The upper and lower bounds on Korenblum's constant

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Abstract

The purpose of this thesis is to study complex analysis, the Bergman space and Korenblum's conjecture. This is done in three parts. In the first part the proofs that the conjecture is true are studied, giving lower bounds of Korenblum's constant. The first proof is explained in detail, to make it as accessible as possible to more people. The main differences for a couple of later proofs that improved the lower bound are presented briefly. In the second part the counter examples to the conjecture for larger radii are presented. The first counter examples are explained briefly. The most recent, with the lowest known upper bound of Korenblum's constant, is presented in great detail. In the third part a couple of attempts of improving the upper bound are discussed. In the first attempt Blaschke products are used, to be able to place zeros of functions anywhere in the unit disc. In the second attempt the upper bound is analyzed as a variational problem. An optimization algorithm is written to find counter examples for as low radii as possible. The algorithm finds counter examples that are close to the best known, but nothing that is better than what already exists.

Sammanfattning

Syftet med denna uppsats är att studera komplex analys, Bergmanrummet och Korenblums förmodan. Detta görs i tre delar. I den första delen studeras bevisen för att förmodan stämmer, vilket resulterar i en nedre begränsning av Korenblums konstant. Det första beviset förklaras detaljerat, så att det är så åtkomligt som möjligt för fler personer. De största skillnaderna för ett par senare bevis som förbättrade den nedre begränsningen förklaras kortfattat. I den andra delen presenteras motexemplen för större radier. De första motexemplen förklaras kortfattat. Det senaste, som ger den lägsta kända övre begränsningen av Korenblums konstant, presenteras mycket detaljerat. I den tredje delen diskuteras ett par försök att förbättra den övre begränsningen. I det första försöket används Blaschkeprodukter, så att funktioners nollställen kan placeras var som helst i enhetsskivan. I det andra försöket analyseras den övre begränsningen som ett variationsproblem. En optimiseringsalgoritm skrivs för att hitta motexempe för så små radier som möjligt. Algoritmen hittar motexempel som är i närheten av de bästa kända, men inget bättre än så.

Preface

This thesis was written in the spring 2020 as a master's thesis in the mathematical engineering program at Lund University. My supervisor, Frank Wikström, proposed it when I asked if he had any idea for a master's thesis. He had previously worked with the problem himself, and had grand visions that I would be able to improve the upper bound.

I would like to thank Frank for supporting me during the thesis, answering all my questions in great detail despite the increase of administrative work due to the covid–19 pandemic and my lack of success in decreasing the upper bound. Our meetings have been a welcome break in the loneliness also caused by the pandemic.

I would also like to thank my family and friends, for being there for me through my life.

Contents

Notations

$\mathbb{C}$	Set of all complex numbers
$\mathbb{N}$	Set of all natural numbers, including zero.
$\mathbb{Z}$	Set of all integers.
$\mathbb{R}$	Set of all real numbers.
$A(r_1, r_2)$	The annulus between r_1 and r_2, $\{z \in \mathbb{C} : r_1 < \|z\| < r_2\}$.
$D(r)$	The disc of radius r, $\{z \in \mathbb{C} : \|z\| < r\}$
$\mathbb{D}$	The unit disc in the complex plane, $\{z \in \mathbb{C} : \|z\| < 1\}$
$\mathbb{T}$	The unit circle in the complex plane, $\{z \in \mathbb{C} : \|z\| = 1\}$

Chapter 1

Background

1.1 The Bergman space

In this thesis we will work in the Bergman space. The Bergman Space $\mathcal{A}^p(\mathbb{D})$ is the set of all analytic functions f on $\mathbb{D}$ satisfying:

$$||f||_{\mathcal{A}^p(\mathbb{D})} = \left(\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{\frac{1}{p}} = \left(\frac{1}{\pi}\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^p r dr d\theta\right)^{\frac{1}{p}} < \infty.$$

In this thesis we will assume $p = 2$. We can actually calculate the Bergman norm immediately from the power series of the function f.

Theorem 1. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ then*

$$||f||^2_{\mathcal{A}^2(\mathbb{D})} = \sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1}.$$

Proof. We have

$$\begin{aligned}
||f||^2_{\mathcal{A}^2(\mathbb{D})} &= \frac{1}{\pi}\int_{\mathbb{D}} f(z)\overline{f(z)} dA(z) \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sum_{k=0}^{\infty} a_k r^k e^{i\theta k} \sum_{l=0}^{\infty} \overline{a_l} r^l e^{-i\theta l} r dr d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sum_{k,l=0}^{\infty} a_k \overline{a_l} r^{k+l} e^{i\theta(k-l)} r dr d\theta \\
&= \frac{1}{\pi}\sum_{k,l=0}^{\infty} a_k \overline{a_l} \int_0^{2\pi} e^{i\theta(k-l)} d\theta \int_0^1 r^{k+l+1} dr.
\end{aligned}$$

The order of summation and integration can be swapped due to uniform convergence on compact subsets of $\mathbb{D}$. If $k \neq l$ then

$$\int_0^{2\pi} e^{i\theta(k-l)} d\theta = \left[\frac{1}{i(k-l)} e^{i\theta(k-l)}\right]_0^{2\pi} = 0.$$

Thus we only get non-zero terms when $k = l$, leaving

$$\begin{aligned}
&\frac{1}{\pi}\sum_{k,l=0}^{\infty} a_k\overline{a_l}\int_0^{2\pi} e^{i\theta(k-l)}d\theta\int_0^1 r^{k+l+1}dr \\
&\quad = \frac{1}{\pi}\sum_{k=0}^{\infty}|a_k|^2\int_0^{2\pi} d\theta\int_0^1 r^{2k+1}dr \\
&\quad = \sum_{k=0}^{\infty}\frac{|a_k|^2}{k+1}
\end{aligned}$$

□

From now, whenever we write $||\cdot||$ we mean $||\cdot||_{\mathcal{A}^2(\mathbb{D})}$.

1.2 Korenblum's conjecture

In 1991 Boris Korenblum [8] conjectured that there exists a constant c, $0 < c < 1$ such that if $|f(z)| \leq |g(z)|$ in the annulus $A(c,1)$ then $||f|| \leq ||g||$. The conjecture was proved by Walter Kurt Hayman [6] in 1999 with $c = 0.04$. We will look at Hayman's proof in Chapter 2. Korenblum provided a counter example when $c > \frac{1}{\sqrt{2}}$, that is $f(z) = 1+\epsilon$, $g(z) = \sqrt{2}z$. Thus there has to be a sharp constant κ such that the conjecture is true for $0 < c < \kappa$ and false for $\kappa < c < 1$. We call this number Korenblum's constant. Both the upper and lower bound of Korenblum's constant have been improved many times over the years, most recently both from below and above by Chunjie Wang [13, 14]. In this thesis we will look at both these results.

The result might at first seem a bit weird. If we only demand that the functions are continuous then f can be arbitrarily large for $|z| < c$, thus giving an arbitrarily large norm. But the fact that we are working with analytic functions restricts functions much more. For analytic functions we have the maximum modulus principle:

Theorem 2. *The modulus of an analytic function can not have a local maximum in the interior of its domain.*

For a proof, see for example Conway [4]. We try to replace the Bergman norm with the infinity norm, $||f||_\infty = \sup_{|z|<1} f(z)$, in Korenblum's conjecture. If $|f(z)| < |g(z)|$ for $c < |z| < 1$ then by the maximum modulus principle

$$||f||_\infty = \sup_{z\in\mathbb{D}}|f(z)| = \lim_{r\to 1^-}\sup_{|z|=r}|f(z)| \leq \lim_{r\to 1^-}\sup_{|z|=r}|g(z)| = \sup_{z\in\mathbb{D}}|g(z)| = ||g||_\infty$$

rendering the conjecture true for any $c \in (0,1)$.

We can also try the Hardy norm,

$$||f||_{H^2} = \sup_{0<r<1}\left(\int_0^{2\pi}|f(re^{i\theta})|^2d\theta\right)^{\frac{1}{2}}$$

where f is analytic. This case is a little bit more tricky. But the conjecture is still true for any $c \in (0,1)$. For this we need to use a theorem on subharmonic functions. A twice continuously differentiable function ϕ is subharmonic if $\Delta\phi \geq 0$. For us the most important observation is that when f is analytic then $|f|^\alpha$ is subharmonic for any $\alpha > 0$.

Theorem 3. *For subharmonic functions ϕ the integral*

$$\int_0^{2\pi} \phi(re^{it})dt$$

is increasing in r.

This is a part of Theorem 2.6.8 from Ransford [10].

