Public Goods on Networks: Statics, Welfare & Mechanisms

Anton Arbman Hansing

LUND UNIVERSITY

Department of Automatic Control

MSc Thesis
TFRT-6099
ISSN 0280-5316

Department of Automatic Control
Lund University
Box 118
SE-221 00 LUND
Sweden

Printed in Sweden by Tryckeriet i E-huset
Lund 2020

Abstract

This thesis studies a network game of heterogeneous and asymmetric public goods. Players allocate their wealth between private and public goods, benefiting from the public goods provisioned by their out-neighbors on the network graph. Utilities are given by a Cobb-Douglas function to capture substitutability and decreasing marginal returns. I prove that the game is well-behaved under a condition relating a simple network characteristic – the spectral radius – to the preferences of the players. Under this assumption, the best response dynamic is guaranteed to converge, and the equilibrium strategy is unique. Equilibrium public good contributions are then linear in the wealth of others contributors. Next, the game is studied through a normative lens. I show that equilibrium outcomes, as a rule, are inefficient with regards to important welfare metrics. Three mechanisms on the game are formalized, drawing on the economic literature of public goods: taxes & subsidies, enforceable contracts, and redistribution. For each mechanism, the scope of attainable welfare improvements is characterized and design considerations discussed.

Acknowledgements

I would like to thank my supervisor Giacomo Como at the Department of Automatic Control, Lund University, for his extensive advice, support, and guidance during my work on this thesis project. He helped me navigate the literature on network games, direct my attention toward the important research questions, and find new approaches when previous methods failed. I am also grateful to Laura Arditti and Luca Damonte, PhD students at Politecnico di Torino, and Rickard Eriksson, senior economist at the Swedish Bankers' Association, for valuable feedback on a draft of the thesis. Lastly, I would like to thank the examiner Prof. Anders Rantzer and opponent Lisa Rygård for engaging so thoroughly with the final report. Any remaining errors are mine.

Contents

1

Introduction

Individuals and organizations commonly purchase goods whose benefits are shared with others. When a firm provides training for its employees, benefits accrue not only to the firm itself, but to the employees and their future employers. If a family renovate their façade, they provide a nicer view for their neighbors. This class of goods, characterized by the inability of the funder to restrict benefits to others, are referred to as *public goods*. In economics, public good situations are go-to examples of how individually rational action may lead to outcomes that are *collectively irrational*; when agents disregard the benefits that their actions confer on others, the resulting outcomes are suboptimal not only for the group as an entity, but for each individual agent. A large number of interventions have been put forward to alleviate this failure of cooperation, often based on rewarding agents that take action to benefit others, or creating structures for enforceable agreements.

Network modeling has been proposed as a tool to understand public goods whose benefits are local to some social or geographical environment. Bramoullé and Kranton (2007) study a game where players exert some costly effort, and benefit from all effort exerted in their neighborhood. They find that equilibrium outcomes can be either *specialized*, with only a subset of players exerting effort, or *distributed*, with some level of effort-sharing by all players. Allouch (2015) models consumers choosing to allocate their income between private and public goods. He proves uniqueness of equilibrium under an assumption relating the network structure to the preferences of the players, and explores how income redistribution affects the amount of public goods provisioned. Elliott and Golub (2019) study the outcomes of a generalized public goods game normatively, and find that the spectral radius of a certain *benefit matrix* uniquely characterizes Pareto-efficient outcomes.

In many real world situations, flows of externalities between agents are messy. Firstly, effects are heterogeneous – some action might greatly benefit one agent, leave another indifferent, and hurt a third. Secondly, pairwise relations are often asymmetrical – the actions of agent A may benefit agent B, even if the actions of B have no impact on A. These differences in the degree and reciprocity of externalities have rarely been captured in the existing literature of public goods on networks.

In this thesis, I consider a game of heterogeneous and asymmetric externalities. Individuals are connected in a directed network, and choose to allocate their wealth between private goods consumption and public goods provision. Players that provide public goods confer benefits heterogeneously among their out-neighbors in the network. The preferences of each player are described by a Cobb-Douglas utility function, where the public goods provisioned in a player's neighborhood is a substitute for her own provision.

Firstly, the thesis describes equilibrium outcomes of the game. Depending on their wealth and placement in the network, some players find it rational to provide public goods while others free-ride entirely on the contributions of their neighbors. I show that under an assumption relating the spectral radius of the network graph to the preferences of the players, the existence of a unique Nash equilibrium is guaranteed. In addition, the best response dynamic of the game will always converge to this strategy. I then characterize public goods contributions in this equilibrium by exploiting a well-known relationship between the inverse of a matrix and a geometric series. This interpretation shows that the network game can be equivalently described as an infinite number of pairwise games of substitutes and complements between neighbors and non-neighbors alike.

Secondly, I examine three possible alterations to the game formulation and their effects on various welfare metrics. The first intervention, taxes & subsidies, aims to increase the public goods provision of particularly central individuals by altering the relative prices that they face between private and public goods. I find a set of individualized taxes and subsidies for which the equilibrium outcome is socially optimal in a utilitarian sense. Next, I explore enforceable contracts: agreements where individuals voluntarily commit to providing more public goods than what is individually rational, as long as their neighbors reciprocate this sacrifice. I characterize a sufficient criterion for the existence of such contracts, and describe the shape of a contract that is optimal in a weak sense. Lastly, I relax the assumption of wealth as exogenous, and consider the effects of redistribution on aggregate welfare. I show that the problem of finding a socially optimal wealth distribution is locally concave, and propose a simple algorithm to find such a distribution.

The rest of the thesis is structured as follows. Section 2 provides a technical background of network modeling, game theory and network games. Section 3 gives an overview of previously proposed network models of public goods. Section 4 presents the game studied in this thesis, derives the individually rational behavior of players, and characterizes the structure of equilibrium outcomes. Section 5 introduces measures of welfare efficiency, briefly discusses the welfare levels of equilibrium outcomes, and presents three mechanisms to improve welfare outcomes compared to the base game. Section 6 summarizes findings and discusses avenues for future research. Proofs and derivations are presented in the appendix.

2

Technical background

This section introduces the basic concepts of network modeling, game theory, and networks games, and presents some of the notation and definitions that will be used throughout the thesis. Readers that are familiar with these areas of study may prefer to skip ahead.

2.1 On network modeling

A network model is a mathematical model aimed to describe and analyze a system of interacting entities. For example, a network model might be used to understand how a pandemic spreads between individuals in a population, how internet users navigate between the pages of a website, or how goods are traded between countries. The variety of problems that can be meaningfully fit into this framework has made network models popular across both natural and social sciences.

The mathematical structure defining a network model is a *graph*. Formally, we define a graph $\mathcal{G}$ as a tuple $(\mathcal{N}, \mathcal{E}, G)$, where:

- $\mathcal{N}$ is a finite set of nodes,
- $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of links, where $e = (i, j) \in \mathcal{E}$ indicates the existence of a link from node i to node j, and
- $G \in \mathbb{R}_+^{\mathcal{N} \times \mathcal{N}}$ is a matrix describing the intensities, or *weights*, of each link.

The set of nodes $\mathcal{N}$ represents the entities of the network model. Depending on the purpose of the model, the entities may represent agents with preferences (such as consumers or companies), objects (such as cities or websites) or more abstract concepts (such as the outcomes of stochastic processes).

The nodes are pairwise connected by a set of links $\mathcal{E}$, where a link from a node i to a node j represents some way in which i is connected to j. If nodes represent cities, a link $e = (i, j)$ may signify that there exists a road from i to j; if nodes represent users of some social medium, the link may indicate that i follows j.

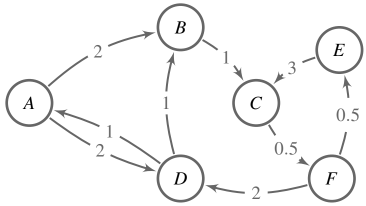

Figure 2.1 A directed, weighted graph of six nodes. Each link (i, j) is represented by an arrow from i to j, labeled with the value of G_{ij}.

Lastly, we may want to attach different *weights* to different links, representing the intensity of the interactions. These weights will be represented by a matrix G, where $G_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. As is conventional, we will assume that links are necessarily between distinct nodes, i.e., (i,i) will not be a link for any node i.

I will now introduce a few definitions that will be used in this thesis.

A graph $\mathcal{G}$ is *undirected* if and only if, for every pair of nodes i and j, $G_{ij} = G_{ji}$.

A graph $\mathcal{G}$ is *unweighted* if and only if, for every pair of nodes i and j, $G_{ij} \in \{0,1\}$.

A node i is *reachable* from a node $j \neq i$ if and only if there exists a sequence of nodes $S = (s_1 = j, s_2, ..., s_k = i)$ such that $(s_h, s_{h+1}) \in \mathcal{E}$ for all $h = 1, ..., k-1$.

A graph $\mathcal{G}$ is *strongly connected* if and only if every node is reachable from every other node. This is the case exactly when G is irreducible.

A node j is an *out-neighbor* (*in-neighbor*) of a node i if and only if $G_{ij} > 0$ ($G_{ji} > 0$). The *(out/in-)neighborhood* of i is the set of neighbors of i.

An *independent set* on a graph $\mathcal{G}$ is a set of nodes $\mathcal{S} \subset \mathcal{N}$ such that no node in $\mathcal{S}$ has a out-neighbor in $\mathcal{S}$.

A *maximal independent set* is an independent set that is not a strict subset of any other independent set.

A graph $\mathcal{G}$ is *bipartite* if and only if $\mathcal{N}$ can be partitioned into two independent sets.

EXAMPLE 1

Figure 2.1 describes a graph $\mathcal{G}$. A link from a node i to a node j is represented by an arrow from i to j, labeled by the weight of the of link G_{ij}. The graph is strongly connected, since every node can be reached from every other node. Node D has an in-neighborhood of $\{A, F\}$ and a out-neighborhood of $\{A, B\}$. Two of the many independent sets on $\mathcal{G}$ are $\{A, C\}$ and $\{B\}$. The set $\{A, C\}$ is a maximal independent set, while $\{B\}$ is not, since it constitutes a strict subset of another independent set: $\{B, F\}$. ꝏ

We will conclude this section with a set of important results relating above-defined network characteristics to the eigenvalues of the weight matrices.

THEOREM 1—PERRON-FROBENIUS[1]
Let $\mathcal{G}$ be a graph characterized by the weight matrix G. Then, G has a real, positive eigenvalue λ_G equal to its spectral radius $\rho(G)$. Furthermore,

1. if $\mathcal{G}$ is strongly connected, then the eigenvector associated with λ_G is element-wise positive and unique up to a scaling factor,
2. if $\mathcal{G}$ is undirected, then every eigenvalue λ of G is real in the interval $-\rho(G) \leq \lambda \leq \rho(G)$, and
3. $\mathcal{G}$ is bipartite if and only if for every eigenvalue λ of G, $-\lambda$ is also an eigenvalue of G. ∞

2.2 On game theory

Game theory studies the behavior of rational agents in strategic interactions. A game, in this formal sense, consists of a set of players who each individually choose some action and obtain a payoff that depends on the actions chosen by themselves and others.

Let $\mathcal{N}$ represent a set of players. Each player i chooses some action a_i from a set of possible actions $\mathcal{A}_i$. A *strategy profile* is a set $\mathbf{a} = \{a_i : i \in \mathcal{N}\}$ that specifies some combination of actions that could be jointly taken by the set of players. A utility function $u_i : \mathcal{A} \to \mathbb{R}$ specifies the payoff attained by i for each given strategy profile, where $\mathcal{A} = \prod_{i \in \mathcal{N}} \mathcal{A}_i$ is the space of all possible strategy profiles.[2] A game can hence be fully defined by a tuple $(\mathcal{N}, \mathcal{A}, \mathcal{U})$, where $\mathcal{U} = \{u_i : i \in \mathcal{N}\}$ is the set of utility functions.

Generally, players will be assumed to act with the goal of maximizing their attained utility, subject to the actions of other players. Let $\mathcal{A}_{\text{-}i}$ represent the set of strategies that can be jointly chosen by all players except i, and $\mathbf{a}_{\text{-}i}$ denote an arbitrary element from this set. The behavior of i can then be said to be governed by a *best response function*.

DEFINITION 1
For each player i and strategy profile $\mathbf{a}_{\text{-}i}$, the best response $\mathcal{B}_i : \mathcal{A}_{\text{-}i} \to \mathcal{P}(\mathcal{A}_i)$ is defined as

$$\mathcal{B}_i(\mathbf{a}_{\text{-}i}) = \underset{a_i \in \mathcal{A}_i}{\arg\max} \{u_i(a_i, \mathbf{a}_{\text{-}i})\}. \tag{2.1}$$

where $\mathbf{a}_{\text{-}i}$ denotes the actions of all players except i, and $\mathcal{P}$ is the power set operator. ∞

[1] For proofs, cf. theorems 4.1.3 and 8.4.4 in Horn and Johnson (2012) and proposition 3.4.1 in Brouwer and Haemers (2011).

[2] Here, we assume that the action space available to each player is not conditional on the choices of the other players. This is the case for all games considered in this thesis.

In the games considered in this thesis, the individually rational behavior of each player will be uniquely determined by the actions of the other players. The best response functions will hence be single-valued. A set of profiles of particular interest are those where every action is a best response to the actions of the other players.

DEFINITION 2
A *(pure-strategy) Nash equilibrium* is a profile $\mathbf{a}^*$ such that, for every player i,

$$a_i^* \in \mathcal{B}_i(\mathbf{a}^*_{-i}). \tag{2.2}$$

ꝏ

In words, a profile is a Nash equilibrium exactly if no player can increase their utility by deviating from their current action, assuming that the other players' actions remain unchanged. This concept can be generalized to stochastic actions, where the strategy of a player is defined as a probability distribution over her permissible actions. A *mixed Nash equilibrium* is, in this case, a vector of such strategies for which no player will benefit from unilaterally changing her own (probabilistic) strategy. This generalization is not insightful to the analysis in this thesis, and we will hence refer to the pure-strategy Nash equilibria as simply Nash equilibria.

