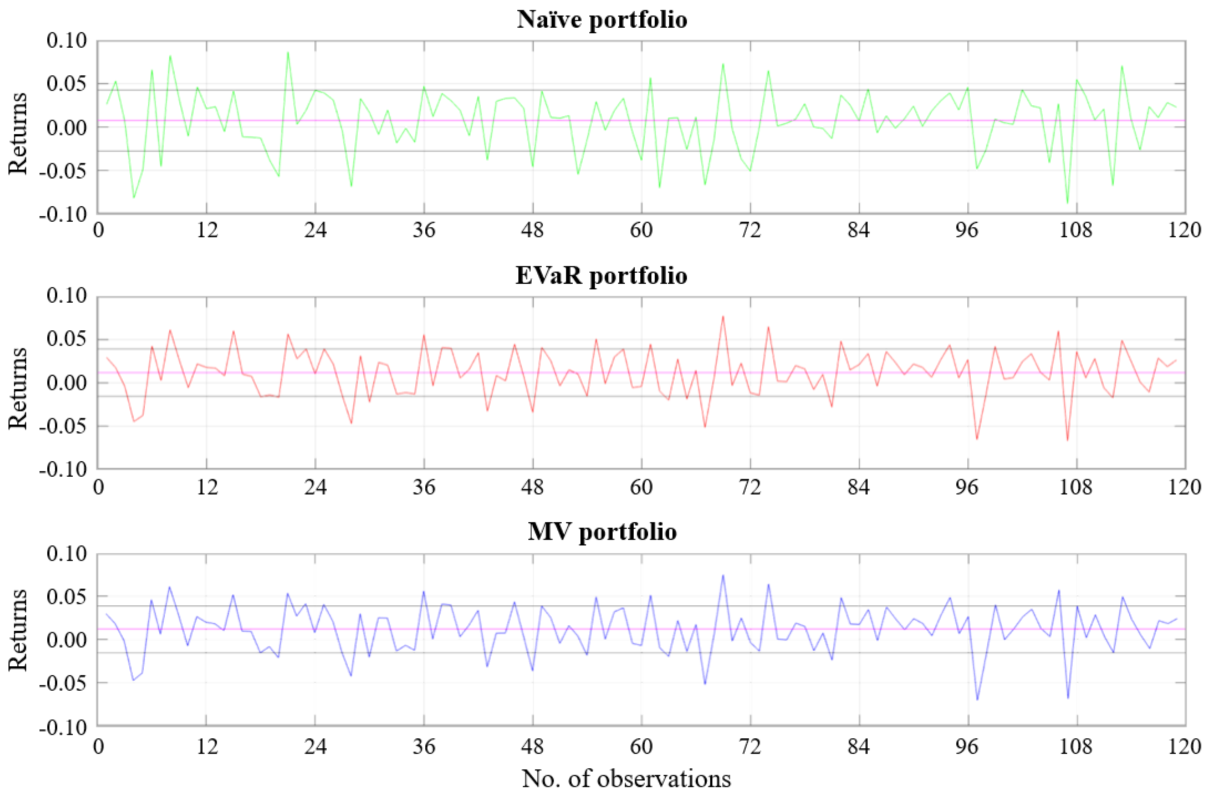
	Equally weighted	*EVaR weighted*	*MV weighted*
Maximum	8.7321	7.7806	7.5085
Minimum	-8.8991	-6.7369	-7.0923
Mean	0.7624	1.1822	1.1826
Standard Deviation	3.5174	2.7191	2.7129
Observations higher than std	14	21	19
Observations lower than std	18	15	16

Notes: The values shown in the first four rows of the table are all denoted in percent (%). The observations higher/lower than std describes how many observations in the respective data vectors deviate higher from the mean than the standard deviation.

Looking at the return structure of the initial portfolio in Figure 6, one can see that the 119 returns vary between 8.73% and -8.90%. The black horizontal lines in the graph show the standard deviation from the mean while the magenta horizontal lines represent the mean of the data. Comparing the returns with the standard deviation one can observe that in the equally weighted portfolio, the data exhibits 32 observations that exceed the standard deviation from the mean (14 positive and 18 negative observations).

