# The Two-Envelope Problem: A Numerical Simulation

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# Abstract

We study a version of the two-envelope problem where a host presents two indistinguishable envelopes to a player. a player is informed that the monetary content of one of the envelopes is twice that of the other and is then allocated one of the envelopes. The player decides whether or not to keep the allocated envelope knowing the content of the allocated envelope and the probability distribution of how the lower value of the envelopes is generated. We obtains conditions for switching envelopes for continuous and discrete distributions and show optimality for each strategy's expected benefit. Numerical simulations for 10,000 instances of the two-envelope game were performed for a sample continuous and a sample discrete distribution. The switching strategy's cumulative and average winnings exceed the non-switching strategy's winnings in both scenarios.
In both the discrete and continuous cases, knowing the distribution of the initial amount and the allocated amount leads to the optimal strategy for switching envelopes.

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# Chapter 1

# Introduction

A host presents two indistinguishable envelopes to a player. One of the envelopes is randomly selected and allocated to the player. Before any interaction with the envelopes, the player is informed that the monetary content of one of the envelopes is twice that of the other. The player is then given the option of retaining the allocated envelope or switching envelopes in order to secure the envelope that contains the larger amount. The aim is to devise a switching strategy (conditions under which it is beneficial to switch the allocated envelope for the complementary one) to realise the greatest expected benefit for each play of the game.

# Chapter 2

# Literature Review

The problem when stated in an ill manner, without any assumption and information, leads to a paradoxical argument. Let $y$ denote the positive amount that is in the allocated envelope. This means that the amount of money in the other envelope is either $2y$ or $\frac{y}{2}$. The probability that it is $2y$ is $\frac{1}{2}$ and the probability that it is $\frac{y}{2}$ is also $\frac{1}{2}$. So the expected amount for switching is

$$\frac{1}{2}\left(2y + \frac{y}{2}\right) = y + \frac{y}{4} = \frac{5y}{4}.$$

The player should swap envelopes since the expected value of the complementary envelope is greater than $y$. But what if the player is given another chance to swap envelopes? The same argument should lead to switching back again. The same argument will lead to an infinite loop of switching. Some may even construe this as the player may gain infinite money if the player is memory-less and is allowed to switched infinitely many times.[9] This is paradoxical as it says "It doesn't matter which envelope you choose initially - you should always switch".[10]

The error in the previous calculation lies in having the same variable $y$ in each term but meaning different things. In particular, the $2y$ represents the expected value in the other envelope only in the specific instance that the allocated envelope has the smaller amount. The other envelope has $\frac{y}{2}$ as its expected value only in the specific instance that the larger amount is in the allocated envelope.

Let $y$ denote the observed amount that is in the allocated envelope and let $c$ be the amount in the complementary envelope. We derive the expected value of $c$, denoted $\mathbb{E}[c]$:

$$\begin{aligned}\mathbb{E}(c) &= \mathbb{E}[c \mid y < c]P(y < c) + \mathbb{E}[c \mid y > c]P(y > c) \\ &= \mathbb{E}[c \mid y < c]\frac{1}{2} + \mathbb{E}[c \mid y > c]\frac{1}{2}\end{aligned}$$

If $y$ were the lower amount then $c = 2y$. Otherwise $c = \frac{y}{2}$ if $y$ were the higher amount. Thus,

$$\begin{aligned}\mathbb{E}[c] &= \mathbb{E}[2y \mid y < c]\frac{1}{2} + \mathbb{E}\left[\frac{y}{2} \mid y > c\right]\frac{1}{2} \\ &= \mathbb{E}[y \mid y < c] + \frac{1}{4}\mathbb{E}[y \mid y > c]\end{aligned}$$

Denote the lower of the two amounts by $s$ then $y = s$ if $y$ were the lower amount. Otherwise $y = 2s$ if $y$ were the higher amount. Hence,

$$\mathbb{E}[c] = s + \frac{1}{4}(2s) = \frac{3}{2}s.$$

Thus the expected value of $c$ is $\frac{3}{2}s$. The same computation gives the same value for $\mathbb{E}[y]$. With the same expected values for $c$ and $y$, there is no reason to switch one envelope to the other.

Another argument leading to the same conclusion considers the total amount of money in both envelopes, denoted by $T$.[1] This amount is fixed and independent of the allocation of the envelope to the player. But the amount in the allocated envelope given that it is the smaller one is $\frac{T}{3}$. And the amount in the allocated envelope given that it is the larger one is $\frac{2T}{3}$. Then the expected gain for switching envelopes is

$$\frac{1}{2}\left(\frac{2T - T}{3}\right) + \frac{1}{2}\left(\frac{T - 2T}{3}\right) = 0.$$

This means there is no advantage or disadvantage to switching, as we have shown previously.

# Chapter 3

# A Strategy Using Randomness

In the previous calculations, the player knows nothing about the amounts in the envelopes except that one amount is twice the other amount. What if the player opens the allocated envelope and observes the amount $y$ before deciding whether to switch or not?

This added piece of information suffices to construct a strategy that improves the player's outcome. If it does not, then it will not matter whether the player sticks with the allocated envelope or switches to the complementary envelope (or it doesn't matter which envelope was allocated at the start).

It can be shown that a strategy that uses randomness and conditioned on the allocated value $y$ leads to the probability of ending up with the larger amount being greater than $\frac{1}{2}$.[7]

Suppose the envelopes contain integer amounts. Then for an observed value $y$, we have the following strategy:

1. Flip a fair coin until it lands tails.

2. If there were at least $y$ flips, switch envelopes. Otherwise, do not switch.

Denote the smaller amount be $s$ and the larger amount be $\ell$. There are two cases for a player to end up with $\ell$:

**Case 1: Allocated envelope has $y = s$ then decision to switch is made.** The probability that this happens is

$$P[y = s \text{ then switch}] = \frac{1}{2}\left(\frac{1}{2^s}\right).$$

**Case 2: Allocated envelope has $y = \ell$ then decision is not to switch.** The probability this happens is

$$P[y = \ell \text{ then did not switch }] = \frac{1}{2}\left(1 - \frac{1}{2^\ell}\right).$$

Hence, the chance to end up with $B$ is

$$\frac{1}{2} + \frac{1}{2^{s+1}} - \frac{1}{2^{\ell+1}}$$

which is indeed larger than $\frac{1}{2}$ since $\frac{1}{2^{s+1}} > \frac{1}{2^{\ell+1}}$ for $s < \ell$.

With the previous strategy depending on flipping coins, it fails to distinguish between non-integer real numbers for $A$ and $B$, like $A = 2.35$ and $B = 2.39$. To remedy this situation with real values, the player samples from a continuous probability density distribution that is a strictly positive over any interval in $\mathbb{R}$, like the standard normal distribution $N(0, 1)$. Suppose the probability distribution function is zero over some interval $[a, b]$, then it could be the case that both $s, \ell \in [a, b]$. If $s$ and $\ell$ were both in the interval of probability zero then $P(w \geq s) = P(w \geq \ell)$. Hence, the probability distribution function must be positive over any interval in $\mathbb{R}$.

The strategy for arbitrary real values $s$ and $\ell$ in the envelopes with $s < \ell$ and observed value $y$ is:

1. Sample $w$ from $N(0, 1)$ (or from any continuous probability density distribution that is a strictly positive over any interval).

2. If $w \geq y$. switch. Otherwise, do not switch.

Similar to the integer case, there are two ways of ending up with the larger amount $\ell$. Hence the chance of ending with the larger amount $\ell$ is

$$\frac{1}{2}P(w \geq s) + \frac{1}{2}(1 - P(w \geq \ell)) = \frac{1}{2} + \frac{1}{2}(P(w \geq s) - P(w \geq \ell)).$$

The right side of the previous equation is greater than $\frac{1}{2}$ because $P(w \geq s) > P(w \geq \ell)$ for $s < \ell$.

The intuition behind this switching strategy is that the player does not switch when $w \leq y$ because the value in the allocated envelope is "large" and the player might think it is unlikely that the complementary envelope contains the even larger amount. Similarly, the player chooses to switch when $w \geq y$ because the allocated value could be construed to be "small".