Since $|f(z)|^2$ is subharmonic we have

$$||f||_{H^2} = \lim_{r\to 1^-} \left(\int_0^{2\pi} |f(re^{i\theta})|^2 \right)^{\frac{1}{2}}.$$

If $|f(z)| \leq |g(z)|$ for all $z \in A(c,1)$ then the integral will be larger for g than for f, and thus the Hardy norm is larger for g than for f.

Theorem 3 will have some additional uses in Chapter 2.

Chapter 2

The lower bound

2.1 Hayman's Proof

In this section we will follow the first proof of Korenblum's conjecture, published by Hayman [6] 1999. Sometimes we will go quicker, and sometimes we will slow down a bit and explain what he does. We start with some definitions. We put $\omega(z) = \frac{f(z)}{g(z)}$. By assumption we have $|f(z)| \leq |g(z)|$ for $z \in A(c, 1)$, and thus $|\omega(z)| \leq 1$ in that region. We also define $\rho_0 = \frac{1}{e}$ and $\omega_0 = \sup_{|z|=\rho_0} |\omega(z)|$. We assume that $c < \rho_0^2$. We will show that

$$\int_{A(\rho_0,1)} \left(|g(z)|^2 - |f(z)|^2\right) \, dA(z) \geq A_1 I_0 \tag{2.1}$$

and

$$\int_{D(c)} \left(|f(z)|^2 - |g(z)|^2\right) \, dA(z) \leq A_2 c^2 I_0, \tag{2.2}$$

where

$$I_0 = (1 - \omega_0^2) \int_0^{2\pi} |g(\rho_0^{3/2} e^{it})|^2 dt,$$

and A_1, A_2 are constants. For the annulus $A(c, \rho_0)$, $|g|$ is greater than $|f|$. Thus with $c = \sqrt{\frac{A_1}{A_2}}$ we get $||f|| \leq ||g||$.

We start by proving (2.1). For that we need Hadamard's three circle theorem.

Theorem 4. *For a holomorphic function f on the annulus $A(r_1, r_2)$, define $M_f(r) = \sup_{|z|=r} |f(z)|$. Then for $r_1 < r < r_2$*

$$\log\left(\frac{r_2}{r_1}\right) \log(M_f(r)) \leq \log\left(\frac{r_2}{r}\right) \log(M_f(r_1)) + \log\left(\frac{r}{r_1}\right) \log(M_f(r_2)).$$

For a proof, see Conway [4].

We define λ such that $\rho_0^\lambda = \omega_0$. Note that $\lambda \geq 0$ since $0 < \rho_0, \omega_0 \leq 1$ Using the above theorem we want to prove that $|\omega(z)| \leq \rho^\lambda$ for $\rho_0 < \rho = |z| < 1$. For ω to be defined we need any zero for g to also be a zero for f with at least the same multiplicity. But $|f(z)| \leq |g(z)|$. Assume g has a zero at z_0 of degree m and f a zero of degree n, $m > n$. Then $f(z) = f_0(z)(z - z_0)^n$ and $g(z) = g_0(z)(z - z_0)^n$, with $f_0(z_0) \neq 0$ and $g_0(z_0) = 0$. Due to continuity there is a point z_1 close to z_0 such that $g_0(z_1) < f_0(z_1)$. But that gives $g(z_1) < f(z_1)$, which is a contradiction. Thus ω is a holomorphic function in $A(c, 1)$. In Theorem 4 we put $r_1 = \rho_0$, $r_2 = 1$, $r = \rho$, $f = \omega$. Then we get

$$\log\left(\frac{1}{\rho_0}\right)\log(M_\omega(\rho)) \leq \log\left(\frac{1}{\rho}\right)\log(M_\omega(\rho_0)) + \log\left(\frac{\rho}{\rho_0}\right)\log(M_\omega(1)).$$

Recall that $\rho_0 = \frac{1}{e}$, and the definitions of ω_0 and λ to get

$$\log(M_\omega(\rho)) \leq \log\left(\frac{1}{\rho}\right)\log\left(\rho_0^\lambda\right) + \log\left(\frac{\rho}{\rho_0}\right)\log(M_\omega(1)).$$

Since $\rho > \rho_0$ and $M_\omega(1) \leq 1$ the rightmost term is less than or equal to zero. That gives

$$\log(M_\omega(\rho)) \leq \lambda \log \rho.$$

From that we get

$$|\omega(z)| \leq \rho^\lambda. \tag{2.3}$$

We have, using (2.3) in the first inequality and Theorem 3 in the second,

$$\begin{aligned}
&\int_{A(\rho_0,1)} \left(|g(z)|^2 - |f(z)|^2\right) \, dA(z) \\
&\quad = \int_{A(\rho_0,1)} |g(z)|^2(1 - |\omega(z)|^2) \, dA(z) \\
&\quad \geq \int_{\rho_0}^1 \int_0^{2\pi} |g(\rho e^{i\theta})|^2(1 - \rho^{2\lambda})\rho \, d\theta \, d\rho \\
&\quad \geq \int_0^{2\pi} |g(\rho_0^{\frac{3}{2}} e^{i\theta})|^2 \, d\theta \int_{\rho_0}^1 (1 - \rho^{2\lambda})\rho \, d\rho.
\end{aligned}$$

We now want to show that

$$1 - \rho^{2\lambda} \geq (1 - \omega_0^2)(1 - \rho) \tag{2.4}$$

for $0 < \rho < 1$. If $\lambda \geq \frac{1}{2}$ the result is easy, since then $1 - \rho^{2\lambda} \geq 1 - \rho$ and $1 - \omega_0^2 < 1$. For $0 < \lambda < \frac{1}{2}$ we first show that $1 - \rho^{2\lambda} \geq 2\lambda(1 - \rho)$. For that, take $h(\rho) = 1 - \rho^{2\lambda} - 2\lambda(1 - \rho)$. Then $h(1) = 0$. Furthermore

$$h'(\rho) = -2\lambda\rho^{2\lambda-1} + 2\lambda = 2\lambda(1 - \rho^{2\lambda-1}) \leq 0$$

since ρ, $2\lambda \leq 1$. Together this implies that h is positive on $0 < \rho < 1$, and thus $1 - \rho^{2\lambda} \geq 2\lambda(1 - \rho)$. Now all that remains is to show that $2\lambda \geq 1 - \omega_0^2$.

Remember the definition of λ as $\rho_0^\lambda = \omega_0$. That gives $2\lambda = \log\left(\frac{1}{\omega_0^2}\right)$. Note that $\omega_0^2 \leq 1$ and define $q(x) = \log\left(\frac{1}{x}\right) - (1-x)$. We have $q(1) = 0$ and

$$q'(x) = 1 - \frac{1}{x} < 0$$

for $0 < x < 1$. Thus $q(x) \geq 0$. Substitute $x = \omega_0^2$ to get $2\lambda \geq 1 - \omega_0^2$. Now we have (2.4) for all values of λ. Then

$$\int_{A(\rho_0,1)} \left(|g(z)|^2 - |f(z)|^2\right)\, dA(z) \geq I_0 \int_{\rho_0}^{1} (1-\rho)\rho\, d\rho > 0.115.$$

Putting $A_1 = 0.115$ gives (2.1). Hayman gets a little better result at $A_1 = 0.138$ by also considering the integral between $\rho_0^{3/2}$ and ρ_0, but this isn't needed to prove the conjecture.

Now we start proving (2.2). For $\rho < \rho_0^{3/2}$ we have

$$\begin{aligned}
&\int_0^{2\pi} \left(|f(\rho e^{i\theta})|^2 - |g(\rho e^{i\theta})|^2\right)\, d\theta \\
&\quad\leq \int_0^{2\pi} \left(|f(\rho e^{i\theta})|^2 - \omega_0^2 |g(\rho e^{i\theta})|^2\right)\, d\theta \\
&\quad\leq \int_0^{2\pi} \left(|f(\rho e^{i\theta})^2 - \omega_0^2 g(\rho e^{i\theta})^2|\right)\, d\theta \\
&\quad\leq \left(\int_0^{2\pi} \left(|f(\rho e^{i\theta}) - \omega_0 g(\rho e^{i\theta})|^2\right)\, d\theta \int_0^{2\pi} \left(|f(\rho e^{i\theta}) + \omega_0 g(\rho e^{i\theta})|^2\right)\, d\theta\right)^{\frac{1}{2}} \\
&\quad\leq \left(\int_0^{2\pi} \left(|f(\rho_0^{3/2} e^{i\theta}) - \omega_0 g(\rho_0^{3/2} e^{i\theta})|^2\right)\, d\theta \int_0^{2\pi} \left((|f(\rho_0^{3/2} e^{i\theta}) + \omega_0 g(\rho_0^{3/2} e^{i\theta})|^2\right)\, d\theta\right)^{\frac{1}{2}}.
\end{aligned}$$