After optimizing the portfolio using the entropic Value-at-Risk and the mean-variance approach by Markowitz (1954) the portfolio performance and the data properties have been calculated. Comparing the performance of the three different weighted portfolios, one can observe that the curve of the two optimized portfolios in Figure 5 appear to be smoother than the curve of the initial portfolio. However the optimized portfolios exhibit a more stable upward trend with a lower first values (observation 1) and a higher last values (observation 120). One can also see that the optimized portfolios are less affected by downward movements like the initial portfolio which especially is shown between the observation 53 and 74 in Figure 5. While the initial portfolio goes down as mentioned above, the optimized portfolios still maintain the positive trend. Therefore, the optimized portfolios resulted in higher monthly average return of 1.18% which is approximately 0.42% higher than the naïve portfolio. Comparing the volatility, the optimized portfolios exhibit lower standard deviations (EVaR = 2.719%; MV = 2.713%), which also lead to lower values for highest and lowest returns over the ten-year period (see Table 3).

Figure 6: Return structure



Analysing the return structure (Figure 6), the optimized portfolios exhibit fewer negative deviations from the mean which exceed the standard deviation than the initial portfolio. In the initial portfolio, 18 observations deviate negatively more from the mean than the standard deviation, while only 15 observations in the EVaR portfolio and 16 observations in the MV portfolio deviate from the respective means. Further, due to the optimization, there are more observations of positive deviations from the mean that are greater than the standard deviation in the optimized portfolios (EVaR = 21 observations; MV = 19 observations), then in the initial portfolio (14 observations).

5 Empirical results

For the application of the utility functions to the two optimized portfolios, the same approach as in Hagströmer et al. (2008) is used. In total, 102 test have been conducted on the two optimized portfolios, 12 times using the exponential utility, 9 times the power utility, 30 times the bilinear utility and 51 times the S-shaped utility. For each test, the individual parameters have been changed and for each change, a value for ΔCE has been obtained.

5.1 Traditional utility functions

For the traditional utility functions, the exponential and power utility function, one can see in .

Table 4 and Table 5, that the ΔCE of the two portfolios is negative for both utilities and for all parameter changes.

Table 4: ΔCE – Exponential utility function

A	ΔCE
0.50	-0.0005
1.00	-0.0006
1.50	-0.0007
2.00	-0.0007
2.50	-0.0007
3.00	-0.0007
3.50	-0.0007
4.00	-0.0006
4.50	-0.0006
5.00	-0.0005
5.50	-0.0004
6.00	-0.0003

Table 5: ΔCE – Power utility function

γ	ΔCE
1.00	-0.0006
1.50	-0.0006
2.00	-0.0006
2.50	-0.0006
3.00	-0.0006
3.50	-0.0006
4.00	-0.0005
4.50	-0.0004
5.00	-0.0003

This means that an investment in the EVaR portfolio under these two utility functions and under all parameter choices, is not favourable in comparison to the MV portfolio. However, the difference between the certainty equivalents is so minor that it seems insignificant. But, looking at how the choice of A and respectively γ influences the certainty equivalent, one can observe, that the highest values of A or γ respectively lead to the difference of the certainty equivalents of the two portfolios that is closest to zero. For an $A = 6$ in the exponential utility function an result of $\Delta CE = -0.0003\%$ was obtained and for an $\gamma = 5.0$ in the power utility function, a value of $\Delta CE = -0.0003\%$ was determined.

The average ΔCE for the exponential utility function resulted in a value of -0.0006% (Table A 2 in the appendix) which indicates that an average investor that has an exponential utility function would invest in the MV portfolio, rather than invest in the EVaR portfolio, regardless the individual risk preference/parameter choice. With an average ΔCE of -0.0005% (Table A 3 in the appendix), an average investor with a power utility function would follow the same approach and invest in the MV portfolio rather than in the EVaR portfolio. However, the obtained values are very small and seem insignificant in determining into which portfolio shall be invested. Hence, to form a justified investment decision, an investor with these types of utility function requires the use of additional information but this analysis goes beyond the scope of this paper.

5.2 Bilinear utility function

In the analysis of the bilinear utility function we varied the parameters x, which resembles the critical return level/kink and P which is the penalty level for returns under the kink. The calculations with $P = 1$ has been excluded from this research since it does not have a kink. Looking at the average ΔCE in Table A 4 in the appendix, one can see that like the traditional utility functions, the bilinear utility has a negative average difference across different parameters ($\Delta CE = -0.0034\%$). But unlike the traditional utility functions, most of the investors under bilinear utility function have a positive ΔCE and would prefer the EVaR portfolio rather than the MV portfolio. In Table 6, it can be seen that the penalty level P has a slight effect on the preferences of the investors, but it is clear that the kink has a bigger influence on it. Only investors with a positive kink located at the values of 0.000 and 0.005 exhibit a negative ΔCE. It was stated by Hagströmer et al. (2008) that the higher the penalty level and the kink level is, the higher is level of risk aversion. It is visible in Table 6 that this statement does not apply across all values of x. At $x = -0.010$ and $P = 10.0$, the investor surely preferred the EVaR optimized portfolio, whereas at $x = 0.000$ and $P = 10.0$ the investor almost with complete certainty prefers MV optimized portfolio over the EVaR portfolio.

