On Levi Decompositions in Finite and Infinite Dimensional Lie Algebras

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January 28, 2021

Populärvetenskaplig Introduktion

Att ändra ordningen på två händelser kan ha väldigt olika effekt på resultatet. Exempelvis är det inte så viktigt om du duschar först, och sedan borstar tänderna, eller om du först borstar tänderna, och sedan duschar på morgonen. Å andra sidan, så blir resultatet väldigt annorlunda om man knäcker ett ägg och sedan steker det, än om man steker ägget, och sedan knäcker det. I matematiska termer kan vi kalla den ena händelsen A och den andra händelsen B. Om först A händer, och sedan B, så kallar vi det för AB, och om de händer i den andra ordningen så kallar vi det för BA. I sådana fall kan vi beskriva att ordningen spelar roll som $AB \neq BA$ eller att den inte spelar roll som $AB = BA$. I matematiska termer kallas A och B kommutativa om $AB = BA$ och ickekommutativa om $AB \neq BA$.

Om vi försöker beskriva detta fenomenet nogrannare får vi dock ett problem. Om A är händelsen att knäcka ett ägg, B är händelsen att steka ägget, och C är händelsen att koka ägget, så är det tydligt att både $AB \neq BA$ och att $AC \neq CA$, men det är svårt att avgöra i vilket fall skillnaden på ordningen är störst. Vi kan alltså säga om två händelser är kommutativa eller inte, men det är svårt att säga *hur* ickekommutativa de är.

Matematiker arbetar oftast inte med vilka händelser som helst, utan med olika matematiska objekt. Ett exempel är matriser, som kan ses som tabeller med tal. Matriser kan adderas och subtraheras som vanligt. De kan också multipliceras, men inte på samma sätt som vi är vana vid. Till exempel visar det sig att om A och B är matriser så är $A \cdot B$ inte nödvändigtvis samma sak som $B \cdot A$. I termer av det vi sa innan så är inte alla matriser kommutativa med varandra. För att undersöka skillnaden mellan $A \cdot B$ och $B \cdot A$ har matematiker kommit på vad som kallas en Lie-algebra.

Det visar sig att vissa sorters Lie-algebror är lättare att arbeta med än andra. En grupp med Lie-algebror som är väldigt lätta att hantera kallas för semisimpla Lie-algebror. År 1905 upptäckte matematikern Eugenio Levi ett sätt att använda semisimpla Lie-algebror för att studera andra algebror. Hans metod fungerar dock bara på Lie-algebror som är tillräckligt små, som matematiker kallar för ändligt dimensionella.

År 1970 studerade fysikern Miguel Virasoro en större, så kallat oändligt dimensionell, Lie-algebra. Han upptäckte att Virasoro-algebran var väldigt viktig i fysik, bland annat i strängteori. I den här kandidatuppsatsen beskriver författaren varför det inte går att använda Levis metod för att undersöka Virasoro-algebran, och visar på så vis att det behövs mer avancerade metoder för att studera den.

Abstract

In this bachelor thesis we introduce Lie algebras, and use Lie algebra cohomology to prove Levi's theorem about splitting of finite dimensional Lie algebras. We then construct the Virasoro algebra, compute its low dimensional cohomology spaces, and use this to demonstrate why Levi's theorem does not hold in the infinite dimensional case.

Acknowledgements

I would like to express my gratitude to my supervisor Arne Meurman for all his support and his patience with my questions. It has been a great opportunity to learn from his experience and knowledge, and his guidance has been invaluable for this thesis.

Contents

Introduction

The goal of this thesis is to present Levi's theorem about splitting of finite dimensional Lie algebras, and present an example showing that the theorem does not hold in the infinite dimensional case. In order to do that, we will define Lie algebras and present some important result about them. We will then introduce the theory of Lie algebra cohomology and use it to prove Levi's theorem. Finally, we will construct the infinite dimensional Virasoro algebra and show that it can't be decomposed as in Levi's theorem.

In the first chapter we will introduce the notion of a Lie algebra and introduce some examples. We will also describe representations of Lie algebras, and define the classes of solvable, nilpotent and semisimple Lie algebras.

The second chapter will build upon the first, and further extend the theory of Lie algebras. In particular, the properties of the classes introduced in the first chapter are studied. While the previous chapter introduced new notions, this one aims to present some deeper results. Some notable results in this chapter include Engel's and Lie's theorems, Cartan's criterion, and the construction of the Casimir element. The content of the first two chapters is elementary and is included in any introductory textbook on Lie algebras. For this reason, some of the proofs in these sections have been left out. The presentation will be based on [2].

Going a bit more specific, chapter three introduces cohomology theory of Lie algebras. After defining the cohomology spaces of a Lie algebra, the main part of this chapter is dedicated to the proofs of the two Whitehead lemmas.

Having developed the cohomology machinery, chapter four is dedicated to the main theorem of the thesis, Levi's theorem. This theorem is proved using techniques from chapter three. We also prove Mal'tsev-Harish-Chandra's theorem, and give a practical example of the Levi decomposition. The theory in chapter three and four is not as standard as in the previous chapters, and is mostly based on [4].

In the final chapter we will further contextualize Levi's theorem by providing a counterexample if the requirement that the Lie algebra is finite dimensional is dropped. For this, we will define the Witt algebra and use it to construct the Virasoro algebra. We will then calculate the first and second cohomology spaces of the Virasoro algebra, and use them to prove that the Virasoro algebra has no Levi decomposition. While all the results of this chapter are well known, the presentation in this paper is independent from other sources.

The article also contains an appendix containing a very technical proof for a theorem in chapter three.

The reader is assumed to be familiar with linear algebra, and further knowledge in abstract algebra can be of help. In addition, the article is not self contained in that the proofs of some theorems in chapter two are omitted. Therefore, a reader looking merely for an introduction to Lie algebras is adviced to refer also to other texts.

1 Lie Algebras

In this chapter we will introduce the notion of a Lie algebra and start building some introductory theory. We will give the definition from an entirely algebraic point of view, ignoring the connection to geometry. This is contrary to the historical development, where the theory of Lie algebras was developed as a toolbox to work with problems concerning Lie groups. Unless otherwise specified, the content of this chapter is based on [2].

We will first introduce some notation. Let V be a vector space over a field F. We denote the set of linear transformations from V to V by $\mathrm{End}(V)$. Note that when $\dim(V) = n < \infty$ we can fix a basis of V and represent the elements of $\mathrm{End}(V)$ with a matrix. $\mathrm{End}(V)$ then becomes a vector space over F with $\dim(\mathrm{End}(V)) = n^2$.

We will now give the definition of a Lie algebra.

Definition 1.1. (Lie algebra)

A **Lie algebra** L is a vector space over a field F equipped with a bilinear operation $L \times L \to L$, denoted $[x, y]$ or just $[xy]$ for $x, y \in L$, satisfying the following axioms:

i. $[xx] = 0 \quad \forall x \in L$

ii. $[x[yz]] + [y[zx]] + [z[xy]] = 0 \quad \forall x, y, z \in L$

The operation is called the **bracket** or the **commutator** and the second axiom is called the **Jacobi identity**.

Analogously to other algebraic structures, a **subalgebra** of a Lie algebra L is a subspace of L that is closed under the bracket. A **homomorphism** from the Lie algebra L to the Lie algebra K over the same field is a linear transformation $\varphi : L \to K$ such that $\varphi([xy]) = [\varphi(x)\varphi(y)]$. A bijective homomorphism is called an isomorphism. A homomorphism from a Lie algebra to itself is called an **endomorphism**. Since these concepts can also refer to maps between vector spaces, unless specifically noted, the notion described in this definition will be the one used.

We will next give some examples of Lie algebras that will be important as we proceed.

Example 1.1. (Abelian Lie algebra)

Any vector space V can trivially be made into a Lie algebra by giving it the Lie bracket $[xy] = 0$ for all $x, y \in L$. A Lie algebra where the commutator is identically zero is called **abelian**.

The next example is arguably the main motivation for the definition of a Lie algebra.

Example 1.2. (General linear algebra)

Let V be a vector space and consider the space $\mathrm{End}(V)$ equipped with the following operation

$$[AB] = AB - BA, \quad A, B \in \mathrm{End}(V) \tag{1}$$

This space is called the **general linear algebra** and is denoted by $\mathfrak{gl}(V)$. Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear algebra** over V. This example gives sense to the term commutator, since it can be interpreted as describing to what extent the two elements A and B commute. In particular, if $AB = BA$ then the commutator will be 0. In light of representation theory of Lie algebras, which will be introduced below, this observation carries over also to other Lie algebras.

Example 1.3. (Special linear algebra)
Let $\mathfrak{gl}(V)$ be a general linear algebra and define the subalgebra $\mathfrak{sl}(V)$ of all elements having trace 0. This is well defined since the trace of a linear transformation is independent of the choice of basis and it is indeed a subalgebra since $\text{tr}(AB) = \text{tr}(BA)$ for all $A, B \in \text{End}(V)$. This Lie algebra is called the **special linear algebra**.

Next, we will introduce the concepts of representations and modules in the context of Lie algebras, and show that they are in fact equivalent.

Definition 1.2. (Lie algebra representation)
Let L be a Lie algebra and V a vector space, and let $\varphi : L \to \mathfrak{gl}(V)$ be a homomorphism. Then φ is called a **representation**. Further, if φ is injective then it is called **faithful**.

Definition 1.3. (Module over Lie algebra)
Let L be a Lie algebra and M be a vector space over the same field F, equipped with an operation from $L \times M$ to M, denoted simply xm or $x \cdot m$ for $x \in L, m \in M$, satisfying the following axioms:

i. $(ax + by)m = a(xm) + b(ym)$

ii. $x(am + bn) = a(xm) + b(xn)$

iii. $[xy]m = x(ym) - y(xm)$

for all $a, b \in F, m, n \in M, x, y \in L$. M is called an **L-module** and the operation is called **module multiplication**. If N is a subspace of M such that $xn \in N$ for all $x \in L, n \in N$ then N is called an **L-submodule** of M. If M has no other L-submodules than 0 and M itself then it is called **irreducible**, and if it is a direct sum of irreducible L-modules it is called **completely reducible**. If M, N are L-modules and $\varphi : M \to N$ satisfies $\varphi(xm) = x\varphi(m)$ for all $x \in L, m \in M$ then φ is called an **L-homomorphism**. If it is bijective it is called an **L-isomorphism**.

Lemma 1.1. *Let L be a Lie algebra and M be a vector space. Then there is a representation $L \to \mathfrak{gl}(M)$ if and only if M can be viewed as an L-module.*

Proof. Let $\varphi : L \to \mathfrak{gl}(M)$ be a representation. Then we can define module multiplication on M by $xm = \varphi(x)(M)$. Indeed, the first two criterions are satisfied by linearity since all involved functions are linear transformations, and the third condition follows from equation (1):

$$\begin{aligned}[xy]m = \varphi([xy])(m) = [\varphi(x)\varphi(y)](m) = \varphi(x)\varphi(y)(m) - \varphi(y)\varphi(x)(m) \\ = x(ym) - y(xm) \end{aligned} \tag{2}$$

Conversely, if M is an L-module then we can define $\varphi : L \to \mathfrak{gl}(M)$ by $\varphi(x)(m) = xm$. Indeed, $\varphi(x) \in \mathfrak{gl}(M)$ by condition (ii), and φ is a linear transformation by condition (i). By (iii), we have that $[xy]m = x(ym) - y(xm)$ so changing the order of the equalities in (2) yields $\varphi([xy])(m) = [\varphi(x)\varphi(y)](m)$, which shows that φ is a homomorphism, and hence a representation. □

Remark 1.1. By the construction in the proof above, the notions of L-modules and representations of L will be used interchangeably as we proceed.

The most important example of a Lie algebra representation for us will be the adjoint representation, which we define below.

Definition 1.4. (Adjoint representation)
Let L be a Lie algebra and $x \in L$. We define the endomorphism $\text{ad}(x) : L \to L, y \mapsto [xy]$. Thus, we have constructed a map $\text{ad} : L \to \mathfrak{gl}(L)$, which we call the **adjoint representation**.

Lemma 1.2. *The adjoint representation is a representation.*

Proof. We must show that the adjoint preserves the bracket. Indeed, for any $z \in L$,

$$\begin{aligned}[\text{ad}(x)\text{ad}(y)](z) = \text{ad}(x)\text{ad}(y)(z) - \text{ad}(y)\text{ad}(x)(z) = [x[yz]] - [y[xz]] \\ = [x[yz]] + [y[zx]] = -[z[xy]] = [[xy]z] = \text{ad}([xy])(z)\end{aligned}$$

as required. □

There are several important classes of Lie algebras, containing some additional structure. We now turn to the definition of some of these classes.

Definition 1.5. (Ideal)
Let I be a subalgebra of a Lie algebra L such that $[xy] \in I$ for all $x \in L, y \in I$. Then I is called an **ideal** in L.

Example 1.4. (Derived algebra)
Let L be a Lie algebra. Then the **derived algebra** of L, which we denote $[LL]$, is the subalgebra generated by all commutators $[xy]$, where $x, y \in L$. This is clearly an ideal.

Example 1.5. (Center of Lie algebra)
Let L be a Lie algebra. Then the **center**

$$Z(L) = \{x \in L : [xy] = 0 \quad \forall y \in L\}$$

is an ideal in L.

Example 1.6. Let K be a subalgebra of the Lie algebra L. Then we define the **normalizer** of K as

$$N_L(K) = \{x \in L : [xK] \subset K\}$$

Then $N_L(K)$ is a subalgebra of L. Indeed, for $x, y \in N_L(K)$,

$$[[xy]K] = [x[yK]] - [y[xK]] \subset K$$

since $[xK], [yK] \subset K$. Further, $N_L(K)$ is the largest subalgebra of L containing K as an ideal.

Analogously to other algebraic structures, ideals are used to construct quotient algebras.

Definition 1.6. (Quotient algebra)
Let I be an ideal in a Lie algebra L, and define the equivalence relation $x \sim y$ if and only if $x - y \in I$. Then the **quotient algebra** L/I is defined as the Lie algebra of equivalence classes with the bracket defined by $[x+I, y+I] = [xy]+I$. Note that this is well defined since if $x - x' \in I$ and $y - y' \in I$, then

$$[xy] - [x'y'] = [xy] - [xy'] + [xy'] - [x'y'] = [x, y - y'] + [x - x', y'] \in I$$

It might come as no surprise that analogues of the classical isomorphism theorems hold also for Lie algebras. This is stated as an exercise in [2], but the details will be carried out here.

Theorem 1.1. ***(Isomorphism theorems)***
Let L, K be Lie algebras and I, J be ideals in L.

i. Let $\varphi : L \to K$ be a homomorphism. Then $\ker\varphi$ is an ideal in L and $L/\ker\varphi \cong im\ \varphi$.

ii. $(I + J)/J \cong I/(I \cap J)$

iii. Suppose that $I \subset J$. Then J/I is an ideal in L/I and $(L/I)/(J/I) \cong L/J$.

