The Existence of Meromorphic Functions with Prescribed Poles and Principal Parts

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Popular Scientific Summary

We summarise some of the important concepts and results.

The Swedish mathematician Gösta Mittag-Leffler (1846-1927) was one of the most influential researchers in Swedish mathematics. His mathematical contributions are connected chiefly with complex analysis. He studied with Karl Weierstrass while he travelled to Berlin and some of his research were built on Weierstrass' work, such as the Mittag-Leffler theorem which is the main topic of this paper.

The Mittag-Leffler theorem proves the existence of meromorphic functions with prescribed poles and principal parts. Interestingly, this study began as an extension of the Weierstrass theorem. As a continuation of Weierstrass's work, it was undertaken with the tools acquired from Weierstrass' lectures. These facts also show that Mittag-Leffler was indeed an important contributor to Weierstrass's research program concerning the foundations of analysis.

Mittag-Leffler published the final version of the theorem in 1884, in his newly-established journal *Acta Mathematica* which served as a comprehensive account of essentially all of his work on the subject. The theorem was seen as an important and fundamental element in complex analysis. It generated a number of research in the following generation of mathematicians, including the well-known figures Picard, Appell, and Poincaré, and it remained popular and widely-studied in different languages, such as French, Russian and German. [6]

Abstract

The Swedish mathematician Gösta Mittag-Leffler (1846-1927) is well-known for founding *Acta Mathematica*, the famous international mathematical journal. His mathematical contributions are connected chiefly with complex analysis. The Mittag-Leffler theorem which built on Karl Weierstrass' work asserts the existence of meromorphic function with prescribed poles and principal parts.

The main purpose of this thesis is to make a well-organised note of the Mittag-Leffler theorem which plays a significant role in complex analysis. The Weierstrass theorem which prescribes the zeros of holomorphic functions will be proved by using the result of the Mittag-Leffler's theorem. Moreover, the concept of the Mittag-Leffler star, a starshaped open domain will also be defined and proved in detail.

Contents

1 Introduction

This paper mainly discusses the results found by Gösta Mittag-Leffler (1846-1927) who was a Swedish mathematician. He was one of the most influential researchers in Swedish mathematics with discoveries in, among other things, theory of functions, which today is called complex analysis. [4] He matriculated at Uppsala University in 1865 and completed his Ph.D in 1872. He was appointed as a docent at the university the same year. He next travelled to Berlin and with his contemporary talents, he became friends with Karl Weierstrass who was a world famous German mathematician.

Mittag-Leffler was a member of Kungliga Vetenskapsakademien (the Royal Swedish Academy of Sciences) from 1883. He collected a large mathematical library in his villa at Djursholm in Sweden and on his 70th birthday in 1916, he established a mathematical foundation under the administration of the Academy of Sciences. Mittag-Leffler and his wife donated their villa and library as the Mittag-Leffler Institute which today is a major mathematical research centre. [2]

The best known result of Mittag-Leffler concerned the analytic representation of a one-valued function and this work culminated in the Mittag-Leffler theorem which will be mainly discussed in this paper. The study began while Mittag-Leffler was studying in Berlin. He attempted to generalise the results from Weierstrass's lectures where Weierstrass had described his theorem on the existence of an entire function with prescribed zeros each with a specified multiplicity. The result that Mittag-Leffler found asserts that we can prescribe the poles and principal parts of a meromorphic functions in complex analysis and this theorem is named after Mittag-Leffler. Moreover, the theorem can also be used to express any meromorphic function as a sum of partial fractions.

Mittag-Leffler received many honours during his life. He was an honorary member of almost every existing scientific society, including the Cambridge Philosophical Society, the Royal Institution, the Royal Irish Academy, the London Mathematical Society, and the Institut de France. He also received honorary degrees from six different universities, and in 1886 was elected a Fellow of the Royal Society of London. [6]

In this paper, the formulation and the proof of the Mittag-Leffler theorem will be shown in detail and the Weierstrass theorem will also be proved by using the results of the Mittag-Leffler's. Moreover, the concept of the Mittag-Leffler star, a starshaped open domain, will be discussed and proved. Firstly, the required definitions and lemmas will be given in order to help proving the theorems later.