2.3 Network games

The traditionally separate fields of network modeling and game theory have in recent years been combined with the goal of studying topics as diverse as juvenile crime [Patacchini and Zenou, 2012], firm R&D investment [Goyal and Moraga-Gonzalez, 2001], and social risk-sharing [Ambrus et al., 2014]. In these *network games*, players are represented by nodes on a graph, while a link (i, j) indicates that the utility of node i depends in some way on the action of node j. For our purposes, a network game is fully characterized by a graph $\mathcal{G}$, an action space $\mathcal{A}$ and a set of utility functions $\mathcal{U}$.

DEFINITION 3
A game $(\mathcal{N}, \mathcal{A}, \mathcal{U})$ is a *network game* on a graph $\mathcal{G}$ if and only if each player $i \in \mathcal{N}$ is represented by a node on $\mathcal{G}$, and each utility function u_i depends only of the actions of i and her out-neighbors in $\mathcal{G}$. ꝏ

A simple example of a network game is a coordination game.

EXAMPLE 2—A NETWORK COORDINATION GAME
Let a set of n nodes $\mathcal{N} = \{1, 2, \ldots, n\}$ play a binary coordination game on a graph $\mathcal{G}$. Each player i chooses a binary action $a_i \in \{0, 1\}$ and receives a payoff of

$$u_i(\mathbf{a}) = -\sum_{j \in \mathcal{N}} G_{ij} |a_i - a_j|, \tag{2.3}$$

where G is the weight matrix of the graph $\mathcal{G}$. ꝏ

In the coordination game, each node i receives 0 utility for each out-neighbor that chooses the same action as her, but a negative utility of G_{ij} if j chooses a different action. Nodes, aiming to maximize their utility, will hence strive to coordinate with (i.e. take the same action as) their out-neighbors, determining the relative importance of each neighbor by the weight of the link between them. Two trivial Nash equilibria of the game are $\mathbf{a} = \mathbf{0}$ and $\mathbf{a} = \mathbf{1}$, but certain network structures may admit additional solutions.

More generally, the coordination game induces a player i to take a higher action (i.e. to choose $a_i = 1$, rather than $a_i = 0$) if more of her neighbors take higher actions. We will refer to this game feature as *complementarity*.

DEFINITION 4
A game exhibits *(strategic) complementarity* if, for every pair of players i, j and strategies $z_i' \geq z_i$, $z_j' \geq z_j$:

$$u_i(z_i', z_j) - u_i(z_i, z_j) \leq u_i(z_i', z_j') - u_i(z_i, z_j'). \tag{2.4}$$

ↀ

Analogously to a coordination game, we can define an *anti-coordination game* as a game where nodes aim to take different actions from their neighbors.

EXAMPLE 3—A NETWORK ANTI-COORDINATION GAME
Let a set of n nodes $\mathcal{N} = \{1, 2, \dots, n\}$ play a binary anti-coordination game on a graph $\mathcal{G}$. Each player i chooses an action $a_i \in \{0, 1\}$ and receives a utility of

$$u_i(\mathbf{a}) = \sum_{j \in \mathcal{N}} G_{ij} |a_i - a_j|. \tag{2.5}$$

where G is the weight matrix of the graph $\mathcal{G}$. ↀ

Unlike the coordination game, this game does not in general allow trivial equilibria. An exception is if the graph is bipartite, i.e. if the set of nodes $\mathcal{N}$ can be partitioned into two independent sets of nodes. In this case, profiles where the set of nodes taking a particular action is exactly one of these subsets constitute equilibria. Additionally, in contrast to the coordination game, a player's optimal strategy in the anti-coordination game depends *negatively* on the strategies of her neighbors. This feature is common in network games and referred to as *substitutability*, since the actions of player's neighbors acts as a substitute for her own action.

DEFINITION 5
A game exhibits *(strategic) substitutability* if, for every pair of players i, j and strategies $z_i' \geq z_i$, $z_j' \geq z_j$:

$$u_i(z_i', z_j) - u_i(z_i, z_j) \geq u_i(z_i', z_j') - u_i(z_i, z_j'). \tag{2.6}$$

ↀ

Whether a real-world phenomenon exhibits complementarity, substitutability, or some combination thereof is an important consideration when designing a game-theoretic model of the phenomenon. Another such consideration is whether the action spaces of players are discrete or continuous. A common game form featuring continuous action spaces is a quadratic game.

Example 4—A Quadratic Game
Let $x_i \in \mathcal{A}_i = \mathbb{R}_+$ represent the level of effort exerted by player i. Then, a quadratic game can be defined by three parameters α, β, γ as

$$u_i(\mathbf{x}) = \alpha x_i + \beta x_i^2 + \gamma \sum_{j \in \mathcal{N}} G_{ij} x_i x_j, \tag{2.7}$$

where $\beta < 0$. ꝏ

Quadratic games on the form of Example 4 are among the most commonly analyzed network games. The character of the game is largely determined by the sign of γ. If γ is positive, the game exhibits complementarity – players will exert higher levels of effort the higher the efforts of their neighbors. If γ is negative, the effort levels of neighbors are instead substitutory – having a neighbor that exerts high levels of effort is a reason to exert less effort yourself.

Despite their popularity, quadratic games are not ideal for the study of public goods provision. The primary reason for this is that they are unsuccessful at simultaneously capturing positive externalities (nodes should always benefit when their neighbors increase their effort levels) and substitutability (nodes should respond to increased provision in their neighborhood by decreasing their own provision). The next section will present a few models that fulfill this desideratum.

3

Literature review

A number of network game models have been proposed to study the private provision of public goods on networks. In this section, I will describe four such models of particular prominence. The models differ in important ways – the action spaces may be discrete or continuous, the purpose of the analysis positive or normative – but several conclusions appear to be robust to this variance. Unless otherwise noted, all the following models assume the network graph to be undirected and unweighted.

3.1 Galeotti et al. (2010)

Galeotti et al. (2010) discuss a binary public goods game in which nodes can choose between being active ($x_i = 1$) or passive ($x_i = 0$). For every node, the action space is hence $\mathcal{A}_i = \{0, 1\}$. Being active incurs a private cost of c (where $0 < c < 1$), while remaining passive is free. However, each node enjoys 1 unit of utility as long as she herself, or at least one of her neighbors, is active. Equivalently, the utility of a node i is defined as

$$u_i(\mathbf{x}) = \begin{cases} 0, & x_i = \sum_{j \in \mathcal{N}} G_{ij} x_j = 0 \\ 1 - c, & x_i = 1 \\ 1, & \text{otherwise.} \end{cases} \tag{3.1}$$

In words, each node prefers to have an active neighbor, but would choose to be active herself if this is not the case. In particular, a Nash equilibrium is any profile where the set of active nodes constitute a maximal independent set on the network graph. In this case, every node is either active or has an active neighbor (but not both).

3.2 Bramoullé and Kranton (2007)

Bramoullé and Kranton (2007) consider a model where agents choose a level of private effort $e_i \in \mathbb{R}_+$, and benefit equally from all effort exerted in their neighborhood. The action space $\mathcal{A}_i$ is hence the set of non-negative real numbers. The utility of each agent is given by

$$u_i(\mathbf{e}) = b\Big(e_i + \sum_{j\in\mathcal{N}} G_{ij}e_j\Big) - ce_i \tag{3.2}$$

where $b(\cdot)$ is some strictly increasing and strictly concave benefit function. The individually optimal response dictates that nodes exert non-zero effort if and only if the aggregate effort in their neighborhood is less than some threshold value $e^* : b'(e^*) = c$. The authors discuss two particular classes of Nash equilibria for the game: *specialized profiles*, where $e_i = e^*$ or $e_i = 0$ for every node i, and *distributed profiles*, where $0 < e_i < e^*$ for every i. They find that the set of specialized equilibria is exactly the set of profiles where the nodes exerting non-zero effort constitute a maximal independent set on the network graph.

The paper also studies game outcomes in terms of welfare, defined as the aggregate utility of all nodes. In particular, they compute welfare-optimal effort profiles for simple graph structures and offer a method for comparing the relative welfare efficiency of different Nash equilibria. They conclude by analyzing the interaction between network structures and equilibrium welfare, noting that the addition of links has an ambiguous effect on aggregate welfare.

3.3 Allouch (2015)

Allouch (2015) studies private provision of public goods when individuals make explicit trade-offs between private goods consumption and public goods provision. Each node i has access to some exogenous wealth w_i, and chooses to spend it on a combination of private goods x_i and public goods q_i, with each unit of either good costing 1 unit of wealth. Possible actions are hence tuples (x_i, q_i) from the action set

$$\mathcal{A}_i = \{x_i,\, q_i \geq 0 : x_i + q_i = w_i\}. \tag{3.3}$$

As in Bramoullé and Kranton (2007), nodes benefit fully from all public goods provisioned in their neighborhoods. Utility functions are strictly increasing and strictly quasiconcave on the form

$$u_i\Big(x_i,\, q_i + \sum_{j\in\mathcal{N}} G_{ij}q_j\Big). \tag{3.4}$$

Allouch finds that the Nash equilibrium profile is unique under a *network normality* assumption, associating the Engel curves of the players[3] to the lowest eigenvalue of the adjacency matrix. As is common in public goods games, the equilibrium

[3] i.e. each player's demand for public goods as a function of her wealth.

generally consists of a set of *contributors* – nodes with non-zero provisioning of public goods, and *non-contributors*, for which $q_i = 0$ in equilibrium.

Inspired by a classic neutrality result from Bergstrom et al. (1986), Allouch moves on to consider the impact of income transfers on the provision of public goods. He finds that in the specified network game, a transfer will change the amount of public goods provided unless the set of contributors constitute a clique[4] on the network graph. In general, changes in aggregate provision are instead related to the Bonacich centrality of the affected nodes.

3.4 Elliott and Golub (2019)

Elliott and Golub (2019) study Pareto efficient outcomes in a general formulation of a public goods game.[5] Nodes can exert some effort $a_i \geq 0$ that is costly to themselves, but beneficial to others in the network. Utility functions $u_i(\mathbf{a})$ are concave, strictly decreasing in a_i, and weakly increasing in a_j for $j \neq i$. As a consequence, the trivial equilibrium of the game is, by design, $\mathbf{a} = \mathbf{0}$. The authors find that the scope for Pareto-efficient cooperation is related to a benefit matrix $B(\mathbf{a})$, consisting of the marginal rates of substitution between one's own effort and those of one's neighbors:

$$B_{ij}(\mathbf{a}) = \begin{cases} -\frac{\frac{\mathrm{d}u_i}{\mathrm{d}a_j}}{\frac{\mathrm{d}u_i}{\mathrm{d}a_i}}, & i \neq j, \\ 0, & i = j. \end{cases} \tag{3.5}$$

In particular, an allocation $\mathbf{a}$ is Pareto efficient if and only if the spectral radius of $B(\mathbf{a})$ is equal to 1. If this is not the case, then there exists a vector $\mathbf{d}$ such that for some small $\varepsilon > 0$,

$$u_i(\mathbf{a} + \varepsilon \mathbf{d}) > u_i(\mathbf{a}) \tag{3.6}$$

for every i. That is, if every node agrees to increase her effort by some tiny amount εd_i, then the utility gain from the increased efforts of others outweigh the personal cost, and every node is made better off.

A particular set of Pareto efficient profiles, *Lindahl outcomes*, are characterized by a centrality property: $B(\mathbf{a})\mathbf{a} = \mathbf{a}$. Elliott and Golub show that these outcomes share a particular microfoundation – they arise when nodes face a certain set of individualized subsidies and taxes reminiscent of Lindahl taxes. This is exemplified by a normative analysis of the model from Bramoullé and Kranton (2007).

[4] A clique is a subset of nodes where every pair of nodes is connected by a link.
[5] For a formal definition of Pareto efficiency, see section 5.1.

4

Game specification and equilibrium analysis

In this section, we will introduce the game considered in this thesis. The game draws on the specification of Allouch (2015), but is more specific in preferences and more general in the structure of the network. We will then derive the individually rational behavior of the players in this game, and characterize equilibrium outcomes. Lastly, we will prove a sufficient condition for the uniqueness of equilibrium, and explore a particular interpretation of the equilibrium under this assumption.

4.1 The game

A set of players $\mathcal{N}$ are connected as nodes on a directed graph characterized by the weight matrix G. The nodes may represent private individuals, firms, or larger, cohesive agents like nation states. Each node i has access to some exogenous wealth $w_i > 0$ that she chooses to spend on a combination of private goods (denoted by $x_i \geq 0$) and public goods provision ($q_i \geq 0$). For our purposes, private goods are goods that primarily benefit the purchaser (e.g. food for an individual, or a new highway for a state), while public goods are goods whose benefits accrue to a larger set of actors (e.g. donations to the local sports team for an individual, or increased funding for climate change research for a state).

Let 1 unit of either good cost 1 unit of wealth, so that the allocation chosen by a node must satisfy a budget constraint of $x_i + q_i = w_i$. Each node benefits from her own private goods consumption only, but from the public goods provisioned by all her out-neighbors. Individual preferences over consumption bundles are described by a Cobb-Douglas utility function

$$u_i(\mathbf{x}, \mathbf{q}) = x_i^\alpha \Big(q_i + \sum_{j \in \mathcal{N}} G_{ij} q_j \Big)^\beta \tag{4.1}$$

where the preference parameters $\alpha, \beta \in (0,1)$ are shared among nodes. A strategy profile is a vector $\mathbf{q} = (q_i : 0 \leq q_i \leq w_i,\ i \in \mathcal{N})$ that describes a certain combination of feasible choices made by the nodes. Since $\mathbf{x}$ can be inferred from the budget constraint, the profile $\mathbf{q}$ sufficiently describes the choice of each node. We will assume throughout that the graph G is strongly connected.