Proof. i. If $\varphi(x) = 0$ then $\varphi([xy]) = [\varphi(x)\varphi(y)] = 0$ so $\ker\varphi$ is indeed an ideal in L. The function $f : L/\ker\varphi \to \text{im}\ \varphi, x + \ker\varphi \mapsto \varphi(x)$ is well defined since if $x - y \in \ker\varphi$ then

$$f(x) - f(y) = \varphi(x) - \varphi(y) = \varphi(x - y) = 0$$

Moreover, f is an isomorphism, as required.

ii. The surjective homomorphism $\varphi : I + J \to I/(I \cap J)$ defined by $\varphi(x+y) = x + (I \cap J), x \in I, y \in J$ is well defined since if $x_1 + y_1 = x_2 + y_2$ for $x_1, x_2 \in I, y_1, y_2 \in J$, then $I \ni x_1 - x_2 = y_2 - y_1 \in J$ so $x_1 - x_2 \in I \cap J$. Since $\ker\varphi = J$, the conclusion follows from part (i).

iii. The surjective homomorphism $\varphi : L/I \to L/J, x + I \mapsto x + J$ is well defined, since $x - y \in I$ implies $x - y \in J$. Since $\ker \varphi = J/I$, the conclusion again follows from part (i).

□

Definition 1.7. (Simple Lie algebra)
A non-abelian Lie algebra L having no non-trivial ideals is called **simple**. Here non-trivial means the zero ideal and L itself.

Definition 1.8. (Solvable Lie algebra)
Let L be a Lie algebra and define the **derived series** of ideals in the following way: $L^{(0)} = L$, $L^{(k)} = [L^{(k-1)}L^{(k-1)}]$, $k = 1, 2, ...$ If there is some k such that $L^{(k)} = 0$ then L is called **solvable**.

The following is for our purposes one of the most important classes of Lie algebras

Definition 1.9. (Semisimple Lie algebra)
A Lie algebra with no non-zero solvable ideals is called **semisimple**.

We will proceed to establish some properties of solvable algebras.

Theorem 1.2. *Let L be a Lie algebra.*

i. Let K be a subalgebra of L, and φ a homomorphism on L. If L is solvable then K and $\varphi(L)$ are solvable.

ii. Let I be a solvable ideal in L such that L/I is solvable. Then L is solvable.

iii. Let I, J be solvable ideals in L. Then $I + J$ is a solvable ideal.

*iv. L has a unique maximal solvable ideal, which we call the **radical** of L and denote $RadL$*

Proof. i. The solvability of both algebras follows by induction, since

$$K^{(k)} = [K^{(k-1)}K^{(k-1)}] \subset [L^{(k-1)}L^{(k-1)}] = L^{(k)}$$

and

$$(\varphi(L))^{(k)} = [(\varphi(L))^{(k-1)}(\varphi(L))^{(k-1)} = [\varphi(L^{(k-1)})\varphi(L^{(k-1)})] = \varphi(L^{(k)})$$

ii. Suppose that $I^{(n)} = 0 = (L/I)^{(m)}$ and let $\varphi : L \to L/I, x \mapsto x + I$. Then $\varphi(L^{(m)}) = (\varphi(L))^{(m)} = 0$ so $L^{(m)} \subset I$. But then

$$L^{(m+n)} = (L^{(m)})^{(n)} = I^{(n)} = 0$$

so L is solvable.

iii. Note that $I/(I \cap J)$ is solvable by applying part (i) to the canonical homomorphism $I \to I/(I \cap J)$. But then by the third isomorphism theorem $(I + J)/J$ is solvable, so by part (ii) also $I + J$ is solvable.

iv. Suppose I is maximal sovable ideal and let J be any other solvable ideal. Then by part (iii), $I + J$ is solvable so by maximality of I, we find that $J \subset I$. This shows that there is no other maximal ideal.

□

Another important class is defined below.

Definition 1.10. (Nilpotent Lie algebra)
Let L be a Lie algebra and define the **lower central series** of ideals in the following way: $L^0 = L, L^k = [LL^{k-1}], k = 1, 2, ...$ If there is some k such that $L^k = 0$, then L is called **nilpotent**.

Theorem 1.3. *Let L be a Lie algebra.*

i. Let K be a subalgebra of L, and φ a homomorphism on L. If L is nilpotent then K and $\varphi(L)$ are also nilpotent.

ii. Suppose $L/Z(L)$ is nilpotent. Then L is nilpotent.

iii. Let I, J be nilpotent ideals in L. Then $I + J$ is nilpotent.

iv. Let L be nilpotent. Then L is solvable.

v. L has a unique maximal nilpotent ideal, which we call the nilradical of L. The nilradical is included in the radical of L.

Proof. i. The proof of this part is exactly the same as the proof of Theorem 1.2 (i) by just exchanging the derived series with the lower central series.

ii. If $L/Z(L)$ is nilpotent then $L^n \subset Z(L)$ for some n. Hence,

$$L^{n+1} = [LL^n] \subset [LZ(L)] = 0$$

iii. We show that $(I+J)^{2n} \subset I^n + J^n$. Indeed, using the bilinearity of the commutator, $(I+J)^{2n}$ is generated by elements of the form $[x_1...[x_{2n-2}[x_{2n-1}x_{2n}]]...]$ where each x_i is in either I or J. But then by the pigeonhole principle each such generator contains at least n x_i's from either I or J. If we assume it is true for I without loss of generality then since I is an ideal this generator is included in I^n. Thus the nilpotency of $I+J$ follows from the nilpotency of I and J.

iv. This follows immediately from the fact that the derived series is a subsequence of the lower central series.

v. The existence of the nilradical follows from the same argument used to prove part (iv) of Theorem 1.2. The fact that the nilradical is included in the radical follows from part (iii).

□

2 Structure of Lie algebras

In order to be able to study the theory of the later chapter, we will need to further understand the structure of Lie algebras. In particular, we will motivate the introduction of solvable, nilpotent and semisimple Lie algebras by discussing far-reaching results about these particular classes. The theory that we develop in this chapter will culminate in the definition of the Casimir element of a representation, which will be crucial in the next chapter. Again, unless otherwise noted, the presentation is based on [2].

Most of the results in this chapter are for our purposes auxiliary results, interesting mostly for their use in the proofs of later results. For this reason, in order to reduce unnecessary repetition, proofs will be omitted unless there is some notable modification in the presentation from the one given in [2].

Throughout this chapter, and in fact for the rest of the paper, F will denote an algebraically closed field of characteristic 0. For some results these limitations are not strictly necessary, but since they will be needed for the theory we are interested in, we will for simplicity make this assumption.

First of all, we will need a result from linear algebra, which can be seen as a generalization of the Jordan canonical form.

Theorem 2.1. ***(Jordan-Chevalley decomposition)***
Let V be a finite dimensional vector space and let $x \in EndV$. Then:

i. x can be decomposed uniquely into $x = x_d + x_n$ where x_d is diagonalizable and x_n is nilpotent, and x_d commutes with x_n.

ii. There are polynomials p, q without constant terms, such that $x_d = p(x)$ and $x_n = q(x)$.

iii. Suppose $A \subset B \subset V$ such that $xB \subset A$. Then $x_sB, x_nB \subset A$.

We will not prove this theorem here. The proof ultimately depends on the Chinese Remainder Theorem and can be found in [2]. We remark that the proof uses the fact that a vector space endomorphism is diagonalisable if and only if the roots of the minimal polynomial (defined as the unique monic polynomial p of smallest degree such that $p(x_d) = 0$, the reader is referred to most books on linear algebra above introductory level) are all distinct. This is only true for vector spaces over algebraically closed fields, so this is one of the crucial points where we need the assumptions in the beginning of the chapter. Since many of the other results depend on this one, this requirement carries over to most of the results that will follow.

The first theorem we will prove is Engel's theorem. We remark that this theorem holds for any field F, in fact without changing the proof. Hence the assumptions above can be omitted, but since we will not need the full generality, we will ignore these considerations. The presentation is based on [2] but has been modified to emphasize the connection with modules and representations. We start with some lemmas.

Lemma 2.1. *Let L be a linear algebra and $x \in L$ be nilpotent as a linear transformation. Then $ad(x)$ is nilpotent.*

Proof. Let n be such that $x^n = 0$. We claim that $\mathrm{ad}(x)^{2n} = 0$. First of all, we show that

$$\mathrm{ad}(x)^n(y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} y x^k \tag{1}$$

Indeed, this is certainly true for $n = 1$, so using induction we find that

$$\begin{aligned}
\mathrm{ad}(x)^n(y) &= \mathrm{ad}(x) \left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^{n-1-k} y x^k \right) \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^{n-k} y x^k - \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} x^{n-k} y x^k \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} y x^k
\end{aligned}$$

This shows (1). Now, expanding $ad(x)^{2n}(y)$ as above, either $2n - k \geq n$ or $k \geq n$ so every term in the sum will be 0, which concludes the proof. □

Lemma 2.2. *Let L be a Lie algebra and M be a non-zero finite dimensional L-module, such that every element of L acts nilpotently on M. Then there is some non-zero $m \in M$ such that $xm = 0$ for any $x \in L$*

Proof. We will use induction on the dimension of L so we start by remarking that the lemma holds for $L = 0$. Let now K be a maximal proper subalgebra of L. We show that K is an ideal in L. Note that L/K exists as a vector space, even though it is not yet established that it is a Lie algebra. Further, L/K is a K-module by the adjoint representation (see Lemma 1.1), where every element of K acts nilpotently on L/K by Lemma 2.1. By the induction hypothesis, since $\dim K < \dim L$, there is some non-zero element $x \in L/K$ such that $yx = 0$ for any $y \in K$. In other words, there is some $x \notin K$ such that $[yx] \in K$ for all $y \in K$. This means K is a proper subspace of $N_L(K)$ so by maximality of K, $N_L(K) = L$, which means that K is an ideal in L.

Now, take some $x \notin K$. Then $K + Fx$ is a subalgebra, since

$$[y_1 + a_1 x, y_2 + a_2 x] = [y_1 y_2] + a_1[x y_1] + a_2[y_2 x] \in K \subset K + Fx$$

By maximality of K, we conclude that $L = K + Fx$. Consider the space $W = \{m \in M : Km = 0\}$. Using the same induction argument as before, since $\dim K < \dim L$ we can assume that W is non-zero. Moreover, if $m \in W$ then $xm \in W$ for all $x \in L$. Indeed, for any $z \in K$,

$$zxm = xzm + [xz]m = 0$$

Finally, since x defined above acts nilpotently on M, and hence also on W, for any non-zero $m \in W$ there is some n such that $x^{n-1}v \neq 0$ but $x^n m = 0$.

Hence, $x^{n-1}m \in W$ so it gets killed by K, and as established above it is also killed by x. Hence, $x^{n-1}m$ gets killed by all of L, which concludes the proof. □

We are now ready to prove Engel's theorem

Theorem 2.2. ***(Engel's Theorem)***

Let L be a finite dimensional Lie algebra such that $ad(x)$ is nilpotent for all $x \in L$. Then L is nilpotent.

Proof. We will again use induction on $\dim L$. Clearly the lemma holds for the base case $L = 0$. We also have that $\text{ad}L$ is a Lie algebra with nilpotent elements acting on the non-zero finite dimensional $\text{ad}L$-module L. Hence, by Lemma 2.2 there is a non-zero $x \in L$ such that $\text{ad}(y)(x) = 0$ for all $y \in L$. Hence, $x \in Z(L)$, so $Z(L)$ is non-zero. Now, $L/Z(L)$ consists of elements with nilpotent adjoints, and $\dim L/Z(L) < \dim L$, so by the induction hypothesis, $L/Z(L)$ is nilpotent, but then by Theorem 1.3, L is also nilpotent. □

The next theorem can be thought of as an extension of Engel's theorem where the nilpotency assumption is exchanged with solvability. In contrast to Engel's theorem, here we will need the assumptions on the field that we gave in the start of the chapter. As before, the theorem is given here in terms of module theory, but another formulation can be found in [2], and the proof is similar.

Lemma 2.3. *Let L be a solvable finite dimensional Lie algebra and M a non-zero finite dimensional L-module. Then there is some non-zero $m \in M$ such that m is an eigenvector of every $x \in L$, i.e. for every $x \in L$ there exists some $\lambda \in F$ such that $xm = \lambda m$.*

Proof. First note that this lemma is very similar to Lemma 2.2. Thus we will try to mimic the technique we used there. Hence, we want to use induction on $\dim L$. As before, the case $L = 0$ is trivial since every element of M will be killed by L. To proceed, we want to find a maximal ideal and show that it has codimension 1. While this required some work under the assumptions of Lemma 2.2, the stronger statement of this lemma simplifies finding such an ideal. Indeed, since L is solvable, $L \neq [LL]$. Hence, the quotient algebra $L/[LL]$ is non-zero. Since it is also abelian, every subspace of $L/[LL]$ is an ideal. Take an ideal K_0 in $L/[LL]$ having codimension one and consider the inverse image $K = \varphi^{-1}(K_0)$ where $\varphi : L \to L/[LL], x \mapsto x + [LL]$ is the canonical map. Then K is an ideal in L since $\varphi([xy]) = [\varphi(x)\varphi(y)] \in K_0$ for $x \in L, y \in K$. Moreover, the codimension of K is one, as required.

Next, K is a solvable Lie algebra of strictly smaller dimensionality than L, so by the induction hypothesis, there is some $m \in M$ and some linear map $\lambda : K \to F$ such that $xm = \lambda(x)m$ for all $x \in K$. We construct the L-submodule

$$W = \{m \in M : xm = \lambda(x)m \quad \forall x \in K\}$$

Note that W is at least one dimensional.

We proceed to show that $LW \subset W$. To this end, let $x \in L, m \in W, y \in K$. We want to show that $yxm = \lambda(y)xm$. However, note that

$$yxm = xym - [yx]m = x\lambda(y)m - \lambda([xy])m = \lambda(y)xm - \lambda([xy])m \qquad (2)$$

so we need to show that $\lambda([xy]) = 0$. For given $m \in W$ and $x \in L$, define recursively $m_0 = m$, $m_k = xm_{k-1}$ and let $W_k = \text{span}(m_0, ..., m_k)$. Then since W is finite dimensional, we can pick n to be the smallest integer such that $W_n = W_{n+1}$. Note that $KW_{k+1} \subset W_{k+1}$, since for every $y \in K$, from equation (2)

$$ym_k = yxm_{k-1} = \lambda(y)xm_{k-1} - \lambda([xy])m_{k-1} = \lambda(y)m_k - \lambda([xy])m_{k-1} \in W_{k+1}$$

The above equation also shows that $ym_k - \lambda(y)m_k \in W_k$, so if we take the basis $\{m_k\}_{k=0}^{n-1}$ for W_n then $y \in K$ acts on W_n as an upper triangular matrix, with $\lambda(y)$ as every diagonal entry, so the trace of this linear transformation will be $n\lambda(y)$. Now, we can consider the action of the element $[xy] \in K$ in this way, having the trace $n\lambda([xy])$. However, both x and y map W_n to itself under multiplication so they can both be considered as endomorphisms of the vector space W_n. But by definition of module multiplication by the bracket of two elements, this means that $[xy]$ acts on W_n as the commutator of the vector space endomorphisms corresponding to x and y, so the trace will be 0. Hence $\lambda([xy]) = 0$ which as remarked above shows that $LW \subset W$.