2 Preliminaries

Definition 2.1. A function $f(z)$ is **analytic** on the open set U if $f(z)$ is (complex) differentiable at each point of U and the complex derivative $f'(z)$ is continuous on U.

Definition 2.2. A function $f(z)$ is **meromorphic** on an open subset D if $f(z)$ is analytic on D except for a set of isolated points, each of which is a pole.

Remark. *If there are infinitely many poles of $f(z)$ in D, then we can arrange them in a sequence that accumulates only at the boundary of D. Otherwise, there would be a point of accumulation in D of the poles of $f(z)$, and this point would not be an isolated singularity of $f(z)$.*

Definition 2.3 (Principal part). The sum of the negative powers,

$$P(z) = \sum_{k=1}^{N} a_k (z - z_0)^{-k} = \frac{a_1}{z - z_0} + \cdots + \frac{a_N}{(z - z_0)^N},$$

is called the **principal part** of $f(z)$ at the pole z_0 if $f(z) - P(z)$ is analytic near z_0.

Lemma 2.1 (Weierstrass M-Test). *Suppose $M_k \geq 0$ and $\sum M_k$ converges. If $g_k(x)$ are complex-valued functions on a set $E \subseteq \mathbb{R}^n$ such that $|g_k(x)| \leq M_k$ for all $x \in E$, then $\sum g_k(x)$ converges uniformly on E.*

Proof. The partial sums of the series are the functions

$$S_n(x) = \sum_{k=0}^{n} g_k(x) = g_0(x) + g_1(x) + \cdots + g_n(x)$$

For each fixed x, the estimate for $g_k(x)$ shows that the series $\sum g_k(x)$ is absolutely convergent, since $|g_k(x)| \le M_k$. The series $\sum g_k(x)$ converges to some complex number $g(x)$, and $|g(x)| \le \sum |g_k(x)| \le \sum M_k$. The same estimate, applied to the tail of the series, shows that

$$|g(x) - S_n(x)| = |\sum_{k=n+1}^{\infty} g_k(x)| \le \sum_{k=n+1}^{\infty} M_k.$$

If we set $\varepsilon_n = \sum_{k=n+1}^{\infty} M_k$, then $\varepsilon_n \to 0$ as $n \to \infty$, and the estimate shows that the partial sums $S_n(x)$ converge uniformly on E to $g(x)$. □

One also needs to prove that a uniform limit of analytic functions is analytic.

Lemma 2.2. *Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D. If the sequence converges uniformly on compacts to $f(z)$ on the domain D, then $f(z)$ is analytic on D.*

To prove $f(z)$ is analytic, one could apply two well-known theorem, Morera's theorem and Cauchy's theorem, which are formulated as following.

Theorem 2.3 (**Morera's theorem** [7])**.** *Let $f(z)$ be a continuous function on a domain D. If $\int_{\partial R} f(z)dz = 0$ for every closed rectangle R contained in*

D with sides parallel to the coordinate axes, then $f(z)$ is analytic on D.

Theorem 2.4 (**Cauchy's theorem**). *Let D be a simply connected bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to ∂D, then*

$$\int_{\partial D} f(z)dz = 0.$$

Now we can prove lemma 2.2. Since analytic functions are continuous, and the limit of a uniformly convergent sequence of continuous functions is continuous, $f(z)$ is continuous. Let E be a closed rectangle contained in D. By Cauchy's theorem, $\int_{\partial E} f_k(z)dz = 0$ for each k. From the lemma above we obtain in the limit that $\int_{\partial E} f(z)dz = 0$. By Morera's theorem, $f(z)$ is analytic.

3 The Mittag-Leffler Theorem

Theorem 3.1 (**The Mittag-Leffler Theorem** [**1**])**.** *Let ζ_j be a sequence of distinct complex numbers such that $|\zeta_j|$ increases to infinity, and let*

$$G_0(z), G_1(z), \dots, G_n(z), \dots$$

be a sequence of rational functions which has the form

$$G_n(z) = \frac{a^{(n)}_{\beta_n}}{(z-\zeta_n)^{\beta_n}} + \cdots + \frac{a^{(n)}_1}{z-\zeta_n}, \qquad n = 0, 1, 2, \dots \tag{1}$$

where $\beta_n \in \mathbb{Z}_+$, so that ζ_n is the unique pole of the corresponding function $G_n(z)$. Then there exists a meromorphic function $f(z)$ in the complex z-plane $\mathbb{C}$ having poles at the points ζ_j with corresponding principal part equals $G_n(z)$, for each n=0,1,2,....