Table 6: ΔCE – Bilinear utility function

	ΔCE		
x	$P = 2.5$	$P = 5.0$	$P = 10.0$
-0.040	0.0026	0.0037	0.0043
-0.035	0.0041	0.0059	0.0067
-0.030	0.0043	0.0060	0.0069
-0.025	0.0027	0.0039	0.0045
-0.020	0.0024	0.0035	0.0040
-0.015	0.0038	0.0054	0.0062
-0.010	0.0022	0.0033	0.0374
-0.005	0.0001	0.0021	0.0057
0.000	-0.0066	-0.0418	-0.0901
0.005	-0.0110	-0.0272	-0.0577

5.3 S-shaped utility function

In the analysis of the S-shaped utility function, several tests were performed at the inflection point z (or critical level of the return) located at 0.00, -0.025 and – 0.05 with varying either the shape parameters γ_1 and γ_2, or the magnitude parameters A and B. In the case of the S-shaped utility function, in Table A 5 in the appendix the average ΔCE of the S-shaped utility function is negative across different parameters ($\Delta CE = -0.0118\%$). The result of the negative average difference was the same as with the traditional and bilinear utility functions. However, in Table 7 and Table 8 it is visible that at the inflection point equal to -0.05 , the ΔCE is positive in both cases and subsequently, the investor would choose the EVaR portfolio over the mean-variance portfolio. In Table 7 with varying magnitude parameters A and B and constant parameters γ_1 and γ_2 at the inflection points equal to -0.025 and -0.05, the mean-variance portfolio outperforms the EVaR portfolio.

Table 7: ΔCE – S-shaped utility function (1)

Parameters		ΔCE		
A	B	$z = 0$	$z = -0.025$	$z = -0.05$
1.50	1.50	-0.0136	-0.0243	0.0024
1.70	1.30	-0.0162	-0.0263	0.0066
1.90	1.10	-0.0160	-0.0289	0.0122
2.10	0.90	-0.0084	-0.0323	0.0202
2.30	0.70	-0.0020	-0.0370	0.0326
2.50	0.50	-0.0052	-0.0433	0.0542
2.70	0.30	-0.0081	-0.0469	0.1010
2.90	0.10	-0.0107	-0.0002	0.2625

Notes: In this application of the utility function the parameters A, B, and z vary while γ_1 and γ_2 are both held constant at 0.5. All values ΔCE for are expressed in percent (%)

The same results appear in Table 8 with varying shape parameters γ_1 and γ_2, at the inflection points equal to -0.025 and -0.05, the mean-variance portfolio outperforms the EVaR portfolio. At the parameter values of $\gamma_1 = 0.05$ and $\gamma_2 = 0.95$, the ΔCE turns positive or equal to zero at all inflection points, which means that the investors who are more risk-averse or more reluctant to the losses either are indifferent between two optimized portfolios or prefer the EVaR optimized portfolio over the MV optimized portfolio. According to the obtained results, the shape parameters γ_1 and γ_2 influence the investment decision in an EVaR optimized portfolio more than in a MV optimized portfolio (See Table A 5 and Table A 6 in the appendix). The results do not correspond to the findings by Hagströmer et al. (2008) who stated that the higher the inflection point is, the higher is the loss aversion. Hence, overall results indicate that investors with low risk aversion. or even risk-loving investors would prefer the EVaR optimized portfolio to the MV optimized portfolio.

