A study incorporating skewness in Expected Shortfall Estimation

Turbulent period versus Tranquil period

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Abstract

Expected Shortfall has become a prominent risk measure after the global financial crisis which hit the economy in 2007. This master thesis examines whether Expected Shortfall (ES) estimation gives better estimates when we incorporate skewness and the impact during turbulent versus tranquil period. This specific analysis scrutinized daily total returns (TR) of three Indexes: Standard & Poor 500 (S&P 500), US benchmark 10 year DS GOVT index (BMUS10Y), and S&P GSCI Gold Total Return - RETURN IND (GSGCTOT). The sample for the estimation was from Jan 2000 to end of Dec 2019, which embrace a turbulent as well as a tranquil period. The Value at Risk (VaR) and Expected Shortfall (ES) forecasts was done with different distributions and evaluated with back testing procedures. However, skewed t-distribution elucidated the main research purpose. The empirical results, gives the idea that ES estimates incorporating skewness helps by retrieving better estimates during turbulent period as well as during tranquil/normal period.

Keywords: Value at Risk, Expected shortfall, normal distribution, student t-distribution, skewed student t-distribution.

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1. Introduction

Risk and its management has become a major necessity of the century due to the volatility of the current market scenarios. All kinds of investments in the market bear risk, which in turn potentially leads to large variation in realized returns. Thus, the investment institutions significant emphasis led to risk identification and management of risk. This insight of risk gave them better possibilities to deal with potential problems. The 2007-08 financial crisis, which emerged in the US sub-prime mortgage sector collapsed the global financial system and impacted in recession. This economic downturn compelled Basel committee on banking supervision (BCBS) to review the good old practices pursued over decades in management of risk, which in turn led to an upward shift in risk management area that is from Value at Risk (VaR) to Expected Shortfall (ES).

Expected Shortfall (ES), otherwise explained as Conditional Value at Risk (CVaR) was recommended by the Basel committee realizing the limitations of VaR in capturing tail risks. The Fundamental review on the trading book (FRTB 2013) gives a good glimpse of Basel committee's decision in order to strengthen the regulatory standard for banking institutions. Alexander and Baptista (2004) analysed the VaR and CVaR constraints on the mean-variance model. In that particular literature they clearly shares the idea that for a particularly given confidence interval CVaR restrictions are tighter than VaR restrictions if CVaR and VaR constraints coincides. Artzner et al. (1997), Acerbi and Tasche (2002) and Mansini and Ogryczak (2007) are few literatures where the authors explained their views about the effects and upper hand of Expected shortfall over Value at Risk. Consequently, VaR and ES play a vital role in the risk management world and to an extent ES has now a days gained popularity because of being a coherent risk measure which in turn can be considered as a desirable property for a risk measure.

Recent developments in focus of risk management has paved way to skewness as an important feature of risk along with Expected Shortfall. Many literatures emphasize the importance of skewness. In probability distribution skewness can be described as a measure of asymmetry around its mean for a set of data. Now a days, many researchers gives light to the focus on skewness, kurtosis in portfolio selection and to see the tendency of risk in a portfolio selection.

Konno, H., Shirakawa, H., and Yamazaki, H. (1993) and Konno and Suzuki (1995) also explains the importance of skewness in selecting optimal portfolio.

The most appropriate idea of skewness for this research was given by the paper-"A new class of multivariate skew densities, with application to GARCH models" by Luc Bauwens and Sebastien Laurent (2002).This paper gave the relevant research ideas of skewed t-distribution considering the leptokurtic behavior and skewness of data for Value at Risk and Expected Shortfall estimations.

1.1 Aim and purpose of the thesis

As aforementioned skewness, the third moment of a distribution becomes prominent mostly when mean and variance fails to meet the requirements. This paper details about the impact of incorporating skewness in Expected Shortfall estimation. To be specific, Risk measures draws attention in turbulent period compared to tranquil/normal period. This idea boosted more attention for the consideration of these periods in the estimation which also led to more evident results. Normally, financial results and risk measures follows a right or left skewed distribution, and not a bell shape curve. This is where usually student t-distribution and skewed distributions outperforms the normal distribution with its fat tails and skewness of data towards left or right according to market behavior.

Daily index data of large, small and commodity index are evaluated over the risk free rate considering 19 years of sample and two periods of estimation each holding three year batch for the thesis. The analyzed periods include the period when recession hit the economy and also a good/normal period of market. The purpose of the thesis also focus on the fact to convey better awareness of the forecasting capability of Expected shortfall estimation incorporating skewness during the crisis time. Usually, during the turbulent period risk measures will have a tendency to fail to incorporate the escalating risk compared to tranquil period.
This research also gives the idea how skewed t-distribution impacts in turbulent as well as tranquil period for a set of data. Here the study aims to investigate whether skewed t-distribution provides better estimates in Expected Shortfall considering the volatility of today's market. Particularly, the research examines the contribution of skewness in ES estimation forecast in two dissimilar states of the economy.

1.2 Outline of the Thesis

The thesis is organized in the following manner. In the Section 2, some preliminary literature and background regarding Value at Risk (VaR), Expected Shortfall (ES) two prominent risk measures are provided. Furthermore in Section 2 importance of skewness as a risk measure is also added. Next towards the Section 3 the methods adapted for estimation of VaR and ES is discussed using different distributions followed by methods to incorporate skewed t-distribution. The last sections (Section 4 & Section 5) concludes with the empirical results, inference and ideas for further research gained from the thesis.

1.3 Limitations

A viable limitation of this particular research is that only one back testing procedure is implemented to test the accuracy and efficiency of the method which is adapted to test. The back testing procedure followed here is the one among the back testing procedure introduced by Acerbi and Szekely (2014). But there is no valuable advantage for this method other than less complexity in computation. For further studies different back testing procedures can be implemented to check the precision of the method.

2. Literature/Theoretical Review

2.1 Theoratical Background

2.1.1 Value at Risk

Definition:

Value at Risk (VaR) can be explained as a statistical measure from the probability of loss distribution and this is defined as a smallest loss in portfolio, such that the probability of a future loss L is larger than the loss l is less than or equal to 1- α.

$$VaR_{\alpha}(L) = min\{\, l : Pr(L > l) \leq 1 - \alpha\} \quad (1)$$

2.1.2 Expected Shortfall

Definition:

Expected Shortfall (ES) can be explained as the average of all losses for all confidence levels larger than or equal to alpha (α).

$$ES_{\alpha}\,(L) = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR_{\alpha}(L)\,dx \qquad (2)$$

2.1.3 Limitations of Value at Risk in Risk management

Value at Risk (VaR) is the measure used around last few decades to estimate the amount of risk and to manage it effectively in order to increase the expected income. Because of its easily understandable nature VaR can be defined as an interesting measure to adapt especially for financial institutions and banks. In risk management point of view, it asks the very simple query "How bad can things get?" (Hull 2018). Mostly, banks and finance related institutions make use of this measure to estimate the profitability and degree of risk of major investments, and equips risk based on VaR.