Proof. The Taylor series expansion

$$G_n(z) = a^{(n)}_0 + a^{(n)}_1 z + \cdots + a^{(n)}_k z^k + \dots, \qquad n = 0, 1, 2, \dots$$

is convergent naturally by analyticity when $|z| < |\zeta_n|$ and uniformly convergent on every smaller disk, particularly on $D_n : |z| < \frac{1}{2}|\zeta_n|$. Take an arbitrary sequence $\{\varepsilon_n\}$ of positive numbers such that

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty. \tag{2}$$

Take integers k_0, k_1, $\ldots$, k_n so that they satisfy

$$|G_n(z) - [\ a_0^{(n)} + a_1^{(n)} z + \cdots + a_{k_n}^{(n)} z^{k_n}] \ | < \varepsilon_n, \qquad n = 0, 1, 2, \ldots \tag{3}$$

$\forall z \in D_n$. Then, let $P_n(z)$ be the polynomials such that

$$P_n(z) = -a_0^{(n)} - a_1^{(n)} z - \cdots - a_{k_n}^{(n)} z^{k_n}, \qquad n = 0, 1, 2, \ldots. \tag{4}$$

Let $K_R : |z| < R$, let $N(R)$ be the smallest integer such that $|\zeta_n| > 2R$ for all $n > N(R)$. Thus we have the series

$$\sum_{n=N(R)+1}^{\infty} [\ G_n(z) + P_n(z)] \ . \tag{5}$$

Note that $K_R \subset D_n$ for all $n > N(R)$, while K_R does not contain any points $\zeta_{N(R)+1}, \zeta_{N(R)+2}, \ldots.$ Now it follows from (3) and (4) that

$$|G_n(z) + P_n(z)| < \varepsilon_n$$

for all $n > N(R)$ and $z \in K_R$.

Therefore, by (2) and Weierstrass M-test (Lemma 2.1), the series (5) is uniformly convergent on K_R, and hence represents an analytic function $h_R(z)$ on K_R. This means that if

$$f(z) = \sum_{n=0}^{\infty} [\ G_n(z) + P_n(z)] \ ,$$

then the function has the representation in the form of a Mittag-Leffler expansion

$$f(z) = f_{N(R)}(z) + h_R(z), \qquad z \in K_R, \tag{6}$$

where $h_R(z)$ is analytic on K_R, and the partial sum

$$f_{N(R)}(z) = \sum_{n=0}^{N(R)} [\, G_n(z) + P_n(z)]$$

is a rational function whose poles in K_R coincide with the points of the sequence in ζ_j, and the principal part at the points $\zeta_n \in K_R$ is precisely $G_n(z)$. Thus the theorem follows at once since K_R has arbitrarily large radius. □

Corollary 3.2. *Suppose $f(z)$ is a meromorphic function whose poles are given by a sequence of distinct numbers ζ_j such that $|\zeta_j|$ goes to infinity, and the corresponding principal parts are $G_n(z)$. Then the representation of the function $f(z)$ can be in the form*

$$f(z) = g(z) + \sum_{n=0}^{\infty} [\, G_n(z) + P_n(z)] \ ,$$

where $P_n(z)$ are polynomials and $g(z)$ is an entire function.

Proof. By using Mittag-Leffler's theorem we can find a function

$$\phi(z) = \sum_{n=0}^{\infty} [\, G_n(z) + P_n(z)]$$

which has the same poles and principal parts as $f(z)$. $f(z) - \phi(z)$ has no singular parts and thus is entire in the whole plane, which denoted by g(z) in the theorem. □

Remark. *The proof of the theorem works with functions with essential singular parts as well as poles.* ***Essential singularities*** *are singularities where the function has no limits at the singularity. A pole has infinity as its limit thus poles are not essential singularities. The principal parts are holomorphic outside z_j with essential singularity at z_j and the proof of the theorem works as for the case for principal parts with poles.*

Examples of essentially singular parts is $e^{g(z)}$, which has an essential singularity at z_0 if $g(z)$ has a pole at z_0. Since $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, we find that $\sin(g(z))$ and $\cos(g(z))$ have essential singularities at $z = 0$ if g(z) has a pole there, for example, $\sin(\frac{1}{z})$ and $\cos(\frac{1}{z})$.