An instance of the game can be fully described by a tuple $(G, \mathbf{w}, \alpha, \beta)$. This game specification has a set of notable properties that shape the behavior of the nodes. Some properties are common in public goods models, while others are more unique to the game considered in this thesis.

Marginal returns. The Cobb-Douglas utility function is strictly increasing in x_i and q_i. Nodes hence exhibit a strict preferences for higher levels of consumption of both private and public goods, regardless of their consumption levels. The marginal returns to each type of good, however, is decreasing in that type of good and increasing in the other type of good. The marginal return to private goods ($\frac{\mathrm{d}u_i}{\mathrm{d}x_i}$), for example, is decreasing in x_i but increasing in q_i. This feature aims to capture the intuition that individuals benefit more from an additional unit the less they have of the good in question.

Positive externalities. Since $G_{ij} \geq 0$ by assumption, increased public goods provision by one node has weakly positive effects on every other node. While the benefits of increasing x_i accrue exclusively to node i, the benefits of increasing q_i are distributed throughout her out-neighborhood. In this context, it is likely that selfish nodes will contribute less to public goods than their neighbors would want them to, and that everyone could be made better off by increased cooperation. This will be a topic of study in a later part of this thesis.

Substitutability. The more public goods are provisioned in a node's neighborhood, the less she will benefit on the margin from increasing her own provision. In particular, G_{ij} denotes node i's rate of substitution between her own public goods provisioning and that of her neighbor j. That is, i would be indifferent between a decrease of G_{ij} units in q_i, and a decrease of 1 unit in q_j. In the important special case where G is a binary matrix, a node is indifferent between an increase in her own public goods provision and an equal increase in the provision of one of her neighbors. This game feature is commonly referred to as *full substitutability*.[6]

Heterogeneity. Unlike most previous network models of public goods, this specification makes no assumption on the relative magnitude of weight links. This heterogeneity is ubiquitous in the real world – some individuals have the opportunity to greatly help others with little effort, while others must make significant sacrifices to confer even small benefits on their fellows. In addition, relations of externalities are asymmetrical as a rule – for example, the way that the actions of a state affects a company bears little similarity to how the actions of the company affects

[6] A recent real-world example of substitutability in the funding of public goods is when the US government on April 14th, 2020 halted its funding of the World Health Organization. Less than 24 hours later, the Bill & Melinda Gates Foundation and the Finnish government had both announced that they would increase their funding in response.

the state. Our specification aims to capture these dynamics better than the norm of unweighted, undirected networks.

4.2 Best response function and existence of equilibrium

Consider a game setting where each node chooses to spend her wealth w_i on some combination of private and public goods. Aiming to maximize utility and taking the actions of other nodes as given, a node i will decide on a level of public goods provision q_i^* as:

$$q_i^* = \underset{0 \leq q_i \leq w_i}{\arg\max} \quad (w_i - q_i)^\alpha \Big(q_i + \sum_{j \in \mathcal{N}} G_{ij} q_j\Big)^\beta \tag{4.2}$$

where $x_i = w_i - q_i$ incorporates the budget constraint. Since u_i is concave in q_i for a fixed w_i, the optimal choice is uniquely defined. We can hence define, for a given $\mathbf{q}_{\text{-}i}$, the best response function $\mathcal{B}_i(\mathbf{q}_{\text{-}i}) = q_i^*$ as the optimal public goods provision for individual i.

PROPOSITION 1
For any node i and profile $\mathbf{q}_{\text{-}i}$, the best response is found as

$$\mathcal{B}_i(\mathbf{q}_{\text{-}i}) = \max\left\{\frac{1}{\alpha+\beta}\left(\beta w_i - \alpha \sum_{j \in \mathcal{N}} G_{ij} q_j\right), 0\right\} \tag{4.3}$$

ꝏ

The derivation of Proposition 1 is found in Section A.1 of the appendix. Notably, the best response is negatively linear in the provision of neighbors down to a saturation at 0. Furthermore, the highest rational public good provision is $\frac{\beta}{\alpha+\beta} w_i$, which is optimal for nodes with no out-neighbors contributing to public goods.

Recall that a Nash equilibrium is a profile $\mathbf{q}^*$ for which $q_i^* = \mathcal{B}_i(\mathbf{q}^*_{\text{-}i})$ for every node i. This means that no node would be better off by changing its strategy unilaterally. The set of such equilibria can be helpfully studied by letting $\mathcal{C} = \{i : q_i^* > 0\}$ denote the set of nodes that contribute to public goods in each equilibrium. Applying the best response condition, we find that a profile $\mathbf{q}^*$ is a Nash equilibrium if and only if

$$\left(I + \frac{\alpha}{\alpha+\beta} G_{\mathcal{C}}\right) \mathbf{q}^*_{\mathcal{C}} = \frac{\beta}{\alpha+\beta} \mathbf{w}_{\mathcal{C}} \tag{4.4}$$

$$\alpha G_{\text{-}\mathcal{C},\mathcal{C}} \mathbf{q}^*_{\mathcal{C}} \geq \beta \mathbf{w}_{\text{-}\mathcal{C}} \tag{4.5}$$

where the subscript $\mathcal{C}$ indicates the submatrix or subvector corresponding to the set of contributors.[7] The first constraint corresponds to the choice of the contributors, and the second to the choice of the non-contributors.

PROPOSITION 2
For any graph structure G, wealth distribution $\mathbf{w}$, and preference constants α, β, there exists at least one Nash equilibrium profile. ↀ

Proof. Let $\mathcal{B}(\mathbf{q})$ denote the vector of best responses for each node in an allocation $\mathbf{q}$. A profile $\mathbf{q}^*$ is then a Nash equilibrium if and only if it is a stationary point on $\mathcal{B}$. Since $\mathcal{B}$ is a continuous function from a compact, convex set to itself, the existence of a stationary point follows directly from Brouwer's fixed point theorem.

4.3 A few examples of equilibria

EXAMPLE 5—SYMMETRIC RING GRAPH
Consider a symmetric ring graph of 4 nodes, where each node is connected to its neighbors by a link with a weight of 1. Let $w_i = 1$ for every i and $\alpha = \beta = 0.5$. Then, the single Nash equilibrium profile is $(x_i^*, q_i^*) = (\frac{3}{4}, \frac{1}{4})$ for every i. Notably, each node provides less public goods than she would in the absence of a network ($\frac{1}{4}$, compared to $\frac{1}{2}$ without a network), but benefits from a larger total neighborhood provision.

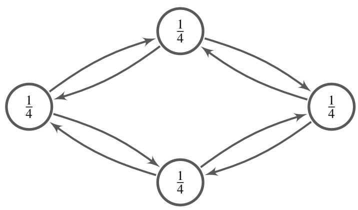

Figure 4.1 Symmetric ring graph. Nodes are labeled by their public goods provision in equilibrium.

EXAMPLE 6—DIRECTED LINE GRAPH
Let n identical nodes be connected through a directed line graph, where each node benefits from the provision of the node to the right (see Figure 4.2). Given the same wealth distribution and preferences as in Example 5, the single equilibrium outcome

[7] Similarly, the subscript -$\mathcal{C}$ denotes non-contributors. $G_{-\mathcal{C},\mathcal{C}}$ is hence the submatrix of G containing the rows corresponding to non-contributors and the columns corresponding to contributors.

is (from left to right):

$$\mathbf{q} = \left[\ldots, \frac{11}{32}, \frac{5}{16}, \frac{3}{8}, \frac{1}{4}, \frac{1}{2} \right] \tag{4.6}$$

The top provider is the only node that does not benefit from the efforts of others. The second-rightmost node, free-riding on the provision of her neighbor, contributes the least. This, in turn, leads the third-rightmost node to provide a higher amount. The oscillation pattern continuous indefinitely, with provision levels approaching $\frac{1}{3}$ asymptotically with each step to the left.

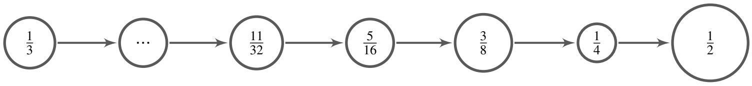

Figure 4.2 Directed line graph. Node labels and sizes correspond to public goods provision in equilibrium.

EXAMPLE 7—MULTIPLE EQUILIBRIA
Consider an undirected star graph of 5 nodes, with the center being node 1. Let $\alpha = \beta = \frac{1}{2}$ and $w_1 = 2$, $w_k = 1$ for $k \geq 2$. This game admits an infinite set of Nash equilibria, with three possible sets of contributing nodes. For any $\theta \in [0,1]$, the profile $q_1 = 1 - \theta$, $q_k = \theta/2$ for $k \geq 2$ constitutes an equilibrium. Notable, the set of equilibria constitutes a convex set. This is, however, not necessarily true in the general case.

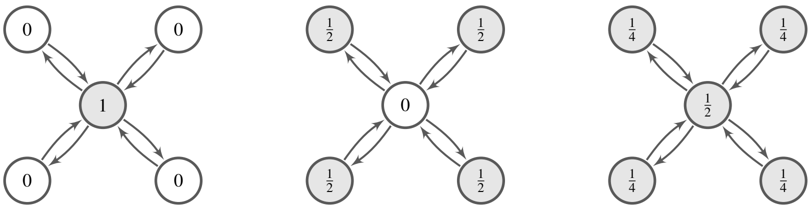

Figure 4.3 Three equilibria for the same game. Contributors in grey.

4.4 Specialized equilibria

A common feature of public goods games on networks, found in e.g. Galeotti et al. (2010) and Bramoullé and Kranton (2007), is the existence of equilibria where the set of contributors constitute a maximal independent set on the network graph. Formally, this means that there exists an equlibrium profile $\mathbf{q}^*$ such that

$$q_i^* = 0 \iff \sum_{j \in \mathcal{N}} G_{ij} q_j^* > 0 \tag{4.7}$$

for every i. This class of outcomes is referred to as *specialized equilibria*, since they involve a set of nodes that specialize in the provision of public goods, and a disjoint set of nodes that free-ride on the provision of their neighbors.

Unlike the games of Galeotti et al. (2010) and Bramoullé and Kranton (2007), the game considered in this thesis rarely admits specialized equilibria when nodes are homogeneous.

PROPOSITION 3
Consider a game on an arbitrary directed graph G. Let $w_k = w$ for every node k and some $w > 0$. Then, G allows a specialized equilibrium if and only if there exists a maximal independent set $\mathcal{C}$ such that

$$\sum_{j \in \mathcal{C}} G_{ij} \geq \frac{\alpha + \beta}{\alpha} \tag{4.8}$$

for every $i \notin \mathcal{C}$. ⚭

This result, derived in Section A.2, places a significant boundary on the existence of specialized equilibria in our game when wealth is equally distributed. In the important special case of an unweighted graph, when $G_{ij} \in \{0, 1\}$, the link-value criterion is effectively a criterion on the number of out-neighbors of non-contributing nodes. If $\alpha = \beta$, for example, each non-contributing node must have at least two contributing neighbors. This appears to be a fairly strict requirement.

Why is it the case that specialized equilibria are seemingly rare in our game, existing only for a particular subset of possible graphs? The informal explanation is that a node with a public goods-providing neighbor is, in some sense, richer than an identical node without such a neighbor. As a consequence of the Cobb-Douglas preferences, a richer node will demand more public goods, and fund the difference between her own demand and the provision of her neighbor from her own wealth. This contrasts with the models of Galeotti et al. (2010) and Bramoullé and Kranton (2007), where the provision level at which a node is not willing to fund additional public goods is instead identical for all nodes.

4.5 Best response dynamic and uniqueness of equilibrium

We have now determined that there exists at least one profile $\mathbf{q}^*$ from which no node would unilaterally deviate. Do we have reason to believe that such an equilibrium outcome would arise naturally, e.g. if nodes interact and update their behavior selfishly for a long time? To investigate this, let $\mathcal{B}$ denote the vector of individual best responses, and define a *best response dynamic* as a sequence $\{\mathbf{q}_t, t = 0, 1, ...\}$ for which

$$\mathbf{q}_t = \begin{cases} \mathbf{q}^0, & t = 0 \\ \mathcal{B}(\mathbf{q}_{t-1}), & t \geq 1, \end{cases} \tag{4.9}$$

for some starting profile $\mathbf{q}^0$. An interpretation of this process is that, at each time step t, every node observes the choices of the other nodes at $t-1$, and updates her action to the optimal response to those choices. Notice that the stationary points of this sequence is exactly the set of Nash equilibria; $\mathbf{q}_t = \mathbf{q}_{t-1}$ if and only if $\mathbf{q}_t$ is an equilibrium of the game. We can hence conclude that if the sequence at any point converges to some profile, then that profile necessarily constitutes an equilibrium of the game. Furthermore, for every equilibrium $\mathbf{q}^*$, there exists at least one starting profile $\mathbf{q}^0$ for which the sequence converges to $\mathbf{q}^*$ – a trivial example is the case where the starting profile is equal to the equilibrium point.

As shown in Example 7, we can find game parameters for which the equilibrium profile is not uniquely defined. If we interpret the set of Nash equilibria as the set of outcome that are, in some sense, likely to arise when nodes interact, this lack of uniqueness hinders analysis. Allouch (2015) constitutes a possible starting point for determining the conditions under which we can guarantee that the equilibrium is unique. He studies a game that is a generalization of our game in terms of possible preferences (see Section 3.3), but more restrictive in terms of the structure of the network. Notably, he considers only undirected and unweighted graphs, whereas we aim to describe arbitrary network structures.