Now, we can write $L = K + Fz$, so the only thing we have left is to find an eigenvector $m \in W$ of z, but since multiplication with z is a linear map from W to W, and the field is algebraically closed, such an eigenvector must exist. Then m is an eigenvector of all of L, which concludes the proof. □

Remark 2.1. The proof of the lemma uses both of the assumptions on the field. Algebraic closure is required in the final step to find an eigenvalue of the linear transformation, and characteristic zero is required in the trace argument slightly before, since in a field of positive characteristic, $n\lambda([xy]) = 0$ does not necessarily imply that $\lambda([xy]) = 0$. Hence, from now on, these assumptions will be crucial for the theory.

Theorem 2.3. ***(Lie's theorem)***

Let L be a solvable finite dimensional Lie algebra. Then there are ideals I_k such that $dim I_k = k$, for $k = 0, 1, ..., n = dim L$, and $I_0 \subset ... \subset I_n$.

Proof. Note first that L is an L-module under the adjoint representation. We will use induction on $\dim L$. For $\dim L = 1$ the theorem is immediate since the chain $0 \subset L$ satisfies the requirements. For an arbitrary dimension of L, by Lemma 2.3 there is some non-zero $y \in L$ such that $xy = [xy] = \lambda(x)y$. Hence, Fy is a one dimensional ideal in L. We take $I_1 = Fy$. Now, the quotient algebra L/I_1 has strictly lower dimensionality than L so by the induction hypothesis there is a chain of ideals $0 = I'_0 \subset ... \subset I'_{n-1} = L/I_1$ with $\dim I'_k = k$. Then just as in the proof of Lemma 2.3, if we let $\varphi : L \to L/I_1, x \mapsto x + I_1$ be the canonical homomorphism then the preimages $I_k = \varphi^{-1}(I'_{k-1})$ are ideals in L such that $0 = I_0 \subset ... \subset I_n = L$ and $\dim I_k = k$. □

Corollary 2.1. *Let L be a solvable finite dimensional Lie algebra. Then $[LL]$ is nilpotent.*

Proof. By Lie's theorem, there is a chain of ideals $I_0 \subset ... \subset I_n$ in L such that $\dim I_k = k$. If we take $e_1, ..., e_n$ to be a basis of L such that $I_k = \text{span}(e_1, ..., e_k)$. Then $\text{ad}(x)$ can be represented as an upper triangular matrix for all $x \in L$. By Lemma 1.2, $\text{ad}([xy]) = [\text{ad}(x), \text{ad}(y)]$. However, direct calculation shows that the commutator of two upper triangular matrices is again upper triangular, with zeroes on the diagonal. Hence, for $x \in [LL]$, $\text{ad}(x)$ is nilpotent, so by Engel's theorem, $[LL]$ is nilpotent. □

Corollary 2.2. *Let L be a finite dimensional Lie algebra. Then $[LRadL]$ is nilpotent.*

Proof. Let $[xy] \in [L, \text{Rad}L]$, with $x \in L, y \in \text{Rad}L$. Note that

$$[K_x, K_x] := [\text{Rad}L + Fx, \text{Rad}L + Fx] \subset [\text{Rad}L, \text{Rad}L] + [Fx, \text{Rad}L] \subset \text{Rad}L$$

Hence, K_x is a solvable subalgebra. By Corollary 2.1, $[K_xK_x]$ is nilpotent, so since $[xy] \in [K_xK_x]$, $\text{ad}([xy])$ is nilpotent on $[K_xK_x]$. However, for any $z \in L$, $\text{ad}^2([xy])(z) \in [\text{Rad}L, \text{Rad}L] \subset [K_xK_x]$ so $\text{ad}([xy])$ is in fact nilpotent on all of L. We can use this argument to show that any element of $[L, \text{Rad}L]$ is nilpotent. Then Engel's theorem asserts that $[L\text{Rad}L]$ is nilpotent. □

We will now proceed to some results that will be given without proofs. Since much of our interest further on will concern solvable or, most notably, semisimple Lie algebras, we would like to find criterions that are easy to use for a Lie algebra to belong to any of these classes. In order to formulate these results, we will introduce the following notion.

Definition 2.1. (Killing form)
Let L be a Lie algebra and define the function $\kappa : L^2 \to F$ by

$$\kappa(x, y) = \text{Tr}(\text{ad}(x)\text{ad}(y))$$

for $x, y \in L$. Then κ is a symmetric bilinear form which we call the **Killing form**.

Definition 2.2. (Non-degenerate bilinear form)
A symmetric bilinear form $\beta : L^2 \to F$ form is called **non-degenerate** if $\beta(x, y) = 0$ for all $x \in L$ implies $y = 0$.

We are now ready to present Cartan's criterion for solvability.

Theorem 2.4. ***(Cartan's criterion)***
Let L be a finite dimensional Lie algebra. If $\kappa(x, y) = 0$ for $x \in [LL], y \in L$, then L is solvable.

The proof can be found in [2]. It is based on first finding a trace criterion for nilpotency of the adjoint of an element in the Lie algebra, and then using the lemma together with Engel's theorem to show that under the assumptions of the theorem, ad$[LL]$ will be nilpotent. Then ad$L$ is solvable by part (iv) of Theorem 1.3. We can conclude that L is solvable by the first isomorphism theorem and part (ii) of Theorem 1.2, since Ker(ad) $= Z(L)$ is solvable. Worth noting is that this means there is nothing special about the adjoint representation. It can be exchanged with any representation φ as long as we add the requirement that Kerφ is solvable.

We proceed now to the criterion for semisimplicity. This is also sometimes known as Cartan's criterion.

Theorem 2.5. *Let L be a finite dimensional Lie algebra. Then L is semisimple if and only if κ is non-degenerate.*

Again, for the proof, the reader is referred to [2].

Remark 2.2. The theorem above can often be used to check for semisimplicity of finite dimensional Lie algebras in practice. Indeed, if we fix a basis $\{e_i\}_{i=1}^n$ for ad$L$, we can consider the matrix K having $\kappa(e_i, e_j)$ as its entry in the position i, j. Then if $x, y \in L$ we get

$$\kappa(x, y) = x^T K y$$

The matrix K is non-degenerate if and only if κ is non-degenerate. To see this, K being non-degenerate means $Ky = 0$ if and only if $y = 0$, and κ being non-degenerate means $x^T Ky = 0$ for all x if and only if $y = 0$. However, $x^T Ky = 0$ for all x is equivalent to $Ky = 0$. Hence, the problem of checking semisimplicity for a finite dimensional Lie algebra has been reduced to the familiar problem of checking non-degeneracy of a matrix.

Next, we will state an important structure theorem for semisimple finite dimensional Lie algebras.

Theorem 2.6. *Let L be a semisimple finite dimensional Lie algebra. Then there are simple ideals $I_1, ..., I_n$ in L such that $L = I_1 \oplus ... \oplus I_n$ as vector spaces. Further, if I is a simple ideal in L then $I = I_k$ for some k. As a consequence, $[LL] = L$, and all ideals of L can be written as a direct sum of some of the I_k's. In particular, all ideals in L and all homomorphic images of L are semisimple.*

As before, this theorem will not be proved here, but can be found in [2].

We are now ready to introduce the Casimir element of a representation. In what we have done so far, a very important role has been played by the adjoint representation. However, for the cohomology theory that we will introduce in the next chapter, we will need to consider any representation of a Lie algebra. The Casimir element turns out to be a very helpful tool for this.

In order to construct the Casimir element, we will start with a generalization of the Killing form.

Definition 2.3. (Trace form)
Let L be a finite dimensional Lie algebra and φ a faithful representation of L. Let $\beta : L^2 \to F$ be the symmetric bilinear form defined by

$$\beta(x, y) = \mathrm{Tr}(\varphi(x)\varphi(y))$$

for $x, y \in L$. We call β the **trace form** of φ.

We remark that the Killing form is the trace form of the adjoint representation.

Lemma 2.4. *Let L be a semisimple finite dimensional Lie algebra and let φ be a faithful representation of L. Then the trace form of φ is non-degenerate.*

Proof. Note first that

$$\begin{aligned}
\beta([xy], z) &= \mathrm{Tr}(\varphi([xy])\varphi(z)) = \mathrm{Tr}([\varphi(x)\varphi(y)]\varphi(z)) \\
&= \mathrm{Tr}(\varphi(x)\varphi(y)\varphi(z)) - \mathrm{Tr}(\varphi(y)\varphi(x)\varphi(z)) \\
&= \mathrm{Tr}(\varphi(x)\varphi(y)\varphi(z)) - \mathrm{Tr}(\varphi(x)\varphi(z)\varphi(y)) \\
&= \mathrm{Tr}(\varphi(x)[\varphi(y)\varphi(z)]) = \mathrm{Tr}(\varphi(x)\varphi([yz])) = \beta(x, [yz])
\end{aligned}$$

Hence, the subspace

$$S = \{x \in L : \beta(x, y) = 0 \quad \forall \quad y \in L\}$$

is an ideal, since if $x \in S$ then $\beta([xy], z) = \beta(x, [yz]) = 0$ for all $y, z \in L$. Further, for $x \in [\varphi(S)\varphi(S)]$ and $y \in \varphi(S)$, there are $x' \in [SS]$ and $y' \in S$ such that $\varphi(x') = x$ and $\varphi(y') = y$. Hence,

$$\mathrm{Tr}(xy) = \mathrm{Tr}(\varphi(x')\varphi(y')) = \beta(x', y') = 0$$

Hence by Cartan's criterion, $\varphi(S)$ is solvable, but $\varphi(S) \cong S$ and since L is semisimple, we conclude that $S = 0$. Therefore if $\beta(x, y) = 0$ for all $y \in L$, then $x = 0$, so β is nondegenerate. $\square$

Pick a basis $\{e_i\}_{i=1}^n$ of L. Then there exists a unique dual basis $\{f_i\}_{i=1}^n$, i.e. a basis satisfying $\beta(e_i, f_j) = \delta_{i,j}$.

Definition 2.4. (Casimir element)
Let L be a semisimple finite dimensional Lie algebra and let φ be a faithful representation of L. Then if $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are dual bases with respect to the trace form of β, as described above, we define the Casimir element

$$c_\varphi = \sum_{i=1}^n \varphi(e_i)\varphi(f_i)$$

Lemma 2.5. *The Casimir element of a representation φ is independent of the choice of basis for L.*

Proof. Suppose $\{e'_i\}_{i=1}^n$ is another basis, with corresponding dual basis $\{f'_i\}_{i=1}^n$. We can write

$$e'_i = \sum_{j=1}^n a_{ij} e_j, \quad f'_i = \sum_{j=1}^n b_{ij} f_j$$

We collect the coefficients in matrices $A = (a_{ij}), B = (b_{ij})$. The dual base property gives us

$$\delta_{ij} = \beta(e'_i, f'_j) = \beta\left(\sum_{k=1}^n a_{ik} e_k, \sum_{k=1}^n b_{jk} f_k\right) = \sum_{k=1}^n a_{ik} b_{jk}$$

In matrix form, we can write this equation as $AB^T = I$. Then also $B^T A = I$ so also

$$\delta_{ij} = \sum_{k=1}^n a_{ki} b_{kj}$$

Putting this into the definition of the Casimir element with respect to the second basis, we get

$$\begin{aligned} c_\varphi &= \sum_{i=1}^n \varphi(e'_i)\varphi(f'_i) = \sum_{i=1}^n \varphi\left(\sum_{j=1}^n a_{ij} e_j\right) \varphi\left(\sum_{k=1}^n b_{ik} f_k\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ij} b_{ik} \varphi(e_j)\varphi(e_k) = \sum_{j=1}^n \sum_{k=1} \delta_{jk} \varphi(e_j)\varphi(e_k) \\ &= \sum_{i=1}^n \varphi(e_i)\varphi(f_i) \end{aligned}$$

which concludes the proof. ☐

Remark 2.3. Fix some $x \in L$, and let a_{ij}, b_{ij} be the constants satisfying

$$[xe_i] = \sum_{j=1}^n a_{ij} e_j, \quad [xf_i] = \sum_{j=1}^n b_{ij} f_j$$

Then

$$\begin{aligned} a_{ik} &= \sum_{j=1}^n a_{ij} \beta(e_j, f_k) = \beta([xe_i], y_k) = -\beta([e_i x], y_k) \\ &= -\beta(x_i, [xy_k]) = -\sum_{j=1}^n b_{kj} \beta(e_i, f_j) = -b_{ki} \end{aligned}$$

Lemma 2.6. *Let L be a semisimple finite dimensional Lie algebra and let φ be a faithful representation of L corresponding to the L-module M. Then the endomorphism c_φ of M commutes with $\varphi(x)$ for all $x \in L$. Further, $Trc_\varphi = dimL$.*

Proof. Let $x \in L$ and let a_{ij}, b_{ij} be as in Remark 2.3. Then

$$\begin{aligned}
[\varphi(x), c_\varphi] &= \sum_{i=1}^{n} \varphi(x)\varphi(e_i)\varphi(f_i) - \sum_{i=1}^{n} \varphi(e_i)\varphi(f_i)\varphi(x) \\
&= \sum_{i=1}^{n} [\varphi(x)\varphi(e_i)]\varphi(f_i) - \sum_{i=1}^{n} \varphi(e_i)[\varphi(x)\varphi(f_i)] \\
&= \sum_{i=1}^{n} \varphi([xe_i])\varphi(f_i) - \sum_{i=1}^{n} \varphi(e_i)\varphi([xf_i]) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\varphi(e_j)\varphi(f_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\varphi(e_i)\varphi(f_j) = 0
\end{aligned}$$

This shows that $\varphi(x)$ commutes with c_φ. For the second statement, we calculate

$$\mathrm{Tr} c_\varphi = \sum_{i=1}^{n} \mathrm{Tr}(\varphi(e_i)\varphi(f_i)) = \sum_{i=1}^{n} \beta(e_i, f_i) = n = \dim L$$

This proves the second statement. □

3 Lie Algebra Cohomology

In this chapter we will introduce a cohomology theory for Lie algebras. An intuitive way to describe this theory is how Lie algebras can be understood by studying the multilinear skew symmetric maps that can be defined on the Lie algebra. As range for our maps, we choose a module, which motivates the emphasis on representations in the previous chapters. In this paper we will aim to give a concrete definition of the cohomology spaces by explicitly working with the functions, following the presentation in [4]. We remark that the theory can also be developed more generally using the concept of derived functors. This connects Lie algebra cohomology to other cohomology theories. We will not pursue this approach here, but the interested reader is referred to [1]. The main goal of the chapter will be to prove Whiteheads two lemmas about the first two cohomology groups. We will also generalize this result to higher dimensions.