An interesting fact is that if $f(z)$ has an essential singularity at z_0, then for any complex number Z there is a sequence $z_j \to z_0$ such that $\lim_{j\to\infty} f(z_j) = Z$. In a more descriptive words, f comes arbitrarily close to any complex value in every neighbourhood of z_0. This is called the **Casorati-Weierstrass theorem** and is easy to prove.

Proof. If w is not a limit point, then by definition there exists an $\varepsilon > 0$ such that $|f(z) - w| > \varepsilon$ when $0 < |z - z_0|$ is small. Then the function $g(z) = 1/(f(z) - w)$ is bounded near z_0, and so by the Riemann theorem we find that $g(z)$ is analytic near z_0. This gives that $f(z) = w + 1/g(z)$ is either analytic or has a pole at z_0. □

Example 3.1. *Given a sequence of distinct ζ_j such that $|\zeta_j|$ increases to infinity, we can find a meromorphic function $f(z)$ with simple poles at ζ_j with corresponding principal parts*

$$G_n = \frac{1}{z - \zeta_n}, \qquad n = 1, 2, \ldots. \tag{7}$$

This is a special case of Mittag-Leffler's theorem, with simple poles and special principal parts, thus the result can be obtained directly from the theorem. Corollary 3.2 gives that,

$$f(z) = g(z) + \sum_{n=1}^{\infty}[\frac{1}{z - \zeta_n} + P_n(z)]$$

is a meromorphic function with poles ζ_j and principal part $\frac{1}{z-\zeta_n}$ where $P_n(z)$ are polynomials and $g(z)$ is entire.

Example 3.2. *Given a sequence of distinct ζ_j such that $|\zeta_j|$ increases to infinity, we can find a meromorphic function $f(z)$ with poles of order 2 at ζ_j and corresponding principal parts*

$$G_n(z) = \frac{1}{(z - \zeta_n)^2} \tag{8}$$

One gets the result using Corollary 3.2 with principal part $G_n(z)$ as in Example 3.1. Alternatively, we can take $f(z) = -F'(z)$ where $F(z)$ is a meromorphic function with singular parts given by (7) in Example 3.1.

3.1 The Weierstrass Theorem

If $f(z)$ is an entire function with zero of order m_j at z_j (with no accumulation point), then Taylor's formula gives

$$f(z) = (z - z_j)^{m_j} g(z) \tag{9}$$

with analytic $g(z)$ near z_j.

Theorem 3.3 (**Weierstrass Theorem**). *Let z_j be a sequence of complex numbers such that $z_j \to \infty$ and let $m_j \in \mathbb{Z}_+$. Then there exists an entire function $f(z)$ such that*

$$f(z) = (z - z_j)^{m_j} g_j(z) \tag{10}$$

close to z_j where $g_j(z) \neq 0$ is analytic near z_j.

Observe that if such a function exists, its logarithmic derivative

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{m_j(z - z_j)^{m_j - 1} g(z) + (z - z_j)^{m_j} g'(z)}{(z - z_j)^{m_j} g(z)} \\ &= \frac{m_j}{z - z_j} + \frac{g'(z)}{g(z)} \end{aligned}$$

satisfies Mittag-Leffler theorem in the case where all the poles are simple and the coefficients m_j are positive integers.

Proof. One can use the Mittag-Leffler theorem to construct a meromorphic function $f(z)$ with singular part $m_j(z - z_j)^{-1}$ near z_j, $j = 1, 2, \ldots$. Then the integral $F(z) = \int_{z_0}^{z} f(z)dz$ is well defined outside z_j modulo $2ik\pi$, which appears when integrating around z_j. In fact, near z_j we have that

$$F(z) = m_j \log(z - z_j) + h(z),$$

where log is the complex logarithm and $h(z)$ is analytic.