Unlike to the advantages, VaR does not give any indication or quiet about the magnitude if the loss is greater than Value at Risk. This could make managers bear higher risk than expected and it enhances the probability to lose more than what taken into consideration. VaR does not support coherent properties (apart from normality) which is also essential for a risk measure. Due to the lack of coherent properties and limitations of VaR capturing the tail risk, ES (Expected shortfall) which satisfies these limitations became an ample option on top of VaR.

2.1.4 Expected Shortfall as a better measure

Expected Shortfall (ES) can be termed as a measure which gives better picture in the risk management for traders than Value at Risk (VaR). "Expected Shortfall can also be defined as the conditional value at risk, conditional tail expectation, or expected tail loss". ES asks "If things do get bad, what is the expected loss?" (Hull 2018). First step in ES calculation is the calculation of VaR, which also roots to the clarification that it considers all the properties of VaR in particular. ES can be explained as the expected loss happened at time T conditional on the loss greater than the VaR (Hull 2018). Expected Shortfall can be described as a more realistic method than VaR when we face an elevated spike in the tail and also provides the probability of loss beyond the confidence interval. Even though, despite of all these advantages this risk measure is not simple to evaluate compared to VaR.

"Value at risk (VaR) and Expected Shortfall (ES) are attempts to provide a single number that summarizes the total risk in a portfolio. It is the measure regulators have traditionally used for many of the calculations they carry out concerned with the setting of capital requirements for market risk, credit risk, and operational risk" (Hull 2018). Now these regulators have shifted to ES which in other words can be defined as an improved form of VaR. Hence the point of study concentrated in Expected Shortfall (ES) and its methods which can take care of risk measure providing better judgements for different market scenarios.

2.2 Expected Shortfall and Skewed distribution

Skewness has showcased its presence in exhibiting different type of risk along with variance. It is considered as a risk measure in the current economy and many studies pointed to the fact that incorporating skewness in evaluations gives sharper picture to the estimations. Scott and

Horvath (1980) suggests that an investor who is risk averse and has constant moment preferences will show a positive attitude towards skewness. Many studies also gives light to the conclusion that negative and positive skewness increases and decreases the probability of losses and profits respectively when the financial returns shows a tendency to deviate from normal distributions. Arditti (1967) explained a theoretical correlation among the second and third moments of the distribution and expected return. Expected utility theory also suggests that prudence and preference for skewness are related (Kimball 1990). Harvey and Siddique (2000) suggests that due to systematic skewness the asset prices follow a trend for a long time and which will impact in a situation where winners outperforms losers with lower skewness.

Wen and Yang (2009) also claimed that inclination of risk is a major cause for the return distribution to be skewed. These researches gives idea regarding skewness in a good way but further scientist researches which happened later in the finance world gave much broader aspect of skewness as an inseparable tool as a consequence of investors risk attitude.

Due to the volatility of the market risk and risk measures tends to follow asymmetrical distribution. Risk measure like VaR and ES gives accurate estimates when it follows a skewed distribution compared to a symmetric distribution (Degiannakis and Potamia 2016). According to the studies, skewed t-distribution contributes better estimates for risk measures compared to some other major distributions which may be because of the truth that in risk measures student t-distributions outperforms normal distribution with its fat tails. Skewed t-distributions are an extension to student distribution with a parameter for skewness. Many literatures shows that skewed student t-distribution provides recommended results under GARCH specifications when compared to a symmetric one. Angelidis, Benos and Degiannakis (2004) also showcased the same information in their literature. All these literatures justifies the intention of this thesis to explore Expected Shortfall, with skewed t-distributions and also to investigate what better outcomes Expected Shortfall contributes incorporating skewness focusing the market in different scenarios.

The highlight of this literature review is to get a glimpse of the significance of study which incorporates skewness in Expected shortfall estimation.

3. Methodology

The methodology described here gives focus to the study implemented in the estimation of ES using parametric methods. The main advantage of using parametric methods is that it helps to study the importance of skewness which is our main focus of study in this research. Non-Parametric methods are usually purely dependent on the sample taken for the estimation, In that case it becomes too slow to reflect and this will end up in taking late decisions according to market volatility. Usually parametric methods takes into account the market flexibility and are easy to evaluate.

The first section describes the data embodied in the study and its transformation to stipulated data for estimation. Following this section, the second section describes the constant and EWMA volatility which are essential for the distributions. Finally, the distributional assumptions related to the research.

3.1 Data Collection & processing:

Here, the sample used in this study comprises of daily returns index of large, small and commodity index. The time frame of the sample consist from 01.01.2000 to 31.12.2019, a period of 19 years and 5218 observations of each index returns.
Details of Index Bonds with Abbreviation:

- US BENCHMARK 10 YEAR DS GOVT. INDEX - CLEAN PRICE INDEX-BMUS10Y - Small stocks
- STANDARD AND POOR, TR INDEX-S&P 500 TR INDEX - Large Stocks
- S&P GSCI GOLD TOTAL RETURN - RETURN IND. (OFCL) - GSGCTOT-Commodity Index

The data is from a single source DataStream, developed by Thomson Reuters.

From the whole sample the main aim of the thesis was to investigate how the ES estimation with skewness helps in forecasting the risk. For that purpose we have split our sample of estimation into two categories. At first turbulent period was considered (01.01.2007 to 31.12.2009) and furthermore a comparison was done with the normal/tranquil test period (01.01.2017 to 31.12.2019) taking skewness into consideration.

Test periods:

- 2007-2009 (Turbulent Period)
- 2017-2019 (Tranquil/Normal Period)

Here the sample data from 2000-2006 and 2010-2016 are considered as 'in sample' for estimation.

3.1.1 *First step:* Loss scenario

In general, the following transformation is used to obtain observations into losses for each holding period assuming $100 is invested in the beginning of each day. For each time period a new loss observation is available. Loss scenario can be obtained as:

$$L_t = -\frac{I_t - I_{t-1}}{I_{t-1}} * 100 \qquad (3)$$

Where I_{t-1} and I_t can be detailed as the previous period and this period observations respectively.

Figure 1, 2 and 3 illustrates the loss scenario over the total stipulated period, in which predominant volatility clustering can be sighted during the global financial crisis period from 2007-2009.