# Chapter 4

# Mathematical Formulation Leading to Switching Strategies

We look further into $\mathbb{E}[c|y = a]$, the expected value of the complementary envelope given the amount $a \in (0, \infty)$ is observed by the player, where the player knows the probability distribution $P(x)$ from which $a$ was sampled from. If $\mathbb{E}[c|y = a] > a$ then switching will be beneficial to the player.

In order to form a better switching strategy, the player should have knowledge of the host's actions. The host of the game is responsible for identifying and assigning amounts to each of the envelopes and allocating an envelope to the player. At the start, the host samples an initial amount from a probability distribution function and assigns the amount to the first envelope.

The information available to the player (who is assumed to be rational and risk-neutral) is listed below [11]:

1. The allocated amount $y$ that is observed by the player
2. The host's strategy for generating the contents of the envelopes
3. The host's strategy for allocating an envelope to the player
4. The distribution function from which the host sampled the initial amount. Otherwise, the player may assume a prior distribution and may refine the prior distribution with each succeeding instance of the game. The density function under consideration for each instance will be taken as given whether or not the player has assumed or derived such a prior distribution.
5. The bounds (if they exist) for the amounts that can be assigned to the envelopes.

From the last item in the list of information known to the player, it is reasonable to examine the case where the host has an infinite amount from which the contents of envelopes are sourced. In that were the case, since the average of all numbers between zero to infinity is infinity, the expected value of the first envelope is infinity. If follows that the expected value of the second envelope is also infinity. Tanke [9] notes that this leads to the paradox of always having to switch. It is also noted that the law of total expectation may not be valid in some cases where the two envelopes both have infinite expected values, it may be the case that two different partitions of the event space give rise to a positive conditional expected gain from a switch in one partition and zero in another partition [12]. Thus, if we take $S$ to be the random variable denoting the lowest value of the envelopes then we may take $\mathbb{E}[S]$ to be finite.

The player can consider the outcome of the host's actions as a triple $\{x_1; \omega_2; \omega_3\}$. The value $x_1$ is the amount assigned to the first envelope and is sampled from a probability distribution $f_{X_1}$ and is between $x_l$ and $x_u$ if bounds exist for $x_1$. We let $\omega_2$ denote how the content of the second envelope is generated, which we obtain by tossing an unbiased coin and enforcing the rule

$$\omega_2 = \begin{cases} 0, & \text{half the initial amount;} \\ 1, & \text{double the initial amount.} \end{cases}$$

with probabilities $P_2(0)$ and $P_2(1)$.

Finally, $\omega_3$ represents how an envelope is allocated to the player from the rule

$$\omega_3 = \begin{cases} 0, & \text{first envelope is allocated ;} \\ 1, & \text{second envelope is allocated.} \end{cases}$$

with probabilities $P_3(0)$ and $P_3(1)$.

Since the outcomes $x_1$, $\omega_2$, and $\omega_3$ are independent from each other, we assume that

$$P\left(\{x_1; \omega_2; \omega_3\}\right) = f_{X_1}\left(x_1\right) P_2\left(\omega_2\right) P_3\left(\omega_3\right) \tag{4.1}$$

We now look at the simplest case where the host generates an amount $s$, allocates this amount to the player, and assigns $2s$ to the other complementary envelope. In our notation, the host's actions are denoted by $\{s; 1; 0\}$, the observed value is $y = s$, and the complementary value is $c = 2s$. In this case,

$$\begin{aligned}
\mathbb{E}[c] &= \mathbb{E}[c|y < c]P(y < c) + \mathbb{E}[c|y > c]P(y > c) \\
&= \mathbb{E}[c|y < c]P(s < 2s) + \mathbb{E}[c|y > c]P(s > 2s) \\
&= \mathbb{E}[c|y < c] \cdot (1) + \mathbb{E}[c|y > c] \cdot 0 \\
&= \mathbb{E}[c|y < c] \\
&= \mathbb{E}[2s|s < 2s] \\
&= 2s
\end{aligned}$$

With the expected value of the complementary envelope being $2s$, switching gives a gain of $s$. Thus, it is always beneficial to switch if the host allocates the generated value to envelope allocated to the player and assigns double to the complementary envelope.

One can ask: "What can we do if we know the probability of the event that the allocated amount is the larger amount given that the allocated amount is $a$?" Let us denote this probability by $P(y > c|y = a)$ and suppose this probability is $p$. Then we compute for the complementary envelope's expected value:

$$\mathbb{E}[c \mid y = a] = \frac{a}{2}P(c = \frac{a}{2} \mid y = a) + 2aP(c = 2a \mid y = a).$$

Since $P(y > c \mid y = a) = P(c = \frac{a}{2} \mid y = a)$,

$$\begin{aligned}\mathbb{E}[c \mid y = a] &= \frac{a}{2}p + 2a(1-p) \\ &= a(2 - \frac{3}{2}p).\end{aligned}$$

Thus, the expected value of the complementary envelope is less than $a$ when $p > \frac{2}{3}$. Consequently, switching is not advised if $p > \frac{2}{3}$. We see that knowing $P(y > c|y = a)$ leads us to a switching strategy.

Let us examine $P(y > c|y = a)$ in the case where the player knows that the host only doubles the generated amount (meaning $\omega_2 = 1$) and that there is equal probability of being allocated the higher or lower amounts (meaning $P(y = l) = P(y = s) = \frac{1}{2}$). In our notation, we have either $\{a; 1; 0\}$ or $\{a; 1; 1\}$, where the observed value is $y = a$, and the complementary value is either $c = 2a$ or $c = \frac{a}{2}$.

Recalling the following from Bayes' theorem

$$P(\beta \mid \alpha) = \frac{P(\beta)P(\alpha \mid \beta)}{P(\beta)P(\alpha \mid \beta) + P(\beta^c) P(\alpha \mid \beta^c)},$$

it follows that

$$\begin{aligned}P(y > c|y = a) &= P(y = l|y = a) \\ &= \frac{P(y = l)P(y = a \mid y = l)}{P(y = l)P(y = a \mid y = l) + P((y = l)^c) P(y = a \mid (y = l)^c)} \\ &= \frac{0.5P(\ell = a)}{0.5P(\ell = a) + 0.5P(\ell = 2a)} \\ &= \frac{P(\ell = a)}{P(\ell = a) + P(\ell = 2a)}.\end{aligned}$$

Now, note that $P(l = a) = P(s = \frac{a}{2})$ since the probability that the observed value $y = a$ is the larger amount is the same as the probability that the smaller amount is $\frac{a}{2}$. Equivalently, $P(l = 2a) = P(s = a)$. Thus, the conditional probability that the allocated amount $y$ is greater than the complementary amount $c$ given that the player observes the allocated amount is $y = a$ can be computed using the probability density function for generating the smaller value, $f_s()$. Thus this conditional probability is given by [6]:

$$P(y > c \mid y = a) = \frac{f_s(\frac{a}{2})}{f_s(\frac{a}{2}) + f_s(a)}.$$

If follows that we also have

$$P(y < c|y = a) = \frac{f_s(a)}{f_s(\frac{a}{2}) + f_s(a)}.$$

We now compute for the expected value in the complementary envelope:

$$\begin{aligned}\mathbb{E}[c|y=a] &= \mathbb{E}[c|y=a]P(y<c|y=a) + \mathbb{E}[c|y>c]P(y>c|y=a)\\ &= 2a\cdot\frac{f_s(a)}{f_s(\frac{a}{2})+f_s(a)} + \frac{a}{2}\cdot\frac{f_s(\frac{a}{2})}{f_s(\frac{a}{2})\ +\ f_s(a)}\\ &= \frac{2af_s(a)+\frac{a}{2}f_s(\frac{a}{2})}{f_s(\frac{a}{2})+f_s(a)}\end{aligned}$$