Then $g(z) = e^{F(z)} = (z - z_j)^{m_j} e^{h(z)}$ satisfies Weierstrass theorem with given zeroes of multiplicity m_j at z_j, $j = 1, 2, \ldots$. □

Example 3.3. *Let* ζ_j *be a sequence such that* $|\zeta_j|$ *increases to infinity, and let* $\{A_n\}$ *be an arbitrary complex sequence, then one can find an entire function* $f(z)$ *such that*

$$f(\zeta_n) = A_n, \qquad n = 1, 2, \ldots. \tag{11}$$

To prove that we use theWeierstrass theorem to get an entire function $g(z)$ with simple zeroes at ζ_j. Then we obtain a sequence of nonzero complex numbers $g'(\zeta_n)$ by first calculating the derivative $g'(z)$ at every point ζ_n, and the corresponding principal part is

$$G_n(z) = \frac{A_n/g'(\zeta_n)}{z - \zeta_n}, \qquad n = 1, 2, \ldots. \tag{12}$$

By using the Mittag-Leffler's theorem, we can construct a meromorphic function

$$\phi(z) = \sum_{n=1}^{\infty} [\ \frac{A_n/g'(\zeta_n)}{z - \zeta_n} + P_n(z)]$$

with principal part $G_n(z)$ at ζ_n which is stated in (12) and $P_n(z)$ are polynomials. Then we get the desired function

$$f(z) = g(z)\phi(z).$$

Moreover, the function $f(z)$ satisfies the condition (11), since

$$\begin{aligned} f(\zeta_n) &= \lim_{z \to \zeta_n} g(z)\phi(z) \\ &= \lim_{z \to \zeta_n} [\ \frac{g(z) - g(\zeta_n)}{z - \zeta_n}\phi(z)(z - \zeta_n)] \\ &= \frac{g'(\zeta_n)A_n}{g'(\zeta_n)} \\ &= A_n, \qquad\qquad n = 1, 2, \ldots. \end{aligned}$$

Since $f(z)$ has a limit at ζ_n, it is analytic by the Riemann Theorem.

4 The Mittag-Leffler Star

A domain $A \subset \mathbf{C}$ is said to be starlike (or starshaped) with respect to a point $a \in A$ if for any point $z \in A$ the line ℓ_z between a and z lies entirely in A. The point a need not be unique, in fact, every convex domain is starlike with respect to any of its points.

An analytic function $f(z)$ defined near a point $a \in \mathbf{C}$ has the property that its Taylor expansion $\sum_{k=0}^{\infty} f^{(k)}(a)(z-a)^k/k!$ converges in an open disk centered at a. For each ray it gives a unique continuation in a domain containning part of the ray and thus a continuation in a starlike domain containing the original disk. This continuation is unique and gives a univalent function in a maximal domain A that is starshaped with respect to a. Observe that this domain is open but could be unbounded, it is called the **Mittag-Leffler star** of $f(z)$.

Example 4.1. *The function* $\frac{1}{1-z}$ *has Taylor expansion* $1 + z + z^2 + \ldots$ *at* $z = 0$ *with radius of convergence equal to 1, but it is analytic in the starshaped domain* $\{z \in \mathbf{C} : z \neq 1\}$*. (This is because the function has a pole at* $z = 1$*.) The complex logarithm* $\log(z)$ *defined by the usual logarithm when* $z > 0$ *can be analytically continued to the starshaped domain* $\{z \in \mathbf{C} : z \not\leq 0\}$*.*

Theorem 4.1 (**The Mittag-Leffler Star**)**.** *An analytic function* $f(z)$ *defined near a point* $a \in \mathbf{C}$ *can be expanded in a series of polynomials in its Mittag-Leffler star* A *with uniform convergence on any compact subset of* A*.*

The expansion, called the Mittag-Leffler expansion, is on the form

$$f(z) = \sum_{n=0}^{\infty} \sum_{j=0}^{k_n} c_j^n f^{(j)}(a)(z-a)^j/j!$$

where the coefficients c_j^n *are positive rational numbers. These numbers and the degree* k_n *of the polynomials are independent of* f*.*

Proof. By a translation, we may assume that $a = 0$ for simplicity. The idea of the proof is to make a Taylor expansion of $f(z)$ stepwise on the line $\ell_z = \{\theta \cdot z : 0 \leq \theta \leq 1\}$ from z backwards until we reach 0. We are going to show that $f(z) = \lim_{n\to\infty} g_n(z)$ uniformly on compact subsets of A, where

$$g_n(z) = \sum_{\lambda_1=0}^{n^2} \sum_{\lambda_2=0}^{n^4} \cdots \sum_{\lambda_n=0}^{n^{2n}} f^{(\lambda_1+\lambda_2+\ldots\lambda_n)}(0) \left(\frac{z}{n}\right)^{\lambda_1+\lambda_2+\cdots+\lambda_n} /\lambda_1!\lambda_2!\ldots\lambda_n! \qquad n \geq 1$$