The inequalities are due to in order $\omega_0 \leq 1$, the triangle inequality, Cauchy–Schwarz inequality and Theorem 3. We analyze these factors separately

$$|f(\rho_0^{3/2} e^{i\theta}) - \omega_0 g(\rho_0^{3/2} e^{i\theta})| = |(\omega(\rho_0^{3/2} e^{i\theta}) - \omega_0) g(\rho_0^{3/2} e^{i\theta})| \leq A(1-\omega_0^2)|g(\rho_0^{3/2} e^{i\theta})|$$

$$|f(\rho_0^{3/2} e^{i\theta}) + \omega_0 g(\rho_0^{3/2} e^{i\theta})| = |(\omega(\rho_0^{3/2} e^{i\theta}) + \omega_0) g(\rho_0^{3/2} e^{i\theta})| \leq 2|g(\rho_0^{3/2} e^{i\theta})|$$

where

$$A = \sup_{|z|=\rho_0^{3/2}} \frac{|\omega(z) - \omega_0|}{1 - \omega_0^2}$$

Using these estimates we get

$$\begin{aligned}
&\int_0^{2\pi} (|f(\rho e^{i\theta})|^2 - |g(\rho e^{i\theta})|^2)\, d\theta \leq 2A(1-\omega_0^2) \int_0^{2\pi} |g(\rho_0^{3/2} e^{i\theta})|^2\, d\theta \\
\iff &\int_0^{2\pi} \int_0^{c} (|f(\rho e^{i\theta})|^2 - |g(\rho e^{i\theta})|^2)\rho\, d\rho\, d\theta \leq c^2 A(1-\omega_0^2) \int_0^{2\pi} |g(\rho_0^{3/2} e^{i\theta})|^2\, d\theta
\end{aligned}$$

If we can find a constant A_2 such that $A \leq A_2$ this gives (2.2). Unfortunately it is quite obvious that there exists no such constant. For any $\epsilon > 0$ take $g(z) = 1$, $f(z) = -1 + \epsilon$. This would give $A = \frac{1(1-\epsilon)}{\epsilon}$. But this function is no counterexample to Korenblum's conjecture. We can actually find an upper bound if we only allow rotations of ω such that $\omega_0 = \omega(\rho_0)$ (two rotations, multiplying z and ω by different $e^{i\theta}$). These rotations will affect neither the absolute value nor the norm, and therefore not Korenblum's conjecture.

We start by showing

$$\left| \frac{\omega(z) - \omega_0}{1 - \omega(z)\omega_0} \right| \leq a < 1 \tag{2.5}$$

for $|z| = \rho_0^{\frac{3}{2}}$. Define $\phi(z) = \frac{\omega(z)-\omega_0}{1-\omega(z)\omega_0}$ and note that $\phi(\rho_0) = 0$. We recognise this as a Blaschke product of $\omega(z)$. We will discuss Blaschke products further in Chapter 3.1. The important property is that a Blaschke product map $\mathbb{D}$ to $\mathbb{D}$. For $\rho_0^2 < |z| < 1$ we have by definition $|\omega(z)| < 1$ and thus $|\phi(z)| < 1$. We will now transform this to a different space. Define

$$\psi(Z) = \phi\left(\exp\left(-1 - \frac{4iZ}{\pi}\right)\right).$$

Then ψ is an analytic function, and $|\psi(Z)| < 1$ for $-\frac{\pi}{4} < \mathrm{Im}(Z) < \frac{\pi}{4}$. For $Z_k = \frac{k\pi^2}{2}$, $k \in \mathbb{Z}$ we have $\psi(Z_k) = 0$. We are now interested in

$$\sup_{0 \leq x \leq \frac{\pi^2}{2}} \psi(x - \frac{i\pi}{8})$$

From Beardon and Minda [1] we define the hyperbolic distance on $\mathbb{D}$ as

$$d(w_1, w_2) = \frac{1}{2} \ln \frac{|1 - w_1\overline{w_2}| + |w_1 - w_2|}{|1 - w_1\overline{w_2}| - |w_1 - w_2|}.$$

For $w_2 = 0$ this becomes

$$d(w_1, 0) = \frac{1}{2} \ln \frac{1 + |w_1|}{1 - |w_1|} = \tanh^{-1} |w_1|. \tag{2.6}$$

It is well known that the hyperbolic distance is invariant under conformal mappings of $\mathbb{D}$ onto itself (see [1] for details). Then we can define the hyperbolic distance on any other simply connected set by mapping the other set conformally and bijectively onto $\mathbb{D}$, and using this definition.

We map the strip $-\frac{\pi}{4} < \mathrm{Im}(Z) < \frac{\pi}{4}$ to $\mathbb{D}$ by

$$w(Z) = \frac{e^Z - e^{-Z}}{e^Z + e^{-Z}} = \tanh(Z)$$

getting

$$d(Z_1, Z_2) = d(w(Z_1), w(Z_2)).$$

Note that we can map the strip onto itself by a translation in the real direction. Thus

$$d(Z_1, Z_2) = d(Z_1 - a, Z_2 - a),\ a \in \mathbb{R}. \tag{2.7}$$

Define $d_k(Z) = d(Z, Z_k)$, and put $Z = x - \frac{i\pi}{8}$ for $x \in [0, \frac{\pi^2}{2}]$.

From Lehto [9] we get the following generalization of Schwarz' Lemma.

Theorem 5. *Let f be analytic in $\mathbb{D}$ with $|f| < 1$ and $f(0) = 0$. If*

$$f(z_1) = f(z_2) = ... = f(z_n) = a$$

then

$$|a| \le \prod_{i=1}^{n} |z_i|.$$

We want to use this theorem with $f = \psi \circ \tanh^{-1}$. Note that f is defined on $\mathbb{D}$, has modulus less than one, and that $f(0) = 0$. We put $a = \psi(Z)$. Since ψ has a period of $\frac{\pi^2}{2}$ this gives

$$|\psi(Z)| \le \prod_k |\tanh(Z - Z_k)|$$

for any finite set of integers k. Using (2.6) and (2.7) we get

$$\begin{aligned} |\tanh(Z - Z_k)| &= \tanh\tanh^{-1} |\tanh(Z - Z_k)| \\ &= \tanh(d(Z - Z_k, 0)) = \tanh(d_k(Z)). \end{aligned}$$

And thus

$$|\psi(Z)| \le \prod_k \tanh(d_k(Z)).$$

In particular we have $|\psi(Z)| \le \tanh(d_0(Z))\tanh(d_1(Z))$. Note that as k goes to positive or negative infinity, $\tanh(d_k(Z))$ goes to 1. If we differentiate $\ln(\tanh(x))$ twice we get $-(\frac{1}{\sinh(x)} + \frac{1}{\cosh(x)}) < 0$. Thus $\ln(\tanh(x))$ is concave, and we have

$$\ln\tanh(d_0(Z)) + \ln\tanh(d_1(Z)) \le 2\ln\tanh\left(\frac{d_0(Z) + d_1(Z)}{2}\right) \iff$$

$$\tanh(d_0(Z))\tanh(d_1(Z)) \le \tanh^2\left(\frac{d_0(Z) + d_1(Z)}{2}\right)$$

Since $\tanh^2$ is an increasing function we will simply find an upper bound of $d_0(Z) + d_1(Z)$. By the triangle inequality

$$d_0(Z) + d_1(Z) \le d(Z_0, x) + 2d(x, Z) + d(x, Z_1)$$

Using (2.7) gives

$$d(Z_0, x) + 2d(x, Z) + d(x, Z_1) = d(x, 0) + 2d(-\frac{i\pi}{8}, 0) + d(\frac{\pi^2}{2} - x, 0).$$

Since $w = \tanh$ maps positive real numbers to positive real numbers we get

$$d(x,0) = \tanh^{-1}|\tanh(x)| = x$$

and

$$d(\frac{\pi^2}{2} - x, 0) = \frac{\pi^2}{2} - x.$$

Finally

$$\left|w\left(-\frac{i\pi}{8}\right)\right| = \left|\frac{e^{\frac{i\pi}{4}} - 1}{e^{\frac{i\pi}{4}} + 1}\right| = \left|\frac{(1-\sqrt{2}+i)(1+\sqrt{2}-i)}{(1+\sqrt{2}+i)(1+\sqrt{2}-i)}\right| = \frac{2\sqrt{2}}{4+2\sqrt{2}} = \sqrt{2}-1$$

which gives

$$d(-\frac{i\pi}{8}, 0) = \frac{1}{2}\ln\frac{\sqrt{2}}{2-\sqrt{2}}.$$

Backtracking, we find

$$\phi(z) \leq \tanh(d_0)\tanh(d_1) \leq \tanh^2\left(\frac{d_0+d_1}{2}\right)$$
$$\leq \tanh^2\left(\frac{\pi^2}{4} + \frac{1}{2}\ln\frac{\sqrt{2}}{2-\sqrt{2}}\right) \leq 0.9882 = a$$

and thus (2.5) is true. Then we have

$$|\omega(z) - \omega_0| \leq a|1 - \omega(z)\omega_0|$$

and want to maximize the left hand side. Substitute $b = \omega(z) - \omega_0$

$$|b| \leq a|1 - \omega_0(\omega_0 + b)| = a|1 - \omega_0^2 - \omega_0 b|.$$

For a given $|b|$ the right hand side will be largest when b is real negative, since $1-\omega_0^2 > 0$. Thus the largest possible value of $|b|$ is when $\omega(z)$ is real and smaller than ω_0, and we have equality.