Table 8: ΔCE – S-shaped utility function (2)

Parameters		ΔCE		
γ_1	γ_2	$z = 0$	$z = -0.025$	$z = -0.05$
0.05	0.95	0.0000	0.0000	0.0151
0.10	0.90	0.0000	-0.2771	0.0241
0.15	0.85	0.0000	-0.2024	0.0260
0.20	0.80	-0.0002	-0.1387	0.0243
0.25	0.75	-0.0004	-0.0950	0.0209
0.30	0.70	-0.0005	-0.0659	0.0168
0.35	0.65	-0.0001	-0.0470	0.0127
0.40	0.60	-0.0209	-0.0350	0.0089
0.45	0.550	-0.0196	-0.0279	0.0054

Notes: In this application of the utility function the parameters γ_1, γ_2 and z vary while A and B are both held constant at 1.5. All values ΔCE for are expressed in percent (%)

6 Conclusion

Conducting this empirical application is a contribution to the paper by Ahmadi-Javid and Fallah-Tafti (2019) which provides a comparison between the EVaR optimized portfolio to the mean-variance optimized portfolio with respect to the utility functions. It further considers the investor preference for the comparison between the two portfolios. Referring to the outcomes of the research it can be argued that the claim by Ahmadi-Javid and Fallah-Tafti (2019) does not hold for all investors and all risk preferences. The results obtained from the analysis show that besides the standard risk measures such as mean and the volatility, the individual risk preference of the investor plays a crucial role in the investment decision in an EVaR optimized portfolio.

Looking at investors with a traditional utility function such as the exponential and the power utility, one can conclude that these investors would not invest in an EVaR portfolio over a MV portfolio. The results exhibit for each parameter change a negative value for the ΔCE. However, the ΔCE was not significantly large and the results are more inclined to the statement that the investors under traditional utility functions are more or less indifferent between EVaR and MV optimized portfolios. Investors with a more detailed utility function such as the bilinear or s-shaped utility function frequently exhibit positive ΔCE values which indicate risk and/or loss aversion behaviour. In regard to the bilinear utility function, the investor's preferences are dependent on the value of the kink/critical level and when it is negative the EVaR optimized portfolio is preferred to the mean-variance optimized portfolio. When discussing the S-shaped utility function, investors with low level of risk aversion (or at $z = -0.05$) prefer to invest in the EVaR optimized portfolio.

However, this result holds only for the confidence level specified in the assumptions and the empirical distribution of the used data set. Therefore, research that analyses other return distributions and confidence levels can complement this paper and further can yield similar or different results. As Hagströmer et al. (2008) argue, challenges still remain in specifying the utility function of the individual investor and therefore are fields that further need to be researched. Overall, the portfolio optimization technique using the Entropic Value-at-Risk as the underlying risk measure as discussed in this paper can be used as a basis for further development and application by asset and risk management professionals.

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Appendix

List of tables in the appendix

Note: Table A 2 to Table A 6 summarize the results of the calculation of the Certainty Equivalent which are denoted as CE_{EVaR} and CE_{MV} for the individual CE of the portfolios and ΔCE as the difference. All values for CE and ΔCE in the Appendix are expressed in percent (%), while the parameters are expressed as decimal numbers.

Table A 1: Portfolio composition and weights

Company Name	*Equal weights*	*EVaR weights*	*MV weights*
3M Co	0.0357	0.0010	0.0015
American Express Co.	0.0357	0.0011	0.0028
Apple Inc.	0.0357	0.0013	0.0016
Boeing Co.	0.0357	0.0086	0.0109
Caterpillar Inc.	0.0357	0.0010	0.0014
Chevron Corp.	0.0357	0.0010	0.0014
Cisco Systems Inc.	0.0357	0.0010	0.0015
Coca Cola Co.	0.0357	0.0011	0.0022
Disney (Walt) Co.	0.0357	0.0026	0.0080
Exxon Mobil Corp.	0.0357	0.0010	0.0014
Goldman Sachs Group Inc.	0.0357	0.0010	0.0014
Home Depot Inc.	0.0357	0.1049	0.1107
Intel Corp.	0.0357	0.0873	0.0446
Intl Business Machines Corp.	0.0357	0.0010	0.0016
Johnson & Johnson.	0.0357	0.0011	0.0024
JPMorgan Chase & Co.	0.0357	0.0010	0.0016
McDonalds Corp.	0.0357	0.2502	0.2614
Merck & Co.	0.0357	0.1179	0.0763
Microsoft Corp.	0.0357	0.0026	0.0195
Nike Inc.	0.0357	0.0029	0.0021
Pfizer Inc.	0.0357	0.0011	0.0019
Procter & Gamble Co.	0.0357	0.1422	0.1469
Raytheon Technologies Corp.	0.0357	0.0010	0.0016
UnitedHealth Group Inc.	0.0357	0.2565	0.2470
Verizon Communications Inc.	0.0357	0.0064	0.0308
Visa Inc.	0.0357	0.0010	0.0014
Walgreens Boots Alliance Inc.	0.0357	0.0010	0.0013
Walmart Inc.	0.0357	0.0011	0.0147