If $G_1(z) = g_1(z)$ and $G_n(z) = g_n(z) - g_{n-1}(z)$ for $n > 1$, then we find $f(z) = \sum_{n=1}^{\infty} G_n(z)$. Thus,

$$c_k^n = \frac{1}{n^k} \sum_{(\lambda)} \frac{1}{\lambda_1!\lambda_2!\ldots\lambda_n!}$$

where the sum is over $\lambda_1 + \lambda_2 + \cdots + \lambda_n = k$ and $(n-1)^{2j} < \lambda_j \leq n^{2j}$, $j \leq n$.

When doing the continuation along the line ℓ_z, we have to estimate the error terms. For that we shall use the Cauchy estimate which we obtain from the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_{|z-\zeta|=\varrho} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Then, by taking derivatives we obtain the estimate

$$|f^{(k)}(z)| \leq F_0 k! \varrho^{-k}$$

where $F_0 = \max_{|z-\zeta|=\varrho} |f(\zeta)|$.

Now, since the line ℓ_z is compact in A and the complement $\mathbf{C} \setminus A$ is closed, we find that the distance $d(w, \ell_z)$ from any $w \notin A$ to ℓ_z has a positive lower bound. (Recall that A contains an open disc at the origin.) Thus there exists $\varrho > 0$ so that $\ell_z^\varrho = \{w : d(w, \ell_z) \leq \varrho\}$ is a compact subset of A. Let

$$F = \max_{\ell_z^\varrho} |f(\zeta)|$$

be the maximum over this compact set. We shall use Cauchy's estimates in ℓ_z^ϱ.

Now for the given $z \in A$ we can find an integer $n \geq 1$ such that $|z/n| < \varrho$, i.e., $n > |z|/\varrho$. But more than that hold true. Let $0 < \alpha < 1$ depend on n so that $\alpha^n \to 1$ when $n \to \infty$, then we may assume that

$$|z/n| < \varrho_n = \alpha^n \varrho$$

i.e., $n > |z|/\varrho\alpha^n$. Later we will take $\alpha = e^{-\frac{1}{n\omega(n)}}$, where $0 < \omega(n) \to \infty$ when

$n \to \infty$.

Next, we let $\xi = z/n$ and $\xi_k = k\xi$, $k = 1, 2, \ldots, n$, so that $\xi_n = z$. Then we have $|\xi| \le \varrho_n < \cdots < \varrho_1 < \varrho$. We shall start by expanding

$$f(z) = \sum_{\lambda_1=0}^{\infty} f^{(\lambda_1)}(\xi_{n-1})(z - \xi_{n-1})^{\lambda_1}/\lambda_1!$$

then Cauchy's estimates gives

$$|f^{(\lambda_1)}(\xi_{n-1})| \le F\lambda_1!\varrho^{-\lambda_1}$$

since $|z - \xi_{n-1}| = |\xi| \le \varrho$, so that $z \in \ell_z^{\varrho}$. Then the first approximation is

$$f(z) = \sum_{\lambda_1=0}^{m_1} f^{(\lambda_1)}(\xi_{n-1})\xi^{\lambda_1}/\lambda_1! + \varepsilon_1$$

with m_1 to be determined later. Here

$$\varepsilon_1 \le F \sum_{\lambda_1=m_1+1}^{\infty} \alpha^{\lambda_1}$$

since we also have $|\xi| \le \varrho_1$ and $\varrho_1/\varrho = \alpha$.