THEOREM 2—ADAPTED FROM ALLOUCH (2015)
For undirected and unweighted graphs $\mathcal{G}$, the Nash equilibrium is unique if

$$|\lambda_{\min}(G)| < \frac{\alpha + \beta}{\alpha} \tag{4.10}$$

where $\lambda_{\min}(G)$ is the lowest eigenvalue of G. ∞

This result serves as a useful precursor to our uniqueness analysis. The lowest eigenvalue is well-defined, since Theorem 1 guarantees that all eigenvalues will be real. As is common in network game theory, however, the proof relies heavily on the symmetry of G, which holds only for undirected networks. To make progress on the more general case of weighted and directed networks, we will make a slightly stricter assumption.

DEFINITION 6
A game specified by the network G and preference constants α, β exhibits *limited network effects* (LME) if and only if

$$\rho(G) < \frac{\alpha + \beta}{\alpha} \tag{4.11}$$

where $\rho(G)$ denotes the spectral radius of G. ↀ

This assumption is stricter than that of Allouch, since $\rho(G) \geq |\lambda|$ for any eigenvalue λ of G. The conditions are equivalent in the special case where the network graph is bipartite, since it follows from Theorem 1 that such a graph has eigenvalues symmetric around 0. Under LME, the best response dynamics has the desirable property of converging toward a certain profile, regardless of starting position.

LEMMA 1
Assume limited network effects. Then, the best response dynamic is contractive; for every graph G there exists a vecor norm $\|\cdot\|$ such that for any pair of profiles $\mathbf{q}^1 \neq \mathbf{q}^2$:

$$\|\mathcal{B}(\mathbf{q}^1) - \mathcal{B}(\mathbf{q}^2)\| < \|\mathbf{q}^1 - \mathbf{q}^2\|. \tag{4.12}$$

ↀ

Consider now that there exist two distinct Nash equilibria, $\mathbf{q}^*$ and $\mathbf{q}^{**}$. Since both are stationary points of the best response, this would imply that:

$$\|\mathcal{B}(\mathbf{q}^*) - \mathcal{B}(\mathbf{q}^{**})\| = \|\mathbf{q}^* - \mathbf{q}^{**}\| \tag{4.13}$$

which contradicts Lemma 1. This proves our uniqueness result.

THEOREM 3
Under limited network effects, the Nash equilibrium is unique for any non-negative weight matrix G. ↀ

As with other key results, the proof of Lemma 1 is provided in the appendix (Section A.3). We have now found that under an assumption only marginally stricter than that of Allouch (2015), the uniqueness result extends to arbitrary networks. Furthermore, the uniqueness criterion is independent of the wealth distribution $\mathbf{w}$. We will now move on to study the shape of the equilibrium outcome under this assumption.

4.6 Geometric series interpretation

In the previous section we found that the LME assumption was sufficient to guarantee uniqueness of equilibrium. In this section, we will explore another implication of the assumption, on the shape of the equilibrium public goods provision. Recall from the equilibrium analysis that a necessary condition for the individual rationality of contributing nodes is

$$\left(I+\frac{\alpha}{\alpha+\beta}G_{\mathcal{C}}\right)\mathbf{q}^*_{\mathcal{C}}=\frac{\beta}{\alpha+\beta}\mathbf{w}_{\mathcal{C}} \tag{4.14}$$

where $\mathcal{C}=\{i: q_i^*>0\}$ indicates the set of contributors in equilibrium. Recall that under LME, $\rho(G)<\frac{\alpha+\beta}{\alpha}$. Since $G_{\mathcal{C}}$ is a submatrix of G, $\rho(G_{\mathcal{C}})\leq\rho(G)$ by necessity. As a result, we find that the matrix

$$\left(I+\frac{\alpha}{\alpha+\beta}G_{\mathcal{C}}\right) \tag{4.15}$$

cannot have 0 among its eigenvalues. The matrix is hence invertible and we can compute $\mathbf{q}^*_{\mathcal{C}}$ as:

$$\mathbf{q}^*_{\mathcal{C}}=\frac{\beta}{\alpha+\beta}\left(I+\frac{\alpha}{\alpha+\beta}G_{\mathcal{C}}\right)^{-1}\mathbf{w}_{\mathcal{C}}. \tag{4.16}$$

Since the equilibrium is unique, an equilibrium profile exists for only a single possible set of contributors $\mathcal{C}$, that depends on the wealth distribution $\mathbf{w}$. For a given equilibrium, however, the public good provision of each contributing node will be a linear combination of the wealth of other contributing nodes.

The equilibrium contribution can be understood further by exploiting a well-known relationship between the inverse of a matrix and the geometric series.

LEMMA 2—HORN AND JOHNSON (2012)[8]
For any square matrix A, it holds that

$$\rho(A)<1 \iff (I-A)^{-1}=\sum_{k=0}^{\infty}A^k. \tag{4.17}$$

ꝏ

Letting $A=-\frac{\alpha}{\alpha+\beta}G_{\mathcal{C}}$, the assumption $\rho(A)<1$ coincides exactly with the LME assumption.

PROPOSITION 4
Assume limited network effects. Then, equilibrium public goods contributions are given by

$$\mathbf{q}^*_{\mathcal{C}}=\frac{\beta}{\alpha+\beta}\left(\sum_{k=0}^{\infty}(-1)^k\left(\frac{\alpha}{\alpha+\beta}\right)^k G_{\mathcal{C}}^k\right)\mathbf{w}_{\mathcal{C}}. \tag{4.18}$$

ꝏ

[8] Theorem 5.6.15.

This result follows directly from applying Lemma 2 to Equation 4.16, letting $A = -\frac{\alpha}{\alpha+\beta}G_{\mathcal{C}}$. Rewriting Proposition 4 node-wise, we find that for each $i \in \mathcal{C}$:

$$q_i^* = \frac{\beta}{\alpha+\beta} \sum_{j \in C} \left(\sum_{k=0}^{\infty} (-1)^k \left(\frac{\alpha}{\alpha+\beta} \right)^k \rho_{jh}^{(k)} \right) w_j \tag{4.19}$$

where $\rho_{jh}^{(k)}$ is the aggregate weight of walks of length k from j to h in the subgraph consisting of only contributors. So, the contributions are a linear combination of the wealth of the contributors, with weights that are larger in magnitude for shorter walks and with alternating signs depending on whether the length is even or odd. When the set of contributors is strongly connected, cycles between them imply that the number of such paths is infinite. The impact of a path on provision is, however, declining exponentially in the length of the path.

This exercise highlights the fact that a game of substitutes on a network is equivalent to an infinite sum of pairwise games of substitutes and complements. Every node plays games of substitutes with their out-neighbors, decreasing their own contributions as their neighbors contribute more. Simultaneously, however, the out-neighbors react the same way to the choices of *their* out-neighbors. So, in effect, the more the neighbors of a node's neighbors contribute, the less her neighbors contribute, and the more she contributes in turn. The efforts of a neighbor k steps away is complementary if k is even, and substitutory if k is odd.

A special case of some interest is when the subgraph of contributors is bipartite, i.e. when the set of contributing nodes can be partitioned into two subsets such that each node has no out-neighbors in its own subset. In this case, the equilibrium contribution of each node is increasing in the wealth of nodes in her own subset, and decreasing in that of the nodes in the other subset.

5

Normative analysis

So far, this thesis has aimed to describe the set of outcomes that are likely to realize when individuals act selfishly, assume others to act selfishly, and have no enforceable means of cooperation. A standard result of the economic literature on public goods is that this individually rational outcome is likely to be collectively inefficient in the presence of externalities.

In this section, I will begin by formalizing two common metrics of welfare efficiency – utilitarian efficiency and Pareto efficiency – and show that equilibrium outcomes of our game are generally inefficient with respect to these metrics. I will then explore how these inefficiencies can be alleviated by respecifying certain aspects of the game. I will refer to these respecifications as *mechanisms*, in a slightly wider definition of the term than that of e.g. Hurwicz and Reiter (2006). The mechanisms will preserve the non-cooperative nature of the game, but alter the optimization problems facing the individuals by changing preferences, constraints or enforceability of cooperation. Specifically, I will consider the effects of taxes & subsidies, enforceable contracts, and redistribution.

5.1 Efficiency metrics

To evaluate the desirability of different outcomes on impartial grounds, we need to define criteria of welfare efficiency. Two commonly used such criteria, suitable for our purposes, are *Pareto efficiency* and *utilitarian efficiency*. I will here define them formally, and make a few remarks about how they relate to the game considered in this thesis.

DEFINITION 7
A profile $\mathbf{q}$ is (weakly) Pareto-efficient if and only if, for every other profile $\tilde{\mathbf{q}}$:

1. there exists an individual i such that $u_i(\mathbf{q}) > u_i(\tilde{\mathbf{q}})$, or
2. $u_j(\mathbf{q}) = u_j(\tilde{\mathbf{q}})$ for every j. ∞

Analogously, I will say that a profile $\mathbf{q}$ constitutes a *Pareto improvement* from $\tilde{\mathbf{q}}$ if and only if $u_i(\mathbf{q}) \geq u_i(\tilde{\mathbf{q}})$ for every i. Pareto efficiency is a weak welfare criterion in the sense that, in many circumstances, the set of efficient outcomes is rather large. An equivalent formulation is that a profile $\mathbf{q}$ is Pareto efficient if and only if it maximizes $\sum_i \theta_i u_i(\mathbf{q})$ for some set of "welfare weights" $\theta_i > 0$, $i \in \mathcal{N}$. This alternative formulation emphasizes that an outcome can be Pareto efficient even if it values the welfare of some nodes much higher than that of other nodes. In fact, this property of not requiring (nor allowing) interpersonal welfare comparisons is often put forward as both the primary virtue of and strongest objection to Pareto efficiency. The Pareto criterion is typically not described as a sufficient condition for a good outcome, but the lack of Pareto efficiency is used to argue that an outcome is obviously bad. Unfortunately, equilibrium outcomes in our game are typically not Pareto efficient.

PROPOSITION 5
An equilibrium profile $\mathbf{q}^*$ is Pareto inefficient if there exists two contributors i and j that are mutually reachable on the subgraph of contributors. ꝏ

The proof (found in Section A.4) builds on the fact that in equilibrium, contributors are almost-indifferent to increasing q_i on the margin, but would strictly prefer for their out-neighbors to increase their public goods provision. So, for any cycle on the subgraph of contributors, there exists an $\varepsilon > 0$ such that if everyone on the cycle increased their public good provision by ε, they would all be made better off. In addition, the nodes that did not form part of the cycle would not be made worse off, since at least as much public goods would be provisioned in their neighborhood as before. This means that such an increase would constitute a Pareto improvement, and hence the equilibrium profile could not be efficient.

An alternative measure of welfare efficiency is what what we will call *utilitarian efficiency*.

DEFINITION 8
A profile $\mathbf{q}$ is *utilitarian-efficient* if and only if for every other profile $\tilde{\mathbf{q}}$:

$$U(\mathbf{q}) = \sum_{i \in \mathcal{N}} u_i(\mathbf{q}) \geq \sum_{i \in \mathcal{N}} u_i(\tilde{\mathbf{q}}) = U(\tilde{\mathbf{q}}) \tag{5.1}$$

ꝏ

I will sometimes refer to a utilitarian-efficient profile as a *social optimum*. The use of this terminology presupposes some additional philosophical assumptions, e.g. that the individual utility functions are cardinal metrics of welfare, that interpersonal utility comparisons are possible and calibrated by the utility functions, and that the social goal is to maximize aggregate welfare.

Utilitarian efficiency is a stricter criterion than Pareto efficiency in the sense that every utilitarian-efficient profile is guaranteed to be Pareto efficient. In fact, using the welfare weights definition of Pareto efficiency described above, the set of utilitarian-efficient outcome is exactly the subset of Pareto-efficient outcomes for which $\theta_i = \theta$ for every i.

By most standards, a Pareto-inefficient outcome is more obviously undesirable than a utilitarian-inefficient outcome. In the latter case, there exists an alternative outcome in which the population *as a whole* can be made better off, but this might come at the expense of certain individuals. In the former case, however, there are alternative outcomes in which *everyone* is better off.

THEOREM 4
If $\alpha + \beta < 1$, then the aggregate welfare function

$$U(\mathbf{q}) = \sum_{i \in \mathcal{N}} u_i(\mathbf{q}) \tag{5.2}$$

is strictly concave over the hyperrectangle $0 \leq \mathbf{q} \leq \mathbf{w}$. ꝏ

This result, derived in Section A.5, follows from the concavity of the individual utility functions. Theorem 4 guarantees that a utilitarian optimum is easy to find for a given game $(G, \mathbf{w}, \alpha, \beta)$ as long as $\alpha + \beta \leq 1$. In addition, the strict concavity of the function and the convexity of the domain implies that the optimal profile is unique.

LEMMA 3
If $\alpha + \beta < 1$, then there exists a unique utilitarian-efficient profile $\mathbf{q}^u$. ꝏ

In the case where $\alpha + \beta = 1$, the function $U(\mathbf{q})$ is only weakly concave. The utilitarian optimum is then not necessarily unique, but the set of optimal profiles is still guaranteed to be convex.

I will now consider three mechanisms commonly proposed to alleviate public goods problems: taxes & subsidies, enforceable contracts, and redistribution. For each mechanism, I will give a brief introduction, present a formalization suited for our network game, and discuss some ways in which outcomes are improved with respect to efficiency metrics.