As a setup, we will consider a Lie algebra L and an L-module M.

Before we get into the definitions, however, we will state a theorem from linear algebra that we will use below. Since the theorem is completely independent from the theory of Lie algebras, it will not be proved in this paper. The proof is based on the fact that a finite dimensional vector space is both noetherian and artinian as a module, and can be found in [3].

Theorem 3.1. ***(Fitting Decomposition)***

Let V be a finite dimensional vector space and $f : V \to V$ a linear transformation. Then there are subspaces V_0 and V' such that $V = V_0 \oplus V'$, where $f(V_0) \subset V_0$, $f(V') \subset V'$, and the restriction of f to V_0 is nilpotent while the restriction to V' is bijective.

We proceed to the definitions required for the cohomology spaces.

Definition 3.1. (Cochain)

For $n > 0$, let $f : \prod_{i=1}^{n} L \to M$ be an n-linear skew symmetric map. Then f is called an **n-dimensional M-cochain**. The vector space of all n-dimensional M-cochains with the usual addition and scalar multiplication is denoted $C^n(L, M)$ or just C^n. In the special case $n = 0$ we define a 0-dimensional M-cochain as a constant function $f : L \to M$, so that every element of L is mapped to some fixed $m \in M$.

Definition 3.2. (Coboundary)

For $n \geq 0$ we define the **coboundary operator** $d : C^n \to C^{n+1}$ by

$$\begin{aligned} df(x_1, ..., x_{n+1}) = &\sum_{j=1}^{n+1} (-1)^{j+1} x_j f(x_1, ..., \hat{x}_j, ..., x_{n+1}) \\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} f([x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+1}) \end{aligned}$$

where the notation $\hat{x}_j$ is used to indicate that x_j is omitted as an argument. We remark that in the case $n = 0$ the above expression is interpreted to mean

$$df(x) = xm, f(x) = m$$

Remark 3.1. There is a slight ambiguity in the definition of the coboundary operator above, in that we use the same notation to describe the coboundary operators for all the different C^n. Most of the time,the dimension will be clear from the context but to avoid ambiguity we will occasionally use the notation $d_n : C^n \to C^{n+1}$ where several coboundary operators are involved.

Remark 3.2. The coboundary operator is well defined in the sense that the image of an n-dimensional cochain is indeed an $(n+1)$-dimensional cochain. To verify this, we must check that df is $(n+1)$-linear and skew symmetric. The $(n+1)$-linearity is easily seen since the bracket is bilinear and f is n-linear, which means each term in each of the sums is in fact $(n+1)$-linear.

The skew symmetry requires a bit more work. We will treat each of the sums separately. Suppose we switch places of x_i and x_j. Then each term where neither x_i nor x_j is omitted will switch sign, by the skew symmetry of f. For the first sum it suffices to show that

$$(-1)^{i+1}x_j f(x_1, ..., \hat{x}_j, ..., x_i, ..., x_{n+1}) = (-1)^j x_j f(x_1, ..., x_i, ..., \hat{x}_j, ...x_{n+1})$$

where x_i is in the j:th position on the right hand side. This identity can be seen by repeatedly shifting x_i to the i:th position, each time using the skew symmetry of f to push x_i one step at a time. This will require $j - i - 1$ shifts, so the coefficient $(-1)^{i+1}$ will become $(-1)^j$

For the second sum we use a similar strategy. Again, for all the terms where none of the exchanged elements are omitted, we can use the skew symmetry of f. Further, if the exchanged elements x_i and x_j are precisely the ones omitted in the term, they both appear in the commutator in the first position, so we can use the skew symmetry of the bracket. Hence, we are left with the case where one of the exchanged elements is also the one being omitted. Hence, suppose x_i and x_j are being exchanged, and consider the term where x_j and x_k are being omitted, with $j < k$. We get two different cases. First, if k is not between i and j we can use the same argument as before to shift x_i one step at a time. If on the other hand k is between i and j, one less shift will be required, but on the other hand the order of the elements in the bracket must be switched, so the skew symmetry still holds.

In light of the previous definition and accompanying remarks, we have constructed a sequence of vector spaces C^n related to each other by the linear transformation d in the following way:

$$\dots \overset{d}{\leftarrow} C^{n+1} \overset{d}{\leftarrow} C^n \overset{d}{\leftarrow} C^{n-1} \overset{d}{\leftarrow} \dots$$

We will now show that the above collection of vector spaces and linear maps is an **cochain complex**, i.e. that $\text{Im} d_{n-1} \subset \text{Ker} d_n$. This is mentioned in [4], but is not proved there.

Theorem 3.2. *The composition of two coboundary operators is identically zero, i.e.* $d^2 = d_{n+1} \circ d_n \equiv 0$.

Proof. The proof of this theorem is very technical, and has therefore been included in the appendix. □

With this result, we are ready to define the most central notion of this chapter.

Definition 3.3. (Cohomology space)
Let f be an n-cochain. Then f is a **cocycle** if $df = 0$ and a **coboundary** if $f = df_0$ for $f_0 \in C^{n-1}$. We denote by Z^n the set of n-cocycles and by B^n the set of n-coboundaries. By Theorem 3.1, $B^n \subset Z^n$ so we can define the n-dimensional **cohomology space** by $H^n = Z^n/B^n$. We remark that if we want to specify the underlying Lie algebra L or L-module M, we will sometimes write $H^n(L, M)$. Note that since the B^0 is not well defined by the above, we take by definition $B^0 = 0$.

We will investigate the cohomology spaces of low dimensionality in some more detail.

Example 3.1. (Zeroth cohomology space)
From the final remark of the definition above, we get that $H^0 = Z^0$, so H^0 is the space of all functions in $f \in C^0$ such that $df = 0$. As we remarked above, a function $f \in C^0$ can be identified with an element $m \in M$, with $f(x) = m$ for all $x \in L$. Then $df(x) = xm$, so H^0 can be identified with the subset of M satisfying $xm = 0$ for all $x \in L$.

Example 3.2. (First cohomology space)
The 1-cocycles are the 1-cochains f satisfying $df(x_1, x_2) = x_1 f(x_2) - x_2 f(x_1) - f([x_1 x_2]) = 0$, that is all linear maps satisfying $f([x_1 x_2]) = x_1 f(x_2) - x_2 f(x_1)$. Such maps are important in many circumstances, and are called **derivations**. The 1-coboundaries are all functions of the form $f(x) = xm$ for some $m \in M$. In particular, if we take $M = L$ as an L-module under the adjoint representation, then $B^1 = \mathrm{ad}L$.

Definition 3.4. (Lie algebra extension)
Let L_1, L_2 and L be Lie algebras and suppose there are homomorphisms f, g such that the sequence

$$0 \to L_2 \xrightarrow{f} L \xrightarrow{g} L_1 \to 0$$

is exact, i.e $\mathrm{Ker}f = 0$, $\mathrm{Im}g = L_1$ and $\mathrm{Im}f = \mathrm{Ker}g$. Then L is called an **extension of L_1 by L_2**.

Remark 3.3. The vector space structure of any Lie algebra extension can be characterized as $L \cong L_1 \oplus L_2$. This can be seen from the above definition by applying the first isomorphism theorem to both f and g. We obtain that $L_2 \cong \mathrm{Im}f$ and that $L/\mathrm{Im}f \cong \mathrm{Im}g = L_1$. Putting it together, we get $L \cong \mathrm{Im}f \oplus L_1 \cong L_1 \oplus L_2$. However, this isomorphism is not necessarily an isomorphism of Lie algebras. In fact, the Lie algebra structure of an extension is not necessarily unique.

Theorem 3.3. ***(Whitehead's first lemma)***

Let L be a finite dimensional semisimple Lie algebra and M a finite dimensional L-module. Then $H^1(L, M) = 0$.

Proof. To start, we remark that $H^1 = 0$ is equivalent to $Z^1 = B^1$, that is, every cocycle is a coboundary. Hence, we must show that if $f : L \to M$ is a linear map satisfying

$$f([xy]) = xf(y) - yf(x) \tag{1}$$

then $f(x) = xm$ for some $m \in M$.

Now, let φ be the representation corresponding to M. In order to introduce the Casimir operator, we need a faithful representation. Therefore, let $L = \text{Ker}\varphi \oplus L_1$. Note that by Theorem 2.6 L_1 can be chosen to be an ideal, which must then be semisimple, also by Theorem 2.6. Now the restriction of φ to L_1 will be faithful so we can construct its corresponding Casimir element c_φ. Recall that $c_\varphi \in \text{End}M$, $\text{Tr}(c_\varphi) = \dim L_1$ and that c_φ commutes with $\varphi(x)$ for $x \in L$.

We will now apply the Fitting decomposition (Theorem 3.1) on M with respect to the linear transformation c_φ. Hence, we get a decomposition $M = M_0 \oplus M'$ where c_φ is nilpotent on M_0 and bijective on M'. We show that M_0 and M' are L-submodules of M. Indeed, let $x \in L_1$ and $m \in M_0$. Then $c_\varphi^n m = 0$ for some n. Further, $c_\varphi^n(xm) = c_\varphi^n \varphi(x)m = \varphi(x)c_\varphi^n m = 0$ so $xm \in M_0$. On the other hand, if $m \in M'$ then since c_φ is a vector space automorphism on M', there is some $m' \in M'$ such that $m = c_\varphi^n m'$. Hence, $xm = xc_\varphi^n m' = c_\varphi^n xm' \in M'$ since if xm' has a part in M_0 then it will be killed by c_φ^n. Thus we have established that M_0 and M' are both L-submodules.

Now, we decompose also the function f into $f = f_0 + f'$, where $f_0 : L \to M_0$ and $f' : L \to M'$. Since M_0 and M' are submodules, both f_0 and f' satisfy the condition in equation (1). Note that if we find some $m_0 \in M_0$ and $m' \in M'$ such that $f_0(x) = xm_0$ and $f'(x) = xm'$, then $f(x) = f_0(x) + f'(x) = xm_0 + xm' = x(m_0 + m')$. Hence, the problem is reduced to the two cases when the Casimir element is nilpotent or a vector space automorphism.

First, we will treat the nilpotent case. Since the trace of a nilpotent vector space endomorphism is 0, $\dim L_1 = \text{Tr}(c_\varphi) = 0$, so $L = \text{Ker}\varphi$ and therefore $f([xy]) = xf(y) - yf(x) = 0$ by the definition of module multiplication. By Theorem 2.6 $[LL] = L$ so f is identically zero, so trivially $f(x) = x0$.

We are left to verify the second case, where c_φ is a vector space automorphism. As in the construction of the Casimir element, take $\{e_i\}_{i=1}^n$ to be a basis of L and $\{f_i\}_{i=1}^n$ to be the dual basis with respect to the trace form generated by φ, with $[xe_i] = \sum_{j=1}^n a_{ij} e_j$ and $[xf_i] = \sum_{j=1}^n b_{ij} f_j$. Define $y = \sum_{i=1}^n f_i f(e_i)$.

Then

$$\begin{aligned}xy &= \sum_{i=1}^n xf_i f(e_i) = \sum_{i=1}^n f_i x f(e_i) + \sum_{i=1}^n [xf_i] f(e_i) \\ &= \sum_{i=1}^n f_i x f(e_i) + \sum_{i=1}^n \sum_{j=1}^n b_{ij} f_j f(e_i) = \sum_{i=1}^n f_i x f(e_i) - \sum_{i=1}^n \sum_{j=1}^n a_{ji} f_j f(e_i) \\ &= \sum_{i=1}^n f_i x f(e_i) - \sum_{j=1}^n f_j f([xe_j]) = \sum_{i=1}^n f_i e_i f(x) = c_\varphi f(x)\end{aligned}$$

Since c_φ commutes with $\varphi(x)$ for $x \in L$, and since c_φ is invertible, this shows that $f(x) = xc_\varphi^{-1} y$ which concludes the proof. □

Theorem 3.4. ***(Whitehead's second lemma)***

Let L be a finite dimensional semisimple Lie algebra and M a finite dimensional L-module. Then $H^2(L, M) = 0$.

Proof. Like in the proof of the first lemma, it is equivalent to show that $Z^2 = B^2$, which we can write as a cocycle condition. So if $f : L \times L \to M$ is a skew-symmetric bilinear map satisfying

$$0 = xf(y,z) + yf(z,x) + zf(x,y) - f([xy],z) - f([yz],x) - f([zx],y) \tag{2}$$

we must show that there is some linear map $g : L \to M$ such that

$$f(x,y) = xg(y) - yg(x) - g([xy]) \tag{3}$$

The strategy is similar to the proof of the first lemma, i.e. we want to use the Fitting decomposition with respect to the Casimir element of the representation φ corresponding to the L-module M. Hence, just as before, we let $L = \mathrm{Ker}\varphi \oplus L_1$ where L_1 is a semisimple ideal in L. Further, $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are dual bases with respect to the trace form of φ, and c_φ is the Casimir element.

Still along the lines of the previous lemma, we decompose $M = M_0 \oplus M'$ where c_φ is nilpotent on M_0 and bijective on M'. As shown above, M_0 and M' are L-submodules of M, so we can decompose the function $f = f_0 + f'$ such that $f_0 : L \times L \to M_0$, $f' : L \times L \to M'$ and f_0 and f' both satisfy equation (2). Thus if we find functions g_0 and g' corresponding to f_0 and f' then their sum satisfies (3), so we are again left with the two cases where c_φ is nilpotent or when c_φ is an automorphism.

We begin with the nilpotent case. We will reduce the problem in this case to the conditions of Whitehead's first lemma, and use it to construct the required g. By the same argument as above, the representation φ is identically zero, so equation (2) reduces to

$$0 = f([xy],z) + f([yz],x) + f([zx],y)$$

and equation (3) becomes

$$f(x,y) = -g([xy])$$

Let V be the vector space of linear transformations $A : L \to M$. We make this into an L-module by defining multiplication as $xA = -A \circ \text{ad}(x)$. For $x \in L$, denote by A_x the mapping in V defined by $A_x(y) = f(x, y)$. Let further $h : L \to V, x \mapsto A_x$. Then

$$h([xy]) - xh(y) + yh(x) = A_{[xy]} + A_y \circ \text{ad}(x) - A_x \circ \text{ad}(y)$$

If we apply this function to any $z \in L$ we obtain

$$\begin{aligned}(A_{[xy]} + A_y \circ \text{ad}(x) - A_x \circ \text{ad}(y))(z) &= f([xy], z) + f(y, [xz]) - f(x, [yz]) \\ &= f([xy], z) + f([zx], y) + f([yz], x) = 0\end{aligned}$$

This shows that $h \in Z^1(L, V)$, but L is finite dimensional and semisimple and V is finite dimensional, so by the first lemma, $h \in B^1(L, V)$. Hence, there is some $g \in V$ such that $h(x) = xg = -g \circ \text{ad}(x)$. Note that

$$f(x, y) = A_x(y) = h(x)(y) = (xg)(y) = -(g \circ \text{ad}(x))(y) = -g([xy])$$

so g satisfies (3).