Table A 2: Certainty Equivalent for each portfolio – Exponential utility function

A	CE_{EVaR}	CE_{MV}	ΔCE
0.50	1.1638	1.1643	-0.0005
1.00	1.1454	1.1460	-0.0006
1.50	1.1269	1.1276	-0.0007
2.00	1.1084	1.1091	-0.0007
2.50	1.0898	1.0905	-0.0007
3.00	1.0711	1.0718	-0.0007
3.50	1.0524	1.0531	-0.0007
4.00	1.0337	1.0343	-0.0006
4.50	1.0148	1.0154	-0.0006
5.00	0.9959	0.9964	-0.0005
5.50	0.9770	0.9774	-0.0004
6.00	0.9580	0.9583	-0.0003
		Average	-0.0006

Table A 3: Certainty Equivalent for each portfolio – Power utility function

γ	CE_{EVaR}	CE_{MV}	ΔCE
1.0	1.1457	1.1462	-0.0006
1.5	1.1273	1.1279	-0.0006
2.0	1.1089	1.1095	-0.0006
2.5	1.0904	1.0910	-0.0006
3.0	1.0719	1.0725	-0.0006
3.5	1.0532	1.0538	-0.0006
4.0	1.0345	1.0350	-0.0005
4.5	1.0158	1.0162	-0.0004
5.0	0.9969	0.9973	-0.0003
		Average	-0.0005

Table A 4: Certainty Equivalent for each portfolio – Bilinear utility function

x	P	CE_{EVaR}	CE_{MV}	ΔCE
-0.040	2.500	-1.9491	-1.9516	0.0026
-0.035	2.500	-1.6704	-1.6746	0.0041
-0.030	2.500	-1.3954	-1.3996	0.0043
-0.025	2.500	-1.1224	-1.1251	0.0027
-0.020	2.500	-0.8505	-0.8529	0.0024
-0.015	2.500	-0.5861	-0.5898	0.0038
-0.010	2.500	-0.3417	-0.3439	0.0022
-0.005	2.500	-0.1105	-0.1105	0.0001
0.000	2.500	0.1020	0.1085	-0.0066
0.005	2.500	-0.0010	0.0100	-0.0110
-0.040	5.000	-3.0070	-3.0108	0.0037
-0.035	5.000	-2.6293	-2.6352	0.0059
-0.030	5.000	-2.2573	-2.2633	0.0060
-0.025	5.000	-1.8889	-1.8929	0.0039
-0.020	5.000	-1.5230	-1.5265	0.0035
-0.015	5.000	-1.1677	-1.1731	0.0054
-0.010	5.000	-0.8402	-0.8435	0.0033
-0.005	5.000	-0.6534	-0.6554	0.0021
0.000	5.000	-1.2289	-1.1871	-0.0418
0.005	5.000	-1.8999	-1.8728	-0.0272
-0.040	10.000	-3.5360	-3.5403	0.0043
-0.035	10.000	-3.1088	-3.1155	0.0067
-0.030	10.000	-2.6883	-2.6952	0.0069
-0.025	10.000	-2.2722	-2.2768	0.0045
-0.020	10.000	-1.8592	-1.8632	0.0040
-0.015	10.000	-1.4585	-1.4647	0.0062
-0.010	10.000	-1.8812	-1.9186	0.0374
-0.005	10.000	-2.8697	-2.8754	0.0057
0.000	10.000	-4.1315	-4.0414	-0.0901
0.005	10.000	-5.5903	-5.5326	-0.0577
			Average	-0.0034

Table A 5: Certainty Equivalent for each portfolio – S-shaped utility function (part 1)