5.2 Taxes & subsidies

The fundamental welfare problem of public goods is that self-interested individuals disregard the effects of their actions on others when choosing which action to take. A common approach to remedy this inefficiency is to design rules that aim to *internalize* these effects into the decision process of each individual, aligning the selfish interest with the public interest. When the decision in question relates to what goods to buy, a natural mechanism is to adjust the relative prices of different goods, reducing the prices of goods with social benefits, and increasing the prices of goods that do harm to others. Several theoretical approaches have been proposed to set such prices[9] and in the real world, many governments subsidize pro-social behavior (such as philanthropic donations) while taxing activities with negative external effects (like alcohol consumption).

Within our game formalization, consider a government planner that aims to maximize aggregate welfare and has the capacity to subsidize and tax the consumption of each individual. Specifically, the planner may set individual prices P_x^i, P_q^i for every i, such that i must pay P_x^i units of wealth for 1 unit of private goods, and P_q^i units of wealth for 1 unit of public goods. Node i will then choose an allocation (x_i^s, q_i^s) as:

$$(x_i^s, q_i^s) = \arg\max_{x_i, q_i} \quad u_i(\mathbf{x}, \mathbf{q}) \tag{5.3}$$

$$= \arg\max_{x_i, q_i} \quad x_i^\alpha \Big(q_i + \sum_{j \in \mathcal{N}} G_{ij} q_j \Big)^\beta \tag{5.4}$$

$$\text{s.t. } P_x^i x_i + P_q^i q_i = w_i \tag{5.5}$$

$$x_i, q_i \geq 0. \tag{5.6}$$

Call this game of individualized prices a *subsidized game*. Denoting the aggregate public goods provisioned in the neighborhood of i by $Q_i \equiv q_i + \sum_{j \in \mathcal{N}} G_{ij} q_j$, with appropriate superscripts, we find a simple relationship between the prices of a subsidized game and the allocations of contributors.

PROPOSITION 6
In any subsidized game, and for any contributor i:

$$\frac{Q_i^s}{x_i^s} = P_i \cdot \frac{\beta}{\alpha}, \quad \text{where } P_i \equiv \frac{P_x^i}{P_q^i}. \tag{5.7}$$

ꝏ

In words, the optimal ratio between private good and public good consumption is linear in the relative price of the two goods. As a result, there is a predictable relationship between the prices set by the planner and the allocation choices of the nodes. Combined with knowledge of the structure of favorable outcomes, this result can be used to set socially preferable prices.

COROLLARY 1
Let $\mathbf{q}^u$ be a utilitarian-efficient profile in a non-subsidized game. Then, for every contributor i:

$$\frac{Q_i^u}{x_i^u} = M_i \cdot \frac{\beta}{\alpha} \tag{5.8}$$

$$\text{where } M_i = 1 + \sum_{j \in \mathcal{N}} G_{ji} \frac{u_j(\mathbf{q}^u)/Q_j^u}{u_i(\mathbf{q}^u)/Q_i^u} \tag{5.9}$$

ꝏ

[9] Notable examples are Lindahl taxes (cf. Roberts (1974)) and Pigouvian taxes (cf. Pigou (1920)).

Here, M_i can be interpreted as a multiplier effect from the public goods provision of i; the social value of q_i on the margin is a factor M_i greater than its value to i. We can note that the optimal ratio between Q_i and x_i is equal to the equilibrium ratio only for nodes with no out-neighbors – the only case when the socially optimal action is identical to the selfish action is in the absence of externalities.

Corollary 1 describes the socially optimal allocation in the utilitarian optimum, and Proposition 6 describes how prices can be set to incentivize arbitrary allocations for any node. Together, these results indicate that the social optimum can always be implemented in an incentive-compatible way through personalized subsidies.

THEOREM 5
Assume that a social optimum $\mathbf{q}^u$ from a non-subsidized game is known. Then $\mathbf{q}^u$ will be a Nash equilibrium of the subsidized game with prices

$$\frac{P_x^i}{P_q^i} = \begin{cases} M_i, & q_i^u > 0 \\ 1, & q_i^u = 0 \end{cases} \tag{5.10}$$

and $P_x^i x_i^u + P_q^i q_i^u = w_i$ for every i. ∞

This result establishes, perhaps unsurprisingly, that a planner with the capacity to set individualized prices, as well as knowledge of a social optimum, can set prices that implement said optimum in an incentive-compatible way. Since the problem of finding a social optimum is concave (see Theorem 4), knowledge of game parameters is in practice sufficient to find this profile.

On a theoretical note, the optimal prices for contributors are the only prices that fulfill the *Samuelson criterion* [Samuelson, 1954] for efficient public goods provision, i.e. that the marginal rate of transformation (in this case, the relative price) is equal to the summed marginal rate of substitution M_i. For non-contributing individuals, any relative price that is sufficiently low to ensure that it is optimal for the node to only produce private goods is sufficient. The final budget constraint ensures that $\mathbf{q}^u$ is in the space of afforded profiles. It also establishes that no individual is net taxed nor subsidized at equilibrium.

In conclusion, individualized taxes and subsidies allow a planner to implement any social optimum, as long as the planner has complete information about the game. The optimal prices subsidize the public goods provision of nodes in relation to the degree of benefit they confer on others.

5.3 Contracts

The previous section on taxes and subsidies emphasized the welfare inefficiency arising in the public good game as a failure of incentives. An alternative perspective on the same problem is that of a failure of cooperation – even if two individuals would both be better off by jointly changing strategy, our game formulation did not provide them with a way to make agreements that they knew would be honored.

Proposition 5 showed that almost any Nash equilibrium is Pareto-inefficient, by noticing that groups of nodes can be made better off by jointly increasing their public goods provision. Without formal means of cooperation, however, these Pareto improvements fail to realize. One possible solution is to allow individuals to enter into enforceable contracts, where each node agrees to increase their public good provision by some amount, conditionally on other nodes also increasing their contributions. Agreements of this sort are commonplace across public good-like problems in the real world – housemates agree to take turns cleaning shared living spaces; firms within an industry formulate joint codes of ethics; countries enter accords to reduce carbon emissions if other countries reciprocate.

Previous game-theoretical models of public goods provision (cf. Bagnoli and Lipman (1989), Tabarrok (1998)) have particularly focused on threshold-based contracts, where individuals pledge to contribute some amount to the funding of a project if the aggregate contribution exceeds some predefined level. In this thesis' game formulation, however, this contract structure is awkward; nodes care not about the aggregate public goods provisioned, but rather about the amount that is provisioned in their neighborhood. The below formalization is adapted to this heterogeneity.

Consider a two-stage game. In the first stage, nodes choose an allocation (x_i^*, q_i^*) individually, resulting in a Nash equilibrium outcome as described in section 4.2. Next, all nodes are offered to enter into an enforceable contract, where each node commits to increasing their public goods provision by some amount. In exchange, each node will also benefit from the increased provision of her neighbors. Furthermore, assume that each node has veto power over the contract. Define a contract as a vector $\mathbf{c}$ of increases in public goods provision, such that the contract stipulates that each node changes her allocation from (x_i^*, q_i^*) to $(x_i^* - c_i, q_i^* + c_i)$ for some $c_i \geq 0$. We will study the set of contracts that constitute a Pareto improvement, and refer to these contracts as *implementable*.

DEFINITION 9
A contract $\mathbf{c}$ is *implementable* if and only if, for each i:

$$u_i(\mathbf{x}^* - \mathbf{c}, \mathbf{q}^* + \mathbf{c}) \geq u_i(\mathbf{x}^*, \mathbf{q}^*).$$ ꝏ

A simple contract structure would be the *uniform contributor contract*, where $c_i = c > 0$ for contributors, and $c_i = 0$ for non-contributors.

PROPOSITION 7
There exists an implementable uniform contributor contract if and only if every contributor has a contributing out-neighbor. ꝏ

This result, proven in Section A.9, follows from the fact the contributors are near-indifferent to increasing their public goods provision on the margin, while benefiting substantially from the increased provision of their out-neighbors. They will hence happily increase their own provision by (at least) some small amount if their out-neighbors reciprocate this action. In fact, they will even accept highly unfair contracts, as long as they are small in magnitude.

PROPOSITION 8
Let $\mathbf{c}$ be any contract such that $c_i = 0$ for non-contributors and $c_i > 0$ for contributors. If every contributor has a contributing out-neighbor, then there exists an $\varepsilon > 0$ such that $\varepsilon\mathbf{c}$ is implementable. ⊕

So, any sufficiently small contract restricted to contributors is implementable. A reasonable next question is: out of all these small contracts $\mathbf{c}$, which would be socially preferred for the group of contributors as a whole? Formally, for a small $\varepsilon > 0$, what contract $\mathbf{c}^*$ solves:

$$\mathbf{c}^* = \underset{\|c\|_1=\varepsilon}{\arg\max} \left\{ \sum_{i \in C} u_i(\mathbf{x}^* - \mathbf{c}, \mathbf{q}^* + \mathbf{c}) \right\} \tag{5.11}$$

such that $c_i = 0$ for every $i \notin \mathcal{C}$? For sufficiently small ε and a contributor i, the implementation of a contract $\mathbf{c}$ yields to a utility change of:

$$\begin{aligned}
\Delta u_i(\mathbf{c}) &\equiv u_i(\mathbf{x}^* - \mathbf{c}, \mathbf{q}^* + \mathbf{c}) - u_i(\mathbf{x}^*, \mathbf{q}^*) && (5.12)\\
&\approx -c_i \frac{\mathrm{d}u_i}{\mathrm{d}x_i} + c_i \frac{\mathrm{d}u_i}{\mathrm{d}q_i} + \sum_{j \in C} c_j \frac{\mathrm{d}u_i}{\mathrm{d}q_j} && (5.13)\\
&= (x_i^*)^{\alpha-1} (Q_i^*)^{\beta-1} \left(c_i(\alpha Q_i^* - \beta x_i^*) + \beta x_i^* \sum_{j \in \mathcal{C}} G_{ij} c_j \right) && (5.14)\\
&= \beta (x_i^*)^{\alpha} (Q_i^*)^{\beta-1} \sum_{j \in \mathcal{C}} G_{ij} c_j && (5.15)
\end{aligned}$$

since $\alpha Q_i^* = \beta x_i^*$ for every contributor. The approximative step consists in disregarding higher-order terms in the Taylor expansion around $\mathbf{q}^*$, that are negligible for sufficiently small ε. The optimal contract $\mathbf{c}^*$ can then be found as:

$$\begin{aligned}
\mathbf{c}^* &= \underset{\|c\|_1=\varepsilon}{\arg\max} \left\{ \sum_{i \in C} u_i(\mathbf{x}^* - \mathbf{c}, \mathbf{q}^* + \mathbf{c}) \right\} && (5.16)\\
&\approx \underset{\|c\|_1=\varepsilon}{\arg\max} \left\{ \sum_{i \in C} \beta (x_i^*)^{\alpha} (Q_i^*)^{\beta-1} \sum_j G_{ij} c_j \right\} && (5.17)\\
&= \underset{\|c\|_1=\varepsilon/\beta}{\arg\max} \left[(x_i^*)^{\alpha} (Q_i^*)^{\beta-1} \right]_{i \in \mathcal{C}}^{\top} (G_C \mathbf{c}_C) && (5.18)\\
&\text{s.t. } i \notin \mathcal{C} \implies c_i = 0 && (5.19)
\end{aligned}$$

where G_C and $\mathbf{c}_C$ denote the submatrix and subvector corresponding to the set of contributors. The optimal contract $\mathbf{c}^*$ is hence one for which the i'th entry of $G_C\mathbf{c}_C^*$ is approximately proportional to $(x_i^*)^\alpha(Q_i^*)^{\beta-1}$. The optimality of this contract is limited, however, in the sense that it fails to take into account the welfare of non-contributors.[10] The optimal contract $\mathbf{c}^*$ is uniquely defined if and only if G_C is invertible. This will generally be the case for the arbitrary, weighted and directed graphs considered in this thesis, e.g. if G_{ij} is sampled from some continuous random distribution. There are, however, special cases where an adjacency matrix is not invertible, for example that of a bipartite graph with an odd number of nodes.[11]

To conclude, this analysis shows that contracts can commonly improve the welfare-efficiency of outcomes in the game on the margin. This conclusion, however, rests on local approximations, and we hence find little indication of the magnitude of potential gains from such cooperation.

5.4 Redistribution

So far, our analysis has assumed the wealth endowments $\mathbf{w}$ to be exogenous. Now, consider the case of a planner with redistributive capacity, aiming to distribute some monetary amount W among nodes to maximize aggregate welfare, without infringing on the individuals' freedom to allocate their wealth. Let $\mathbf{x}^w$ and $\mathbf{q}^w$ denote the Nash equilibrium profiles for a given $\mathbf{w}$. Furthermore, assume limited network effects to ensure that this equilibrium is unique for every $\mathbf{w}$. The planner's problem is then to find $\mathbf{w}^*$:

$$\mathbf{w}^* \in \underset{\mathbf{w}\geq\mathbf{0}}{\arg\max} \quad U(\mathbf{w}) \tag{5.20}$$

$$= \underset{\mathbf{w}\geq\mathbf{0}}{\arg\max} \quad \sum_{i\in\mathcal{N}} u_i(\mathbf{x}^w, \mathbf{q}^w) \tag{5.21}$$

$$\text{s.t.} \sum_{i\in\mathcal{N}} w_i = W. \tag{5.22}$$

How tractable is it to find a solution $\mathbf{w}^*$? As we have seen, the mapping from w_i to the allocation q_i^w depends on numerous aspects – the network structure in i's neighborhood, the wealth and choices of her neighbors, and (indirectly) the behavior of unconnected nodes. As a consequence, $U(\mathbf{w})$ is generally not concave. Since the equilibrium is unique for every $\mathbf{w}$, however, we know that every $\mathbf{w}$ gives rise to a uniquely determined set of contributors $\mathcal{C}_w$, such that $q_i^w > 0$ if and only if $i \in \mathcal{C}_w$. The function $U(\mathbf{w})$ turns out to be well-behaved over each such set.