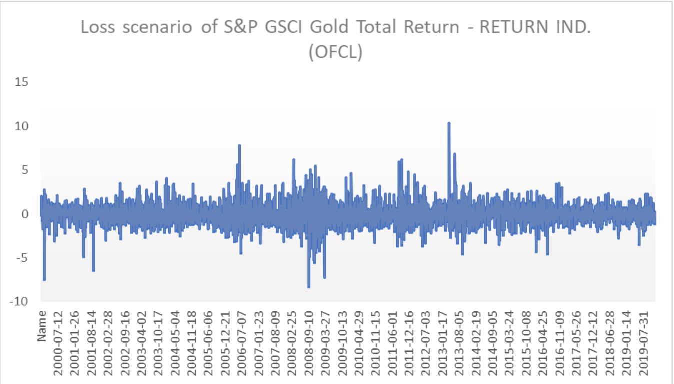


Figure 1: Loss scenario of S&P GSCI Gold Total Return

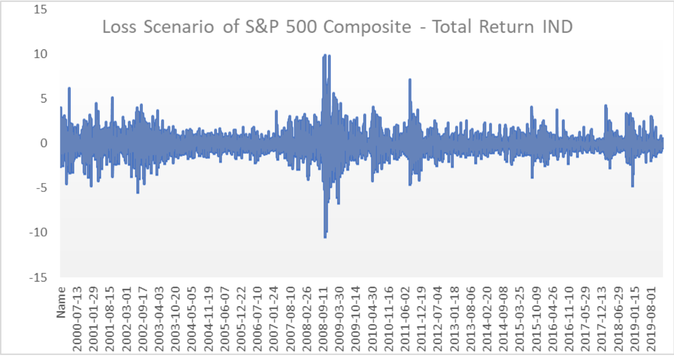


Figure 2: Loss scenario of S&P 500 Composite

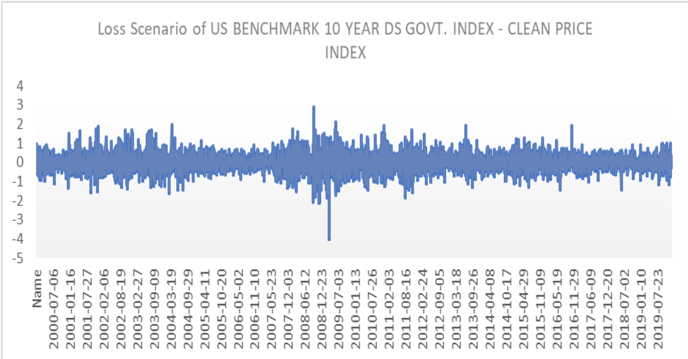


Figure 3: Loss scenario of US Benchmark 10 year DS Govt Index – Clean price index

Descriptive statistics which gives an overview of the information of large amount of data (here losses)in a more accessable and sensible way is also deliberated to have a comprehensive overview of data. The descriptive properties of the review period gives an indication that turbulent period is highly volatile compared to tranquil period. This also provides the statistics that turbulent is not so bad in terms of average losses(gains), but the calculation of standard deviation gives the information of high volatility during the bad period compared to normal period (Volatility is almost double during the turbulent period).

Descriptive statistics S&P500 Gold Index turbulent period versus tranquil period				Descriptive statistics S&P 500 TR Index Turbulent versus tranquil period				Descriptive statistics US Benchmark 10 yrs GOVT.Index turbulent versus tranquil period.			
Mean	***-0,031***	***Mean***	***-0,054***	***Mean***	***0,039***	***Mean***	***-0,05***	***Mean***	***-0,007***	***Mean***	***-0,004***
Median	*-0,022*	*Median*	*-0,053*	*Median*	*-0,051*	*Median*	*-0,06*	*Median*	*0*	*Median*	*0*
Mode	*0*	*Mode*	*0*	*Mode*	*0*	*Mode*	*0*	*Mode*	*0*	*Mode*	*0*
Standard Deviation	***0,668***	***Standard Deviation***	***1,466***	***Standard Deviation***	***1,857***	***Standard Deviation***	***0,796***	***Standard Deviation***	***0,618***	***Standard Deviation***	***0,328***
Variance	*0,446*	*Variance*	*2,151*	*Variance*	*3,449*	*Variance*	*0,633*	*Variance*	*0,382*	*Variance*	*0,108*
Kurtosis	*2,087*	*Kurtosis*	*3,198*	*Kurtosis*	*6,423*	*Kurtosis*	*5,779*	*Kurtosis*	*3,204*	*Kurtosis*	*0,892*
Skewness	*-0,075*	*Skewness*	*-0,086*	*Skewness*	*0,404*	*Skewness*	*0,799*	*Skewness*	*-0,228*	*Skewness*	*-0,111*
Range	*5,774*	*Range*	*14,454*	*Range*	*20,301*	*Range*	*9*	*Range*	*6,887*	*Range*	*2,436*
Minimum	*-3,45*	*Minimum*	*-8,233*	*Minimum*	*-10,38*	*Minimum*	*-4,73*	*Minimum*	*-3,971*	*Minimum*	*-1,424*
Maximum	*2,325*	*Maximum*	*6,221*	*Maximum*	*9,921*	*Maximum*	*4,273*	*Maximum*	*2,915*	*Maximum*	*1,011*
Count	*782*	*Count*	*784*	*Count*	*784*	*Count*	*782*	*Count*	*784*	*Count*	*782*

Table 1: Descriptive statistics

3.1.2 *Second step*: Constant volatility and EWMA volatility:

Practically, volatility symbolizes measure of dispersion of returns around the mean price and it is considered as an inevitable feature for the asset returns. These can be termed as optimum characteristic to be implemented in the model. Usually financial data shows a tendency of volatility clustering due to the deviation in asset returns and due to variability in time. As the first procedure constant volatility which can be otherwise represented as standard deviation is estimated and these values are utilized in order to arrive at EWMA volatilities.

EWMA (Exponentially weighted moving average) volatility is a case where the α_i, the weights shows an exponentially declining characteristic over time. Particularly, $\alpha_{i+1} = \lambda\alpha_i$, where λ ranges from zero and one (Hull 2015). This can also be expressed as a particular case of GARCH (1,1) – Generalized Autoregressive Conditional Heteroscedasticity, where this fit as a best model to volatility clustering satisfying the condition by allowing the conditional variance to dependent on the previous values. The variance equation used in GARCH (1, 1) can be defined as:

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta\sigma_{t-1}^2 \qquad (4)$$

The variance equation of GARCH (1, 1) by using EWMA, which considers the volatility variations can be rewritten as,

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1 - \lambda)u_{t-1}^2 \quad (5)$$

Where $\omega = 0, \alpha = 1 - \lambda$ and $\beta = \lambda$.

Here In general practice, μ=0 (mean of the data is assumed as zero) and sample standard deviation derived from the respective in sample of both turbulent as well as tranquil period then,

λ=0.94 and 1-$\lambda = 0.06$, EWMA with λ=0.94 is taken into account to capture volatilities, because of the fact that among multiple market variables, this particular value chosen to λ=0.94 gives forecasts of the variance rate that come closest to the realized variance rate (JP MORGAN 1996).

$$\sigma_t^2 = 0.94 * \sigma_{t-1}^2 + 0.06 * u_{t-1}^2 \qquad (6)$$

And EWMA volatility is calculated as a square root of EWMA.

$$\sigma_{t+1} = \sqrt{0.06 * u_t^2 + 0.94 * \sigma_t^2} \qquad (7)$$

In this research EWMA volatility is considered for normal, student t and skewed t-distributions.