It is beneficial to switch if $\mathbb{E}[c|y=a] > a$, or equivalently,

$$\begin{aligned}\frac{2af_s(a)+\frac{a}{2}f_s(\frac{a}{2})}{f_s(\frac{a}{2})+f_s(a)} &> a\\ 2af_s(a)+\frac{a}{2}f_s(\frac{a}{2}) &> a(f_s(\frac{a}{2})+f_s(a))\\ f_s(a) &> \frac{1}{2}f_s(\frac{a}{2})\\ 2f_s(a) &> f_s(\frac{a}{2})\end{aligned}$$

*Example*: Let $q\in(0,1)$ and consider the discrete distribution that attains values at powers of 2 and vanishes elsewhere:

$$P(2^t) = \frac{q^t}{1-q}\ t\in\{0,1,2,\dots\}.$$

In this arrangement, a player can deduce the contents of both envelopes only if the observed allocated value is 1. Switching may bring a gain or a loss for any other value. So for an observed value $a=2^t$, we have

$$\begin{aligned}P(\frac{a}{2}) &= P(\frac{2^t}{2}) = P(2^{t-1}) = \frac{q^{t-1}}{1-q}\\ 2P(a) &= 2P(2^t) = 2\frac{q^t}{1-q} = 2q\frac{q^{t-1}}{1-q} = 2qP(\frac{a}{2})\end{aligned}$$

Hence, $2P(a) > P(\frac{a}{2})$ if and only if $q>\frac{1}{2}$.

# Chapter 5

# Computing the Benefit for the Continuous Cases

It is apparent from the previous computations that discrete random variables were considered. Tyler [11] considers a variant where the observed amount is centered on an interval with endpoints $y-\varepsilon$ and $y+\varepsilon$, the length of which goes to zero. These intervals are considered to take advantage of the ease of dealing with continuous variables and probabilities associated with intervals.

We can use the following to exploit the limit as $\varepsilon$ goes to zero:

$$\begin{aligned} f_X(x) &= \frac{dF_X(x)}{dx} \\ &= \lim_{\varepsilon \to 0} \frac{F_X(x+\varepsilon) - F_X(x-\varepsilon)}{2\varepsilon} \end{aligned}$$

Thus,

$$\begin{aligned} f_X(x)2\varepsilon &\approx F_X(x+\varepsilon) - F_X(x-\varepsilon) \\ &= P(x-\varepsilon \le X \le x+\varepsilon) \\ &= P(X \le x+\varepsilon) - P(X \le x-\varepsilon) \\ &= \int_{y-\varepsilon}^{y+\varepsilon} f_{X_1}(x_1)\, dx_1 \end{aligned}$$

The intuition using these intervals comes from imagining that the observed amount $a$ in the allocated envelope is rounded off in such a way that differences smaller than an infinitesimal quantity $h$ are not noticeable, even though actually it varies continuously [5]. The probability that the smaller amount is in an interval around $a$ of length $h$ and that the allocated amount is the smaller amount is approximately $f(a)h(\frac{1}{2})$. The probability that the larger amount is in an interval around $a$ of length $h$ corresponds to the smaller amount being in an interval of length $\frac{h}{2}$ around $\frac{a}{2}$. Hence the probability that the larger amount of money is in a small interval around $a$ of length $h$ and the allocated amount is the larger amount is approximately $f(\frac{a}{2})(\frac{h}{2})(\frac{1}{2})$. Thus, if the allocated amount is observed to be $a$, the probability the allocated amount is the smaller amount is roughly

$$\frac{hf(a)(\frac{1}{2})}{f(\frac{a}{2})(\frac{h}{2})(\frac{1}{2}) + f(a)(h)(\frac{1}{2})} = \frac{2f(a)}{f(\frac{a}{2}) + 2f(a)}.$$

We denote by $B(\{c\})$ the benefit of switching envelopes when the player experiences the event $c = \{x_1 : \omega_2, \omega_3\}$. This is the expected value of the the complementary envelope minus the value of the allocated envelope. Thus, if the host generates the initial value to be $a$ then we have

$$b(\{a : 1 : 1\}) = a \text{ and } b(\{a : 1 : 0\}) = -a.$$

If we consider the sets

$$C_a = \bigcup_{\omega_3 \in \{0;1\}} \left\{ \{x_1, 1, \omega_3\} : 2^{-\omega_3}(a - \varepsilon) < x_1 \leq 2^{-\omega_3}(a + \varepsilon) \right\},$$

it follows from (4.1) that expected benefit from switching envelopes given that the player observes the allocated value $a$ is

$$\begin{aligned}
E(B \mid y = a) &= \lim_{\varepsilon \to 0} E(B \mid a - \varepsilon < y \leq a + \varepsilon) \\
&= \lim_{\varepsilon \to 0} E\left(B \mid C_a\right) \\
&= \lim_{\varepsilon \to 0} \frac{\int_{C_a} b(\{c\}) f_{X_1}(x_1)\, dx_1}{\int_{C_a} f_{X_1}(x_1)\, dx_1} \\
&= \lim_{\varepsilon \to 0} \frac{-\int_{\frac{a}{2}-\frac{\varepsilon}{2}}^{\frac{a}{2}+\frac{\varepsilon}{2}} x_1 f_{X_1}(x_1)\, dx_1 + \int_{a-\varepsilon}^{a+\varepsilon} x_1 f_{X_1}(x_1)\, dx_1}{\int_{\frac{a}{2}-\frac{y}{2}}^{\frac{a}{2}+\frac{a}{2}} f_{X_1}(x_1)\, dx_1 + \int_{a-\varepsilon}^{a+\varepsilon} f_{X_1}(x_1)\, dx_1} \\
&= \lim_{\varepsilon \to 0} \frac{-\frac{a}{2} f_{X_1}\left(\frac{a}{2}\right)\varepsilon + a f_{X_1}(a) 2\varepsilon}{f_{X_1}\left(\frac{a}{2}\right)\varepsilon + f_{X_1}(a) 2\varepsilon} \\
&= \frac{-\frac{a}{2} f_{X_1}\left(\frac{a}{2}\right) + 2a f_{X_1}(a)}{f_{X_1}\left(\frac{a}{2}\right) + 2 f_{X_1}(a)} \qquad (5.1)
\end{aligned}$$

Since this is the benefit from switching envelopes, the player should switch if $E(B \mid y = a)$ is positive. The previous equation implies that switching should happen if $-\frac{a}{2} f_{X_1}\left(\frac{a}{2}\right) + 2a f_{X_1}(a) > 0$, or equivalently

$$4 f_{X_1}(a) > f_{X_1}\left(\frac{a}{2}\right).$$

*Example:* Consider the distribution function $f_{X_1}(x_1) = \frac{1}{(x_1+1)^2} \quad \forall x_1 > 0$.

Using (5.1),

$$\begin{aligned}
E(B \mid Y = y) &= \frac{-\frac{y}{2} f_{X_1}\left(\frac{y}{2}\right) + 2y f_{X_1}(y)}{f_{X_1}\left(\frac{y}{2}\right) + 2 f_{X_1}(y)} \\
&= y \frac{(y+2)^2 - (y+1)^2}{(y+2)^2 + 2(y+1)^2} \\
&= \frac{2y^2 + 3y}{3y^2 + 8y + 6}
\end{aligned}$$

This value is positive for all positive values of $y$, implying a positive benefit from switching envelopes if the initial amount is selected using the probability distribution function $f_{X_1}(x_1) = \frac{1}{(x_1+1)^2} \quad \forall x_1 > 0$. Therefore, we should always switch for any amount observed.

From the previous example, it is natural to ask "Doesn't this lead to the paradox of always switching?" This led to defining paradoxical distributions [3]:

**Definition 5.0.1.** *A distribution $f$ on $X$ is* ***paradoxical*** *if*

1. $4f(x) > f\left(\frac{x}{2}\right)$ *for all* $x \in \operatorname{supp}(X)$ *for continuous distribution* $f$
2. $2f(x) > f\left(\frac{x}{2}\right)$ *for all* $x \in \operatorname{supp}(X)$ *for discrete distribution* $f$

Broome noticed that if a distribution has finite expectation then the distribution is not paradoxical [3]. Tzur and Jacobi later proved the contrapositive for the discrete case [12], which we give here:

**Lemma 5.0.2.** *Let $f$ be a discrete paradoxical distribution on $X$, then the expectation $\boldsymbol{E}[X]$ is infinite.*

*Proof.* Let $x$ be a possible realization of $X$, then $2^k x$ with $k \in \mathbb{N}$ are possible realizations of $X$ in the setting of the two envelope problem.