We proceed to show the case when c_φ is a vector space automorphism. In this case we will set $z = f_i$ in equation (2), then multiply by e_i and sum for all i. This will give us

$$\begin{aligned}0 &= \sum_{i=1}^{n} e_i x f(y, f_i) + \sum_{i=1}^{n} e_i y f(f_i, x) + \sum_{i=1}^{n} e_i f_i f(x, y) \\ &\quad - \sum_{i=1}^{n} e_i f([xy], f_i) - \sum_{i=1}^{n} e_i f([y f_i], x) - \sum_{i=1}^{n} e_i f([f_i x], y) \\ &= \sum_{i=1}^{n} [e_i x] f(y, f_i) + \sum_{i=1}^{n} x e_i f(y, f_i) + \sum_{i=1}^{n} [e_i y] f(f_i, x) \\ &\quad + \sum_{i=1}^{n} y e_i f(f_i, x) + c_\varphi f(x, y) - \sum_{i=1}^{n} e_i f([xy], f_i) \\ &\quad - \sum_{i=1}^{n} e_i f([y f_i], x) - \sum_{i=1}^{n} e_i f([f_i x], y) \end{aligned} \tag{4}$$

where we used property *(iii)* of L-modules and the definition of the Casimir element. Recall that $[e_i x] = \sum_{j=1}^{n} a_{ij} e_j$, $[f_i x] = \sum_{j=1}^{n} b_{ij} f_j$ where $a_{ij} = -b_{ji}$. Hence,

$$\begin{aligned}\sum_{i=1}^{n} [e_i x] f(y, f_i) &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} e_j f(y, f_i) = -\sum_{j=1}^{n} e_j f\left(y, \sum_{i=1}^{n} b_{ji} f_i\right) \\ &= \sum_{j=1}^{n} e_j f([f_j x], y)\end{aligned}$$

Similarly,

$$\sum_{i=1}^{n}[e_i y]f(f_i,x) = \sum_{i=1}^{n} e_i f([yf_i],x)$$

Hence, four of the sums in expression (4) above cancel, and we are left with

$$-c_\varphi f(x,y) = x\sum_{i=1}^{n} e_i f(y,f_i) - y\sum_{i=1}^{n} e_i f(x,f_i) - \sum_{i=1}^{n} e_i f([xy],f_i) \qquad (5)$$

Then since c_φ is invertible, we can define the function

$$g(x) = -c_\varphi^{-1}\sum_{i=1}^{n} e_i f(x,f_i)$$

With this function, equation (3) becomes exactly equation (5), which completes the proof. □

There is a generalization of Whitehead's lemmas to the cohomology spaces of higher dimensions. However, it requires some more assumptions. Conceptually, the proof for the nilpotent case does not carry over to higher dimensions, but the bijective case does. This theorem is stated in [4], but is not proved there.

Theorem 3.5. ***(Whitehead's Theorem)***
Let L be a finite dimensional semisimple Lie algebra and M a finite dimensional irreducible L-module such that $LM \neq 0$. Then $H^k(L,M) = 0$ for all $k > 0$.

Proof. We define c_φ, $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ like in the proof of the first or second lemma.

We can now apply the Fitting decomposition, but as established in the proof of the first lemma, L acts trivially on the nilpotent part of M. However, the set $\{m \in M : Lm = 0\}$ is a submodule so since M is irreducible and $LM \neq 0$, $Lm = 0$ if and only if $m = 0$. This means that the nilpotent part of M is identically 0, so c_φ must be a vector space automorphism on all of M. Hence, what we need to prove is that if $f : \prod_{i=1}^k L \to M$ is a k-linear skew symmetric map such that

$$\begin{aligned} 0 = &\sum_{j=1}^{k+1}(-1)^{j+1}x_j f(x_1,...,\hat{x}_j,...,x_{k+1}) \\ &+ \sum_{1\le i<j\le k+1}(-1)^{i+j} f([x_i x_j],x_1,...,\hat{x}_i,...,\hat{x}_j,...,x_{k+1}) \end{aligned} \qquad (6)$$

then there is a $(k-1)$-linear skew symmetric map $g : \prod_{i=1}^{k-1} L \to M$ such that

$$\begin{aligned} f(x_1,...,x_k) = &\sum_{j=1}^{k}(-1)^{j+1}x_j g(x_1,...,\hat{x}_j,...,x_k) \\ &+ \sum_{1\le i<j\le k}(-1)^{i+j} g([x_i x_j],x_1,...,\hat{x}_i,...,\hat{x}_j,...,x_k) \end{aligned} \qquad (7)$$

We use a similar argument to the one in the second lemma. Indeed, if we take $x_{k+1} = f_l$ in equation (6), then multiply by e_l and sum over l, we get

$$\begin{aligned} 0 &= \sum_{l=1}^{n} \sum_{j=1}^{k+1} (-1)^{j+1} e_l x_j f(x_1, ..., \hat{x}_j, ..., x_k, f_l) \\ &\quad + \sum_{l=1}^{n} \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} e_l f([x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_k, f_l) \\ &= \sum_{l=1}^{n} \sum_{j=1}^{k} (-1)^{j+1} ([e_l x_j] + x_j e_l) f(x_1, ..., \hat{x}_j, ..., x_k, f_l) \\ &\quad + \sum_{l=1}^{n} \sum_{1 \leq i < j \leq k} (-1)^{i+j} e_l f([x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_k, f_l) \\ &\quad + (-1)^k c_\varphi f(x_1, ..., x_k) + \sum_{l=1}^{n} \sum_{i=1}^{k} (-1)^{i+k+1} e_l f([x_i f_l], x_1, ..., \hat{x}_i, ..., x_k) \end{aligned}$$

Using exactly the same argument as in the second lemma, we have for each $j = 1, ..., k$

$$\sum_{l=1}^{n} (-1)^{j+1} [e_l x_j] f(x_1, ..., \hat{x}_j, ..., x_k, f_l) = \sum_{l=1}^{n} (-1)^{j+k} e_l f([x_j f_l], x_1, ..., \hat{x}_j, ..., x_k)$$

so after cancellation we have

$$\begin{aligned} c_\varphi f(x_1, ..., x_k) &= \sum_{l=1}^{n} \sum_{j=1}^{k} (-1)^{j+k} x_j e_l f(x_1, ..., \hat{x}_j, ..., x_k, f_l) \\ &\quad + \sum_{l=1}^{n} \sum_{1 \leq i < j \leq k} (-1)^{i+j+k+1} e_l f([x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_k, f_l) \end{aligned}$$

which is equation (7) if we take

$$g(x_1, ..., x_{k-1}) = (-1)^{k+1} c_\varphi^{-1} \sum_{l=1}^{n} e_l f(x_1, ..., x_{k-1}, f_l)$$

This concludes the proof. □

4 Levi's Theorem

When studying Lie algebras, it is often useful to try to reduce the problem to studying Lie algebras of the well understood classes that we have dicussed before, most notably semisimple Lie algebras. One tool in this strategy is Levi's theorem, which allows the splitting of a finite dimensional into a direct sum of a solvable and a semisimple subalgebra. The aim of this chapter will be to state and prove this theorem. To this end, we will use the cohomology theory introduced in the previous chapter. We will also prove Mal'tsev-Harish-Chandra's theorem which relates different Levi decompositions to one another. The theory in this chapter is taken from [4], with minor adjustments.

We will start with an example.

Example 4.1. Consider the subalgebra L of $\mathfrak{sl}(\mathbb{C}^3)$ consisting of linear maps having matrices of the form

$$\begin{bmatrix} a & b & d \\ c & -a & e \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to some fixed basis of $\mathbb{C}^3$. Since the composition of two maps in L is again in L, we see that L is indeed a subalgebra since it is closed under the commutator. Define now two subalgebras of L in the following way:

$$L_1 = \left\{ \begin{bmatrix} 0 & 0 & d \\ 0 & 0 & e \\ 0 & 0 & 0 \end{bmatrix}, d, e \in \mathbb{C} \right\}, S = \left\{ \begin{bmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{bmatrix}, a, b, c \in \mathbb{C} \right\} \cong \mathfrak{sl}(\mathbb{C}^2)$$

Since $L = L_1 \oplus S$, the structure of L_1 and S gives us information about the structure of L. First, we claim that S is semisimple. By Theorem 2.5 it is sufficient to check that the Killing form of S is non-degenerate. If we take as basis elements of $\mathfrak{sl}(\mathbb{C}^2)$

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then we compute

$$[e_1 e_2] = 2e_2, \quad [e_1 e_3] = -2e_3, \quad [e_2 e_3] = e_1$$

$$\mathrm{ad}(e_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathrm{ad}(e_2) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}(e_3) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

so the Killing form of S becomes

$$\kappa = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

which is nondegenerate, which shows the semisimplicity of S. We turn now to L_1. A simple calculation shows that L_1 is an ideal. We claim now that L_1 is solvable, which would imply $L_1 = \text{Rad}L$. This can be seen easily, since $[L_1L_1] = 0$. Thus, we have constructed a decomposition $L = \text{Rad}L \oplus S$, where S is semisimple. As we will see, such a decomposition can be made for every finite dimensional Lie algebra.

Before stating Levi's theorem, we will need a lemma.

Lemma 4.1. *Let L be a finite dimensional Lie algebra and suppose $S \subset L$ is a subalgebra such that $S \cong L/RadL$. Then S is semisimple and $L = S \oplus RadL$.*

Proof. Let I be a solvable ideal in S. Then by the given isomorphism we can find some solvable ideal $J/\text{Rad}L$ in $L/\text{Rad}L$, where $\text{Rad}L \subset J$. By maximality we must have $\text{Rad}L = J$ so $I = 0$. This proves the semisimplicity of S. Further, $S \cap \text{Rad}L$ is contained in $\text{Rad}L$, so it is solvable, but it is also contained in S, so $S \cap \text{Rad}L = 0$. Note that

$$\text{dim}L = \text{dimRad}L + \text{dim}L/\text{Rad}L = \text{dimRad}L + \text{dim}S$$

This shows that $L = S \oplus \text{Rad}L$. □

We will now proceed to show the main result of this chapter, and arguably of the entire paper.

Theorem 4.1. ***(Levi's theorem)***
*Let L be a finite dimensional Lie algebra. Then there is a semisimple subalgebra S of L such that $L = RadL \oplus S$. The subalgebra S is called a **Levi factor** of L.*

Proof. We will first prove the result in the case when $\text{Rad}L$ is abelian and then use this to show the general case. For abelian $\text{Rad}L$, we define $\overline{L} = L/\text{Rad}L$ and denote $\overline{x} = x + \text{Rad}L \in \overline{L}$. Note that $\overline{L}$ is semisimple by Lemma 4.1. We make $\text{Rad}L$ into an $\overline{L}$-module by defining the multiplication $\overline{x}y = [xy]$, which is well defined by the assumption that $\text{Rad}L$ is abelian.

Let S be a subspace of L such that $L = \text{Rad}L \oplus S$. Then by the first isomorphism theorem for vector spaces there is an isomorphism from $\overline{L}$ to S induced by the homomorphism $L \to \overline{L}, x \mapsto \overline{x}$. Let $\sigma : \overline{L} \to L$ be the extension of this isomorphism to L. If $x = y + z$, where $x \in L, y \in \text{Rad}L, z \in S$, then

$$\overline{\sigma(\overline{x})} = \overline{z} = \overline{x} \tag{1}$$

Thus, our choice of S determines a map σ satisfying equation (1). Conversely, given a linear injective map $\sigma : \overline{L} \to L$ that satisfies equation (1), then $L = \text{Rad}L \oplus \sigma(\overline{L})$. Indeed, by the first isomorphism theorem for vector spaces, $\overline{L} \cong \sigma(\overline{L})$, but if $x \in \text{Rad}L \cap \sigma(\overline{L})$ then $x = \sigma(\overline{y})$ for some $\overline{y}$ but $\overline{y} = \overline{x} = 0$ by (1) so $x = 0$. Further, if $x \in L, y \in \text{Rad}L$ then

$$\overline{x}y = [\sigma(\overline{x})y]$$

Now, define the bilinear map $f : \overline{L} \times \overline{L} \to \text{Rad}L, (\overline{x}_1, \overline{x}_2) \mapsto [\sigma(\overline{x}_1)\sigma(\overline{x}_2)] - \sigma([\overline{x}_1\overline{x}_2])$. This is well defined since

$$\begin{aligned}\overline{f(\overline{x}_1, \overline{x}_2)} &= \overline{[\sigma(\overline{x}_1)\sigma(\overline{x}_2)]} - \overline{\sigma([\overline{x}_1\overline{x}_2])} = [\overline{\sigma(\overline{x}_1)}, \overline{\sigma(\overline{x}_2)}] - \overline{[x_1x_2]} \\ &= [\overline{x}_1\overline{x}_2] - [\overline{x}_1\overline{x}_2] = 0\end{aligned}$$

so indeed, $f(\overline{x}_1, \overline{x}_2) \in \text{Rad}L$.