3.2 Normal, student & skewed distribution (In parametric approaches)

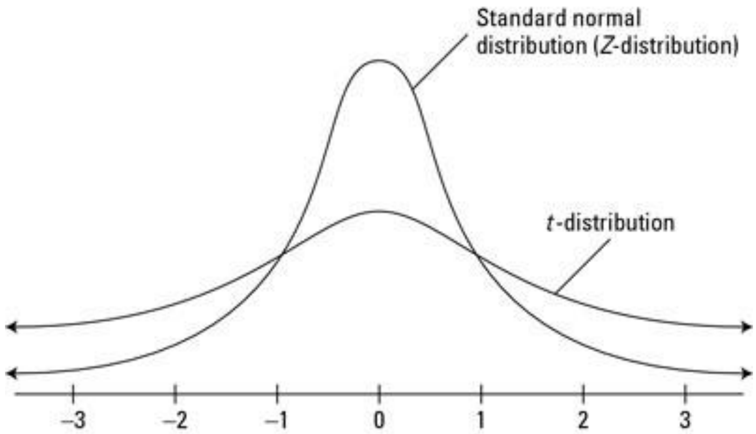


Figure 4: Normal & student t-distribution

The normal distribution can be explained as a restricted form of student t-distribution. Both distributions shows a bell shaped curve, but t-distribution has a fatter tail than Normal distribution. According to the law of large numbers, t-distribution approaches to normal distribution when the considered sample size increases. The main characteristics of normal distribution can be termed as its “bell shaped” curve and symmetry towards the average value. Mainly four central moments are popular among studies, the mean and standard deviation are termed as the first two central moments. Likewise skewness and kurtosis are explained as the third and fourth moments. A symmetrical distribution will have zero skewness and kurtosis as 3.Skewed distribution always have one longer tail towards left or right. Positively skewed distribution has tail extended towards right and negatively skewed distribution has tail extended towards left.

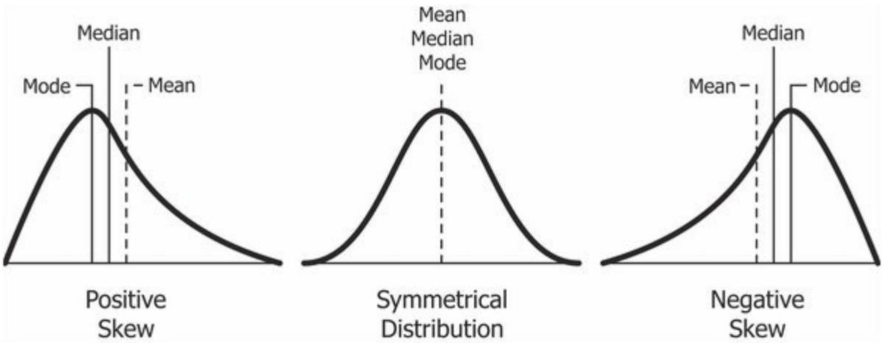


Figure 5: Positive skew, Symmetrical Distribution & Negative skew

4. Analysis & Discussion

In analysis part the normal distribution, one of the parametric approaches is assumed as the first step followed by student t-distribution using both EWMA and kurtosis. Further skewed t-distribution is considered in the final part in order to check whether skewed t-distribution can give better estimates in turbulent as well as tranquil period. A back testing procedure was also done to check the efficiency of the stipulated method. Here alpha is considered as 2.5% which gives a confidence interval of 97.5% according to BCBS suggestion on FRTB (2019).

4.1 Normal/Gaussian distribution with ES

According to definition, VaR and ES can be estimated using the formula:

$$VaR_\alpha (L) = \mu + \sigma z_\alpha ,$$

$$ES_\alpha (L) = \mu + \sigma \frac{f_{std}(z_\alpha)}{1-\alpha} \qquad (8)$$

Where mean is μ and σ is the standard deviation, with z_α as α quantile of the standard normal distribution.

When time varying volatility is considered for a conditional model, where μ as sample mean and σ_{t+1} is the forecasted volatility using EWMA. The VaR and ES formula under normal distribution can be rewritten as:

$$VaR_\alpha (L) = \mu + \sigma_{t+1} z_\alpha ,$$

$$ES_\alpha (L) = \mu + \sigma_{t+1} \frac{f_{std}(z_\alpha)}{1-\alpha} \qquad (9)$$

4.2 Student t-distribution with ES

Student t-distribution plays an inevitable role in financial returns, with its fat long tails and ability to accommodate kurtosis greater than three. Risk also follows similar pattern as financial returns. Normal distributions underestimates financial risk if kurtosis is greater than three. Kurtosis is captured by the degrees of freedom v which is normally used in student t-distribution. The degrees of freedom (v) is the rate of change of value of the portfolio with regard to the change in underlying asset price.

$$For\ v > 3, \quad v = \frac{4k-6}{k-3} \qquad (10)$$

The VaR and ES can be estimated using the formula under student t-distribution:

$$VaR_\alpha(L) = \mu + \sqrt{\frac{v-2}{v}}\ \sigma t_{\alpha,v}\ ,$$

$$ES_\alpha(L) = \mu + \sqrt{\frac{v-2}{v}}\ \sigma \frac{f^*_{std}(t_{\alpha,v})}{1-\alpha} \left(\frac{v+t^2_{\alpha,v}}{v-1}\right) \qquad (11)$$

Here f^*_{std} is the pdf for standard t-distribution with mean as zero and standard deviation as one. Considering the conditional model, with time varying volatility, where μ as sample mean and σ_{t+1} is the forecasted volatility using EWMA. The VaR and ES formula under normal distribution can be rewritten as:

$$VaR_\alpha(L) = \mu + \sqrt{\frac{v-2}{v}}\ \sigma_{T+1} t_{\alpha,v}\ ,$$

$$ES_\alpha(L) = \mu + \sqrt{\frac{v-2}{v}}\ \sigma_{T+1} \frac{f^*_{std}(t_{\alpha,v})}{1-\alpha} \left(\frac{v+t^2_{\alpha,v}}{v-1}\right) \qquad (12)$$

4.3 Skewed t-distribution with ES

As aforementioned, skewed t-distribution can be explained as a t-distribution with a parameter of skewness. The main idea for skewed t-distribution was stated from the core discussion paper by Luc Bauwens and Sebastien Laurent (2002) – "A new class of multivariate skew densities, with application to GARCH models".