[10] It is, however, guaranteed to be a weak Pareto improvement even for these nodes.

[11] It follows from Theorem 1 that the weight matrices of bipartite graphs have eigenvalues that are symmetric around 0, so if the number of nodes is odd, then 0 must be an eigenvalue of the matrix.

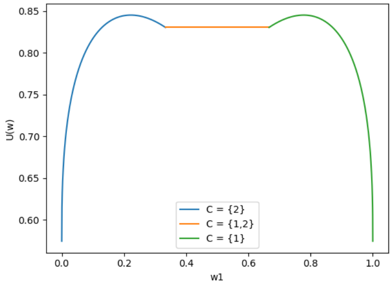

Figure 5.1 $U(\mathbf{w})$ from Example 8, $\alpha = \beta = 0.4$.

PROPOSITION 9
Let $\alpha + \beta \leq 1$. Then, for any $\mathcal{C}$, $U(\mathbf{w})$ is concave over the set $\{\mathbf{w} : \mathcal{C}_w = \mathcal{C}\}$. ꝏ

This proposition is helpfully understood by considering an example.

EXAMPLE 8
Consider the simple case of two nodes, 1 and 2, connected by links $G_{12} = G_{21} = 1$. Let $W \equiv w_1 + w_2 = 1$ and $\alpha = \beta$. The possible sets of contributors are $\{1\}$, $\{2\}$, and $\{1,2\}$. Figure 5.1 shows $U(\mathbf{w})$ for different wealth combinations, with w_1 on the horizontal axis. As we can see, the function is concave over each set of contributors, but not globally. Somewhat counterintuitively, the socially preferable wealth distribution is not symmetric. The reason for this is that relative wealth equality in our game setting leads to higher levels of free-riding than unequal distributions. So, if $w_1 \approx w_2$, then the aggregate public goods funding is lower than if the difference in wealth levels is larger. ꝏ

Proposition 9 hints at a possible algorithm to find $\mathbf{w}^*$. For each $\mathcal{C} \subseteq \mathcal{N}$:

1. Compute the vectors $\mathbf{w}$ for which $\mathcal{C}_w = \mathcal{C}$.
2. Find a closed-form expression for $U(\mathbf{w})$ and the partial derivatives $\frac{\mathrm{d}U}{\mathrm{d}w_i}$ for every i.
3. If there exists a point $\mathbf{w}$ for which $\mathcal{C}_w = \mathcal{C}$ and $\frac{\mathrm{d}U}{\mathrm{d}w_i} = \frac{\mathrm{d}U}{\mathrm{d}w_j}$ for every i and j, then compute $U(\mathbf{w})$ at this point. If this is higher than any previously computed value, set $\bar{\mathbf{w}} = \mathbf{w}$.

I conjecture that this algorithm guarantees that $\bar{\mathbf{w}} = \mathbf{w}^*$ after a full iteration. For a formal proof it would be sufficient to show that the set $\{\mathbf{w} : \mathcal{C}_w = \mathcal{C}\}$ is convex for any $\mathcal{C}$ and that the optimal solution is interior (i.e. that no node is indifferent between contributing and not contributing on the margin).

Let us exemplify the algorithm using Example 8. It follows from the best responses that (recall that $\alpha = \beta$):

$$\mathcal{C}_w = \begin{cases} \{2\}, & w_1 \leq \frac{1}{3}, \\ \{1,2\}, & \frac{1}{3} < w_1 < \frac{2}{3}, \\ \{1\}, & w_1 \geq \frac{2}{3}, \end{cases} \tag{5.23}$$

Now, if $\mathcal{C}_w = \{1\}$, then only node 1 will contribute to the public goods. The aggregate utility becomes:

$$\begin{aligned} U(\mathbf{w}) &= \sum_{i=1}^{2} u_i(\mathbf{x}^w, \mathbf{q}^w) && (5.24) \\ &= u_1(\mathbf{x}^w, \mathbf{q}^w) + u_2(\mathbf{x}^w, \mathbf{q}^w) && (5.25) \\ &= x_1^\alpha q_1^\beta + x_2^\alpha q_1^\beta && (5.26) \\ &= \left(\frac{\alpha}{\alpha+\beta} w_1\right)^\alpha \left(\frac{\beta}{\alpha+\beta} w_1\right)^\beta && (5.27) \\ &\quad + (1-w_1)^\alpha \left(\frac{\beta}{\alpha+\beta} w_1\right)^\beta && (5.28) \\ &= \left(\frac{w_1^2}{4}\right)^\alpha + \left(\frac{w_1 - w_1^2}{2}\right)^\alpha, && (5.29) \end{aligned}$$

which has a stationary point on the allowed interval for any $\alpha = \beta < 0.5$. Similar computations can be made for $\mathcal{C}_w = \{2\}$ and $\mathcal{C}_w = \{1,2\}$. The maximum aggregate utility for each set can then be computed and compared, and the optimal wealth distribution $\mathbf{w}$ found by comparing the maxima across contributor sets.

Example 8 showed that the welfare-optimal wealth distribution has a tendency towards inequality. Is this a consequence of our particular game formulation, or does it capture an important dynamic of the public goods problem? On the one hand, it seems plausible that a billionaire in the real world (on average) will spend a larger proportion of her wealth on public goods, funding universities, museums or charities. On the other hand, this does not clearly imply that transferring welfare to rich individuals is welfare-increasing on aggregate. In our game, the optimal degree of redistribution is likely sensitive to the magnitude of $\alpha + \beta$. For example, doubling both the private and public goods available to a node will increase her utility by a factor $2^{\alpha+\beta}$. The empirical literature of subjective well-being, in contrast, generally finds that reported happiness is logarithmic in wealth, i.e. that a doubling of consumption leads to a constant increase in welfare.[12] In this way, our game might

[12] For high levels of wealth, the relationship may be even weaker, see e.g. Kahneman and Deaton (2010).

overestimate welfare gains from increased wealth, leading the optimal redistribution to overly favor unequal outcomes.

Notably, the redistribution mechanism bears some similarity to the price mechanism from section 5.2. Both aim to increase aggregate public good provision – the tax system by incentivizing nodes to allocate their wealth pro-socially, and the redistributive system by transferring wealth to central nodes so that they will selfishly demand high levels of public goods. Still, the two interventions are orthogonal – while either can be implemented without the other, they can equally well be combined. In this highly regulated economy, the planner of some society with aggregate wealth W may effectively implement any allocation $(\mathbf{x}, \mathbf{q})$ satisfying $\sum_i x_i + q_i = W$.

6

Conclusions

Many issues facing individuals, organizations and governments can be understood as public goods problems. Some are trivial, like the everyday struggle of dividing household chores, while others are urgent and far-reaching in their stakes, such as ensuring that countries coordinate to mitigate harmful climate change. Understanding how self-interested agents behave in the presence of public goods – and how different mechanisms can increase cooperation – may help us achieve better outcomes across a wide range of domains. Network models may be well-suited to study these scenarios without making overly simplistic assumptions about the nature of interactions between agents.

In this thesis, I present a network public goods game of asymmetric, heterogeneous externalities. The uniqueness, emergence and structure of equilibrium outcomes are described in detail and contrasted with other game formulations from the literature. I then move on to study the game through a normative lens. Defining metrics to compare the welfare efficiency of different outcomes, I find that equilibrium outcomes are inefficient as a rule. Drawing on commonly proposed solutions to the public goods problem, I formalize three interventions on the base game: two that are associated with a centralized planner (taxing & subsidizing, and redistribution) and one based on voluntary, decentralized agreements between individuals. For each mechanism, I discuss design considerations and the scope of welfare gains that are attainable compared to equilibrium outcomes.

The application of mechanism design to public goods on networks in this thesis is highly exploratory, covering only a small subset of possible mechanisms in rather narrow formalizations. Given the rich body of literature on mechanism design for public good provision, I expect a more rigorous translation of non-network mechanisms to a network setting to be enlightening. Such an analysis could also consider a broader scope of social goals than this thesis, including societal preferences for low-inequality outcomes or Rawlsian concerns.

Another direction of future study would be to make weaker informational assumptions than the complete-information case covered in this thesis. Uncertainty could be defined over the structure of the graph or the distribution of wealth, and affect either the players or the designer of some mechanism.

A

Proofs and derivations

A.1 Proposition 1

The node aims to choose x_i and q_i as to optimize $u_i(x_i, \mathbf{q})$, taking her own wealth w_i and the actions of other nodes as exogenous. Using the budget constraint $x_i + q_i = w_i$, this problem can be reduced to one dimension:

$$q_i^* = \underset{0 \leq q_i \leq w_i}{\arg\max} \quad u_i(w_i - q_i, \mathbf{q}) \tag{A.1}$$

$$= \underset{0 \leq q_i \leq w_i}{\arg\max} \quad (w_i - q_i)^{\alpha}(q_i + Q_{\text{-}i})^{\beta}, \tag{A.2}$$

$$\text{where } Q_{\text{-}i} \equiv \sum_{j \in \mathcal{N}} G_{ij} q_j. \tag{A.3}$$

The utility function is strictly concave for $0 < q_i < w_i$. This follows from the fact that every term in the second derivative

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}q_i^2} = -(w_i - q_i)^{\alpha-2}(q_i + Q_{\text{-}i})^{\beta-2} \tag{A.4}$$

$$\cdot \Big[\alpha(1-\alpha)(q_i + Q_{\text{-}i})^2 \tag{A.5}$$

$$+ \beta(1-\beta)(w_i - q_i)^2 \tag{A.6}$$

$$+ 2\alpha\beta(w_i - q_i)(q_i + Q_{\text{-}i})\Big] \tag{A.7}$$

is strictly negative whenever $(w_i - q_i)$ and $(q_i + Q_{\text{-}i})$ are both strictly positive[13]. Therefore, the optimal allocation must be either the unique stationary point (if it exists on the permissible interval), or a boundary point. Since $q_i = w_i$ yields $u_i = 0$, the global minimum, the only relevant boundary is $q_i = 0$. We can hence deduce that:

$$q_i^* = \max\left\{q_i : \frac{\mathrm{d}u_i}{\mathrm{d}q_i} = 0, 0\right\} \tag{A.8}$$

[13] Recall that $\alpha, \beta < 1$

Lastly, the stationary point is found as:

$$\frac{\mathrm{d}u_i}{\mathrm{d}q_i} = -\alpha(w_i - q_i)^{\alpha-1}(q_i + Q_{\text{-}i})^{\beta} \tag{A.9}$$

$$+\beta(w_i - q_i)^{\alpha}(q_i + Q_{\text{-}i})^{\beta-1} = 0 \tag{A.10}$$

$$\iff q_i = \frac{1}{\alpha+\beta}\left(\beta w_i - \alpha Q_{\text{-}i}\right) \tag{A.11}$$

which yields the result:

$$q_i^* = \max\left\{\frac{1}{\alpha+\beta}\left(\beta w_i - \alpha Q_{\text{-}i}\right), 0\right\}. \tag{A.12}$$

A.2 Proposition 3

Consider a game setting where each node has equal wealth: $w_i = w$ for some $w > 0$. Assume that, for some profile $\mathbf{q}$, the set of contributing nodes $\mathcal{C} = \{i : q_i > 0\}$ is a maximal independent set. Then, $\mathbf{q}$ is a NE if and only if:

1. for every node $i \in \mathcal{C}$, $q_i = \frac{\beta}{\alpha+\beta}w$, and
2. for every $i \notin \mathcal{C}$, $\sum_{j\in\mathcal{C}} G_{ij} > \frac{\alpha+\beta}{\alpha}$.

I will show this equivalence, and then conclude that the existence of a specialized NE is equivalent to the existence of a maximal independent set $\mathcal{C}$ for which (2) holds.

Proof: If $\mathbf{q}$ *is a NE, the (1) and (2) hold.* Let $i \in \mathcal{C}$. Since $\mathcal{C}$ is an independent set, i has no contributing out-neighbor and $\sum_{j\in\mathcal{N}} G_{ij}q_j = 0$. So, the fact that $\mathbf{q}$ is a NE means that (1) is fulfilled, since it follows from the individual rationality of i.

Now, let $i \notin \mathcal{C}$. If i has total links worth $g_i \equiv \sum_{j\in\mathcal{C}} G_{ij}$ to neighbors in $\mathcal{C}$, that all contribute the above amount given by (1), then the best response implies that:

$$\mathcal{B}_i(\mathbf{q}) = \max\left\{\frac{1}{\alpha+\beta}\left(\beta w - \alpha\sum_{j\in\mathcal{C}} G_{ij}q_j\right), 0\right\} \tag{A.13}$$

$$= \max\left\{\frac{1}{\alpha+\beta}\left(\beta w - \alpha\left(g_i \cdot \frac{\beta}{\alpha+\beta}w\right)\right), 0\right\} \tag{A.14}$$

$$= 0 \iff g_i \geq \frac{\alpha+\beta}{\alpha} \tag{A.15}$$

So, i's strategy is optimal if and only if $g_i > \frac{\alpha+\beta}{\alpha}$. Since $\mathbf{q}$ is a NE, this must necessarily hold.

Proof: If (1) and (2) hold, then $\mathbf{q}$ *is a NE.* This amounts to showing that if (1) and (2) hold, then each node is playing its best response.

(1) implies individual rationality for every contributing node, which is necessary for NE. It was also shown that under (1), every $q_i = 0$ is individually rational for non-contributors if and only if their outdegree towards contributors is at least $\frac{\alpha+\beta}{\alpha}$. So, (2) and (1) jointly imply NE.

This proves the existence of (and describes precisely) a specialized equilibrium on any graph where there exists an independent subset $\mathcal{C}$ such that every node not in $\mathcal{C}$ has links of at least $\frac{\alpha+\beta}{\alpha}$ to a node in $\mathcal{C}$.