A	B	γ_1	γ_2	z	CE_{EVaR}	CE_{MV}	ΔCE
1.5	1.5	0.50	0.50	0.000	0.4124	0.4260	-0.0136
1.7	1.3	0.50	0.50	0.000	0.2720	0.2882	-0.0162
1.9	1.1	0.50	0.50	0.000	0.1275	0.1435	-0.0160
2.1	0.9	0.50	0.50	0.000	0.0143	0.0227	-0.0084
2.3	0.7	0.50	0.50	0.000	-0.0060	-0.0040	-0.0020
2.5	0.5	0.50	0.50	0.000	-0.0343	-0.0291	-0.0052
2.7	0.3	0.50	0.50	0.000	-0.0768	-0.0687	-0.0081
2.9	0.1	0.50	0.50	0.000	-0.1270	-0.1163	-0.0107
1.5	1.5	0.50	0.50	-0.025	0.4020	0.4263	-0.0243
1.7	1.3	0.50	0.50	-0.025	0.2960	0.3223	-0.0263
1.9	1.1	0.50	0.50	-0.025	0.1546	0.1835	-0.0289
2.1	0.9	0.50	0.50	-0.025	-0.0432	-0.0109	-0.0323
2.3	0.7	0.50	0.50	-0.025	-0.3385	-0.3015	-0.0370
2.5	0.5	0.50	0.50	-0.025	-0.8225	-0.7792	-0.0433
2.7	0.3	0.50	0.50	-0.025	-1.7135	-1.6666	-0.0469
2.9	0.1	0.50	0.50	-0.025	-2.5016	-2.5014	-0.0002
1.5	1.5	0.50	0.50	-0.050	0.6589	0.6565	0.0024
1.7	1.3	0.50	0.50	-0.050	0.6220	0.6155	0.0066
1.9	1.1	0.50	0.50	-0.050	0.5720	0.5598	0.0122
2.1	0.9	0.50	0.50	-0.050	0.5000	0.4798	0.0202
2.3	0.7	0.50	0.50	-0.050	0.3879	0.3553	0.0326
2.5	0.5	0.50	0.50	-0.050	0.1890	0.1348	0.0542
2.7	0.3	0.50	0.50	-0.050	-0.2606	-0.3617	0.1010
2.9	0.1	0.50	0.50	-0.050	-2.2031	-2.4655	0.2625

Table A 6: Certainty Equivalent for each portfolio – S-shaped utility function (part 2)

A	B	γ_1	γ_2	z	CE_{EVaR}	CE_{MV}	ΔCE
1.5	1.5	0.05	0.95	0.000	0.0000	0.0000	0.0000
1.5	1.5	0.10	0.90	0.000	0.0000	0.0000	0.0000
1.5	1.5	0.15	0.85	0.000	-0.0001	-0.0001	0.0000
1.5	1.5	0.20	0.80	0.000	-0.0009	-0.0006	-0.0002
1.5	1.5	0.25	0.75	0.000	-0.0018	-0.0014	-0.0004
1.5	1.5	0.30	0.70	0.000	-0.0017	-0.0012	-0.0005
1.5	1.5	0.35	0.65	0.000	-0.0003	-0.0001	-0.0001
1.5	1.5	0.40	0.60	0.000	0.0787	0.0995	-0.0209
1.5	1.5	0.45	0.55	0.000	0.2680	0.2876	-0.0196
1.5	1.5	0.05	0.95	-0.025	-2.5000	-2.5000	0.0000
1.5	1.5	0.10	0.90	-0.025	-2.4030	-2.1259	-0.2771
1.5	1.5	0.15	0.85	-0.025	-1.5101	-1.3077	-0.2024
1.5	1.5	0.20	0.80	-0.025	-0.8587	-0.7200	-0.1387
1.5	1.5	0.25	0.75	-0.025	-0.3974	-0.3024	-0.0950
1.5	1.5	0.30	0.70	-0.025	-0.0756	-0.0097	-0.0659
1.5	1.5	0.35	0.65	-0.025	0.1429	0.1899	-0.0470
1.5	1.5	0.40	0.60	-0.025	0.2841	0.3192	-0.0350
1.5	1.5	0.45	0.55	-0.025	0.3663	0.3942	-0.0279
1.5	1.5	0.05	0.95	-0.050	-0.6228	-0.6380	0.0151
1.5	1.5	0.10	0.90	-0.050	-0.1561	-0.1802	0.0241
1.5	1.5	0.15	0.85	-0.050	0.1620	0.1360	0.0260
1.5	1.5	0.20	0.80	-0.050	0.3770	0.3527	0.0243
1.5	1.5	0.25	0.75	-0.050	0.5188	0.4979	0.0209
1.5	1.5	0.30	0.70	-0.050	0.6079	0.5911	0.0168
1.5	1.5	0.35	0.65	-0.050	0.6584	0.6457	0.0127
1.5	1.5	0.40	0.60	-0.050	0.6798	0.6709	0.0089
1.5	1.5	0.45	0.55	-0.050	0.6786	0.6732	0.0054
						Average	-0.0118