Next, we want to approximate the partial expansion at ξ_{n-1} by a partial expansion at ξ_{n-2}. In order to do that, we take z and z_1 so that $|z_1 - \xi_{n-2}| \le \varrho_1$ and $|z - z_1| \le \varrho - \varrho_1$ which implies $|z - \xi_{n-2}| \le \varrho$. By expanding $f(z) = \sum_{\lambda_1=0}^{\infty} f^{(\lambda_1)}(z_1)(z - z_1)^{\lambda_1}/\lambda_1!$ we obtain from Cauchy's estimate that

$$|f^{(\lambda_1)}(z_1)| \le F\lambda_1!(\varrho - \varrho_1)^{-\lambda_1}$$

which gives

$$|f^{(\lambda_1)}(z_1)(z_1-\xi_{n-2})^{\lambda_1}/\lambda_1!| \le F\varrho_1^{\lambda_1}(\varrho-\varrho_1)^{-\lambda_1} = F\left(\frac{\alpha}{1-\alpha}\right)^{\lambda_1}$$

since $|z_1-\xi_{n-2}| \le \varrho_1$.

Now expand again

$$f^{(\lambda_1)}(z_1)(z_1-\xi_{n-2})^{\lambda_1}/\lambda_1! = \sum_{\lambda_2=0}^{\infty} f^{(\lambda_1+\lambda_2)}(\xi_{n-2})(z_1-\xi_{n-2})^{\lambda_1+\lambda_2}/\lambda_1!\lambda_2!$$

and then Cauchy's estimate gives

$$|f^{(\lambda_1+\lambda_2)}(\xi_{n-2})/\lambda_1!\lambda_2!| \le F\left(\frac{\alpha}{1-\alpha}\right)^{\lambda_1}\varrho_1^{-\lambda_1-\lambda_2}$$

and

$$|f^{(\lambda_1+\lambda_2)}(\xi_{n-2})\xi^{\lambda_1+\lambda_2}|/\lambda_1!\lambda_2! \le F\left(\frac{\alpha}{1-\alpha}\right)^{\lambda_1}\alpha^{\lambda_1+\lambda_2}$$

since $|\xi| \le \varrho_2$. This gives the second approximation

$$\sum_{\lambda_1=0}^{m_1} f^{(\lambda_1)}(\xi_{n-1})\xi^{\lambda_1}/\lambda_1! = \sum_{\lambda_2=0}^{m_2}\sum_{\lambda_1=0}^{m_1} f^{(\lambda_1+\lambda_2)}(\xi_{n-2})\xi^{\lambda_1+\lambda_2}/\lambda_1!\lambda_2! + \varepsilon_2$$

with m_2 to be determined later, where

$$\varepsilon_2 \le F\sum_{\lambda_1=0}^{m_1}\sum_{\lambda_2=m_2+1}^{\infty}\frac{\alpha^{2\lambda_1+\lambda_2}}{(1-\alpha)^{\lambda_1}} = F\frac{1-(\frac{1-\alpha}{\alpha^2})^{m_1+1}}{1-\frac{1-\alpha}{\alpha^2}}\left(\frac{\alpha^2}{1-\alpha}\right)^{m_1}\frac{\alpha^{m_1+m_2}}{1-\alpha}$$

We shall proceed like this, and choose z, $z_1\ldots,z_k$ so that $|z_k-\xi_{n-k-1}| \le$

ϱ_k, $|z_{k-1}-z_k| \le \varrho_{k-1}-\varrho_k, \dots, |z-z_1| \le \varrho-\varrho_1$ which implies $|z-\xi_{n-k-1}| \le \varrho$. Observe that we get $|z_{n-1}| \le \varrho_{n-1}$ when $k = n-1$. Then we obtain the approximation

$$f(z) = \sum_{\lambda_1=0}^{m_1} \cdots \sum_{\lambda_n=0}^{m_n} f^{(\lambda_1+\cdots+\lambda_n)}(0)\xi^{\lambda_1+\cdots+\lambda_n}/\lambda_1!\dots\lambda_n! + \varepsilon_1 + \cdots + \varepsilon_n$$

where

$$\begin{aligned} \varepsilon_k &\le F \sum_{\lambda_1=0}^{m_1} \cdots \sum_{\lambda_{k-1}=0}^{m_{k-1}} \sum_{\lambda_k=m_k+1}^{\infty} \frac{\alpha^{k\lambda_1+(k-1)\lambda_2+\cdots+\lambda_k}}{(1-\alpha)^{\lambda_1+\cdots+\lambda_{k-1}}} \\ &= F\, \frac{\prod_{j=1}^{k-1}\left(1-(\frac{1-\alpha}{\alpha^{k-j+1}})^{m_j+1}\right)}{\prod_{j=1}^{k-1}\left(1-\frac{1-\alpha}{\alpha^{k-j+1}}\right)} \frac{\alpha^{m_1+1}}{1-\alpha} \frac{\alpha^{m_1+m_2}}{(1-\alpha)^{m_1}} \cdots \frac{\alpha^{m_1+\cdots+m_k}}{(1-\alpha)^{m_{k-1}}} \qquad k \le n \end{aligned}$$

with $m_1, \dots, m_n$ to be determined later, see [3].