A.3 Lemma 1

Recall the best response function for node i:

$$\mathcal{B}_i(\mathbf{q}) = \max\left\{h_i(\mathbf{q}), 0\right\}, \text{ where} \tag{A.16}$$

$$h_i(\mathbf{q}) = \frac{1}{\alpha+\beta}\left(\beta w_i - \alpha \sum_{j\in\mathcal{N}} G_{ij} q_j\right). \tag{A.17}$$

Denote by $\mathcal{B}(\mathbf{q})$ and $\mathbf{h}(\mathbf{q})$ the corresponding vectors. For a given G, we want to find a vector norm $\|\cdot\|_G$ for which:

$$\|\mathcal{B}(\mathbf{q}^1) - \mathcal{B}(\mathbf{q}^2)\|_G \leq \|\mathbf{h}(\mathbf{q}^1) - \mathbf{h}(\mathbf{q}^2)\|_G < \|\mathbf{q}^1 - \mathbf{q}^2\|_G \tag{A.18}$$

for every distinct pair $\mathbf{q}^1, \mathbf{q}^2$. Let us start by finding sufficient conditions for the left-most and right-most inequalities, in that order. For the left-most inequality, a sufficient condition is that the norm $\|\cdot\|_G$ is *monotone*, meaning that for any $\mathbf{y}, \mathbf{z}$:

$$|y_i| \leq |z_i| \quad \forall i \implies \|\mathbf{y}\|_G \leq \|\mathbf{z}\|_G \tag{A.19}$$

To see why this is sufficient, note that:

$$|\mathcal{B}_i(\mathbf{q}^1) - \mathcal{B}_i(\mathbf{q}^2)| = \left|\max\left\{h_i(\mathbf{q}^1), 0\right\} - \max\left\{h_i(\mathbf{q}^2), 0\right\}\right| \tag{A.20}$$

$$\leq |h_i(\mathbf{q}^1) - h_i(\mathbf{q}^2)|. \tag{A.21}$$

For the right-most inequality, the norm must fulfill

$$\|\mathbf{h}(\mathbf{q}^1) - \mathbf{h}(\mathbf{q}^2)\|_G < \|\mathbf{q}^1 - \mathbf{q}^2\|_G \tag{A.22}$$

for every distinct pair $\mathbf{q}^1, \mathbf{q}^2$. This simplifies as:

$$\|\mathbf{h}(\mathbf{q}^1) - \mathbf{h}(\mathbf{q}^2)\|_G = \frac{1}{\alpha+\beta} \| \left(\beta \mathbf{w} - \alpha G \mathbf{q}^1\right) - \left(\beta \mathbf{w} - \alpha G \mathbf{q}^2\right) \|_G \tag{A.23}$$

$$= \frac{\alpha}{\alpha+\beta} \|G(\mathbf{q}^2 - \mathbf{q}^1)\|_G \tag{A.24}$$

$$< \|\mathbf{q}^1 - \mathbf{q}^2\|_G \tag{A.25}$$

Assuming that $\|\cdot\|_G$ is even[14], this condition is equivalent to

$$|||G|||_G < \frac{\alpha+\beta}{\alpha} \tag{A.26}$$

where $|||\cdot|||_G$ is the matrix norm induced by $\|\cdot\|_G$. From LME, this holds if $|||G|||_G = \rho(G)$. In conclusion, a norm $\|\cdot\|_G$ fulfills our criteria if it is even, monotone, and induces a matrix norm for which $|||G|||_G = \rho(G)$.

Denoting by $\mathbf{v}$ the dominant eigenvector of G, define a candidate norm as:

$$\|\mathbf{z}\|_G = \|V^{-1}\mathbf{z}\|_\infty = \max_i \left\{ \left| \frac{z_i}{v_i} \right| \right\}, \quad V = \text{diag}(\mathbf{v}). \tag{A.27}$$

From Theorem 1, $\mathbf{v}$ is entry-wise positive, so the norm is well-defined.[15] It should be clear by inspection that this norm is even and monotone. As for the matrix norm, let

$$A = V^{-1}GV \tag{A.28}$$

and find $|||A|||_\infty$ as the maximal absolute entry of $A\mathbf{1}$, where

$$A\mathbf{1} = V^{-1}GV\mathbf{1} = V^{-1}G\mathbf{v} = V^{-1}\rho(G)\mathbf{v} = \rho(G)\mathbf{1}, \tag{A.29}$$

yielding $|||A|||\infty = \rho(G)$. This, in turn, means that for any vector $\mathbf{z}$:

$$\|G\mathbf{z}\|_G = \|V^{-1}G\mathbf{z}\|_\infty = \|AV^{-1}\mathbf{z}\|_\infty \leq \rho(G)\|V^{-1}\mathbf{z}\|_\infty = \rho(G)\|\mathbf{z}\|_G \tag{A.30}$$

so $|||G|||_G \leq \rho(G)$. In addition, for any matrix norm it holds that $|||G||| \geq \rho(G)$, so we can conclude that $|||G|||_G = \rho(G)$. Since the proposed norm fulfills all three criteria, we can conclude that the best response dynamic is indeed contractive for this norm.

[14] This ensures that $\|\mathbf{q}^1 - \mathbf{q}^2\|_G = \|\mathbf{q}^2 - \mathbf{q}^1\|_G$.

[15] Notce that G is non-negative and irreducible.

A.4 Proposition 5

We want to show that if there exists a pair of contributors that are mutually reachable on the subgraph of contributors, then $\mathbf{q}^*$ is Pareto inefficient. This is equivalent to the existence of a cycle on the set of contributors, i.e. that there exists a sequence of contributors $\mathcal{S} = \{1, 2, ..., k\}$ such that $G_{i,i+1} > 0$ for each $i < k \in S$ and $G_{k,1} > 0$.

To show that $\mathbf{q}^*$ is Pareto inefficient, it is sufficient to show that there exists a different profile for which everyone is at least as well off, and at least one node is better off. Consider the alternative allocation $\mathbf{q}^+$:

$$q_i^+ = \begin{cases} q_i^* + \varepsilon, & i \in \mathcal{S} \\ q_i^*, & i \notin \mathcal{S} \end{cases} \tag{A.31}$$

for some $\varepsilon > 0$. For every node $i \notin S$, the allocation $\mathbf{q}^+$ is at least as good as $\mathbf{q}^*$; these nodes retain the same quantity of private goods, and enjoy at least as much public goods. So, $\mathbf{q}^+$ is a Pareto improvement from $\mathbf{q}^*$ as long as every node in $\mathcal{S}$ is better off in $\mathbf{q}^+$ than in $\mathbf{q}^*$. For $i \in S$, a Taylor expansion around $u_i(\mathbf{x}^*, \mathbf{q}^*)$ yields:

$$u_i(\mathbf{x}^+, \mathbf{q}^+) - u_i(\mathbf{x}^*, \mathbf{q}^*) = \varepsilon \left(-\frac{\mathrm{d}u_i}{\mathrm{d}x_i} + \frac{\mathrm{d}u_i}{\mathrm{d}q_i} + \sum_{j \in \mathcal{S}} \frac{\mathrm{d}u_i}{\mathrm{d}q_j} \right) + \mathcal{O}(\varepsilon^2) \tag{A.32}$$

$$= \varepsilon \sum_{j \in \mathcal{S}} \frac{\mathrm{d}u_i}{\mathrm{d}q_j} + \mathcal{O}(\varepsilon^2) \tag{A.33}$$

$$= \varepsilon \sum_{j \in \mathcal{S}} \beta G_{ij} (x_i^*)^{\alpha} (Q_i^*)^{\beta - 1} + \mathcal{O}(\varepsilon^2) \tag{A.34}$$

where $\mathcal{O}(\varepsilon^2)$ is a sum of higher-degree differential terms proportional to ε^k for $k \geq 2$.[16] By assumption, every $i \in \mathcal{S}$ has an out-neighbor is $\mathcal{S}$, so this change in utility is the sum of a strictly positive term that is linear in ε, and a term that is no greater than proportional to ε^2. We can hence find an $\varepsilon > 0$ for which the change in utility is strictly positive for $i \in \mathcal{S}$. The equilibrium profile $\mathbf{q}^*$ must then be Pareto inefficient, since $\mathbf{q}^+$ yields a lower utility for no one, and a strictly higher utility for every node in $\mathcal{S}$.

A.5 Theorem 4

We will begin by showing that $u_i(\mathbf{x}, \mathbf{q})$ is strictly concave for a fixed $\mathbf{w}$, assuming $\alpha + \beta < 1$. First, note that we can write this function as a composition

[16] Notice that since $\mathbf{q}^*$ is an equilibrium and i is a contributor, $\frac{\mathrm{d}u_i}{\mathrm{d}x_i} = \frac{\mathrm{d}u_i}{\mathrm{d}q_i}$ when evaluated at $\mathbf{q}^*$.

$$u_i(\mathbf{x},\mathbf{q}) = u_i(\mathbf{w}-\mathbf{q},\mathbf{q}) = (w_i - q_i)^{\alpha}(q_i + Q_{\text{-}i})^{\beta}, \tag{A.35}$$

$$\text{where } Q_{\text{-}i} = \sum_{j\in\mathcal{N}} G_{ij}q_j. \tag{A.36}$$

This function $u_i(q_i, Q_{\text{-}i})$ is concave if and only if the Hessian H_i

$$H_i(q_i, Q_{\text{-}i}) = \begin{bmatrix} \frac{\partial^2 u_i}{\partial q_i^2} & \frac{\partial^2 u_i}{\partial q_i\,\partial Q_{\text{-}i}} \\ \frac{\partial^2 u_i}{\partial q_i\,\partial Q_{\text{-}i}} & \frac{\partial^2 u_i}{\partial Q_{\text{-}i}^2} \end{bmatrix} \tag{A.37}$$

is negative semi-definite. A general matrix $M = [m_1\, m_2; m_3\, m_4]$ is negative semidefinite if $m_1, m_4 \leq 0$ and $m_1 \cdot m_4 - m_2 \cdot m_3 \geq 0$. Computing the second-order derivatives for H_i, this is equivalent to:

$$\alpha \leq 1, \quad \beta \leq 1, \quad 1-\alpha-\beta \geq 1 \tag{A.38}$$

$$\iff \alpha + \beta \leq 1 \tag{A.39}$$

If the inequalities are strict ($\alpha + \beta < 1$), then the Hessian is negative definite. Now, notice that $Q_{\text{-}i}$ is a non-decreasing, linear function in $\mathbf{q}$. This means that $u_i(q_i, Q_{\text{-}i})$ is the composition of a concave function and a concave and non-decreasing function for $\alpha + \beta \leq 1$. It is hence concave (and strictly so for $\alpha + \beta < 1$). This attribute is inherited by the sum $U(\mathbf{x},\mathbf{q}) = \sum_{i\in\mathcal{N}} u_i(\mathbf{x},\mathbf{q})$.

A.6 Proposition 6

Each contributor solves the maximization problem

$$(x_i^s, q_i^s) = \underset{x_i, q_i}{\arg\max} \quad x_i^{\alpha}\Big(q_i + \sum_{j\in\mathcal{N}} G_{ij}q_j\Big)^{\beta} \tag{A.40}$$

$$\text{s.t. } P_x^i x_i + P_q^i q_i = w_i \tag{A.41}$$

$$x_i, q_i \geq 0. \tag{A.42}$$

Let $\hat{x}_i = P_x^i x_i$ and $\hat{q}_i = P_q^i q_i$:

$$(\hat{x}_i^s, \hat{q}_i^s) = \underset{\hat{x}_i, \hat{q}_i}{\arg\max} \quad \left(\frac{\hat{x}_i}{P_x^i}\right)^{\alpha} \Big(\frac{\hat{q}_i}{P_q^i} + \sum_{j\in\mathcal{N}} G_{ij}q_j\Big)^{\beta} \tag{A.43}$$

$$\text{s.t. } \hat{x}_i + \hat{q}_i = w_i \tag{A.44}$$

$$\hat{x}_i, \hat{q}_i \geq 0. \tag{A.45}$$

Since at any point, one unit of $\hat{x}_i$ is exchangeable for one unit of $\hat{q}_i$, a necessary condition for an interior solution (i.e. that i is a contributor) is that

$$\frac{\mathrm{d}u_i}{\mathrm{d}\hat{x}_i} = \frac{\mathrm{d}u_i}{\mathrm{d}\hat{q}_i} \iff P_q^i \cdot \frac{\mathrm{d}u_i}{\mathrm{d}x_i} = P_x^i \cdot \frac{\mathrm{d}u_i}{\mathrm{d}q_i} \tag{A.46}$$

$$\iff P_q^i \cdot \alpha x_i^{\alpha-1} Q_i^\beta = P_x^i \cdot \beta x_i^\alpha Q_i^{\beta-1} \tag{A.47}$$

$$\iff \frac{Q_i}{x_i} = P_i \cdot \frac{\beta}{\alpha} \tag{A.48}$$

where $Q_i \equiv q_i + \sum_{j\in\mathcal{N}} G_{ij} q_j$. This concludes the proof.