$$\omega_0 - \omega(z) = a(1 - \omega(z)\omega_0)$$

In this case we have

$$\omega(z) = \frac{\omega_0 - a}{1 - a\omega_0}$$

and hence

$$\omega_0 - \omega(z) = \frac{a(1-\omega_0^2)}{1 - a\omega_0}$$

which is a maximum value for $|\omega_0 - \omega(z)|$. Then

$$A \leq \frac{a(1-\omega_0^2)}{(1-a\omega_0)(1-\omega_0^2)} \leq \frac{a}{1-a} \leq 84 = A_2.$$

This proves (2.2), and for $c = \sqrt{\frac{A_1}{A_2}} = 0.037...$ finalizes the proof of Korenblum's conjecture.

2.2 Improvements by Hinkkanen and Wang

In the same year that Hayman proved Korenblum's conjecture, Hinkkanen [7] improved the bound to $c = 0.15724$. He uses very similar techniques. We will not go through the proof in detail, but instead point out some similarities and differences to what Hayman did. Hinkkanen still compares the integrals

$$\int_{A(\rho_0,1)} \left(|g(z)|^2 - |f(z)|^2\right) \, dA(z)$$

and

$$\int_{D(c)} \left(|f(z)|^2 - |g(z)|^2\right) \, dA(z)$$

for $r < c < \rho$. However he compares them directly instead of via I_0. Hence he gets a different relation, that is

$$\int_0^{2\pi} |f(re^{i\theta})|^2 - |g(re^{i\theta})|^2 d\theta \leq 2\gamma(\rho) \int_0^{2\pi} |g(\rho e^{i\theta})|^2 - |f(\rho e^{i\theta})|^2 d\theta.$$

Here

$$\gamma(\rho) = \sup_{|z|=\rho} \frac{|\omega(z) - \omega_0(\rho)|}{1 - |\omega(z)|^2}.$$

We define $\omega_0(\rho)$ as the maximum modulus for each radius ρ, instead of only at ρ_0.

We put $c < r_1 < r_2 < 1$, and $\gamma = \sup_{\rho \in (r_1, r_2)} \gamma(\rho)$. Integrating each side first with respect to r from 0 to c and then to ρ from r_1 to r_2 gives

$$\int_{D(c)} |f(z)|^2 - |g(z)|^2 dA(z) \leq \frac{2c^2\gamma}{r_2^2 - r_1^2} \int_{A(r_1,r_2)} |g(z)|^2 - |f(z)|^2 dA(z).$$

Now all that remains is to find constants r_1, r_2, c such that $\frac{2c^2\gamma}{r_2^2 - r_1^2} \leq 1$.

Just as Hayman did, Hinkkanen also finds an upper bound of γ by using Theorem 5 and the expression $\left|\frac{\omega(z)-\omega_0}{1-\omega_0\omega(z)}\right|$. Since we simultaneously have to decide the values of r_1, r_2, c the calculations become much more complex.

Wang [14] further improves the lower bound to $c = 0.28185$. He does this by first proving that $f^2 - g^2$ has at least two zeros (with multiplicity) in $D(c)$. Then he gets that

$$\int_c^1 \frac{\rho}{\gamma(\rho) \int_0^c (\frac{r+c}{\rho + \frac{cr}{\rho}})^2 r dr} \int_{D(c)} |f(z)|^2 - |g(z)|^2 dA(z)$$
$$\leq \int_{A(r_1,r_2)} |g(z)|^2 - |f(z)|^2 dA(z).$$

where

$$\gamma(\rho) = \sup_{|z|=\rho} \frac{|1 - \omega(z)|}{1 - |\omega(z)|}$$

After that the proof is the same as Hinkkanen's proof. The same method works to bound also this γ from above.

Chapter 3

The upper bound

3.1 History of the upper bound

The upper bound is a very different problem than the lower bound. To improve the upper bound it is enough to find functions f and g that contradicts Korenblum's conjecture for some value of c. Wang [11, 12, 13] has improved the upper bound several times, first using singular inner functions, and later simple Blaschke products. The singular inner functions are defined as

$$S_a(z) = \exp\left(-a\frac{1+z}{1-z}\right).$$

For the first improvement to $\kappa < 0.6947117$, Wang [11] puts

$$f(z) = e^{-a}S_a(z^n)$$

$$g(z) = \frac{\exp\left(-\frac{2a}{1+c^n}\right)}{c}z.$$

If we put $a = -\frac{1+c^n}{1-c^n}\log c > 0$ then

$$\phi(r) = \sup_{|z|=r}\left|\frac{f(z)}{g(z)}\right| = \frac{f(-r)}{g(r)}.$$

since g is a monomial and f takes its maximum along a circle at the negative real axis. We see that $\phi(c) = \lim_{r\to 1^-}\phi(r) = 1$. Then the maximum modulus principle gives $|f(z)| < |g(z)|$ for $z \in A(c, 1)$. Further calculations show that if $n = 14$ then we can choose $c = 0.6947116...$

In his second improvement Wang [12] instead puts

$$f(z) = S_{a+b}(z^n)$$

$$g(z) = zS_b(z^n),$$

with the same definition of a. The exact same result holds for ϕ, and choosing b, n correctly gives $\kappa < c = 0.685086$.

3.2 Blaschke products and Wang's function

Blaschke products are a family of analytic functions on the unit disc. A finite Blaschke product of degree n is defined as

$$B(z) = \gamma \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}$$

where $\gamma \in \mathbb{T}$ and $z_k \in \mathbb{D}$. In an infinite Blascke product we have an infinite product instead. In that case we also demand that $\sum_k (1 - |z_k|)$ is finite, since that means the infinite product converges on $\mathbb{D}$ as well.

The Blaschke functions are analytic on $\mathbb{D}$. They have zeros at z_k, and they map $\mathbb{D}$ to itself exactly n times. What makes them extra useful for us is that they map $\mathbb{T}$ to itself.

What we will look at in this thesis is actually quotients of Blaschke products,

$$\frac{B_1(z)}{B_2(z)} = \prod_{k=1}^{n_1} \frac{z - z_k}{1 - \overline{z_k} z} \prod_{l=1}^{n_2} \frac{1 - \overline{\lambda_l} z}{z - \lambda_l}.$$

We don't care about the unimodular constant γ. We will not use B_1 and B_2 as f and g in Korenblum's conjecture, but instead we will use the product of numerators as f and the product of denominators as g. That is

$$f(z) = \prod_{k=1}^{n_1} (z - z_k) \prod_{l=1}^{n_2} (1 - \overline{\lambda_l} z)$$

$$g(z) = \prod_{k=1}^{n_1} (1 - \overline{z_k} z) \prod_{l=1}^{n_2} (z - \lambda_l).$$

This assures that $|f(z)| = |g(z)|$ for $|z| = 1$. If we for some radius $|z| = c < 1$ have $|f(z)| \leq |g(z)|$, and that $g(z)$ has no zeros in $c \leq |z| \leq 1$ then by the maximum modulus principle on $\frac{f}{g}$ we have $|f(z)| \leq |g(z)|$ on the annulus $A(c, 1)$. Now what Wang [13] does is to look at quite simple Blaschke products. We put $B_1(z) = \frac{a+z^m}{1+az^m}$ and $B_2(z) = z$ where $a \in \mathbb{R}$, $m \in \mathbb{Z}, m \geq 2$. Here B_1 is a Blaschke product of degree m. Then $f(z) = a + z^m$ and $g = z(1 + az^m)$. We want $||f||^2 = ||g||^2$. By Theorem 1 we have $||f||^2 = a^2 + \frac{1}{m+1}$, $||g||^2 = \frac{1}{2} + \frac{a^2}{m+2}$. Solving $||f||^2 = ||g||^2$ for a gives

$$a = \sqrt{\frac{(m+2)(m-1)}{2(m+1)^2}}.$$

Theorem 6. *For $0 < r, a < 1$ we have*

$$\sup_{|z|=r} \left| \frac{a + z^m}{z(1 + az^m)} \right| = \frac{a + r^m}{r(1 + ar^m)}$$