In order to approach the skewed t-distribution of Expected Shortfall, the skew t-distribution with parameter of skewness is considered as the first step and which is denoted as,

$$z_t \sim SKST\ (\mu, \sigma, \xi, v) \quad (13)$$

Where z_t can be assumed as a standardized data with mean μ, standard deviation σ, v degrees of freedom and ξ as the parameter for skewness. Here, Maximum likelihood which maximizes the likelihood of a set of parameters θ is used to estimate the appropriate parameters. The skew t-distribution parameters which has to be estimated are θ=(μ, σ, ξ, v) and log likelihood function for a student t-distribution which has to maximized with respect to the parameters mentioned above is given by:

$$l_t(\theta) = ln\left(\frac{2}{\xi + \frac{1}{\xi}}\right) + ln\Gamma\ \left(\frac{v+1}{2}\right) - \frac{1}{2} ln\big(\pi\ (v-2)\big) - ln\,\Gamma\ \left(\frac{v}{2}\right)$$
$$+ ln\left(\frac{s}{\sigma}\right) - \frac{1}{2}(v+1)\ ln\left[1 + \frac{(sz_t + m)^2\ \xi^{-2I_t}}{v-2}\right] \qquad (14)$$

with $z_t = (y_t - \mu_t)/\ \sigma_t$ and

$$I_t = \begin{cases} 1\ if\ z_t \geq -\frac{m}{s} \\ -1\ if\ z_t < -\frac{m}{s} \end{cases}$$

m and s^2 are two constants which does not varies with time and can be calculated using the formula given below:

$$m = \frac{\Gamma\left(\frac{v-1}{2}\right)\sqrt{v-2}}{\sqrt{\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(\xi - \frac{1}{\xi}\right)$$

&

$$s^2 = \left(\xi^2 + \frac{1}{\xi^2} - 1\right) - m^2 \qquad (15)$$

After maximizing the parameters with respect to time t, the estimated values are obtained.

Expected shortfall in other words can be considered as the average value of Value at Risk due to its properties for capturing tail risk. In this estimation the estimated value obtained after maximizing the parameters are employed in the quantile function in order to obtain VaR estimates for different confidence intervals. In this case confidence intervals which are considered for the quantile function are 97.5, 97.7, 97.9, 98.1, 98.3, 98.5, 98.7, 98.9, 99.1, 99.3, 99.5, 99.7 and 99.9 in order to obtain ES-0.975.

The quantile function formula for different confidence intervals are given as:

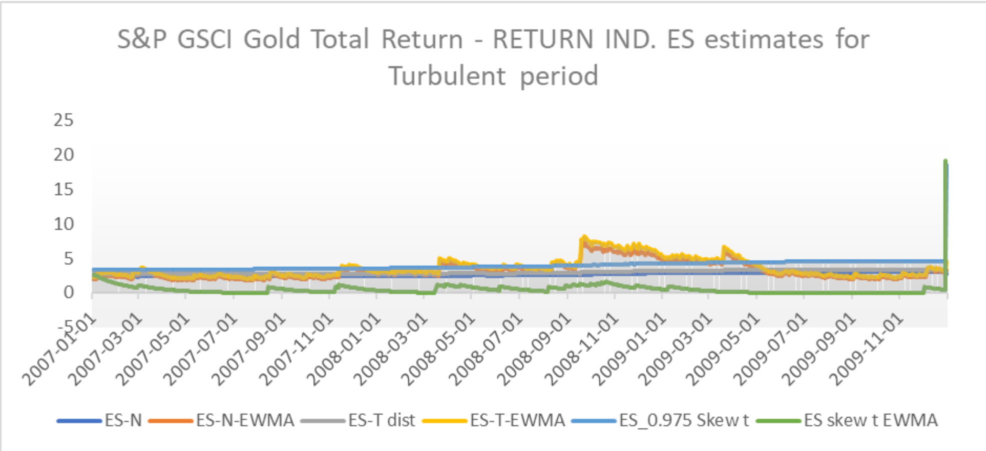
$$F^{-1}(p|\,\xi) = \begin{cases} \dfrac{\frac{1}{\xi}G^{-1}\left(\frac{p}{2}(1+\xi^2)\right) - m}{s} & if\ p < \dfrac{1}{1+\xi^2} \\ \dfrac{-\xi G^{-1}\left(\frac{1-p}{2}(1+\xi^{-2})\right) - m}{s} & if\ p \geq \dfrac{1}{1+\xi^2} \end{cases} \qquad (16)$$

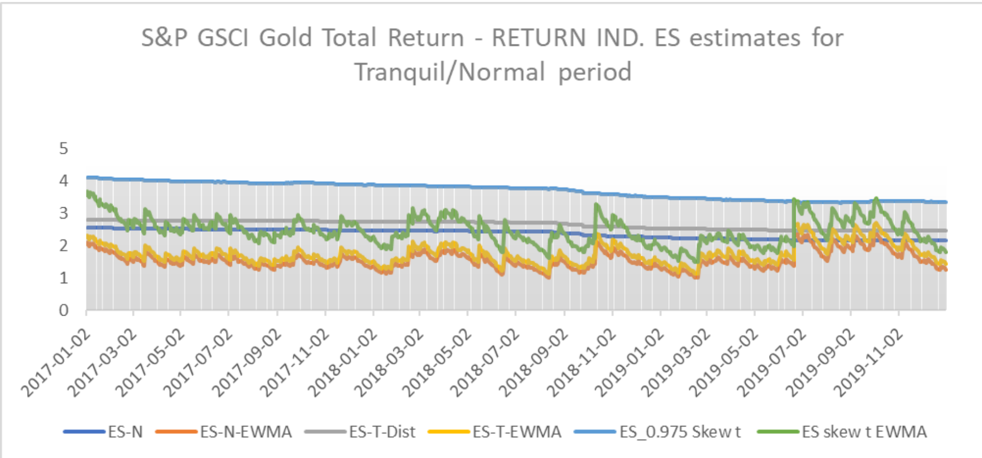
The VaR for the skew t-distribution can be estimated using the formula:

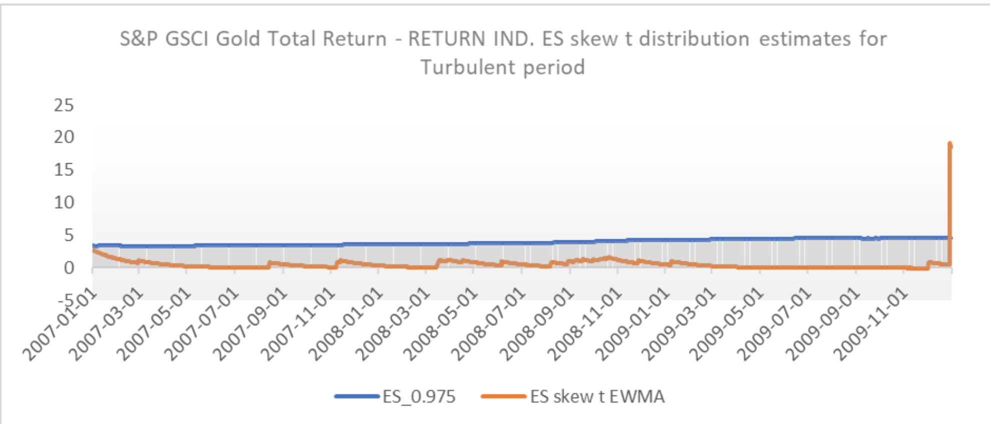
$$VaR_\alpha = \mu + F^{-1}(p|\xi)\,\sigma \qquad (17)$$

And in this case the average value of VaR with alpha from 97.5 to 99.9 gives the Expected Shortfall estimate for a confidence interval of 97.5.