Since $f$ is paradoxical, $2^{k+1}xf(2^{k+1}x) > 2^k xf(2^k x)$ holds for all $x \in \mathbb{N}$. Due to the fact that $2^k xf(2^k x)$ is increasing in $k$, $\sum_{k=0}^{\infty} 2^k xf(2^k x) = \sum_{k=0}^{\infty} kf(k) = \mathbb{E}[x]$. Implying an unbounded $\mathbb{E}[x]$. □

In our example for the discrete case, $f(2^t) = \frac{q^t}{1-q}$ $t \in \{0, 1, 2, \dots\}$ and zero elsewhere, we have a paradoxical distribution if $q \in (\frac{1}{2}, 1)$ because the sum of the geometric series is $+\infty$.

We give Broome's proof which holds both for continuous and discrete cases:

**Lemma 5.0.3.** *If a distribution has finite $\boldsymbol{E}[X]$ then the distribution is not paradoxical.*

*Proof.* Consider a bivariate distribution over $X$ and $Y$. To show that it is not paradoxical, we prove that there is an $X$ such that $\mathbb{E}[Y|X]$ is not greater than $X$, equivalently $\mathbb{E}[Y|X] - X$ is not always positive.

Since the difference is a function of $X$ and since $X$ is a random variable, we can take the expectation of the difference and then use the fact that the expectation of a difference is the difference in expectations to get

$$\mathbb{E}[\mathbb{E}[Y|X] - X] = \mathbb{E}[\mathbb{E}[Y|X]] - \mathbb{E}[X].$$

Since the expectation of a conditional expectation is the unconditional expectation,

$$\mathbb{E}[\mathbb{E}[Y|X]] - \mathbb{E}[X] = \mathbb{E}[Y] - \mathbb{E}[X].$$

Since the envelopes do not give a hint about which contains the larger amount, the distribution is symmetric, that is $\mathbb{E}[X] = \mathbb{E}[Y]$. It follows that

$$\mathbb{E}[\mathbb{E}[Y|X] - X] = \mathbb{E}[Y] - \mathbb{E}[Y] = 0.$$

This means that the difference of the expectation of $Y$ given $X$ and $X$ itself is zero on average. Therefore, if this difference is positive for some value of $X$ then it must be negative for some other value. □

# Chapter 6

# Maximizing the gain from the switching strategy

Suppose we know the probability density function from which the host picks the lower amount $A$ of the two envelopes. Can we maximize the gain from the previous switching strategies?

## 6.1 Optimal switching in the continuous case

McDonnell et al [4] gives an optimal strategy for a more general context. They look at a non-negative continuous random variable $X$, with finite mean, representing the smaller amount $X$ having probability $p$ of being allocated to the player and the larger amount $2X$ having probability $1-p$ of being allocated to the player, and with the observed value $y$:

**Theorem 6.1.1.** *Let*

$$g(x) = pf_X(x) - \frac{1-p}{4} f_X\left(\frac{x}{2}\right).$$

*Define the switching function on the amount $y$ to be*

$$P_{\mathrm{S}}^*(y) = \begin{cases} 1 & g(y) \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

*That is, if $P_S^*(y) = 1$ then the player must switch envelopes. Otherwise, the player stays with the allocated amount $y$. This switching function is optimal.*

*Proof.* We give McDonnell et al's proof of optimality [8][4].

Suppose on one instance of the game, an amount $x$ is selected from the distribution $f_X$ and placed in one envelope, while $2x$ is placed in the other envelope. With the amount observed by the player in the opened envelope denoted by $y$, let $P(y = x) = p$ and $P(y = 2x) = 1 - p$.

Let the probability of switching be a function of the observed amount $y$, i.e. $P_{\mathrm{S}}(y) \in [0, 1]$. The probability that the player ends the instance of the game with amount $x$ is

$$P_x = p\,(1 - P_{\mathrm{S}}(x)) + (1-p)P_{\mathrm{S}}(2x)$$

and the probability of finishing with $2x$ is

$$P_{2x} = pP_{\mathrm{S}}(x) + (1-p)\,(1 - P_{\mathrm{S}}(2x))\,.$$

Hence, the expected return for $X = x$ is

$$R(x) = xP_x + 2xP_{2x} \tag{6.1}$$
$$= x(2-p) + x\left(p\left(P_{\mathrm{S}}(x) + P_{\mathrm{S}}(2x)\right) - P_{\mathrm{S}}(2x)\right) \tag{6.2}$$
$$= x(2-p) + p\left(x\left(P_{\mathrm{S}}(x) + P_{\mathrm{S}}(2x)\right) - xP_{\mathrm{S}}(2x)\right) \tag{6.3}$$

The average return over the distribution of $X$ is then

$$\begin{aligned} R &= \int_0^\infty f_X(x)R(x)\mathrm{d}x \\ &= \int_0^\infty f_X(x)\left(x(2-p) + p\left(x\left(P_{\mathrm{S}}(x) + P_{\mathrm{S}}(2x)\right) - xP_{\mathrm{S}}(2x)\right)\right)\mathrm{d}x \\ &= (2-p)\mathrm{E}[X] + \int_0^\infty f_X(x)\left(p(x) + P_{\mathrm{S}}(2x)\right) - xP_{\mathrm{S}}(2x)\Big)\Big)\,\mathrm{d}x \end{aligned}$$

If the player never switches, then this strategy is simply $P_{\mathrm{S}}(y) = 0$ for all observed values $y$. Substituting this into the expression for $R$, the benchmark base return is

$$R_{\mathrm{B}} := (2-p)\mathrm{E}[X].$$

This allows us to define the average gain $G$ incurred by switching function $P_{\mathrm{S}}(y)$ to be

$$G := R - R_{\mathrm{B}} = \int_0^\infty xf_X(x)\left[pP_{\mathrm{S}}(x) - (1-p)P_{\mathrm{S}}(2x)\right]\mathrm{d}x.$$

If we take $g(\phi) = pf_X(\phi) - \frac{1-p}{4}f_X\left(\frac{\phi}{2}\right)$, then

$$G = \int_0^\infty \phi P_{\mathrm{S}}(\phi)g(\phi)\mathrm{d}\phi.$$

Changing variables allows us to rewrite $G$ as

$$G = \int_{\phi=0}^{\phi=\infty} \phi P_{\mathrm{S}}(\phi)\left[pf_X(\phi) - \frac{1-p}{4}f_X\left(\frac{\phi}{2}\right)\right]\mathrm{d}\phi.$$

We can interpret the average gain $G$ to be the difference between the average return $R$ when using the switching strategy $P_{\mathrm{S}}(y)$ and the benchmark base return $R_B$ when the player never switches.

The switching strategy $P_S^*(y)$ can be shown to be optimal via contradiction. Suppose another switching function is optimal, say there exists a $y_0$ such that $P_{\mathrm{S}}(y_0) \neq P_{\mathrm{S}}^*(y_0)$.

For $y_0$, we look at what $P_{\mathrm{S}}(y_0)$ contributes to $G$. There are two cases to consider:

**(1)** $pf_X(y_0) \geq \frac{1-p}{4}f_X\left(\frac{y_0}{2}\right)$**:** In this case, $P_{\mathrm{S}}^*(y_0) = 1$. Since $P_{\mathrm{S}}(y_0) \neq P_{\mathrm{S}}^*(y_0)$, the value of $P_{\mathrm{S}}(y_0)$ may be increased. This implies that $G$ can also be increased, which contradicts the optimality of $P_{\mathrm{S}}$.