Proof. We show inequality in both directions. To get $\geq$ we just put $z = r$. The other direction is more difficult. We have under the condition $|z| = r$

$$\begin{aligned}
\sup_{|z|=r} \left| \frac{a + z^m}{z(1 + az^m)} \right| &\leq \frac{a + r^m}{r(1 + ar^m)} \\
&\iff |a + z^m|(1 + ar^m) \leq |1 + az^m|(a + r^m) \\
&\iff |a + z^m|^2(1 + ar^m)^2 \leq |1 + az^m|^2(a + r^m)^2 \\
&\iff (a^2 + 2\operatorname{Re}(az^m) + r^{2m})(1 + ar^m)^2 \\
&\quad \leq (1 + 2\operatorname{Re}(az^m) + a^2r^{2m})(a + r^m)^2.
\end{aligned}$$

Now we substitute $2\operatorname{Re}(az^m)$ as x, and $|z| = r$ becomes $|x| \leq 2ar^m$. We get

$$\begin{aligned}
&(a^2 + x + r^{2m})(1 + ar^m)^2 \leq (1 + x + a^2r^{2m})(a + r^m)^2 \\
&\quad\iff x\left(1 + a^2r^{2m} - a^2 - r^{2m}\right) \leq 2ar^m\left(1 + a^2r^{2m} - a^2 - r^{2m}\right) \\
&\quad\iff 1 + a^2r^{2m} \geq a^2 + r^{2m}
\end{aligned}$$

The last inequality follows from the rearrangement inequality, since $a, r \leq 1$. □

To use Theorem 6 we need to show that $a < 1$. But for $m \geq 1$ we have $2(m+1) > m+2$ and $m+1 > m-1$, giving $a < 1$. Now we only need to find r such that $\frac{a+r^m}{r(1+ar^m)} = 1$. Then we can change a by an arbitrarily small ϵ to get a counterexample to Korenblum's conjecture for any $c > r$. When we numerically solve the polynomial equation $a + r^m = r(1 + ar^m)$ for different m up to 15 we find the smallest solution in the range $(0, 1)$ is for $m = 10$, $r = 0.679501....$ For $m > 15$ we have

$$r > r\frac{1 - r^{m-1}}{1 - r^{m+1}} = a = \sqrt{\frac{(m+2)(m-1)}{2(m+1)^2}} \geq \sqrt{\frac{(15+2)(15-1)}{2(15+1)^2}} = 0.68179...$$

The last inequality is proved by calculating $\frac{d}{dm}a^2 = \frac{d}{dm}\frac{(m+2)(m-1)}{2(m+1)^2} = \frac{m+5}{2(m+1)^3}$, and thus a is increasing in m. This r is an upper bound of Korenblum's constant. In Figure 3.1 we can see the graph of the function $|f(z)| - |g(z)|$. We can see the ten zeros of f close to the unit circle, and the single zero of g at the origin. We want the graph to be smaller than zero outside of the red circle.

3.3 Pressing the result slightly with a modifying function

In this section we will follow Wang [13] and push the upper bound slightly lower. The idea is to multiply f and g by another analytic function h. That won't affect when $|f(z)| < |g(z)|$, but it will change the norms. In that way we

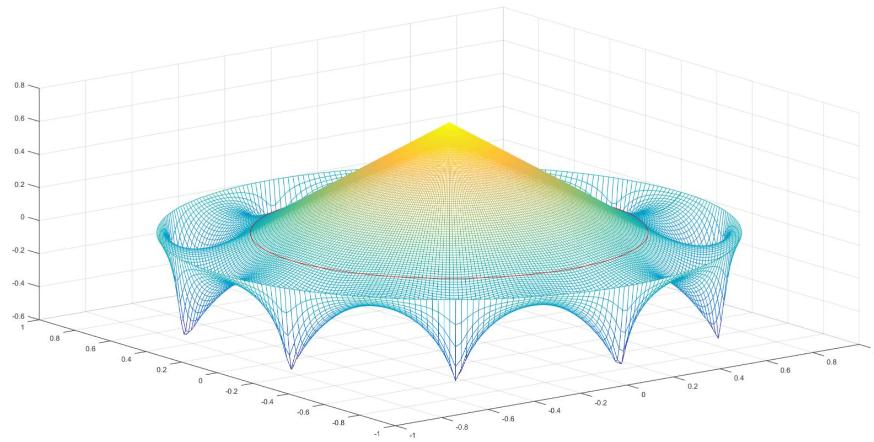

Figure 3.1: The function $|f(z)| - |g(z)|$. The red circle is at radius 0.68.

can chose a slightly different value of a and thus get a different value of r. So we redefine

$$f(z) = (z^m + a)h(z)$$

$$g(z) = z(1 + az^m)h(z)$$

where $a \in \mathbb{R}$, $m \geq 3 \in \mathbb{Z}$ First we will prove that if $a \leq \sqrt{\frac{m-2}{2m-2}}$ then $||f|| \leq ||g||$ for any analytic function h. Further equality will only occur if $a = \sqrt{\frac{m-2}{2m-2}}$ or if $h \equiv 0$. After that we find h such that $||f|| = ||g||$ for $a = \sqrt{\frac{m-2}{2m-2}}$.

Define

$$h(z) = \sum_{k=0}^{\infty} c_k z^k$$

Then we get

$$f(z) = \sum_{k=0}^{\infty} ac_k z^k + \sum_{k=0}^{\infty} c_k z^{k+m}$$

$$g(z) = \sum_{k=0}^{\infty} c_k z^{k+1} + \sum_{k=0}^{\infty} ac_k z^{k+m+1}$$

Using Theorem 1 we get

$$\begin{aligned}||f|| - ||g|| &= \sum_{k=0}^{\infty} |c_k|^2 \left(\frac{a^2}{k+1} + \frac{1}{k+m+1} - \frac{1}{k+2} - \frac{a^2}{k+m+2} \right) \\ &\quad + \sum_{k=0}^{\infty} \left(\frac{2a \operatorname{Re}(\overline{c_k} c_{k+m})}{k+m+1} - \frac{2a \operatorname{Re}(\overline{c_k} c_{k+m})}{k+m+2} \right) \\ &= \sum_{k=0}^{\infty} \left(|c_k|^2 \left(\frac{a^2(m+1)}{(k+1)(k+m+2)} - \frac{m-1}{(k+2)(k+m+1)} \right) \right. \\ &\quad \left. + \frac{2a \operatorname{Re}(\overline{c_k} c_{k+m})}{(k+m+1)(k+m+2)} \right) \end{aligned} \tag{3.1}$$

By the AM-GM inequality we have for $b > 0$

$$2a \operatorname{Re}(\overline{c_k} c_{k+m}) \leq 2a|c_k c_{k+m}| \leq ab\frac{k+2m}{k+m}|c_k|^2 + \frac{a(k+m)}{b(k+2m)}|c_{k+m}|^2. \tag{3.2}$$

That gives

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{2a \operatorname{Re}(\overline{c_k} c_{k+m})}{(k+m+1)(k+m+2)} \\ &\quad \leq \sum_{k=0}^{\infty} \frac{ab(k+2m)}{(k+m)(k+m+1)(k+m+2)}|c_k|^2 \\ &\qquad + \sum_{k=m}^{\infty} \frac{ak}{b(k+m)(k+1)(k+2)}|c_k|^2 \\ &\quad \leq \sum_{k=0}^{\infty} |c_k|^2 \left(\frac{ab(k+2m)}{(k+m)(k+m+1)(k+m+2)} + \frac{ak}{b(k+m)(k+1)(k+2)} \right). \end{aligned} \tag{3.3}$$

Using this we continue

$$\begin{aligned}&||f|| - ||g|| \\ &\quad \leq \sum_{k=0}^{\infty} |c_k|^2 \left(\frac{a^2(m+1)}{(k+1)(k+m+2)} - \frac{m-1}{(k+2)(k+m+1)} \right. \\ &\qquad \left. + \frac{ab(k+2m)}{(k+m)(k+m+1)(k+m+2)} + \frac{ak}{b(k+m)(k+1)(k+2)} \right) \\ &\quad = \sum_{k=0}^{\infty} \frac{|c_k|^2}{(k+1)(k+m+2)} \left(a^2(m+1) - \frac{(m-1)(k+1)(k+m+2)}{(k+2)(k+m+1)} \right. \\ &\qquad \left. + \frac{ab(k+2m)(k+1)}{(k+m)(k+m+1)} + \frac{ak(k+m+2)}{b(k+m)(k+2)} \right). \end{aligned}$$