Figure 6, 7 & 8 shows the graphical representation of ES estimates obtained by Normal, Student t and Skew t-distributions to show that the skew t-distribution gives better picture of ES estimates considering turbulent as well as tranquil period.







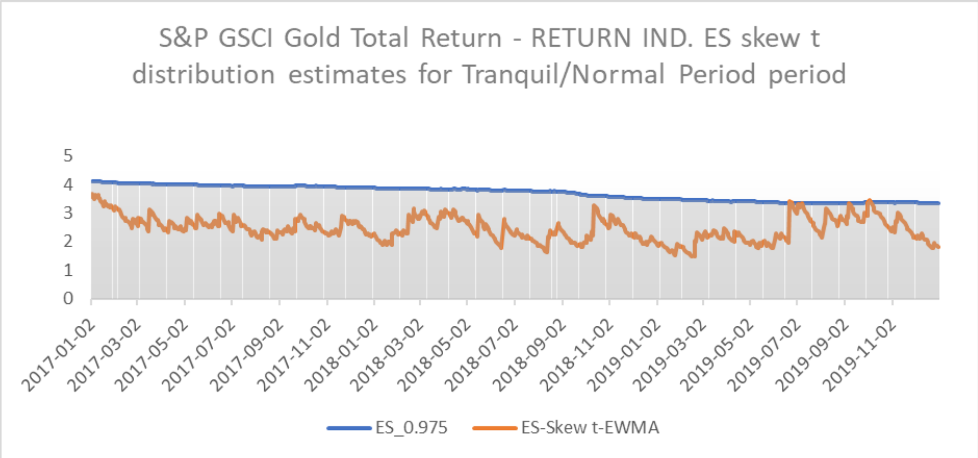
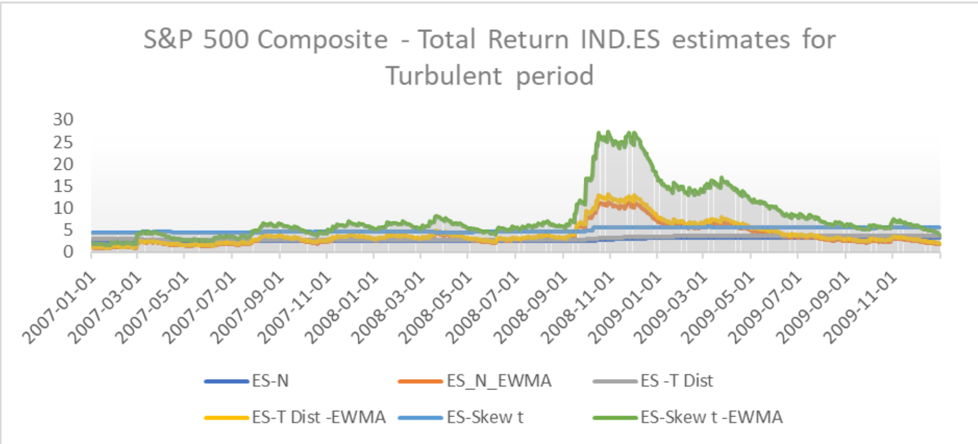
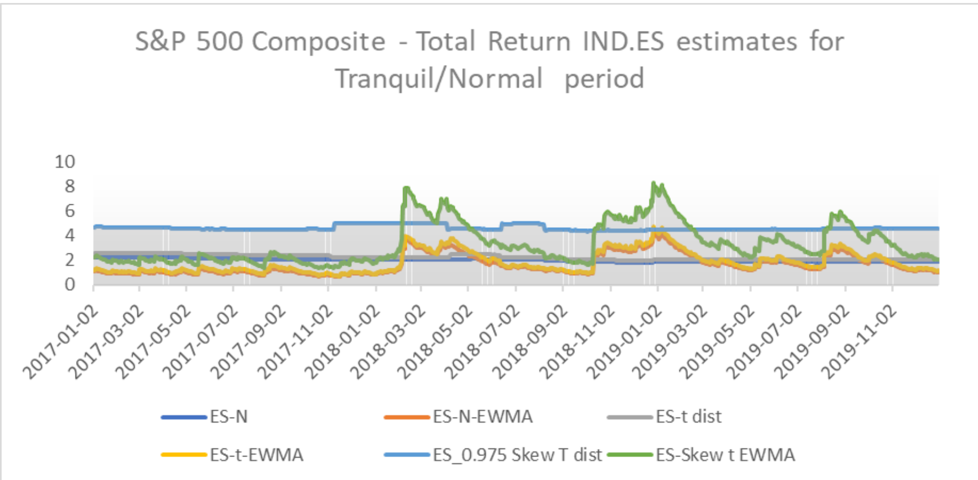
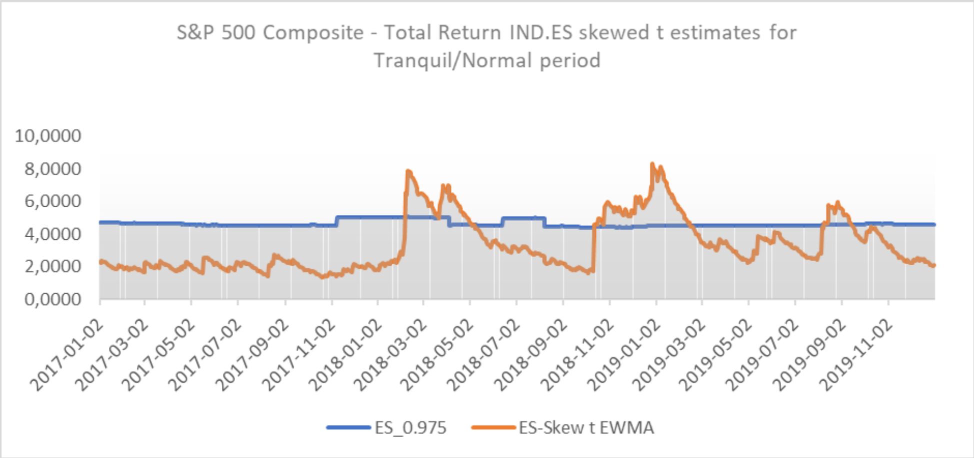


Figure 6: ES estimates for turbulent period of S&P GSCI Gold Total return under various