**(2)** $pf_X(y_0) \geq \frac{1-p}{4} f_X\left(\frac{y_0}{2}\right)$**:** In this case, $P^*_{\mathrm{S}}(y_0) = 0$. Since $P_{\mathrm{S}}(y_0) \neq P^*_{\mathrm{S}}(y_0)$, the value of $P_{\mathrm{S}}(y_0)$ may be increased. This implies that G can also be increased, which contradicts the optimality of $P_{\mathrm{S}}$.

□

Brams and Kilgour [2] note that if $L$ is a random variable representing the larger amount and $S$ is a random variable representing the smaller amount then, due to the fact that $S = \frac{L}{2}$, a prior distribution for $L$ defines a prior distribution for $S$ such that

$$\mathrm{P}(L \leq x) = \Pr(S \leq \tfrac{x}{2}), 0 < x < \infty.$$

For our model where the envelope is allocated to the player uniformly, we have $p = \frac{1}{2}$. Hence, the theorem's condition that $g(y) > 0$ is equivalent to

$$4f_{X_1}(y) > f_{X_1}(\frac{y}{2}).$$

This is the same condition we have in our computations for the continuous case.

## 6.2 Optimal switching in the discrete case

For the case that $X$ is a discrete random variable with distribution $f_X$, Tanke [9] obtains an expression similar to (6.3) for the expected return for $X = x$ for the switching function $P_S(x)$:

$$\mathbb{E}[B \mid X = x] = x(2-p) + pxP_S(x) - x(1-p)P_S(2x)$$

The expected return over the distribution is therefore

$$\begin{aligned}\mathbb{E}[R] &= \sum_x f(x)\mathbb{E}[R \mid X = x] \\ &= (2-p)\mathbb{E}[X] + \sum_x xf(x)(pP_S(x) - (1-p)P_S(2x)).\end{aligned}$$

If $p = \frac{1}{2}$, we can define the average gain incurred by switching function $P_S(y)$ to be

$$G = \tfrac{1}{2}\sum_x xf(x)(P_S(x) - P_S(2x)).$$

We then use a change of variables to write

$$G = \frac{1}{2}\sum_\theta \theta P_S(\theta)(f(\theta) - \tfrac{1}{2}f(\tfrac{\theta}{2})).$$

This implies that for a positive gain on the observed value $y$, the condition that must be satisfied is

$$2f(y) > f(\frac{y}{2}).$$

This is also what we calculated previously in the discrete case.

# Chapter 7

# Switching Strategies for Selected Distributions

We present some calculations for some selected discrete and continuous distributions to demonstrate envelope switching strategies.

## 7.1 Uniform distribution

Suppose the smaller amount $s$ is uniformly distributed on the interval $(a, a+t)$ where $2a < \frac{a+t}{2}$, that is

$$f_s(x) = \begin{cases} \frac{1}{t} & a < x < a+t \\ 0 & \text{otherwise} \end{cases}$$

For the allocated amount $y$, the player should switch if $4f_s(y) \geq f(\frac{y}{2})$. We look at the possible values of $y$:

**$a < y \leq 2a$:** Then $4f(y) = \frac{4}{t}$. Since $\frac{a}{2}$ will be between $\frac{a}{2}$ and $a$, we have $f(\frac{y}{2}) = 0$. Hence the switching condition is satisfied and, thus switching should occur.

**$2a < y < \frac{a+t}{2}$:** Then $4f_s(y) = \frac{4}{t}$. Since $\frac{y}{2}$ will be between $a$ and $\frac{a+t}{4}$, we have $f_s(\frac{y}{2}) = \frac{1}{t}$. With the switching condition satisfied, switching should occur.

**$2a < y < a+t$:** Since $\frac{y}{2}$ will be between $a$ and $\frac{a+t}{2}$, we have $4f_s(y) = \frac{4}{t}$ and $f_s(\frac{y}{2}) = \frac{1}{t}$. With the switching condition satisfied, switching should occur.

**$a+t < y$:** Then $\frac{y}{2} > a + \frac{t}{2}$. If follows that $4f_s(y) = 0$ and $f(\frac{y}{2}) = \frac{1}{t}$. Hence, the player should not switch envelopes because switching condition is not satisfied. Unless $y$ is very close to $a + \frac{t}{2}$ then it´s more likely to get the larger amount.

## 7.2 A discrete distribution

Suppose the host randomly samples the smaller amount from the following distribution:

$$f_s(x) = \begin{cases} \frac{2^n}{3^{n+1}} & \text{if } x = 2^n \text{ where } n \text{ is a nonnegative integer} \\ 0 & \text{otherwise} \end{cases} .$$

For an observed allocated amount $y > 0$, switching is advised if $2f_s(y) > f_s(\frac{y}{2})$. Since $y = 2^k$ for some $k$, the switching condition is equivalent to

$$2\frac{2^k}{3^{k+1}} > \frac{2^{k-1}}{3^k}.$$

But this condition is always satisfied since we can simplify the inequality to $4 > 3$. This means we should always switch for any observed allocated amount $y$.

By definition, this distribution is paradoxical. One can check that the expectation is infinite since the sum $\sum_{n=0}^{\infty}(2^n \frac{2^n}{3^{n+1}})$ diverges.

## 7.3 Another discrete distribution

Suppose the host randomly samples the smaller amount from the following distribution:

$$f_s(x) = \begin{cases} \frac{9^{n-1}}{10^n} & \text{if } x \text{ is the positive integer } n \\ 0 & \text{otherwise} \end{cases}.$$

One can interpret this distribution as the host picks an integer $n$ with probability $\frac{9^{n-1}}{10^n}$.

For an observed allocated amount $y > 0$, switching is advised if $2f_s(y) > f_s(\frac{y}{2})$. Note that if $y$ is an odd integer for some $k$, then $f_s(\frac{y}{2}) = 0$. Consequently, the switching condition is satisfied when $y$ is an odd.

For even values of $y$, the switching condition is equivalent to

$$2\frac{9^{y-1}}{10^y} > \frac{9^{\frac{y}{2}-1}}{10^{\frac{y}{2}}},$$

which can be written as

$$2 > \frac{9^{\frac{y}{2}-1}10^y}{9^{y-1}10^{\frac{y}{2}}} = \left(\frac{10}{9}\right)^{\frac{y}{2}}.$$

After taking logarithms, we conclude that for even values of $y$, the player should switch envelopes if

$$y < \frac{2\ln 2}{\ln \frac{10}{9}} \approx 13.1576.$$

## 7.4 Weibull distribution

Suppose the host randomly samples the smaller amount from a Weibull distribution with parameters $k \geq 0$ and $\lambda > 0$ , that is

$$f_s(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(\frac{x}{\lambda})^k} \text{ for } x \geq 0.$$

For an observed allocated amount $y > 0$, switching is advised if $4f_s(y) > f_s(\frac{y}{2})$. Equivalently,

$$4\frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{k-1} e^{-(\frac{y}{\lambda})^k} > \frac{k}{\lambda}\left(\frac{y}{2\lambda}\right)^{k-1} e^{-(\frac{y}{2\lambda})^k}.$$

We simplify the inequality further,

$$2^2 \left(\frac{y}{\lambda}\right)^{k-1} e^{-(\frac{y}{\lambda})^k} > \frac{1}{2^{k-1}} \left(\frac{y}{\lambda}\right)^{k-1} e^{-(\frac{y}{2\lambda})^k}$$
$$2^{k+1} \left(\frac{y}{\lambda}\right)^{k-1} e^{-(\frac{y}{\lambda})^k} > \left(\frac{y}{\lambda}\right)^{k-1} e^{-(\frac{y}{2\lambda})^k}.$$

Since $\left(\frac{y}{\lambda}\right)^{k-1}$ and $e^{-(\frac{y}{\lambda})^k}$ are positive,

$$2^{k+1} > \frac{e^{-(\frac{y}{2\lambda})^k}}{e^{-(\frac{y}{\lambda})^k}}$$
$$2^{k+1} > e^{-(\frac{y}{\lambda})^k\left(\frac{1}{2^k}-1\right)}.$$

Taking logarithms,

$$(k+1)\ln 2 > -(\frac{y}{\lambda})^k \left(\frac{1-2^k}{2^k}\right).$$

Since $k > 0$, $1 - 2^k$ is negative. Thus we have

$$\frac{2^k \lambda^k (k+1)\ln 2}{2^k - 1} > y^k.$$

This means the player should switch if

$$y < 2\lambda \left(\frac{(k+1)\ln 2}{2^k - 1}\right)^{\frac{1}{k}}.$$

# Chapter 8

# Numerical Simulations

We perform numerical simulations to test the optimal switching strategies we have seen in the preceding sections. The computer code, with comments in the code, is provided in the appendix.