Remember that $a \leq \sqrt{\frac{m-2}{2m-2}}$. Define $b = \sqrt{\frac{2}{(m-1)(m-2)}}$. Then $ab \leq \frac{1}{m-1}$, $\frac{a}{b} \leq \frac{m-2}{2}$.

$$
\begin{aligned}
&||f|| - ||g|| \\
&\quad \leq \sum_{k=0}^{\infty} \frac{|c_k|^2}{(k+1)(k+m+2)} \left(\frac{(m+1)(m-2)}{2m-2} - \frac{(m-1)(k+1)(k+m+2)}{(k+2)(k+m+1)} \right. \\
&\qquad \left. + \frac{(k+2m)(k+1)}{(m-1)(k+m)(k+m+1)} + \frac{(m-2)k(k+m+2)}{2(k+m)(k+2)} \right) \qquad (3.4) \\
&\quad = \sum_{k=0}^{\infty} \frac{|c_k|^2}{(k+1)(k+m+2)} \left(\frac{m^2-m-2}{2m-2} - (m-1)\left(1 - \frac{m}{(k+2)(k+m+1)}\right) \right. \\
&\qquad \left. + \frac{1}{m-1}\left(1 - \frac{m^2-m}{(k+m)(k+m+1)}\right) + \frac{m-2}{2}\left(1 - \frac{2m}{(k+m)(k+2)}\right) \right) \\
&\quad = \sum_{k=0}^{\infty} \frac{|c_k|^2}{(k+1)(k+m+2)} \left(\frac{m^2-m-2-2(m-1)^2+2+m^2-3m+2}{2m-2} \right. \\
&\qquad \left. + \frac{m(m-1)(k+m) - m(k+2) - m(m-2)(k+m+1)}{(k+2)(k+m)(k+m+1)} \right) \\
&\quad = 0
\end{aligned}
$$

and thus $||f|| \leq ||g||$. Now we want to find nontrivial h such that we have equality for $a = \sqrt{\frac{m-2}{2m-2}}$. Wang [13] just propose $h(z) = \frac{1}{(1-bz^m)^2}$, but we can derive why it has to be exactly this function. For (3.4) to be equal we simply need $a = \sqrt{\frac{m-2}{2m-2}}$. For (3.3) to be equal we need

$$\sum_{k=1}^{m-1} |c_k|^2 \frac{ak}{b(k+m)(k+1)(k+2)} = 0 \iff c_k = 0,\ 1 \leq k \leq m-1.$$

Finally for (3.2) to be equal we need $|c_{k+m}| = |c_k| b \frac{k+2m}{k+m}$. This gives $c_k \neq 0 \iff m|k$. By induction $|c_{mn}| = c_0 b^n (n+1)$ for $n \in \mathbb{N}$, and otherwise $c_k = 0$. If we put $c_k = |c_k|$ we recognise this as the Maclaurin expansion of

$$h(z) = \frac{c_0}{2(1-bz^m)^2}.$$

The constant $\frac{c_0}{2}$ will not affect the sign of $||f|| - ||g||$, or where $|f| \leq |g|$. Note that from (3.1) we get that $||f|| - ||g||$ is an increasing function of a. So if we pick a slightly larger than $\sqrt{\frac{m-2}{2m-2}}$ Then we get a counter example to Korenblum's conjecture for the real solution to the equation $a + r^m = r + ar^{m+1}$ on the interval $(0,1)$. The same methods as without the modifying function h gives a minimum for $m = 10$. Then we have $r = 0.6778994$.

Chapter 4

Numerical methods

4.1 Blaschke products

Inspired by the success of Blaschke products thus far we try some new ones. When $f = a + z^m$ it has m evenly spaced zeros at the circle $|z| = \sqrt[m]{a}$. When $m = 10$, $a = \sqrt{\frac{m-2}{2m-2}}$ we have $\sqrt[m]{a} \approx 0.96$. With the Blaschke products we can place the zeros for f and g wherever we want. We just have to be careful to see that the norms of f and g are the same.

One idea is to put the zeros of g just inside the zeros of f. Then it is reasonable that outside the rings of zeros we are closer to the zeros of f and thus $|f(z)|$ is smaller than $|g(z)|$. We try to put $f(z) = (a + z^m)(1 + bz^m)$ and $g(z) = (b + z^m)(1 + az^m)$. If we put $||f|| = ||g||$ then by Theorem 1 we get

$$|a|^2 + \frac{|ab+1|^2}{m+1} + \frac{|b|^2}{2m+1} = |b|^2 + \frac{|ab+1|^2}{m+1} + \frac{|a|^2}{2m+1} \iff |a| = |b|.$$

Unfortunately this means for $a, b \in \mathbb{R}^+$ that $f = g$, which is a trivial and not interesting case.

But what if we add m more zeros for f? We put $f(z) = (a + z^{2m})(1 + bz^m)$ and $g(z) = (b + z^m)(1 + az^{2m})$. Suppose that $a, b \in \mathbb{R}^+$ Then by putting $||f|| = ||g||$ we get

$$a^2 + \frac{a^2b^2}{m+1} + \frac{1}{2m+1} + \frac{b^2}{3m+1} = b^2 + \frac{1}{m+1} + \frac{a^2b^2}{2m+1} + \frac{a^2}{3m+1}$$

Solving for b^2 we get

$$b^2 = \frac{\frac{3ma^2}{3m+1} - \frac{m}{(m+1)(2m+1)}}{\frac{3m}{3m+1} - \frac{ma^2}{(m+1)(2m+1)}}$$

We try for $m = 5$. Then we find that b is defined and the radius of the zeros for g, $\sqrt[m]{b}$, is less than 0.68 when a^2 is between approximately 0.081 and 0.102. None of these values are promising when we graph the function. An example for

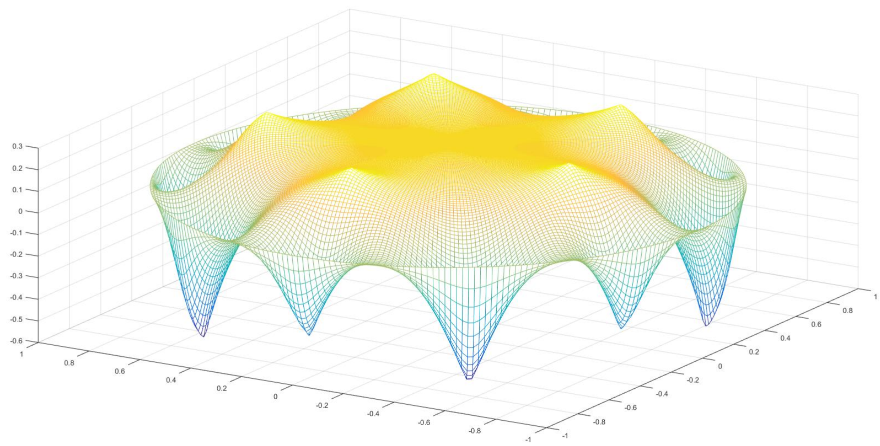

Figure 4.1: The function $|f(z)| - |g(z)|$ for $a^2 = 0.09$.

$a^2 = 0.09$ can be seen in Figure 4.1. We can't even see the red circle at radius 0.68.

It hurts a bit that we can not force the radius of the zeros of both f and g. But what if we force the radius of the zeros of g and the zeros of f that are at the same angles, thus letting the other zeros of f vary such that the norms are the same. We put

$$f(z) = (a + z^m)(1 + bz^m)(c - z^m),$$
$$g(z) = (1 + az^m)(b + z^m)(1 - cz^m).$$

Assuming $||f|| = ||g||$ gives

$$a^2c^2 + \frac{(c(1+ab) - a)^2}{m+1} + \frac{(cb - (ab+1))^2}{2m+1} + \frac{b^2}{3m+1}$$
$$= b^2 + \frac{(cb - (ab+1))^2}{m+1} + \frac{(c(1+ab) - a)^2}{2m+1} + \frac{a^2c^2}{3m+1}.$$

Writing this as a polynomial in c we get

$$c^2\left(\frac{a^2 3m}{3m+1} + \left((a+ab)^2 - b^2\right)\frac{m}{(m+1)(2m+1)}\right)$$
$$+c\left(\frac{2m(b-a)(ab+1)}{(m+1)(2m+1)}\right) + \left(\frac{m(a^2 - (ab+1)^2)}{(m+1)(2m+1)} - \frac{3mb^2}{3m+1}\right) = 0$$

The graph of one of these pairs of functions is shown in Figure 4.2. We can see only parts of the important red line at radius 0.68.

Unfortunately none of these attempts lead to anything better than the best known results. The reader is invited to try different parameters and functions. The matlab code is found in Appedix A.

