The surprising behaviour of dynamical systems

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Dynamical systems is the study of how complex systems evolve in time according to deterministic rules. In this thesis we investigate a connection between a certain dynamical system and the so-called ergodic hypothesis, a more than 100 years old hypothesis from statistical physics.

Imagine you have a piece of string, and there is an ant somewhere on the string. The ant moves along the string, but not randomly, according to a specific mathematical rule. The rule is a function that takes the ant's current position, and based on that, tells the ant which position to move to next. This kind of situation, where something evolves in time according to a specific rule, is what mathematicians call a dynamical system. A common question about dynamical systems is: what is the average position of the system after a long time?

This question was studied by the physicist Boltzmann in the 1800's, in the context of statistical physics, a branch of physics dealing with very large systems, such as gases and fluids. He was led to formulate the *ergodic hypothesis*: the time a system spends in a given region of space is proportional to the volume of the region. It means, for example, that a gas molecule in a room will eventually visit every region of the room, and the bigger the region, the more often it will be visited. It is often assumed to be true in thermodynamics, although there are physical systems that do not satisfy it. Mathematicians became interested in finding conditions for the hypothesis to be satisfied, and in 1931, Birkhoff provided an answer in his important *ergodic theorem*.

Inspired by the ergodic hypothesis, we can ask ourselves: will the ant spend an equal amount of time in the left and right halves of the string? The answer of course depends on the rule that determines how it moves. This thesis is about a certain rule that gives a very surprising answer to the question. The ergodic hypothesis would suggest that the ant is in the left half of the string as often as in the right half, but this is not what happens. We instead find that sometimes, the ant has spent most of its time in the left half of the string, and sometimes it has spent most of its time in the right half, and the system never converges to a stable average position. In this thesis we show why this happens, using the fascinating theory of dynamical systems.