First, we recall the following optimal strategies for a player being allocated the value $y$ with the given probability distribution $f$ of generating the lower value in the two envelopes:

**Continuous Case:** Switch if $4f(y) > f(\frac{y}{2})$

**Discrete Case:** Switch if $2f(y) > f(\frac{y}{2})$

For the continuous case, we will look at the uniform distribution on the interval $[7, 37]$. For the discrete case, we will consider the distribution on the integers given by $f(n) = \frac{9^{n-1}}{10^n}$. The switching strategies for these chosen distributions were discussed in the preceding section.

In each case, we will compare the switching strategy with the strategy of never switching with 10,000 simulations of repeated instances of the two-envelope game. We will compare them three ways:

**Cumulative Winnings Difference:** We plot the difference of the cumulative winnings as a function of the number of simulations of the two-envelope game. At the n-th instance, we take the difference of the player's total winnings using the optimal switching strategy for all the n games played minus the total winnings for the n games played using the the never switching strategy. We then plot the point whose x-coordinate is $n$ and y-coordinate is the cumulative sum of the winnings differential for all of the n games.

**Average Winnings:** We plot the average winnings as a function of the number of simulations of the two-envelope game. For each of the two strategies, we take the average winnings over all of the n instances of the game (which is just the sum of all the winnings from the n games divided by the number of games n). We then plot the point whose x-coordinate is $n$ and y-coordinate is average winnings for the n games. The plots for both strategies are contained in the same axis for our comparison.

**Boxplots and Histograms of 100 Samples:** For each strategy, we will take 100 samples where each sample will have 10,000 instances of the two-envelope game. For each sample, we will take note of the total winnings over the 10,000 games in the sample. We then make a box plot to described the distribution of the winnings for 100 samples of each strategy. For completion, we will also supply the histogram for the samples.

## 8.1 Continuous Case

We give the following results for the uniform distribution on the interval $[7, 37]$:

**Cumulative Winnings Difference:** We see that as we increase n, the cumulative differential of the winnings increase in a linear fashion.

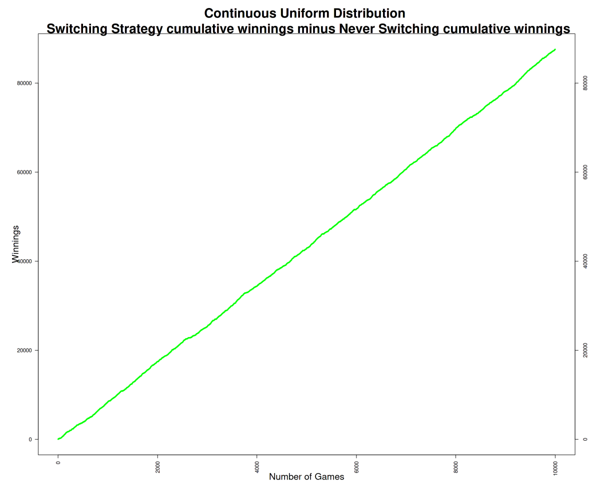


Figure 8.1: Continuous gain graph

This shows that, in the long run, the switching strategy cumulative winnings exceed the winnings from the never switching strategy.

**Average Winnings:** We see that the average winnings for the switching strategy exceeds the average winnings for the never switching strategy.

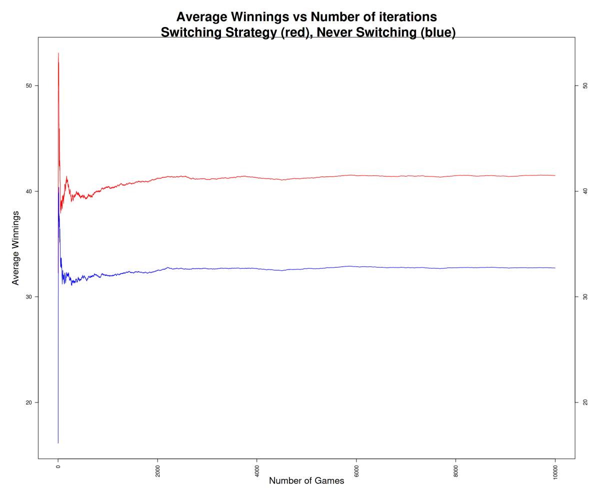


Figure 8.2: Average winnings vs number of iterations plot for continuous case

In the long run as the number of games increase, the difference between the average winnings seems like a constant (via visual inspection of the plot). This accounts for the somewhat linear plot we see for the cumulative winnings difference.

**Boxplots and Histograms of 100 Samples:** The boxplots show that the switching strategy's minimum value of the winnings over 100 samples is greater than the maximum value of the winnings of 100 samples of the never switching strategy.

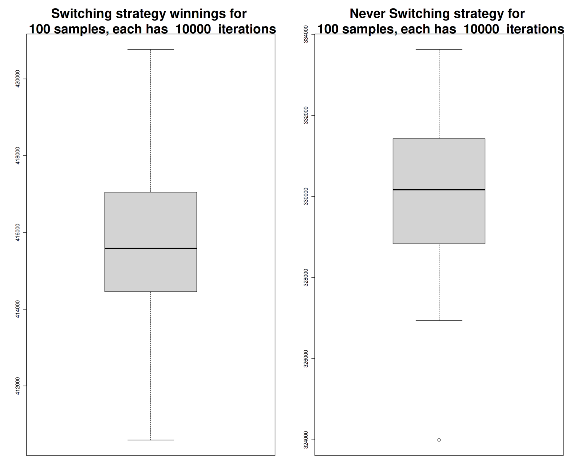


Figure 8.3: Box plot of 100 samples for continuous case.

The following shows the histogram of the 100 samples of each strategy:

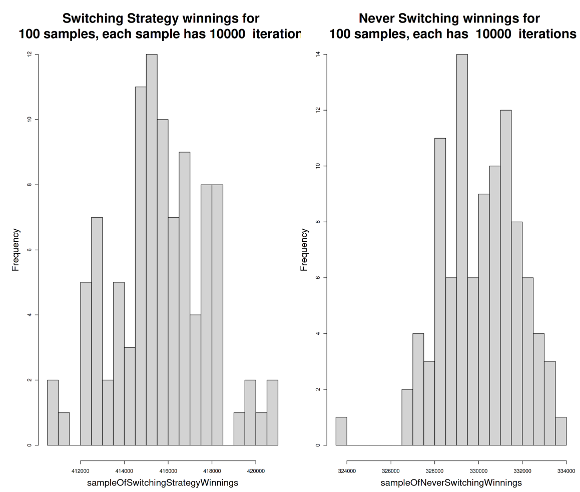


Figure 8.4: Histogram of 100 samples for continuous case.

## 8.2 Discrete Case

We give the following results for the the distribution on the integers given by $f(n) = \frac{9^{n-1}}{10^n}$.

**Cumulative Winnings Difference:** We see that as we increase n, the cumulative differential of the winnings increase in a linear fashion.