We will now show that $f \in Z^2(\overline{L}, \text{Rad}L)$. Indeed, f is skew symmetric since

$$\begin{aligned}f(\overline{x}_1, \overline{x}_2) + f(\overline{x}_2, \overline{x}_1) &= f(\overline{x}_1 + \overline{x}_2, \overline{x}_1 + \overline{x}_2) \\ &= [\sigma(\overline{x}_1 + \overline{x}_2)\sigma(\overline{x}_1 + \overline{x}_2)] - \sigma([\overline{x}_1 + \overline{x}_2, \overline{x}_1 + \overline{x}_2]) = 0\end{aligned}$$

We are left to show the cocycle condition. Note that

$$[\sigma(\overline{x}_1)\sigma(\overline{x}_2)] = \sigma([\overline{x}_1\overline{x}_2]) + f(\overline{x}_1, \overline{x}_2)$$

Taking the bracket on both side with $\sigma(\overline{x}_3)$ we obtain

$$\begin{aligned}[[\sigma(\overline{x}_1)\sigma(\overline{x}_2)]\sigma(\overline{x}_3)] &= [\sigma([\overline{x}_1, \overline{x}_2])\sigma(\overline{x}_3] + [f(\overline{x}_1\overline{x}_2)\sigma(\overline{x}_3)] \\ &= \sigma([[\overline{x}_1\overline{x}_2]\overline{x}_3]) + f([\overline{x}_1\overline{x}_2], \overline{x}_3) + [f(\overline{x}_1, \overline{x}_2)\sigma(\overline{x}_3)] \qquad (2)\end{aligned}$$

However, recall that

$$[[\sigma(\overline{x}_1)\sigma(\overline{x}_2)]\sigma(\overline{x}_3)] + [[\sigma(\overline{x}_2)\sigma(\overline{x}_3)]\sigma(\overline{x}_1)] + [[\sigma(\overline{x}_3)\sigma(\overline{x}_1)]\sigma(\overline{x}_2)] = 0$$

and

$$\sigma([[\overline{x}_1\overline{x}_2]\overline{x}_3] + \sigma([[\overline{x}_2\overline{x}_3]\overline{x}_1] + \sigma([[\overline{x}_3\overline{x}_1]\overline{x}_2] = 0$$

Hence if we sum both sides of equation (2) over the permutations (x_1, x_2, x_3), (x_2, x_3, x_1), (x_3, x_1, x_2) we obtain

$$\begin{aligned}0 &= f([\overline{x}_1\overline{x}_2], \overline{x}_3) + f([\overline{x}_2\overline{x}_3], \overline{x}_1) + f([\overline{x}_3\overline{x}_1], \overline{x}_2) \\ &\quad + [f(\overline{x}_1, \overline{x}_2)\sigma(\overline{x}_3)] + [f(\overline{x}_2, \overline{x}_3)\sigma(\overline{x}_1)] + [f(\overline{x}_3, \overline{x}_1)\sigma(\overline{x}_2)]\end{aligned}$$

which is the required cocycle condition with respect to the module multiplication defined above.

Now, $\overline{L}$ is a semisimple Lie algebra and Rad$L$ is a finite dimensional $\overline{L}$-module, so by Whitehead's second lemma, $H^2(\overline{L}, \text{Rad}L) = 0$. Hence, $f \in B^2(\overline{L}, \text{Rad}L)$ which means there is some linear map $g : \overline{L} \to \text{Rad}L$ that satisfies

$$f(\overline{x}_1, \overline{x}_2) = [\sigma(\overline{x}_1)g(\overline{x}_2)] - [\sigma(\overline{x}_2)g(\overline{x}_1)] - g([\overline{x}_1\overline{x}_2])$$

We want to find necessary and sufficient conditions for $\sigma(\overline{L})$ to be a subalgebra. Note that if it is, then it is semisimple and the theorem is proved. Suppose $\sigma(\overline{L})$ is a subalgebra. Then $[\sigma(\overline{x}_1)\sigma(\overline{x}_2)] \in \sigma(\overline{L})$, and since $[\overline{x}_1, \overline{x}_2] \in \overline{L}$, also $\sigma([\overline{x}_1, \overline{x}_2]) \in \sigma(\overline{L})$. Hence $f(\overline{x}_1, \overline{x}_2) \in \text{Rad}L \cap \sigma(\overline{L}) = 0$. On the other hand if $f \equiv 0$ then $[\sigma(\overline{x}_1)\sigma(\overline{x}_2)] = \sigma([\overline{x}_1\overline{x}_2]) \in \sigma(\overline{L})$. Thus we can conclude that $\sigma(\overline{L})$ is a subalgebra if and only if $f \equiv 0$.

If $\sigma(\overline{L})$ is not a subalgebra we claim that $\tau(\overline{L})$ is one, where $\tau = \sigma - g : \overline{L} \to L$. Then τ also has the property that

$$\overline{\tau(\overline{x})} = \overline{\sigma(\overline{x})} + \overline{f(\overline{x})} = \overline{x} + 0 = \overline{x}$$

Therefore, as shown above, it is sufficient to check that $[\tau(\overline{x}_1)\tau(\overline{x}_2)] - \tau([\overline{x}_1\overline{x}_2]) = 0$. Indeed,

$$\begin{aligned}
[\tau(\overline{x}_1)\tau(\overline{x}_2)] - \tau([\overline{x}_1\overline{x}_2]) &= [\sigma(\overline{x}_1) - g(\overline{x}_1), \sigma(\overline{x}_2) - g(\overline{x}_2)] - \sigma([\overline{x}_1\overline{x}_2]) \\
&\quad + g([\overline{x}_1\overline{x}_2]) \\
&= [\sigma(\overline{x}_1)\sigma(\overline{x}_2)] - [\sigma(\overline{x}_1)g(\overline{x}_2)] - [g(\overline{x}_1)\sigma(\overline{x}_2)] \\
&\quad + [g(\overline{x}_1)g(\overline{x}_2)] - \sigma([\overline{x}_1\overline{x}_2]) + g([\overline{x}_1\overline{x}_2]) \\
&= [\sigma(\overline{x}_1)\sigma(\overline{x}_2)] - \sigma([\overline{x}_1\overline{x}_2]) - f(\overline{x}_1, \overline{x}_2) + [g(\overline{x}_1)g(\overline{x}_2)] \\
&= [g(\overline{x}_1)g(\overline{x}_2)] = 0
\end{aligned}$$

Thus, taking $S = \tau(\overline{L})$ completes the proof in the abelian case.

For the general case we will use induction on $\dim L$. Suppose $(\mathrm{Rad}L)^2 \neq 0$ and define $\overline{L} = L/(\mathrm{Rad}L)^2$. Then $\dim\overline{L} < \dim L$ so by the induction hypothesis we can assume the theorem holds for the Lie algebra $\overline{L}$. Now, $\mathrm{Rad}L/(\mathrm{Rad}L)^2$ is solvable by part (i) of Theorem 1.2, since it is a homomorphic image of $\mathrm{Rad}L$. Further, if $L_1/(\mathrm{Rad}L)^2$ is solvable then by part (ii) of the same theorem, L_1 is also solvable. Hence, we must have that $\mathrm{Rad}\overline{L} = \mathrm{Rad}L/(\mathrm{Rad}L)^2$. By the third isomorphism theorem, $\overline{L}/(\mathrm{Rad}\overline{L}) \cong L/\mathrm{Rad}L$. Now, by the induction hypothesis, as remarked above, there is a semisimple subalgebra $\overline{S} \subset \overline{L}, \overline{S} \cong L/\mathrm{Rad}L$. We write $\overline{S} = L_1/(\mathrm{Rad}L)^2$, with $(\mathrm{Rad}L)^2 \subset L_1 \subset L$. Since $(\mathrm{Rad}L)^2 \subset \mathrm{Rad}L$ and $L_1/(\mathrm{Rad}L)^2 \cong L/\mathrm{Rad}L$, we conclude that $\dim L_1 < \dim L$. Hence by the induction hypothesis L_1 has a subalgebra $S \cong L/\mathrm{Rad}L$, but this is also a subalgebra of L, which concludes the proof. □

The following corollary will be uselful below.

Corollary 4.1. *Let L be a finite dimensional Lie algebra. Then the radical of $L^{(1)}$ is nilpotent.*

Proof. Let $L = \mathrm{Rad}L \oplus S$ be a Levi decomposition. Then

$$L^{(1)} = [LL] = [\mathrm{Rad}L \oplus S, \mathrm{Rad}L \oplus S] = [SS] + [L\mathrm{Rad}L]$$

Since the radical of $L^{(1)}$ is $L^{(1)} \cap \mathrm{Rad}L$, we get

$$L^{(1)} \cap \mathrm{Rad}L = [SS] \cap \mathrm{Rad}L + [L\mathrm{Rad}L] \cap \mathrm{Rad}L = [L\mathrm{Rad}L]$$

However, by Corollary 2.2, $[L\mathrm{Rad}L]$ is nilpotent, which concludes the proof. □

The strength of Levi's theorem lies in the rich theory about semisimple Lie algebras. Since there are many results about this particular class of Lie algebras, the fact that there is always a Levi decomposition allows this theory to some extent to be used when studying general finite dimensional Lie algebras. A natural follow-up question would be whether the decomposition is unique. The radical of a Lie algebra is always uniquely determined, as established in part (*iv*) of Theorem 1.2, so the question reduces to whether the Levi factor S is unique. In general, this is not true, but we can nevertheless give some conditions for how different Levi components relate to one another. First, we will need a definition

Definition 4.1. Let L be a Lie algebra and $x \in L$ be such that $\mathrm{ad}(x)$ is nilpotent. Then we define

$$\exp(\mathrm{ad}(x)) = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}(x))^k}{k!}$$

This is well defined since the nilpotency of $\mathrm{ad}(x)$ means the sum will have a finite number of terms.

Lemma 4.2. *Let L be a Lie algebra and $x \in L$ be such that $ad(x)$ is nilpotent. Then $exp(ad(x))$ is an automorphism of L.*

Proof. We start by showing that $\exp(\mathrm{ad}(x))$ is a vector space isomorphism, i.e. invertible. Take n to be the smallest integer such that $(\mathrm{ad}(x))^n = 0$ and let $\eta = \exp(\mathrm{ad}(x)) - 1$. Then $\sum_{k=0}^{n-1}(-1)^k$ is a linear map on L and

$$\exp(\mathrm{ad}(x)) \sum_{k=0}^{n-1}(-1)^k\eta^k = (1+\eta)\sum_{k=0}^{n-1}(-1)^k\eta^k = 1 + (-1)^{n-1}\eta^n = 1$$

so $\exp(\mathrm{ad}(x))$ is invertible with inverse $\sum_{k=0}^{n-1}(-1)^k\eta^k$, and hence a vector space automorphism. We proceed to show that $\exp(\mathrm{ad}(x))$ preserves the bracket. In other words, we need to show that

$$\exp(\mathrm{ad}(x))([yz]) = [\exp(\mathrm{ad}(x))(y), \exp(\mathrm{ad}(x))(z)] \tag{3}$$

First of all, we can rewrite the Jacobi identity as

$$\mathrm{ad}(x)([yz]) = [y, \mathrm{ad}(x)(z)] + [\mathrm{ad}(x)(y), z]$$

Iterating the above equation, we get

$$(\mathrm{ad}(x))^m([yz]) = \sum_{k=0}^{m} \frac{m!}{(k!)(m-k)!}[(\mathrm{ad}(x))^k(y), (\mathrm{ad}(x))^{m-k}(z)]$$

The above equation is sometimes referred to as the **Leibniz rule**. We use this to derive equation (3).

$$\begin{aligned}
[\exp(\mathrm{ad}(x))(y), \exp(\mathrm{ad}(x))(z)] &= \left[\sum_{k=0}^{n-1} \frac{(\mathrm{ad}(x))^k(y)}{k!}, \sum_{k=0}^{n-1} \frac{(\mathrm{ad}(x))^k(z)}{k!}\right] \\
&= \sum_{m=0}^{2n-2} \sum_{k=0}^{m} \left[\frac{(\mathrm{ad}(x))^k(y)}{k!}, \frac{(\mathrm{ad}(x))^{n-k}(z)}{(n-k)!}\right] \\
&= \sum_{m=0}^{2n-2} \frac{(\mathrm{ad}(x))^m([yz])}{m!} = \sum_{m=0}^{n-1} \frac{(\mathrm{ad}(x))^m([yz])}{m!} \\
&= \exp(\mathrm{ad}(x))([yz])
\end{aligned}$$

This completes the proof. □

Definition 4.2. (Inner automorphism)

Let L be a Lie algebra and consider the subgroup of Aut$L$ generated by all elements of the form $\exp(\mathrm{ad}(x))$ where $\mathrm{ad}(x)$ is nilpotent. This subgroup is denoted Int$L$ and its elements are called **inner automorphisms**.

Theorem 4.2. ***(Mal'tsev-Harish-Chandra's theorem)***

Let L be a finite dimensional Lie algebra and let $L = S \oplus RadL$ be a Levi decomposition. If $S_1 \subset L$ is a semisimple subalgebra, then there is some $A \in IntL$ such that $A(S_1) \subset S$

Proof. Define the projections $\rho : S_1 \to \mathrm{Rad}L$ and $\sigma : S_1 \to S$ such that $x = \rho(x) + \sigma(x)$ for all $x \in S_1$. Since $S_1 \subset L = S \oplus \mathrm{Rad}L$, these maps exist and are unique. Furthermore, since S_1 is semisimple, $\mathrm{Ker}\sigma = S_1 \cap \mathrm{Rad}L = 0$, so σ is injective. Now, for $x, y \in S_1$ we can expand $[xy]$ in two different ways:

$$\sigma([xy]) + \rho([xy]) = [xy] = [\sigma(x)\sigma(y)] + [\sigma(x)\rho(y)] + [\rho(x)\sigma(y)] + [\rho(x)\rho(y)]$$

Since the radical is an ideal, we can thus conclude that

$$\sigma([xy]) = [\sigma(x)\sigma(y)]$$

and

$$\rho([xy]) = [\sigma(x)\rho(y)] + [\rho(x)\sigma(y)] + [\rho(x)\rho(y)] \tag{4}$$

Hence, $\rho([xy]) \in [L\mathrm{Rad}L]$, but by Corollary 2.2 $[L\mathrm{Rad}L]$ is nilpotent. Thus if we let N be the nilradical of L, then $\rho([xy]) \in N$. By Theorem 2.6 $[S_1S_1] = S_1$, so $\rho(x) \in N$ for all $x \in S_1$. This shows that $S_1 \subset S \oplus N$.

Our goal now is to find automorphisms $A_k \in \mathrm{Int}L$ such that $A_k(S_1) \subset S \oplus N^{(k+1)}$. If we can do that then we are done since the map $A_n \in \mathrm{Int}L$ satisfies $A_n(S_1) \subset S \oplus N^{(n)} = S$, if we take n to be the smallest number such that $N^{(n)} = 0$, which exists since N is solvable by part (*iv*) of Theorem 1.3.

We construct these maps inductively. We can do this since we have already established that $S_1 \subset S \oplus N$. Thus, assuming that A_{k-1} maps S_1 into $S \oplus N^{(k)}$, we can assume without loss of generality that $S_1 \subset S \oplus N^{(k)}$, and construct an automorphism A'_k mapping it into $S_1 \subset S \oplus N^{(k+1)}$. The general situation is then also done, by taking $A_k = A'_k \circ A_{k-1}$.

In order to construct the map A'_k, we will use Whitehead's first lemma. To do that, we will need a semisimple Lie algebra and a module. For the semisimple Lie algebra, we pick the natural choice S_1. For the module first note that we can turn $N^{(k)}$ into an S_1-module by defining the multiplication $xm = [\sigma(x)m]$ for $x \in S_1$ and $m \in N^{(k)}$. This will turn $N^{(k+1)}$ into an S_1-submodule, so the factor module $N^{(k)}/N^{(k+1)}$ is also an S_1-module with the multiplication $x(m + N^{(k+1)}) = [\sigma(x)m] + N^{(k+1)}$. This is the S_1-module we will apply Whitehead's lemma on. Indeed, S_1 is a semisimple Lie algebra and $N^{(k)}/N^{(k+1)}$ is a finite dimensional S_1-module, so $H^1(S_1, N^{(k)}/N^{(k+1)}) = 0$.