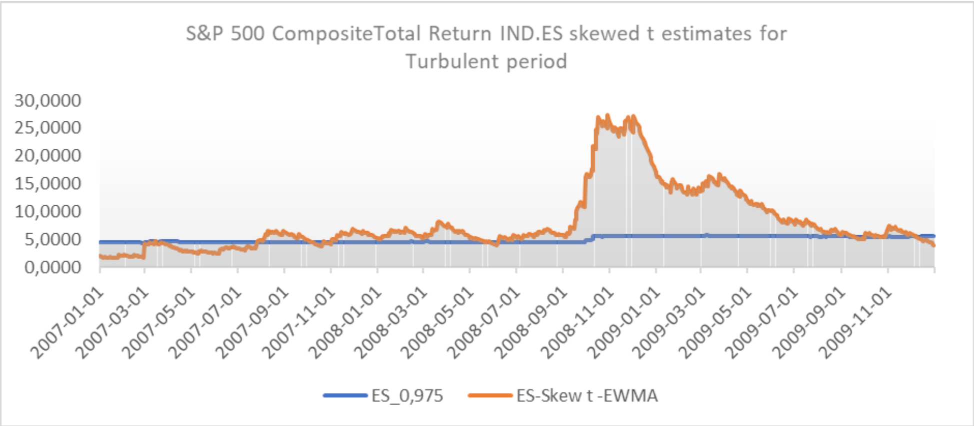
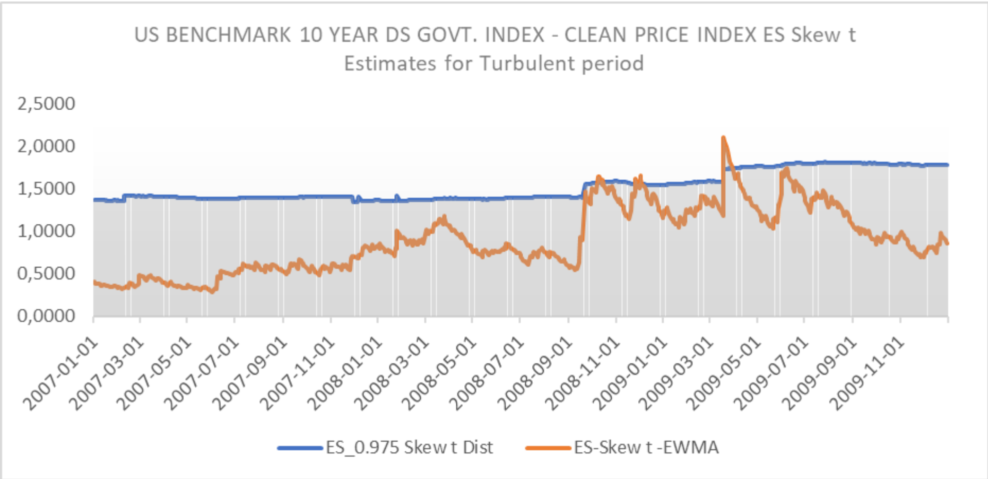


Figure 7: ES estimates for turbulent period of S&P 500 composite TR Index daily under various distributions



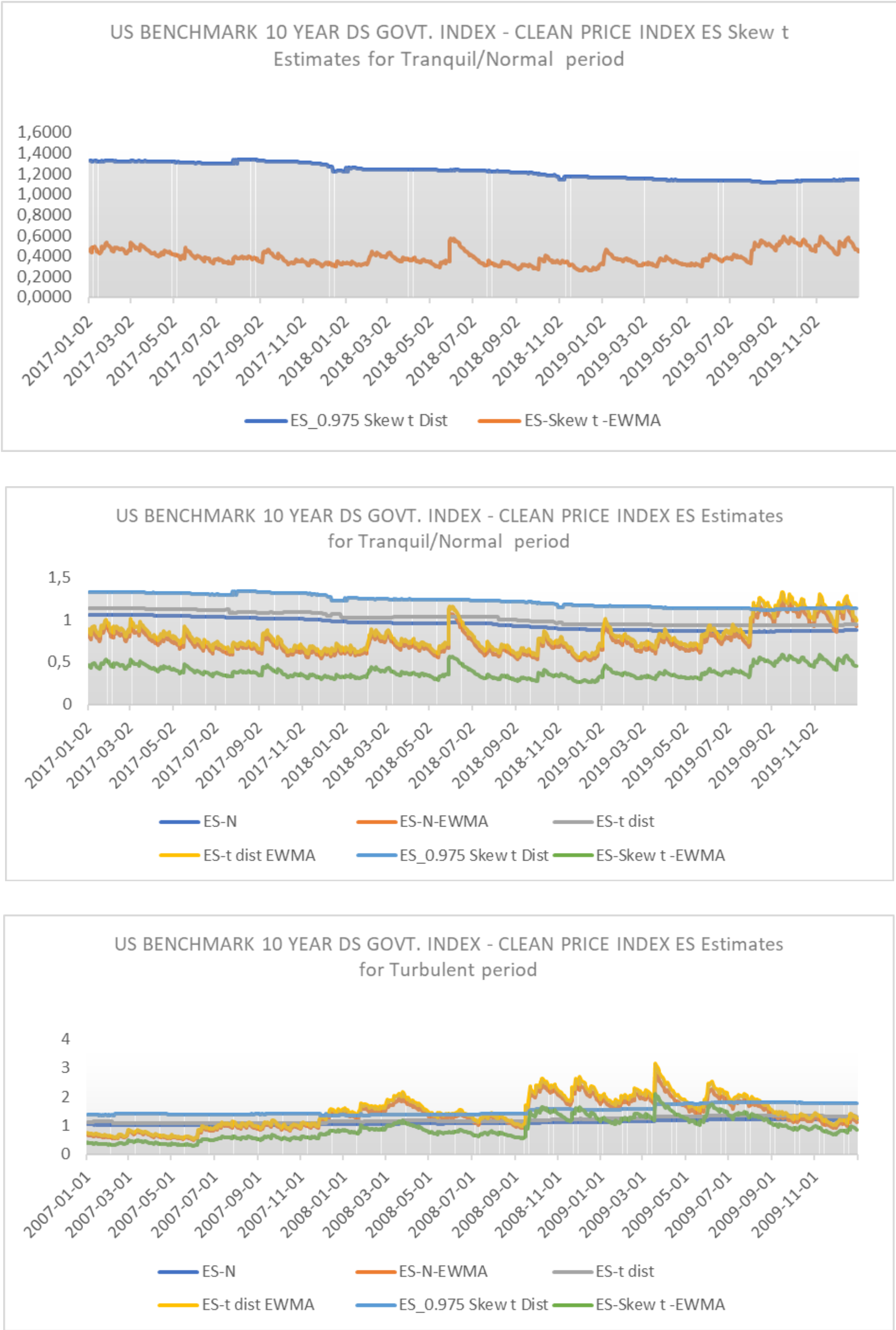


Figure 8: ES estimates for turbulent period of US Benchmark 10 yrs DS Govt Index under various distributions

In terms of software application in this study, all investigations are done with the help of Ms-Excel and maximization of parameters for the skewed t-distribution is done using Python.

4.4 Back Testing

The backtesting approach is to demonstrate the accuracy and efficiency of the method which is adapted to test. In risk management backtesting plays an inevitable role. VaR which was followed by our financial institutions before the financial crisis is easy to backtest. For the observed day the VaR violations usually occurs when the observed loss goes beyond the estimated VaR (Hull 2006). This may be one of the reason many financial institutions showed hesitance to follow Expected Shortfall despite of all the advantages ES has over VaR after the Basel committee's recommendation. Many mentioned different back testing procedures in order to check the efficiency of Expected Shortfall but the back test introduced by Acerbi and Szekely (2014) turned to be more applicable one.