Now we are going to choose

$$\alpha = e^{-1/n\omega(n)}$$

where $0 < \omega(n) \to \infty$ when $n \to \infty$ (for example $\omega(n) = n^\delta$ where $0 < \delta < 1$). We have that $\alpha^n = e^{-1/\omega(n)} \nearrow 1$ as $n \to \infty$ and $1-\alpha = 1 - e^{-1/n\omega(n)} < 1/n\omega(n)$. When $\lambda \le n$ we have $\alpha^\lambda = e^{-\lambda/n\omega(n)} \ge e^{-1/\omega(n)}$ so that $\frac{1-\alpha}{\alpha^\lambda} < \frac{e^{1/\omega(n)}}{n\omega(n)}$. This gives

$$\frac{1}{1-\frac{1-\alpha}{\alpha^\lambda}} < \frac{1}{1-\frac{e^{1/\omega(n)}}{n\omega(n)}}$$

and thus

$$\frac{1}{\prod_{j=1}^{k-1}\left(1-\frac{1-\alpha}{\alpha^{k-j+1}}\right)} < \left(1-\frac{e^{1/\omega(n)}}{n\omega(n)}\right)^{-n} \searrow 1 \qquad n \geq k \to \infty$$

so we can find a majorant $\mu > 1$ of the left hand side for any k. We also have that $\prod_{j=1}^{k-1}\left(1-(\frac{1-\alpha}{\alpha^{k-j+1}})^{m_j+1}\right) < 1$.

Now we shall choose $m_1 \geq 2n\omega(n)\log(n\omega(n))$, $m_2 \geq m_1 n\omega(n)\log(n\omega(n))$, $m_1 + m_3 \geq m_2 n\omega(n)\log(n\omega(n))$ and

$$m_1 + m_2 + \cdots + m_{k-2} + m_k \geq m_{k-1} n\omega(n)\log(n\omega(n)) \qquad k \leq n$$

If we take $\omega(n) = n^\delta$ where $0 < \delta < 1$, then we may choose $m_1 = n^2$, $m_2 = n^4$ and $m_k = n^{2k}$ for $k \leq n$ and $n \gg 1$.

We find $\alpha^{m_1} \leq \alpha^{2n\omega(n)\log(n\omega(n))} = (n\omega(n))^{-2}$, $\alpha^{m_2} \leq \alpha^{m_1 n\omega(n)\log(n\omega(n))} = (n\omega(n))^{-m_1}$, $\alpha^{m_1+m_3} \leq \alpha^{m_2 n\omega(n)\log(n\omega(n))} = (n\omega(n))^{-m_2}$ and

$$\alpha^{m_1+m_2+\cdots+m_{k-2}+m_k} \leq (n\omega(n))^{-m_{k-1}} \qquad k \leq n$$

when $n \gg 1$. We also find that $(\frac{\alpha}{1-\alpha})^{m_k} < (n\omega(n))^{m_k}$ since $\frac{1-\alpha}{\alpha} = \frac{1}{e^{1/n\omega(n)}-1} < n\omega(n)$.

Putting these estimates together, we find that

$$\begin{aligned}\varepsilon_k &\leq F\mu\alpha^{m_1}\alpha^{m_2}\ldots\alpha^{m_1+m_3}\ldots\alpha^{m_1+m_2+\cdots+m_{k-2}+m_k}\left(\frac{\alpha}{1-\alpha}\right)^{1+m_1+\cdots+m_{k-1}} \\ &\leq \frac{F\mu}{n\omega(n)}\end{aligned}$$

for $k \leq n$ and $n \gg 1$. This gives that $\sum_{k=1}^{n} \varepsilon_k < F\mu/\omega(n) \to 0$ when $n \to \infty$, which proves the result.

$\square$

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