Now, for $x, y \in S_1 \subset S \oplus N^{(k)}$, we have that $\rho([xy]) \in N^{(k)}$ so under the canonical homomorphism from $N^{(k)}$ to $N^{(k)}/N^{(k+1)}$, equation (4) becomes

$$\begin{aligned}\rho([xy]) + N^{(k+1)} &= [\sigma(x)\rho(y)] + [\rho(x)\sigma(y)] + [\rho(x)\rho(y)] + N^{(k+1)} \\ &= [\sigma(x)\rho(y)] + [\rho(x)\sigma(y)] + N^{(k+1)}\end{aligned}$$

Hence, if we define $f : S_1 \to N^{(k)}/N^{(k+1)}, x \mapsto \rho(x) + N^{(k+1)}$, then the equation above shows that f is a 1-cocycle. Thus f must be a coboundary as established above. In other words, there is some $m \in N^{(k)}$ such that $f(x) = x(m + N^{(k+1)})$, which means that $\rho(x) - [\sigma(x)m] \in N^{(k+1)}$. Take now $A'_k = \exp(\mathrm{ad}(m))$. Then since $[m[mx]] \in N^{(k+1)}$,

$$\begin{aligned}A'_k(x) + N^{(k+1)} &= \sum_{i=0}^{n} \frac{(\mathrm{ad}(m))^i(x)}{i!} + N^{(k+1)} = x + [mx] + N^{(k+1)} \\ &= \sigma(x) + \rho(x) + [m\sigma(x)] + [m\rho(x)] + N^{(k+1)} \\ &= \sigma(x) + [\sigma(x)m] + [m\sigma(x)] + [m[\sigma(x)m]] + N^{(k+1)} \\ &= \sigma(x) + N^{(k+1)}\end{aligned}$$

Hence, A'_k maps S_1 to $S + N^{(k+1)}$ which is exactly what we needed, as we established before. Hence, this concludes the proof. □

Corollary 4.2. *Let L be a finite dimensional Lie algebra and let $L = S_1 \oplus RadL = S_2 \oplus RadL$ be two Levi decompositions of L. Then there is an isomorphism $A \in IntL$ between S_1 and S_2.*

Proof. By Mal'tsev-Harish-Chandra's theorem there is an automorphism of L mapping S_1 into S_2. Since $\dim S_1 = \dim S_2$, this means S_1 and S_2 are isomorphic. □

5 The Virasoro Algebra

In this chapter we will introduce the Virasoro algebra, which will serve as an example to show that the assumption about finite dimensionality in Levi's theorem is necessary. The Virasoro algebra will be constructed in this chapter, relying only on the definition of the Witt algebra. This means that this chapter will be more independent from other works, even though none of the results presented here can be considered new. The examples provided in this chapter will serve to illustrate the applications, but also the limits of the theory presented in the previous chapters.

We start by defining the Witt algebra.

Definition 5.1. (Witt algebra)
Consider a vector space over $\mathbb{C}$ with a countable basis $\{d_n\}_{n\in\mathbb{Z}}$. We equip this vector space with a bilinear form defined by $[d_m d_n] = (m-n)d_{m+n}$. The resulting algebra is called the **Witt algebra** and is denoted by W.

Lemma 5.1. *The Witt algebra is a Lie algebra under the bracket indicated in the definition above.*

Proof. The defined commutator is a bilinear form satisfying $[xx] = 0$ for all $x \in W$. We are left to check the Jacobi identity. By bilinearity it is enough to verify for basis elements.

$$\begin{aligned}
&[d_k[d_m d_n]] + [d_m[d_n d_k]] + [d_n[d_k d_m]] \\
&\quad = (m-n)[d_k d_{m+n}] + (n-k)[d_m d_{n+k}] + (k-m)[d_n d_{k+m}] \\
&\quad = ((m-n)(k-m-n) + (n-k)(m-n-k) + (k-m)(n-k-m))d_{k+m+n} \\
&\quad = 0
\end{aligned}$$

This concludes the proof. □

Remark 5.1. The Witt algebra can arise in several different ways. One way, which can serve as geometric motivation for the construction is taking the elements of the Witt algebra to be analytic vector fields on the punctured complex plane, and choose the basis $d_n = -z^{n+1}\frac{\partial}{\partial z}$ for $n \in \mathbb{Z}$. Then taking the commutator to be defined by $[xy] = xy - yx$ we obtain

$$\begin{aligned}
[d_m d_n] &= z^{m+1}\frac{\partial}{\partial z} z^{n+1}\frac{\partial}{\partial z} - z^{n+1}\frac{\partial}{\partial z} z^{m+1}\frac{\partial}{\partial z} \\
&= z^{m+1}\left((n+1)z^n\frac{\partial}{\partial z} + z^{n+1}\frac{\partial^2}{\partial z^2}\right) \\
&\quad - z^{n+1}\left((m+1)z^m\frac{\partial}{\partial z} + z^{m+1}\frac{\partial^2}{\partial z^2}\right) \\
&= -(m+1)z^{m+n+1}\frac{\partial}{\partial z} + (n+1)z^{m+n+1}\frac{\partial}{\partial z} \\
&= (n-m)z^{m+n+1}\frac{\partial}{\partial z} = (m-n)d_{m+n}
\end{aligned}$$

which is exactly the commutator relation in the definition of the Witt algebra. We will not discuss the details of this construction since we are avoiding the geometric connections. Nevertheless, it can serve as a motivation for studying the Witt algebra.

Lemma 5.2. *The Witt algebra is simple. In particular, since W is not abelian,* $[WW] = W$

Proof. Let I be a non-zero ideal in W. We show that $d_k \in I$ for some $k \in \mathbb{Z}$. Let $0 \neq x \in I$. Then we can write $x = \sum_{i=1}^{n} a_i d_{k_i}$ for some n, where $a_i \in \mathbb{C}$ and $k_i \in \mathbb{Z}$ for $i = 1, ..., n$. We proceed by induction on n. If $n = 1$ then $0 \neq x = a_1 d_{k_1}$ so $d_{k_1} \in I$, which proves the base step. For the inductive step, note that

$$[d_0 x] - k_n x = \sum_{i=1}^{n} k_i a_i d_{k_i} - \sum_{i=1}^{n} k_n a_i d_{k_i} = \sum_{i=1}^{n-1} (k_i - k_n) a_i d_{k_i} \in I$$

Further, $[d_0 x] - k_n x \neq 0$ since the indices k_i are pairwise different. Since we have now constructed an element in the ideal with $n-1$ basis components, so by the induction hypothesis we can find some k such that $d_k \in I$. But then if $k \neq 0$ then

$$d_0 = \frac{1}{2k}[d_k d_{-k}] \in I$$

and from this we get that

$$d_n = \frac{1}{n}[d_n d_0] \in I$$

for all $n \neq 0$. Thus we conclude that $I = W$. □

We will now introduce a special case of Lie algebra extensions that will be relevant in constructing the Virasoro algebra.

Definition 5.2. (Central extension)
Let the Lie algebra L be an extension of the Lie algebra L_1 by the Lie algebra L_2, i.e. f and g are homomorphisms such that the sequence

$$0 \to L_2 \xrightarrow{f} L \xrightarrow{g} L_1 \to 0$$

is exact. If $\mathrm{Ker} g \subset Z(L)$ the extension is called **central**.

Definition 5.3. (Graded Lie algebra)
Let L be a Lie algebra that can be decomposed as a direct sum of vector spaces $L = \bigoplus_{k \in \mathbb{Z}} V_k$, where $[V_m V_n] \subset V_{m+n}$. Then L is called a graded Lie algebra. An element $x \in V_n$ is said to be of **degree** n.

We will now study the central extensions of the Witt algebra by the one dimensional Lie algebra $\mathbb{C}$. Note that a one dimensional Lie algebra is always abelian. According to Remark 4 in Chapter 3, if V is a central extension of W by $\mathbb{C}$ then

$$V \cong W \oplus \mathbb{C} \cong \bigoplus_{k \in \mathbb{Z}} \mathbb{C} L_k \oplus \mathbb{C} c \tag{1}$$

as vector spaces, where $g : L_k \mapsto d_k$. Thus, we must only investigate the commutation relations on the resulting space.

Lemma 5.3. *Let V be a graded central extension of W by $\mathbb{C}$ such that the element c in equation* (1) *has grade 0 and L_k has grade k. Then there are constants $a, b \in \mathbb{C}$ such that V has commutation relations*

$$[L_m L_n] = (m-n)L_{m+n} + (am + bm^3)\delta_{m,-n}c \tag{2}$$

$$[cL_n] = 0$$

for all $m, n \in \mathbb{Z}$. Conversely, for any $a, b \in \mathbb{C}$, the vector space in (1) *equipped with the commutator defined in* (2) *is a central extension of W by $\mathbb{C}$*

Proof. The fact that $[cL_n] = 0$ for all $n \in \mathbb{Z}$ follows from the fact that $c \in Z(V)$ since the extension is central and $\mathrm{Im} f = \mathrm{Ker} g$. For the commutation relations of the elements L_k, we inherit some structure from the Witt algebra. Indeed,

$$g([L_m L_n]) = [g(L_m)g(L_n)] = [d_m d_n] = (m-n)d_{m+n} = (m-n)g(L_{m+n})$$

Hence, by the first isomorphism theorem,

$$[L_m L_n] - (m-n)L_{m+n} \in \mathrm{Ker} g = \mathbb{C}c$$

Thus, the commutation relations can be written as

$$[L_m L_n] = (m-n)L_{m+n} + \alpha_{m,n}c$$

By the assumptions on the gradation, using the notation of Definition 5.3, we get that $V_0 = \mathbb{C}L_0 \oplus \mathbb{C}c$ and $V_k = \mathbb{C}L_k$ for $k \neq 0$. Since $[L_m L_n] \in V_{m+n}$, we must have $\alpha_{m,n} = 0$ if $m \neq -n$. Thus, we are left to find the constants $\alpha_k = \alpha_{k,-k}$. First of all, by skew symmetry we find that

$$\alpha_k c = [L_k L_{-k}] - 2kL_0 = -[L_{-k}L_k] + (-2k)L_0 = -\alpha_{-k}c$$

so $\alpha_k = -\alpha_{-k}$ which means we must only investigate the sequence for $k > 0$. We now apply the Jacobi identity to the elements L_m, L_n and L_{-m-n}.

$$\begin{aligned}0 &= [L_m[L_n L_{-m-n}]] + [L_n[L_{-m-n}L_m]] + [L_{-m-n}[L_m L_n]] \\ &= (m+2n)[L_m L_{-m}] - (2m+n)[L_n L_{-n}] + (m-n)[L_{-m-n}L_{m+n}] \\ &= ((m+2n)\alpha_m - (2m+n)\alpha_n - (m-n)\alpha_{m+n})c\end{aligned}$$

This leads us to the question of solving the recurrence relation

$$(m+2n)\alpha_m - (2m+n)\alpha_n - (m-n)\alpha_{m+n} = 0 \tag{3}$$

We will use generating functions to solve this recurrence. In order to simplify computations, we will first fix $m = 1$. Define the formal power series $f(x) =$

$\sum_{n=0}^{\infty} \alpha_n x^n$. Then if we multiply equation (3) by x^n and sum for n from 0 to ∞, we obtain

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(1+2n)\alpha_1 x^n - \sum_{n=0}^{\infty}(2+n)\alpha_n x^n - \sum_{n=0}^{\infty}(1-n)\alpha_{1+n}x^n \\
&= \frac{\alpha_1}{1-x} + \frac{2\alpha_1 x}{(1-x)^2} - 2f(x) - xf'(x) - \frac{2f(x)}{x} + f'(x) \\
&= \frac{\alpha_1(1+x)}{(1-x)^2} - \left(\frac{2}{x}+2\right) f(x) - (x-1)f'(x)
\end{aligned}$$

So we have reduced the problem to solving the differential equation

$$\left(\frac{2}{x}+2\right) f(x) + (x-1)f'(x) = \frac{\alpha_1(1+x)}{(1-x)^2} \tag{4}$$

We will solve this using integrating factors. If we rewrite the equation as

$$\left(\frac{2(x+1)}{x(x-1)}\right) f(x) + f'(x) = -\frac{\alpha_1(1+x)}{(1-x)^3}$$

then we can multiply with the integrating factor

$$e^{\int \frac{2(x+1)}{x(x-1)}dx} = e^{4\log(1-x)-2\log(x)} = \frac{(1-x)^4}{x^2}$$

to obtain

$$\left(\frac{(1-x)^4}{x^2}f(x)\right)' = \frac{2(x+1)(x-1)^3}{x^3}f(x) + \frac{(1-x)^4}{x^2}f'(x) = -\frac{\alpha_1(1-x^2)}{x^2}$$

which gives us

$$f(x) = \frac{\alpha_1 x(x^2-4x+1)}{(1-x)^4} + \frac{Cx^2}{(1-x)^4}$$

Now, f is analytic in a neighbourhood around 0 and has power series expansion

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\left(\alpha_1\left(\binom{n}{3} - 4\binom{n+1}{3} + \binom{n+2}{3}\right) + C\binom{n+1}{3}\right)x^n \\
&= \sum_{n=0}^{\infty}\frac{1}{6}(Cn(n^2-1) - 2\alpha_1 n(n^2-4))x^n
\end{aligned}$$

Thus we can conclude that $C = \alpha_2$ and that

$$\alpha_k = \frac{1}{6}(\alpha_2 n(n^2-1) - 2\alpha_1 n(n^2-4)) = \frac{-2\alpha_1+\alpha_2}{6}n^3 + \frac{8\alpha_1-\alpha_2}{6}n$$

This shows that any solution to equation (3) has the form $\alpha_k = ak + bk^3$. However, since we restricted the argument to the case $m = 1$, this solution might be too general. We show that this is not the case by simply substituting this solution into equation (3), to get

$$\begin{aligned}
&(m+2n)(am+bm^3)-(2m+n)(an+bn^3)-(m-n)(a(m+n)+b(m+n)^3) \\
&\quad = m^2(a+bm^2)+2bm^3n-n^2(a+bn^2)-2bmn^3 \\
&\qquad -(m^2-n^2)(a+b(m+n)^2) \\
&\quad = (m^2-n^2)(a+b(m^2+n^2)+2bmn)-(m^2-n^2)(a+b(m+n)^2) \\
&\quad = 0 \qquad (5)
\end{aligned}$$

This shows also the converse part of the lemma. □

Lemma 5.4. *Let V_1, V_2 be two graded central extensions of W by $\mathbb{C}$ with commutation relations as in Lemma 5.3, with constants a_1, b_1 and a_2, b_2, respectively. If $b_1 = b_2 = 0$ then $V_1 \cong V_2$. In this case, we call the extensions **trivial**. Further, if $b_1 \neq 0 \neq b_2$ then also $V_1 \cong V_2$.*