A.7 Corollary 1

Recall that a utilitarian-efficiency profile is a profile that maximizes

$$U(\mathbf{q}) = \sum_{i\in\mathcal{N}} u_i(\mathbf{w}-\mathbf{q},\mathbf{q}). \tag{A.49}$$

In particular, since every node can exchange one unit of x_i for one unit of q_i. In an interior solution (i.e. for $i \in \mathcal{C}$), a necessary condition for optimality is that

$$\frac{\mathrm{d}U}{\mathrm{d}x_i} = \frac{\mathrm{d}U}{\mathrm{d}q_i} \tag{A.50}$$

$$\iff \frac{\mathrm{d}u_i}{\mathrm{d}x_i} = \frac{\mathrm{d}u_i}{\mathrm{d}q_i} + \sum_{j\in\mathcal{N}} \frac{\mathrm{d}u_i}{\mathrm{d}q_i} \tag{A.51}$$

$$\iff \alpha x_i^{\alpha-1} Q_i^\beta = \beta \left(x_i^\alpha Q_i^{\beta-1} + \sum_{j\in\mathcal{N}} G_{ji} x_j^\alpha Q_j^{\beta-1} \right) \tag{A.52}$$

$$\iff \frac{Q_i}{x_i} = \frac{\beta}{\alpha} \left(1 + \sum_{j\in\mathcal{N}} G_{ji} \frac{u_j(\mathbf{q})/Q_j}{u_i(\mathbf{q})/Q_i} \right) \tag{A.53}$$

$$= M_i \cdot \frac{\beta}{\alpha} \tag{A.54}$$

using the fact that $u_i(\mathbf{q}) = x_i^\alpha Q_i^\beta$. The final step of the proof is to rule out the possibility that $q_i^u = w_i$ for some contributor i, in which case the above argument need not apply. This is done by noticing that, since $\alpha < 1$,

$$\frac{\mathrm{d}U}{\mathrm{d}x_i} \to \infty \text{ as } x_i \to 0 \tag{A.55}$$

if $q_i^u > 0$, while the $\frac{\mathrm{d}U}{\mathrm{d}q_i}$ remains finite. It will hence never be socially preferable for any node to provide only public goods.

A.8 Theorem 5

We want to show that the Nash equilibrium $\mathbf{q}^s$ for the subsidized game coincides exactly with the social optimum $\mathbf{q}^u$. To this end, we must show that no node i benefits by deviating from the socially optimal action q_i^u, for the given prices.

Start by considering the nodes for which $q_i^u = 0$. For them, it is not socially worthwhile to contribute, so it is necessarily the case that

$$\frac{\mathrm{d}U}{\mathrm{d}x_i} = \frac{\mathrm{d}u_i}{\mathrm{d}x_i} \geq \frac{\mathrm{d}u_i}{\mathrm{d}q_i} + \sum_{j \in \mathcal{N}} \frac{\mathrm{d}u_i}{\mathrm{d}q_j} = \frac{\mathrm{d}U}{\mathrm{d}q_i} \tag{A.56}$$

$$\implies \frac{\mathrm{d}u_i}{\mathrm{d}x_i} \geq \frac{\mathrm{d}u_i}{\mathrm{d}q_i}. \tag{A.57}$$

Since their relative price is $P_i = 1$, they will hence not find it worthwhile to spend wealth on q_i rather than x_i, given that $\mathbf{q}^s_{\text{-}i} = \mathbf{q}^u_{\text{-}i}$.

For the nodes where $q_i^u > 0$, it follows from Proposition 6 that

$$\frac{Q_i^s}{x_i^s} = P_i \cdot \frac{\beta}{\alpha} = M_i \cdot \frac{\beta}{\alpha}. \tag{A.58}$$

Since $\mathbf{q}^s_{\text{-}i} = \mathbf{q}^u_{\text{-}i}$, the price constraint $P_x^i x_i^u + P_q^i q_i^u = w_i$ means that this can only be true if $q_i^s = q_i^u$. It is hence not individually rational for any node to deviate from $\mathbf{q}^s$ if $\mathbf{q}^s = \mathbf{q}^u$.

A.9 Proposition 7 and 8

We want to show that there exists a scalar $c > 0$ such that for the contract

$$\mathbf{c} : c_i = \begin{cases} c, & q_i^* > 0 \\ 0, & q_i^* = 0 \end{cases} \tag{A.59}$$

it is the case that

$$u_i(\mathbf{x}^* - \mathbf{c}, \mathbf{q}^* + \mathbf{c}) \geq u_i(\mathbf{x}^*, \mathbf{q}^*) \tag{A.60}$$

if and only if every contributor has a contributing out-neighbor, i.e.

$$\forall i : q_i^* > 0 \implies \sum_{j \in \mathcal{N}} G_{ij} q_j^* > 0. \tag{A.61}$$

First, notice that the contract does not require non-contributors to change their allocation. Furthermore, since $q_i^* + c_i \geq q_i^*$, the public goods provisioned in the neighborhood of each node will be no lower after the contract is enforced than before. All non-contributors will therefore enjoy as much private good consumption after the contract as before and at least as much public goods, so they will not be made worse off.

Now, consider a contributor i without contributing out-neighbors. The public goods provisioned in her neighborhood will not increase through the contract. Any change of allocation will therefore be costly for her, as

$$u_i(\mathbf{x}^*, \mathbf{q}^*) = (x_i^*)^\alpha (q_i^* + 0)^\beta < (x_i^* - c)^\alpha (q_i^* + c + 0)^\beta = u_i(\mathbf{x}^* - \mathbf{c}, \mathbf{q}^* + \mathbf{c}) \quad \text{(A.62)}$$

for any $c > 0$. This is because (x_i^*, q_i^*) is defined as the allocation that maximizes the above utility.

Lastly, view the case of a contributor with at least one contributing out-neighbor, i.e. a node i for which $q_i^* > 0$ and $\sum_{j \in \mathcal{N}} G_{ij} q_j^* > 0$. The Taylor expansion of u_i around equilibrium is

$$\Delta u_i(\mathbf{c}) \equiv u_i(\mathbf{x}^* - \mathbf{c}, \mathbf{q}^* + \mathbf{c}) - u_i(\mathbf{x}^*, \mathbf{q}^*) \quad \text{(A.63)}$$

$$= -c_i \cdot \frac{\mathrm{d}u_i}{\mathrm{d}x_i} + c_i \cdot \frac{\mathrm{d}u_i}{\mathrm{d}q_i} + \sum_{j \in \mathcal{N}} c_j \frac{\mathrm{d}u_i}{\mathrm{d}q_j} + \mathcal{O}(c_k c_h) \quad \text{(A.64)}$$

where $\mathcal{O}(c_k c_h)$ is some sum of finite, higher-order differential terms proportional to the product of at least two elements of $\mathbf{c}$. Letting $c_i = c_j = c$ for all $i, j \in \mathcal{C}$:

$$\Delta u_i(\mathbf{c}) = c \left(-\frac{\mathrm{d}u_i}{\mathrm{d}x_i} + \frac{\mathrm{d}u_i}{\mathrm{d}q_i} + \sum_{j \in \mathcal{C}} \frac{\mathrm{d}u_i}{\mathrm{d}q_j} \right) + \mathcal{O}(c^2) \quad \text{(A.65)}$$

$$= c \sum_{j \in \mathcal{C}} \frac{\mathrm{d}u_i}{\mathrm{d}q_j} + \mathcal{O}(c^2) \quad \text{(A.66)}$$

Now, notice that

$$\frac{\mathrm{d}u_i}{\mathrm{d}q_j} = \beta G_{ij} (x_i^*)^\alpha (q_i^* + \sum_{j \in \mathcal{N}} G_{ij} q_j^*)^{\beta - 1} > 0 \quad \text{(A.67)}$$

exactly if j is an out-neighbor of i. So, if i has at least one contributing out-neighbor, then $\Delta u_i(\mathbf{c})$ can be written as the sum of a term that is strictly positive and linear in c, and a term that is no greater than quadratic in c. Hence, there exists a $c > 0$ for which $\Delta u_i(\mathbf{c}) > 0$.

For Proposition 8, consider an arbitrary contract for which $c_i > 0$ if and only if $i \in \mathcal{C}$. We want to show that, for any such fixed $\mathbf{c}$, there exists an $\varepsilon > 0$ such that $\varepsilon \mathbf{c}$ is implementable. The arguments for non-contributing nodes and contributing nodes without neighbors still apply. For contributing nodes with neighbors, we can find $\Delta u_i(\varepsilon \mathbf{c})$ as:

$$\Delta u_i(\varepsilon \mathbf{c}) = \sum_{j \in \mathcal{C}} \varepsilon c_j \frac{\mathrm{d}u_i}{\mathrm{d}q_j} + \mathcal{O}(\varepsilon^2 c_k c_h) \tag{A.68}$$

$$= \sum_{j \in \mathcal{C}} \varepsilon c_j \frac{\mathrm{d}u_i}{\mathrm{d}q_j} + \tilde{\mathcal{O}}(\varepsilon^2) \tag{A.69}$$

which analogously to the previous proof is greater than 0 for small enough ε, as long as i has a contributing out-neighbor.

A.10 Proposition 9

Assume that a wealth profile $\mathbf{w}$ gives rise to a set of contributors in equilibrium $\mathcal{C}$. Then, by assumption:

$$x_i^w = \begin{cases} w_i, & i \notin \mathcal{C} \\ \frac{\alpha}{\alpha+\beta}(w_i + Q_{-i}^w), & i \in \mathcal{C} \end{cases} \tag{A.70}$$

$$q_i^w = w_i - x_i^w \tag{A.71}$$

$$\text{where } Q_{-i}^w \equiv \sum_{j \in \mathcal{N}} G_{ij} q_j^w. \tag{A.72}$$

We can hence find the individual utilities as:

$$u_i(\mathbf{x}^w, \mathbf{q}^w) = (x_i^w)^\alpha \Big(q_i^w + \sum_{j \in \mathcal{N}} G_{ij} q_j^w \Big)^\beta \tag{A.73}$$

$$= \begin{cases} w_i^\alpha (Q_{-i}^w)^\beta, & i \notin \mathcal{C} \\ \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta}} (w_i + Q_{-i}^w)^{\alpha+\beta}, & i \in \mathcal{C}. \end{cases} \tag{A.74}$$

Next, notice that for a fixed $\mathcal{C}$, Q_{-i}^w is linear in $\mathbf{w}$. Each utility function u_i is hence a composition of linear function and a power function of exponent at most $\alpha + \beta \leq 1$, and therefore concave.

Bibliography

Allouch, N. (2015). “On the private provision of public goods on networks”. *Journal of Economic Theory* **157**, pp. 527–552.

Ambrus, A., M. Mobius, and A. Szeidl (2014). “Consumption risk-sharing in social networks”. *American Economic Review* **104**:1, pp. 149–82.

Bagnoli, M. and B. L. Lipman (1989). “Provision of public goods: fully implementing the core through private contributions”. *The Review of Economic Studies* **56**:4, pp. 583–601.

Bergstrom, T. C., L. E. Blume, and H. R. Varian (1986). “On the private provision of public goods”. *Journal of Public Economics* **29**, pp. 25–49.

Bramoullé, Y. and R. Kranton (2007). “Public goods in networks”. *Journal of Economic Theory* **135**:1, pp. 478–494.

Brouwer, A. E. and W. H. Haemers (2011). *Spectra of graphs*. Springer Science & Business Media.

Elliott, M. and B. Golub (2019). “A network approach to public goods”. *Journal of Political Economy* **127**:2, pp. 730–776.

Galeotti, A., S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv (2010). “Network games”. *The review of economic studies* **77**:1, pp. 218–244.

Goyal, S. and J. L. Moraga-Gonzalez (2001). “R&D networks”. *Rand Journal of Economics* **32**, pp. 686–707.

Horn, R. A. and C. R. Johnson (2012). *Matrix analysis*. 2nd ed. Cambridge university press, p. 350.

Hurwicz, L. and S. Reiter (2006). *Designing economic mechanisms*. Cambridge University Press.

Kahneman, D. and A. Deaton (2010). “High income improves evaluation of life but not emotional well-being”. *Proceedings of the national academy of sciences* **107**:38, pp. 16489–16493.

Patacchini, E. and Y. Zenou (2012). “Juvenile delinquency and conformism”. *The Journal of Law, Economics, & Organization* **28**:1, pp. 1–31.

Pigou, A. C. (1920). *The economics of welfare*. Palgrave Macmillan.

Roberts, D. J. (1974). “The Lindahl solution for economies with public goods”. *Journal of Public Economics* **3**:1, pp. 23–42.

Samuelson, P. A. (1954). “The pure theory of public expenditure”. *The review of economics and statistics*, pp. 387–389.

Tabarrok, A. (1998). “The private provision of public goods via dominant assurance contracts”. *Public Choice* **96**:3-4, pp. 345–362.

Lund University
Department of Automatic Control
Box 118
SE-221 00 Lund Sweden

Document name
MASTER'S THESIS

Date of issue
September 2020

Document Number
TFRT-6099

Author(s)
Anton Arbman Hansing

Supervisor
Giacomo Como, Dept. of Automatic Control, Lund University, Sweden
Anders Rantzer, Dept. of Automatic Control, Lund University, Sweden (examiner)

Title and subtitle
Public Goods on Networks : Statics, Welfare & Mechanisms

Abstract

This thesis studies a network game of heterogeneous and asymmetric public goods. Players allocate their wealth between private and public goods, benefiting from the public goods provisioned by their out-neighbors on the network graph. Utilities are given by a Cobb-Douglas function to capture substitutability and decreasing marginal returns. I prove that the game is well-behaved under a condition relating a simple network characteristic – the spectral radius – to the preferences of the players. Under this assumption, the best response dynamic is guaranteed to converge, and the equilibrium strategy is unique. Equilibrium public good contributions are then linear in the wealth of others contributors. Next, the game is studied through a normative lens. I show that equilibrium outcomes, as a rule, are inefficient with regards to important welfare metrics. Three mechanisms on the game are formalized, drawing on the economic literature of public goods: taxes & subsidies, enforceable contracts, and redistribution. For each mechanism, the scope of attainable welfare improvements is characterized and design considerations discussed.

Keywords

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title	*ISBN*
0280-5316	

Language	*Number of pages*	*Recipient's notes*
English	1-51	
Security classification		

http://www.control.lth.se/publications/