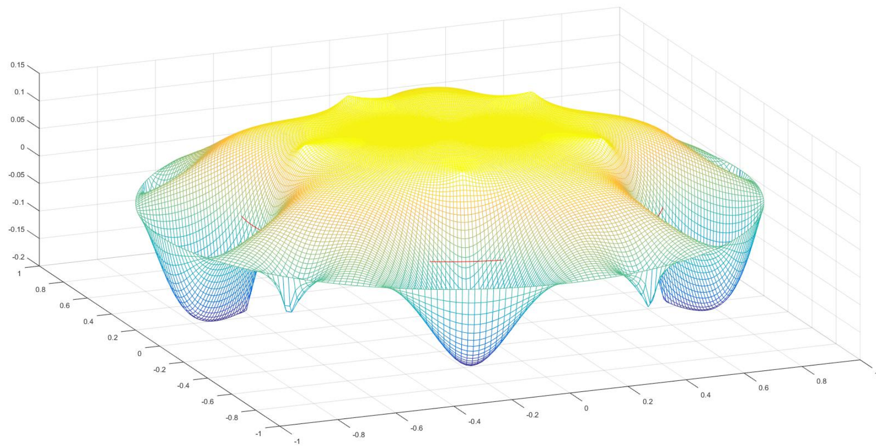

Figure 4.2: The function $|f(z)| - |g(z)|$. The zeros of g are at radius 0.5 and the zeros of f at the same angles are at radius 0.7. The remaining zeros of f are at radius 0.9924

4.2 Optimization algorithm

Inspired by Chakraborty [3, p. 50] we consider Korenblum's conjecture as a variational problem as follows.

For $0 < c < 1$ find

$$F(c) = \inf_{f,g \in \mathcal{T}} \sup_{z \in A(c,1)} \left| \frac{f(z)}{g(z)} \right|$$

where $\mathcal{T} = \{f \in \mathcal{A}^2(\mathbb{D}) : ||f|| = 1\}$.

It is clear that $F(c) \leq 1$, since we can choose $f = g$. Furthermore, iff $c > \kappa$ then $F(c) < 1$. We will develop an optimization method where we can try classes of polynomials. It would be quite time consuming to search the whole annulus for the optimal value. But the maximum modulus principle gives us better methods. If we assume that g has no zeros in the annulus, then the function $\omega(z) = \frac{f(z)}{g(z)}$ is analytic. The function ω will take its maximum at either $|z| = c$ or $|z| = 1$. Unfortunately it is difficult to find the maximum of a rational function along a circle, since the modulus of a rational function can change arbitrarily quickly close to poles and zeros. But what we are really interested in is whether there exists functions such that $F(c) < 1$. But that is equivalent to $\sup_{z \in A(c,1)} |f(z)|^2 - |g(z)|^2 < 0$. It is also enough to examine this expression at the two circles. If we only allow f and g to be polynomials of degree n then $|f(z)|^2 - |g(z)|^2 = f(z)\overline{f(z)} - g(z)\overline{g(z)}$ along the unit circle is a function on the form

$$h(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}.$$

Along the circle with radius c this instead becomes

$$h(\theta) = \sum_{k=-n}^{n} a_k c^k e^{ik\theta}$$

which by incorporating c^k into a_k becomes exactly the same problem. Now this problem is close to what is presented by Green [5]. The difference is that our function can have negative values, while he only maximizes $|f(z)|^2$. He solves this using Stečkin's lemma:

Theorem 7. *Suppose we have a real valued polynomial on the form*

$$h(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}$$

If $h(\theta_0) = ||h||_\infty$ *then* $h(\theta_0 + s) \geq h(\theta_0)\cos(ns)$ *for* $|s| \leq \frac{\pi}{n}$.

Green observes that if h is a positive function then by using Theorem 7 we get that if $h(\theta)$ takes its maximum in the interval $[\theta_k - s, \theta_k + s]$ then $h(\theta_k) \geq ||h||_\infty \cos(ns)$. We divide $[0, 2\pi]$ into k intervals with width s, and calculate h at the midpoint of each. We denote the maximum of these values by $\tilde{h}$. Then $\tilde{h} \leq ||h||_\infty \leq \tilde{h}\sec(ns)$. This leads to the algorithm

```
Set s = π/m, for m > 2n.
Divide [0, 2π] into m intervals I_k each of width 2s
while h̃(sec(ns) − 1) > ε do
    Calculate h at the midpoints θ_k of I_k.
    Set h̃ to the max of these values.
    Reject I_k with h(θ_k) < h̃ cos(ns).
    Divide the remaining intervals in half.
    Halve s.
end
return h̃
```

Algorithm 1: Greens algorithm

We can change $|f(z)|^2 - |g(z)|^2$ by simply adding a constant so that it is positive, or at least the function has its maximum modulus at a positive point. Suppose that the maximum modulus is at a negative point. Then by applying Theorem 7 and the previous discussion on $|g(z)|^2 - |f(z)|^2$ we get a lower bound on $|f(z)|^2 - |g(z)|^2$ by $\tilde{h}\sec(ns)$. Adding this number can be done in the first iteration of the while-loop in the algorithm. To add a too large number would make $\tilde{h}$ larger and thus the stopping criterion harder. This would make the algorithm unnecessarily slow.

Whenever we evaluate $|f(z)|^2 - |g(z)|^2$ we first check whether $g(z)$ has any zeros in $A(c, 1)$. If there are zeros we return a large value, so that the algorithm knows that it should stop searching here.

So now we can evaluate if a pair of functions contradicts Korenblum's conjecture for a given radius c. The next issue is how to find good candidates of functions f and g. We do this by considering the variational problem as a multi-dimensional optimization problem in the polynomial's non constant coefficients. To do this we need to try a specific pair of polynomials at a time, for example

$$f(z) = a_1 z^2 + a_2 z + a_3,\ g(z) = a_4 z^2 + a_5 z + a_6,\ a_i \in \mathbb{R}.$$

We want both polynomials to have the same norm. For a given set of values of a_1, a_2, a_4, a_5 we can use Theorem 1 to calculate a_3 and a_6 such that both f and g have norm one. Thus we can get $\sup_{z \in A(c,1)} |f(z)|^2 - |g(z)|^2$ as a function of these coefficients. Unfortunately we know very little about how this function behaves. We don't know its derivatives, and we have no idea about the number of local minima. But we can still try to solve this using a simple optimization algorithm. For this the cyclic coordinate algorithm [2] is used. That means we optimize, or line search, for one coefficient at a time, keeping the others constant. We continue cycling through all coefficients until there is no improvement for any coefficient. When line searching we simply start with a small step (10^{-6}) and double it until the value no longer decreases. This is a variant of Armijo's rule [2]. Then we try many random starting values. If we have complex coefficients we simply treat the real and imaginary parts as two different optimization dimensions. We can always assume that the constant coefficients are real, since we can always get that by multiplying the whole polynomial by a number $\xi \in \mathbb{T}$ without affecting the absolute value.

Instead of trying the algorithm for different radii c manually we can incorporate this in the code as well. Whenever we find a combination of functions and radius that contradict Korenblum's conjecture we decrease c by a tiny step (10^{-4}). We continue using the same constants, because it is likely that they will work for many radii in a row. We terminate the algorithm when we have tried 200 starting values for the final radius. The matlab code for all the different parts of the algorithm is found in Appendix B. A new value function has to be written to try each new class of function pairs f and g.

4.2.1 Discussion on the step size

If we want the algorithm to have an precision of 10^{-5}, then what is an appropriate step size? From Theorem 1 the value of a coefficient of degree n is at most $\sqrt{n+1}$ if the norms are 1. Then a change in the coefficient with absolute value ϵ will affect the norm by at most $\frac{2\epsilon}{\sqrt{n+1}} + \mathcal{O}(\epsilon^2)$. To adjust for this change the constant coefficient have to change at most $\sqrt{\frac{2\epsilon}{\sqrt{n+1}}} + \mathcal{O}(\epsilon)$. Since $|z| \le 1$ this means that the value of f or g will change by at most $\sqrt{\frac{2\epsilon}{\sqrt{n+1}}} + \mathcal{O}(\epsilon)$ and thus f^2 or g^2 by at most $\frac{2\epsilon}{\sqrt{n+1}} + \mathcal{O}(\epsilon\sqrt{\epsilon})$. Thus 10^{-6} is a good minimum step size.

4.2.2 Results

The algorithm has been run for many different combinations of polynomials. For example when run for the optimal Blaschke products from Chapter 3.1, that is $f(z) = a_1 z^{10} + a_2$, $g(z) = a_3 z^{11} + a_4 z$, we find counterexamples to Korenblum's conjecture for $c = 0.6799$. This is very close to the result we got in Chapter 3.1. More results for different functions are shown in Table 4.1. An interesting result is that the algorithm finds worse results for the second pair of functions $f(z) = a_1 z^{10} + a_2 z^5 + a_3$ and $g(z) = a_4 z^{11} + a_5 z^6 + a_6 z$, despite we could always get the same result by putting $a_2 = a_5 = 0$. We conclude that the algorithm suffers heavily from more variables to optimize for.