Proof. We first show that changing the constant a yields an isomorphic Lie algebra, that is, if $b_1 = b_2$ then $V_1 \cong V_2$. In particular, this shows the trivial case. Note that since isomorphism is transitive, we can assume without loss of generality that $a_2 = 0$. Then we can define the linear map $\varphi : V_1 \to V_2$ by $\varphi(c) = c$ and $\varphi(L_k) = L_k - \frac{1}{2}a_1\delta_{0,k}c$. This is bijective, so we must verify that it is a homomorphism. Indeed,

$$\begin{aligned}
\varphi([L_mL_n]_1) &= \varphi((m-n)L_{m+n} + (a_1m + b_1m^3)\delta_{m,-n}c) \\
&= (m-n)L_{m+n} - (m-n)\frac{1}{2}a_1\delta_{m,-n}c + (a_1m+b_1m^3)\delta_{m,-n}c \\
&= (m-n)L_{m+n} + b_2m^3\delta_{m,-n}c = [L_m - \frac{1}{2}a\delta_{0,m}c, L_n - \frac{1}{2}a\delta_{0,n}c]_2 \\
&= [\varphi(L_m)\varphi(L_n)]_2
\end{aligned}$$

as required. Thus we can from now on assume $a_1 = a_2 = 0$. We must show that if b_1 and b_2 are both non-zero then $V_1 \cong V_2$. For this, we can pick the linear map $\psi : V_1 \to V_2$ defined by $\psi(L_k) = L_k$ and $\psi(c) = \frac{b_2}{b_1}c$. Since b_1 and b_2 are non-zero this is a bijective map. We show that it is a homomorphism

$$\begin{aligned}
\psi([L_mL_n]_1) &= \psi((m-n)L_{m+n} + b_1m^3\delta_{m,-n}c) \\
&= (m-n)L_{m+n} + b_2m^3\delta_{m,-n}c = [L_mL_n]_2 = [\psi(L_m)\psi(L_n)]_2
\end{aligned}$$

This concludes the proof. □

This shows that the non-trivial graded (according to the assumptions of Lemma 5.3) central extension of the Witt algebra by $\mathbb{C}$ is unique up to isomorphism. Thus, we can make the following definition.

Definition 5.4. (Virasoro algebra)
The unique non-trivial central extension from Lemma 5.4 is called the **Virasoro algebra** and is denoted V. In physics, it is a convention to take the constants to be $a = \frac{1}{12}$ and $b = -\frac{1}{12}$.

In fact, as we will see below, the assumptions on the gradation of the central extension above can be omitted. In order to see this, we will study the cohomology spaces of the Witt algebra with respect to the one-dimensional W-module $\mathbb{C}$, where multiplication is defined trivially as $W\mathbb{C} = 0$.

Lemma 5.5. $H^1(W, \mathbb{C}) = 0$.

Proof. Let $f \in Z^1(W, \mathbb{C})$. The 1-cocycle condition becomes

$$df(x, y) = xf(y) - yf(x) - f([xy]) = -f([xy]) = 0$$

By Lemma 5.2, this means that $f \equiv 0$. We conclude that $Z^1 = 0$ so clearly $H^1 = 0$ as well. □

Lemma 5.6. *$H^2(W, \mathbb{C})$ is one-dimensional.*

Proof. As in the previous lemma, we start by taking some $f \in Z^2(W, \mathbb{C})$, which means f must satisfy

$$\begin{aligned} df(x, y, z) &= xf(y, z) + yf(z, x) + zf(x, y) - f([xy], z) - f([yz], x) - f([zx], y) \\ &= -f([xy], z) - f([yz], x) - f([zx], y) = 0 \qquad (6) \end{aligned}$$

Meanwhile, a function $g \in B^2(W, \mathbb{C})$, as we saw in the proof of Lemma 5.5, must satisfy

$$g(x, y) = h([xy])$$

where $h : W \to \mathbb{C}$ is a linear map. In particular, the linear map h defined by

$$h(d_0) = -\frac{1}{2}f(d_1, d_{-1}), \quad h(d_k) = -\frac{1}{k}f(d_k, d_0), \quad k \neq 0$$

gives rise to the 2-coboundary $g(x, y) = h([xy])$. Then the 2-cocycle $f' = f + g$ differs from f only by a 2-coboundary, so $f \equiv f'$ in the cohomology space $H^2(W, \mathbb{C})$ with

$$f'(d_1, d_{-1}) = f'(d_k, d_0) = 0, \quad k \in \mathbb{Z}$$

Now, substituting d_0, d_m, d_n into equation (6) with the function f' we obtain

$$\begin{aligned} 0 &= f'([d_0 d_m], d_n) + f'([d_m d_n], d_0) + f'([d_n d_0], d_m) \\ &= -mf'(d_m, d_n) + nf'(d_n, d_m) = (m + n)f'(d_n, d_m) \end{aligned}$$

so $f'(d_m, d_n) = 0$ unless $m = -n$. Further, for $n > 2$ we can substitute d_{n-1}, d_1, d_{-n} into equation (6) with the function f', to get

$$\begin{aligned} 0 &= f'([d_{n-1}d_1], d_{-n}) + f'([d_1 d_{-n}], d_{n-1}) + f'([d_{-n}d_{n-1}], d_1) \\ &= (n - 2)f'(d_n, d_{-n}) + (1 + n)f'(d_{1-n}, d_{n-1}) + (1 - 2n)f'(d_{-1}, d_1) \\ &= (n - 2)f'(d_n, d_{-n}) + (1 + n)f'(d_{1-n}, d_{n-1}) \end{aligned}$$

so by induction f' is uniquely determined by $f'(d_2, d_{-2})$, i.e. by a single parameter. This shows that $H^2(W, \mathbb{C})$ is at most one-dimensional. To show that it is exactly one-dimensional it is sufficient to find a 2-cocycle that is not a 2-coboundary. As such, we can take for instance f defined by

$$f(d_n, d_{-n}) = n - n^3, \quad f(d_m, d_n) = 0, \quad m \neq -n$$

This is indeed a skew symmetric bilinear form. We show that it satisfies equation (6)

$$\begin{aligned} &f([d_k d_m], d_n) + f([d_m d_n], d_k) + f([d_n d_k], d_m = \\ &\quad (k-m) f(d_{k+m}, d_n) + (m-n) f(d_{m+n}, d_k) + (n-k) f(d_{n+k}, d_m) \end{aligned}$$

In order for not every term to be zero, we assume without loss of generality that $k = -m - n$, so the expression above becomes

$$(-2m - n) f(d_{-n}, d_n) + (m-n) f(d_{m+n}, d_{-m-n}) + (2n+m) f(d_{-m}, d_m)$$

which is equal to 0 as shown in equation (5). However, if f is a 2-coboundary then there is some linear map $g : W \to \mathbb{C}$ such that $f(x, y) = g([xy])$. But then we would have

$$2f(d_1, d_{-1}) = 4g(d_0) = f(d_2, d_{-2})$$

which is not the case. Hence, f is not a 2-coboundary, which concludes the proof. □

Remark 5.2. The above lemma serves as an example showing that Whitehead's second lemma does not hold in the infinite dimensional case. Indeed, the Witt algebra is simple by Lemma 5.2 so in particular it is semisimple, and $\mathbb{C}$ is a finite dimensional W-module. The only criterion that is not satisfied is finite dimensionality of W, but it is enough for $H^2(W, \mathbb{C})$ to be non-zero.

Remark 5.3. The fact that the same equations show up in the proofs of Lemmas 5.3 and 5.6 is no coincidence. In fact, if we in Lemma 5.3 drop the requirements on the gradation we will get the commutation relations

$$[L_m L_n] = (m-n) L_{m+n} + \alpha(L_m, L_n) c, \quad [c L_n] = 0$$

for all $m, n \in \mathbb{Z}$, where $\alpha : W^2 \to \mathbb{C}$ is a bilinear map. Since the bracket is skew symmetric, so is α. Hence, α is a 2-cochain. Further, by the Jacobi identity on the commutation relations above, we get equation (6). This shows that every central extension determines uniquely a 2-cocycle.

Lemma 5.7. *Let V_1, V_2 be two central extensions of the Witt algebra by $\mathbb{C}$ determined by the 2-cocycles α_1, α_2, respectively. If α_1 and α_2 are cohomologous, then V_1 and V_2 are isomorphic.*

Proof. The function $g = \alpha_2 - \alpha_1$ is a 2-coboundary. Then $g(x, y) = h([xy])$ for some linear map $h : W \to \mathbb{C}$. Define the bijective map $\varphi : W_1 \to W_2$ by

$$\varphi(L_k) = L_k + h(L_k) c, \quad \varphi(c) = c$$

We show that φ is a homomorphism. Indeed,

$$\begin{aligned}\varphi([L_m L_n]_1) &= (m-n)(L_{m+n} + h(L_{m+n})c) + \alpha_1(L_m, L_n)c \\ &= (m-n)L_{m+n} + g(L_m, L_n)c + \alpha_1(L_m, L_n)c \\ &= (m-n)L_{m+n} + \alpha_2(L_m, L_n)c = [\varphi(L_m)\varphi(L_n)]_2\end{aligned}$$

as required. □

Lemma 5.8. *The Virasoro algebra is the unique non-trivial central extension of the Witt algebra by* $\mathbb{C}$*, up to isomorphism.*

Proof. As shown in the proof of Lemma 5.6, the cohomology space $H^2(W, \mathbb{C})$ is spanned by the map defined by

$$\alpha(L_n, L_{-n}) = n - n^3, \quad \alpha(L_m, L_n) = 0, \quad m \neq -n$$

By Lemma 5.4 the extensions corresponding to these 2-cocycles are isomorphic to the Virasoro algebra, except in the trivial case. But the 2-cocycle of any central extension differs from one of the maps above by a 2-coboundary, so by Lemma 5.7 any non-trivial central extension is isomorphic to the Virasoro algebra. □

Theorem 5.1. *The Virasoro algebra has no Levi decomposition.*

Proof. Since the Witt algebra is simple, we have that $\mathrm{Rad}V = \mathbb{C}c$ which means that $W = V/\mathrm{Rad}V$. If we follow the proof of Levi's theorem, we construct the W-module $\mathbb{C}$ by multiplication in V, i.e. $xy = [xy] = 0$ for $x \in W, y \in \mathbb{C}c$. This means that the module we are interested in is in fact the module we have been investigating for most of this chapter. Using the fact that the radical is abelian, we can follow the proof of Levi's theorem perfectly. In this proof, we showed that the existence of a Levi decomposition was equivalent to Whitehead's second lemma, i.e. that $H^2(W, \mathbb{C}) = 0$. However, according to Lemma 5.6 this is not the case, which concludes the proof. □

This final result proves what we set out to do, namely that the assumption of finite dimensionality is crucial for Levi's theorem to hold. This hints towards the fact that the study of infinite dimensional Lie algebras require more involved techniques than what is presented in this paper. With this final remark, we conclude the article.

Appendix

In the appendix, we will give the proof of Theorem 3.2. The proof is not difficult, merely technical. Therefore, the reader is encouraged to check the cancellation, since it will most likely be easier to follow than the messy calculations below. Nevertheless, the proof is included for completeness.

Proof. We will start by proving the 0-dimensional case. Let $f \in C^0$ be given by $f(x) = m$. Then as remarked above, $df(x) = xm$, and so we find that

$$d^2 f(x_1, x_2) = x_1 x_2 m - x_2 x_1 m - [x_1 x_2]m = 0$$

by the definition of an L-module.

We proceed to the higher dimensional case. Let $f \in C^n$. The following expression is a mess, but as we will see, it all cancels

$$\begin{aligned}
d^2 f(x_1, ..., x_{n+2}) &= \sum_{j=1}^{n+2} (-1)^{j+1} x_j df(x_1, ..., \hat{x}_j, ..., x_{n+2}) \\
&+ \sum_{1 \le i < j \le n+2} (-1)^{i+j} df([x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2}) \\
&= \sum_{1 \le i < j \le n+2} (-1)^{i+j} x_j x_i f(x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2}) && (1) \\
&+ \sum_{1 \le i < j \le n+2} (-1)^{i+j+1} x_i x_j f(x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2}) && (2) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k+1} x_k f([x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (3) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k} x_j f([x_i x_k], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (4) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k+1} x_i f([x_j x_k], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (5) \\
&+ \sum_{1 \le i < j \le n+2} (-1)^{i+j} [x_i x_j] f(x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2}) && (6) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k} x_k f([x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (7) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k+1} x_j f([x_i x_k], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (8) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k} x_i f([x_j x_k], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (9) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k+1} f([[x_j x_k] x_i], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (10) \\
&+ \sum_{1 \le i < j < k \le n+2} (-1)^{i+j+k} f([[x_i x_k] x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ... x_{n+2}) && (11)
\end{aligned}$$

$$+ \sum_{1\le i<j<k\le n+2} (-1)^{i+j+k+1} f([[x_i x_j] x_k], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ...x_{n+2}) \quad (12)$$

$$+ \sum_{1\le i<j<k<l\le n+2} (-1)^{i+j+k+l} f([x_i x_j], [x_k x_l], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ..., \hat{x}_l, ...x_{n+2}) \quad (13)$$

$$+ \sum_{1\le i<j<k<l\le n+2} (-1)^{i+j+k+l+1} f([x_i x_k], [x_j x_l], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ..., \hat{x}_l, ...x_{n+2}) \quad (14)$$

$$+ \sum_{1\le i<j<k<l\le n+2} (-1)^{i+j+k+l} f([x_i x_l], [x_j x_k], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ..., \hat{x}_l, ...x_{n+2}) \quad (15)$$

$$+ \sum_{1\le i<j<k<l\le n+2} (-1)^{i+j+k+l} f([x_j x_k], [x_i x_l], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ..., \hat{x}_l, ...x_{n+2}) \quad (16)$$

$$+ \sum_{1\le i<j<k<l\le n+2} (-1)^{i+j+k+l+1} f([x_j x_l], [x_i x_k], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ..., \hat{x}_l, ...x_{n+2}) \quad (17)$$

$$+ \sum_{1\le i<j<k<l\le n+2} (-1)^{i+j+k+l} f([x_k x_l], [x_i x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., \hat{x}_k, ..., \hat{x}_l, ...x_{n+2}) \quad (18)$$

From here, it is despite the mess fairly easy to see that everything cancels out. Lines (1), (2) and (6) cancel out since $x_j x_i - x_i x_j = -[x_i x_j]$, lines (3), (4) and (5) cancel lines (7), (8) and (9), lines (10), (11) and (12) cancel by the Jacobi identity, and lines (13), (14), and (15) cancel lines (18), (17) and (16) respectively, by skew symmetry. This concludes the proof. □

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