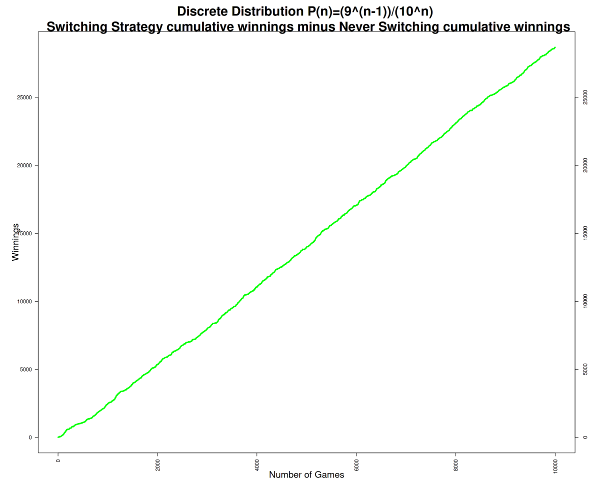


Figure 8.5: Discrete gain graph

This shows that, in the long run, the switching strategy cumulative winnings exceed the winnings from the never switching strategy.

**Average Winnings:** We see that the average winnings for the switching strategy exceeds the average winnings for the never switching strategy.

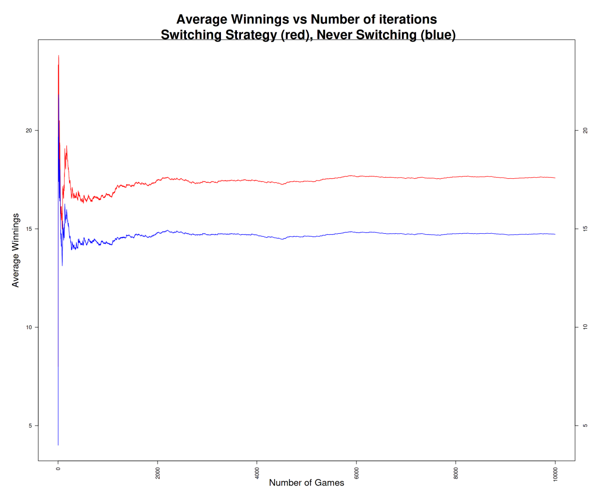


Figure 8.6: Average winnings vs number of iterations plot for discrete case

In the long run as the number of games increase, the difference between the average winnings seems like a constant (via visual inspection of the plot). This accounts for the somewhat linear plot we see for the cumulative winnings difference.

**Boxplots and Histograms of 100 Samples:** The boxplots show that the switching strategy's minimum value of the winnings over 100 samples is greater than the maximum value of the winnings of 100 samples of the never switching strategy.

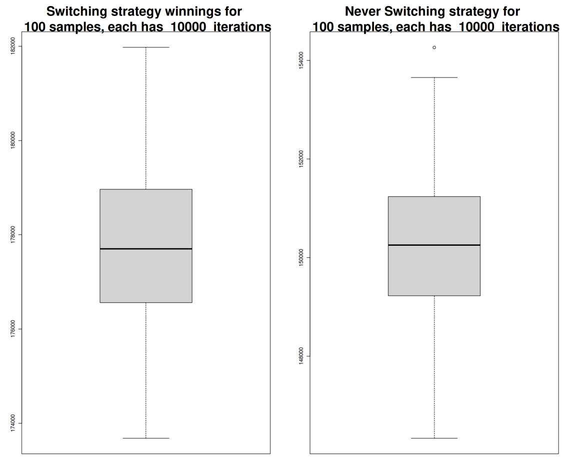


Figure 8.7: Box plot of 100 samples for discrete case.

The following shows the histogram of the 100 samples of each strategy:

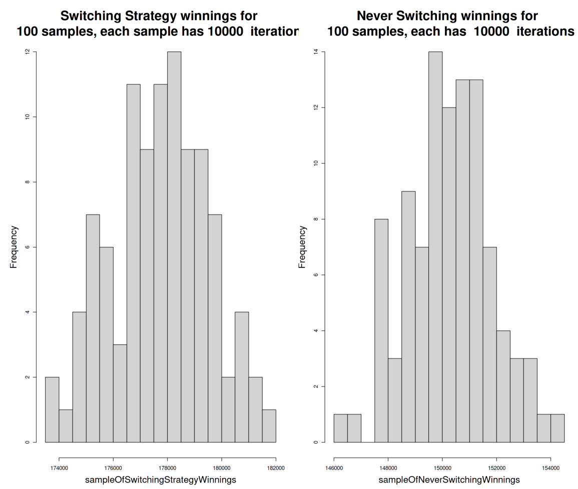


Figure 8.8: Histogram of 100 samples for discrete case.

# Chapter 9

# Conclusions

We have seen that knowing the contents of the allocated envelope provides actionable information for the player's decision whether or not to switch. Furthermore, knowing the distribution of the initial generated amount gives conditions that allow the player to decide to switch for a positive gain. However, one must also recognize whether the distribution is paradoxical.

We have also run numerical simulations for 10,000 instances of the two envelope game and compared the cumulative and average winnings of the optimal switching strategy and the never switching strategy. The switching strategy's cumulative and average winnings exceed the never switching strategy's winnings in both scenarios.

In both the discrete and continuous cases, knowing the distribution of the initial amount and the allocated amount leads to the optimal strategy for switching envelopes.

# References


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# Appendix A

# Computer Codes

The following codes were done in R.