Acerbi and Szekely (2014) introduced three back tests procedures for ES which are nonparametric and easy to compute considering the power of ES over VaR. Out of three the second test "Testing ES directly" turned to be more attractive compared to other two which works with the help of Monte Carlo simulation of the distribution in order to arrive the p value.

In this particular research the second method proposed by him "Testing ES directly" was taken into account among three methods of back testing. The second method test static-Z_2 requires only two parameters for estimation, one is the one day ahead estimated $ES_{\alpha,t}$ and the other one is the magnitude $L_t \, I_t$ of a $VaR_{\alpha,t}$ violation. Here L_t is denoted as the loss and it is known as the indicator variable at time t:

$$I_t = \begin{cases} 1, & L_t > VaR_{\alpha,t} \\ 0, & otherwise \end{cases} \quad (18)$$

The Expected Shortfall formula ES_α, can be written in a simplified form if the loss distribution is continuous as shown below,

$$ES_\alpha = E(L{:}\, L > VaR_{\alpha,t}) = \frac{E(L_t \, I_t)}{1-\alpha} \quad (19)$$

Now the test static-Z_2 can be defined in this way:

$$Z_2 = \sum_{t=1}^{T} L_t \, I_t \, / \; T(1 -< \alpha) + 1 \quad (20)$$

Back-testing is the procedure where the accuracy of the test method is evaluated and it also helps to make sure whether the model do not yield underestimated results. Here in this case the one sided test is performed, where the null hypothesis is such that the underlying Expected Shortfall model is correct, also conveys that it provides efficient ES estimates. If the null hypothesis is rejected the alternative hypothesis conveys that the method underestimates ES.

4.5 Results

As aforementioned, in this research "Testing ES directly" is taken into account and according to Acerbi and Szekely (2014) the critical value across many distributional assumption is set at 5% confidence interval and this value is estimated by the author as -0.70. So the actual test static is compared to the critical value and if it is less than the estimated critical value the method is rejected.

Table 2, 3 and 4 gives the information of acceptance and rejection of null hypothesis for the data employed in this research.

Turbulent Period	Z value	Accept/Reject
ES-Normal	-2,137625098	Reject
ES-N-EWMA	-0,777125028	Reject
ES-t-dist	-1,514568872	Reject
ES-t-EWMA	-0,263829014	Accept
ES-Skew t	-0,08708513	Accept

Tranquil /normal period	Z value	Accept/Reject
ES-Normal	0,66805997	Accept
ES-N-EWMA	-0,4148342	Accept
ES-t-dist	0,86164571	Accept
ES-t-EWMA	-0,186423	Accept
ES-Skew t	1	Accept

Table 2: S&P GSCI GOLD TOTAL RETURN - RETURN IND. (OFCL)

Turbulent Period	Z Value	Accept/Reject
ES-Normal	-3,628317	Reject
ES-N-EWMA	-1,4323	Reject
ES-t-dist	-3,054238	Reject
ES-t-EWMA	-0,928782	Reject
ES-Skew t	-0,313073	Accept

Tranquil /normal period	Z Value	Accept/Reject
ES-Normal	-0,9802795	Reject
ES-N-EWMA	-0,86790131	Reject
ES-t-dist	-0,69981819	Accept
ES-t-EWMA	-0,56765548	Accept
ES-Skew t	0,916960139	Accept

Table 3: S&P 500 COMPOSITE - TOT RETURN IND

Turbulent Period	Z Value	Accept/Reject
ES-Normal	-2,021487824	Reject
ES-N-EWMA	-0,239996407	Accept
ES-t-dist	-1,65048294	Reject
ES-t-EWMA	-0,102468678	Accept
ES-Skew t	-0,402636335	Accept

Tranquil /normal period	Z Value	Accept/Reject
ES-Normal	0,421266362	Accept
ES-N-EWMA	-0,202756317	Accept
ES-t-dist	0,465105834	Accept
ES-t-EWMA	-0,076557206	Accept
ES-Skew t	0,779003971	Accept

Table 4: US BENCHMARK 10 YEAR DS GOVT. INDEX - CLEAN PRICE INDEX

Risk measures always plays a main role in turbulent period compared to tranquil/normal period, where the risk also follows a smooth curve. In this research the result clearly indicates that the test static gets accepted most of the methods in tranquil period compared to turbulent period.

In turbulent period of S&P GSCI GOLD Total Return the result evidently interprets that only skewed - distribution gives a Z value which is greater than the critical value, so in this case only skewed t-distribution method is getting accepted, where normal distribution and student t- distribution are rejected. But at the same time the data employed accepts all the methods used for this research in tranquil/normal period.

In the case of S&P 500 the results resembles the S&P GSCI GOLD Total Return in turbulent period, whereas in tranquil period the data shows a tendency to reject the first two methods used for the studies.

For US benchmark 10 year DS GOVT Index, the results shows a peculiar path in turbulent period. The data accepted all the methods with EMWA volatility in first two, but it shows the other way around in skewed t-distribution method. At the same time the data exercised accepted all the methods used for the estimation.

From these results it is evident that the data used for the studies indicate a positive approach towards the skewed t-distribution method to produce better ES estimates compared to all other distributions considered in the research. Our research also gives the idea that it produces more appropriate ES estimates in all three sample of observations considered in study in two dissimilar states of economy (Turbulent as well as Tranquil period).

5. Conclusion

The motive of this research paper was to examine whether incorporating skewness in ES estimates produce better estimates that really helps during turbulent period as well as during tranquil/normal period.

To conduct the test daily total returns of three indexes were considered: S&P500, US benchmark 10 year DS GOVT Index and S&P GSCI Gold Total Return. The period used for analysis is during Jan 2000 to end of Dec 2019, where the estimation period were divided into turbulent and tranquil period which is from 2007-09 and 2017-19 respectively.

From the research results, it is clear that ES estimates gives a better picture when we integrate skewed t-distribution in turbulent period as well as tranquil period. Student t-distribution which is known as the better distribution among other distribution for the financial data outperformed with skewed distribution to give better estimates for Expected Shortfall in our data employed for the research. In order to get idea regarding how ES incorporating skewness produces better estimates, all the major distributions like Normal, Student t and skew t-distributions were implemented in the research. The results derived from the statistical inference and back testing suggests that skew t-distribution model can produce better estimates in different market scenarios. This also shed light to the fact that the conditional methods does not improve the efficiency of the ES estimates especially during turbulent period.

5.1 Future Research

Skewness and its contributions are always a topic of interest in financial world. The paper by Luc Bauwens and Sebastien Laurent on “A new class of multivariate skew densities, with application to GARCH models” has provided a great idea to this research topic specifically for the concept of univariate skewed t-distribution. It is indubitably fascinating to track future studies which will further expand skewed distributions unavoidable contribution for obtaining better ES estimates. I certainly thinks a ray of hope that by repeating the methodology and assumptions of this paper other researches can follow the concept and can notice whether skewness produce better ES estimates in all market scenarios using other skewed distributions as well.

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