The functions f and g	Best radius c.	Time to run
$f(z) = a_1 z^{10} + a_2$ $g(z) = a_3 z^{11} + a_4 z$	$c = 0.6799$	258 s.
$f(z) = a_1 z^{10} + a_2 z^5 + a_3$ $g(z) = a_4 z^{11} + a_5 z^6 + a_6 z$	$c = 0.6900$	900 s.
$f(z) = a_1 z^{20} + a_2 z^{10} + a_3$ $g(z) = a_4 z^{21} + a_5 z^{11} + a_6 z$	$c = 0.6798$	1623 s.
$f(z) = a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5$ $g(z) = a_6 z^4 + a_7 z^3 + a_8 z^2 + a_9 z + a_{10}$	$c = 0.7075$	2454 s.
$f(z) = a_1 z + a_2$ $g(z) = a_3 z + a_4$	$c = 0.7072$	14 s.
$f(z) = a_1 z^2 + a_2 z + a_3$ $g(z) = a_4 z^2 + a_5 z + a_6$	$c = 0.7072$	2003 s.
$f(z) = a_1 z^{12} + a_2 z^{10} + a_3 z^8 + a_4 z^6 + a_5 z^4 + a_6 z^2 + a_7$ $g(z) = a_8 z^{13} + a_9 z^{11} + a_{10} z^9 + a_{11} z^7 + a_{12} z^5 + a_{13} z^3 + a_{14} z$	$c = 0.7047$	12829 s.
$f(z) = a_1 z^{15} + a_2 z^5 + a_3$ $g(z) = a_4 z^{16} + a_5 z^6 + a_6 z$	$c = 0.6854$	2286 s.
$f(z) = a_1 z^{42} + a_2 z^{28} + a_3 z^{14} + a_4$ $g(z) = \sqrt{2} z$	$c = 0.6989$	10843 s.
$f(z) = a_1 z^{40} + a_2 z^{30} + a_3 z^{20} + a_4 z^{10} + a_5$ $g(z) = \sqrt{2} z$	$c = 0.7018$	24951 s.

Table 4.1: Radius for different families of polynomials f and g. Each function is run until 200 iterations of the algorithm does not improve the radius. Compare these to the best known counter example $c = 0.6778994$ and $\frac{1}{\sqrt{2}} \approx 0.707107$

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Appendix A

Visualisation of Blaschke products

```
m = 5;
a = 0.7^m
b = 0.5^m
divide = a^2*3*m/(3*m+1) + ((1 + a*b)^2 - b^2)*m/((m+1)*(2*m+1));

p = 2*(b-a)*(a*b+1)*m/((m+1)*(2*m+1))/divide;
q = ((a^2-(a*b+1)^2)*m/((m+1)*(2*m+1)) - b^2*3*m/(3*m+1))/divide;
c = -p/2 + sqrt(p^2/4-q)
cm = nthroot(c, m)
f = @(z) abs((a + z.^m).*(1 + b*z.^m).*(c - z.^m));
g = @(z) abs((b + z.^m).*(1 + a*z.^m).*(1 - c*z.^m));
figure
[R,Phi] = meshgrid(0:0.01:1,0:0.02:2*pi);
X = R.*cos(Phi);
Y = R.*sin(Phi);
Z = f(X + 1i*Y) - g(X +1i*Y);

mesh(X,Y,Z)
hold on
phi = 0:0.01:2*pi;
plot3(0.68*sin(phi), 0.68*cos(phi), 0*phi, 'r')

%% Check that the norms are the same
a^2*c^2 + (a*b*c+c-a)^2/(m+1) + (b*c - a*b - 1)^2/(2*m + 1) + b^2/(3*m+1)
a^2*c^2/(3*m+1) + (a*b*c+c-a)^2/(2*m+1) + (b*c - a*b - 1)
```

ˆ2/(m + 1) + bˆ2

Appendix B

Matlab code for the optimization algorithm

```
%Change the value function to test other families of functions
%f(z) = a_1z^10 + a_2z^5 + a_3, g(z) = a_4z^11 + a_5z^6 + a_6z
Value = @(const, c) Blaschke5_10Value(const, c);

%The number of coefficients to optimize for. Number of coefficients in
%total minus two.
len = 4;
%The square root of the higest degree coefficient plus one.
largestCoefficient = sqrt(12);

c = 0.72;

epsilon = 1e-6;

bestTotal = 1e10;
bestCoefficients = 0;
k = 0
tic
while k < 200

    coefficients = largestCoefficient*rand(1, len);
    while(Value(coefficients, c) > 1e9)
        coefficients = largestCoefficient*rand(1, len);
    end
```

```
    k = k+1
    c;
    stop = 0;
    while stop == 0
        stop = 1;
        for j = 1:len
            d = zeros(1, len);
            d(j) = 1;
            best = Value(coefficients, c);
            %Modified Armijo's rule
            if Value(coefficients + d*epsilon, c) < best
                stop = 0;
                pot = 1;
                next = Value(coefficients + d*pot*epsilon, c);
                while next < best
                    best = next;
                    pot = 2*pot;
                    next = Value(coefficients + d*pot*epsilon, c);
                end
                coefficients = coefficients + d*pot*epsilon/2;
            elseif Value(coefficients - d*epsilon, c) < best
                stop = 0;
                pot = 1;
                next = Value(coefficients - d*pot*epsilon, c);
                while next < best
                    best = next;
                    pot = 2*pot;
                    next = Value(coefficients - d*pot*epsilon, c);
                end
                coefficients = coefficients - d*pot*epsilon/2;
            end
            while best < -1e-5
                c = c - 0.0001
                bestTotal = best;
                bestCoefficients = coefficients;
                best = Value(coefficients, c);
                k = 0;
                stop = 0;
            end
```

```
            end
        end
end
c = c + 0.0001
bestTotal
bestCoefficients
toc
```

```
function val = Blaschke5_10Value(coefficients, c)
%VALUE for the functions f(z) = a_1z^10 + a_2z^5 + a_3, g(z) = z(a_4z^10 + a_5z^5 + a_6)
%   Detailed explanation goes here

const1 = 1 - coefficients(1)^2/11 - coefficients(2)^2/6;
const2 = 1 - coefficients(3)^2/12 - coefficients(4)^2/7;

if const1 < 0 || const2 < 0
    val = 1e10;
    return
end
const1 = sqrt(const1);
const2 = sqrt(2*const2);

f = [0 coefficients(1), 0, 0, 0, 0, coefficients(2), 0, 0, 0, 0, const1];
g = [coefficients(3), 0, 0, 0, 0, coefficients(4), 0, 0, 0, 0, const2, 0];

r = roots(g);
if any(abs(r) >= c & abs(r) <= 1)
    val = 1e10;
else
    val = max(Greene(f, g, c), Greene(f, g, 1));

end

end
```

```
function Max = Greene(f, g, c)
%Greene Takes two complex polynomials as vectors, and
%returns the maximum modulus of |f|^2 - |g|^2 at radius c
    N = length(f);
    for i = 0:N-1
        f(N-i) = f(N-i)*(c^i);
        g(N-i) = g(N-i)*(c^i);
    end
```

```
    con = conj(f);
    q = conv(f, flip(con));
    con = conj(g);
    q = q - conv(g, flip(con));

    M = 2*N+5;
    h = pi/M;

    T = 2*h*(1:M)-h;

    q_vals = EvalPoly(q, T, N);
    q_min = min(q_vals);
    remember = 0;
    if(q_min < 0)
        q_min = q_min*sec(N*h);
        q(N) = q(N) - q_min;
        remember = 1;
    end

    while true
        q_vals = EvalPoly(q, T, N);
        q_max = max(q_vals);
        if abs(q_max*(sec(N*h) - 1)) < 1e-5
            break
        end
        T = [T(q_vals >= q_max*cos(N*h)) - h/2, T(q_vals >= q_max*cos(N*h)) + h/2];
        h = h/2;
    end
    if remember
        q_max = q_max + q_min;
    end
    Max = q_max;

end
```

```
function val = EvalPoly(q, T, N)
%EVALPOLY Evaluates the polynomial q at e^(it).
%q on the form a_1e^iNt + a_2e^i(N-1)t + ... + a_N + ... a_2Ne^-iNt

val = zeros(size(T));
for i = 1:2*N-1
    val = val + q(i)*exp(1i*T*(N-i));
end
%The value should be real already.
```

```
val = real(val);
end
```