## A.1 Code for the Continuous Distribution

```
set.seed(22)

# the following code is a simulation of playing the two-envelope problem with a uniform distribution
numberOfGames = 10000

#create data frame
envelopes <- data.frame(matrix(ncol = 0, nrow = numberOfGames))
#sample from a uniform distribution on an interval of length T with minimum value A and maximum value A+T where 2A < (A+T)/2
A = 7
T = 30
if (2*A < ((A+T)/2)) {
    print(2*A < ((A+T)/2))
    sampleFromUniformDistribution <- as.matrix(runif(numberOfGames, min = A, max = A+T))
    } else {
    sampleFromUniformDistribution <- 0
    }

#fill the envelopes, first column contains the sampled values, second column contains twice the sampled values
envelopes <- as.data.frame(cbind(sampleFromUniformDistribution,2*sampleFromUniformDistribution))
#envelopes #this line, if uncommented, checks the contents of the envelopes

#shuffle the entries of each row
perms<- t(replicate(nrow(envelopes), sample(ncol(envelopes))))
shuffledEnvelopes <- t(sapply(seq_len(nrow(perms)),
           function(i, perms, mat) mat[i, perms[i,]],
           mat = envelopes, perms = perms))
#each row of shuffledEnvelopes is an instance of the game, the first entry is the allocated envelope, and
#the second envelope is the complementary envelope

#allocate the first envelopes to the player and the complementary values
envelopes$allocatedValues <- shuffledEnvelopes[,c(1)]
envelopes$complementaryValues <- shuffledEnvelopes[,c(2)]
#envelopes#this line, if uncommented, checks the contents of the shuffled envelopes

#player knows that the host uses the uniform distribution as defined above
#player must check for the probability that the allocated value is selected
#this is easy since if allocated value is within the bounds of the distribution then probability is 1/t
#if allocated value is outside the bounds of the distribution then probability if zero
# if allocated value is greater than A+T or less than A then we do not switch
envelopesNotForSwitching <- which((envelopes$allocatedValues < A) | (envelopes$allocatedValues > (A + T)))
#envelopesNotForSwitching#this line, if uncommented, gives the rows where switching should not be done

#valuesAfterSwitchingStrategy[!c(envelopesNotForSwitching)] <- shuffledEnvelopes[!c(envelopesNotForSwitching),2]
envelopes$valuesAfterSwitchingStrategy <- shuffledEnvelopes[,2]
envelopes$valuesAfterSwitchingStrategy[c(envelopesNotForSwitching)] <- envelopes$allocatedValues[c(envelopesNotForSwitching)]
#envelopes

#get the sums of the non-switch strategy and the switching strategy
envelopes$valuesAfterSwitchingStrategy <- {as.numeric(envelopes$valuesAfterSwitchingStrategy)}
print('With the Switching Strategy, the total value of all the envelopes is ')
sum(as.vector(envelopes$valuesAfterSwitchingStrategy))
envelopes$allocatedValues <- {as.numeric(envelopes$allocatedValues)}
print('With the Never Switching strategy, the total value of all the envelopes is ')
sum(envelopes$allocatedValues)

print('With the Switching Strategy, the number of times the value after switching was greater than the allocated value is')
```

```
length(which(envelopes$valuesAfterSwitchingStrategy > envelopes$allocatedValues))

print('With the Switching Strategy, the number of times the value after switching is less than the allocated value is')
length(which(envelopes$allocatedValues > envelopes$valuesAfterSwitchingStrategy))

print('With the Switching Strategy, the number of times the value after switching is equal to the allocated value is')
length(which(envelopes$allocatedValues == envelopes$valuesAfterSwitchingStrategy))


# vector whose ith entry is the the difference of the cumulative winnings of the switchingStrategy minus
# the cumulative winnings of the never-switching strategy at the ith iteration
differenceOfCumulativeWinnings <- cumsum(envelopes$valuesAfterSwitchingStrategy) - cumsum(envelopes$allocatedValues)

#specify width and height of plot
options(repr.plot.width=18, repr.plot.height=15)

#we plot the difference of the cumulative winnings of the Switching strategy minus the Never Switching Strategy
par(cex.main=2.4,cex.lab=1.7)
plot(1:numberOfGames,differenceOfCumulativeWinnings, lwd=3, col="green", ann=FALSE, las=2,type='l')
 #mtext("Number of games", side=1, line=2)
 title("Continuous Uniform Distribution \n Switching Strategy cumulative winnings minus Never Switching cumulative winnings",
 xlab="Number of Games", ylab="Winnings")
 axis(4)

#we plot the average gain over the number of iterations
games = seq(1, numberOfGames, by=1)
movingAverageWinningsSwitchingStrategy <- cumsum(envelopes$valuesAfterSwitchingStrategy) / games
movingAverageWinningsNeverSwitching <- cumsum(envelopes$allocatedValues) / games
rangeOfAverageWinnings <- range(c(movingAverageWinningsSwitchingStrategy,movingAverageWinningsNeverSwitching))

plot(range(games),range(rangeOfAverageWinnings),type='n', ann=FALSE, las=2)
    lines(1:numberOfGames,movingAverageWinningsSwitchingStrategy, type='l', col="red")
    lines(1:numberOfGames,movingAverageWinningsNeverSwitching, col="blue", type='l')
    title("Average Winnings vs Number of iterations\n Switching Strategy (red), Never Switching (blue)",xlab="Number of Games", ylab="Average Winnings'
    axis(4)


#lets make 100 samples, where each sample has 10000 iterations
samples = 100;
sampleOfSwitchingStrategyWinnings <- numeric(samples)
sampleOfNeverSwitchingWinnings <- numeric(samples)
for (i in 1:samples) {
    envelopes <- data.frame(matrix(ncol = 0, nrow = numberOfGames))
    #sample from continuous distribution
    sampleFromContinuousDistribution <- as.matrix(runif(numberOfGames, min = A, max = A+T))
    envelopes <- as.data.frame(cbind(sampleFromContinuousDistribution,2*sampleFromContinuousDistribution))
    perms<- t(replicate(nrow(envelopes), sample(ncol(envelopes))))
    shuffledEnvelopes <- t(sapply(seq_len(nrow(perms)),
               function(i, perms, mat) mat[i, perms[i,]],
               mat = envelopes, perms = perms))
    envelopes$allocatedValues <- as.numeric(shuffledEnvelopes[,c(1)])

    envelopesNotForSwitching <- which((envelopes$allocatedValues < A) | (envelopes$allocatedValues > (A + T)))
    envelopes$valuesAfterSwitchingStrategy <- shuffledEnvelopes[,2]
    envelopes$valuesAfterSwitchingStrategy[c(envelopesNotForSwitching)] <- envelopes$allocatedValues[c(envelopesNotForSwitching)]

    #append the vector of winnings for the Switching Strategy
    envelopes$valuesAfterSwitchingStrategy <- {as.numeric(envelopes$valuesAfterSwitchingStrategy)}
    sampleOfSwitchingStrategyWinnings[i] <- sum(as.vector(envelopes$valuesAfterSwitchingStrategy))

    #append the vector of winnings for the Never Switching Strategy
    envelopes$allocatedValues <- {as.numeric(envelopes$allocatedValues)}
    sampleOfNeverSwitchingWinnings[i] <- sum(envelopes$allocatedValues)

}


#sampleOfSwitchingStrategyWinnings #uncomment this to reveal the contents of the vector
#sampleOfNeverSwitchingWinnings #uncomment this to reveal the contents of the vector

par(mfrow=c(1,2))

boxplot(sampleOfSwitchingStrategyWinnings, main = paste("Switching strategy winnings for \n 100 samples, each has ", numberOfGames," iterations"))
boxplot(sampleOfNeverSwitchingWinnings, main = paste("Never Switching strategy for \n 100 samples, each has ", numberOfGames," iterations"))

hist(sampleOfSwitchingStrategyWinnings, breaks = 19, main = paste("Switching Strategy winnings for \n 100 samples, each sample has",
numberOfGames," iterations"))

hist(sampleOfNeverSwitchingWinnings, breaks = 19, main = paste("Never Switching winnings for \n 100 samples, each has ", numberOfGames," iterations"))
```

## A.2 Code for the Discrete Distribution

```
set.seed(22)

# the following code is a simulation of playing the two-envelope problem with a distribution on the integers with P(n)=(9^(n-1))/(10^n)
outcomes = 1:8000
probabilities = (1/9)*((0.9)^outcomes)

#we check the
sum(probabilities)

#get the index of the last nonzero element of the "probabilities vector"
max(which(probabilities>0))

#get the probability corresponding to the last nonzero element of the "probabilities" vector
probabilities[max(which(probabilities>0))]
#to doublecheck, get the probability corresponding to the next entry to the last nonzero element of the "probabilities" vector
probabilities[1+max(which(probabilities>0))]
#we see that R can only get values 1 up to 7051 because all the succeeding integers will have probability 0


#define a discrete distribution on the integers where sampling is done with replacement
sampleDist = function(n) {
    sample(x = outcomes, n, replace = T, prob = probabilities)
}

#############################
numberOfGames = 10000

#create data frame
envelopes <- data.frame(matrix(ncol = 0, nrow = numberOfGames))
#we take a sample from our discrete distribution
sampleFromDiscreteDistribution <- as.matrix(sampleDist(numberOfGames))

#fill the envelopes, first column contains the sampled values, second column contains twice the sampled values
envelopes <- as.data.frame(cbind(sampleFromDiscreteDistribution,2*sampleFromDiscreteDistribution))
#envelopes #this line, if uncommented, checks the contents of the envelopes

#shuffle the entries of each row
perms<- t(replicate(nrow(envelopes), sample(ncol(envelopes))))
shuffledEnvelopes <- t(sapply(seq_len(nrow(perms)),
           function(i, perms, mat) mat[i, perms[i,]],
           mat = envelopes, perms = perms))
#each row of shuffledEnvelopes is an instance of the game, the first entry is the allocated envelope,
#and the second envelope is the complementary envelope

#allocate the first envelopes to the player and the complementary values
envelopes$allocatedValues <- as.numeric(shuffledEnvelopes[,c(1)])
envelopes$complementaryValues <- as.numeric(shuffledEnvelopes[,c(2)])
#envelopes#this line, if uncommented, checks the contents of the shuffled envelopes

#player knows that the host uses the discrete distribution as defined above
#player must check for the probability that the allocated value is selected
#if allocated value is odd, switching is advised
#if allocated value is even, switching is advised if the allocated value is less than (2 ln 2)/(ln (10/9))
# hence, we don't switch if the allocated value is even AND the allocated value is greater than or equal to (2 ln 2)/(ln (10/9))
allocatedValue.is.even <- sapply(envelopes$allocatedValues, function(i) i %% 2 == 0)
#allocatedValue.is.even,this line, if uncommented, checks the contents of the allocatedValue.is.even
switchingStrategyCutoffValue = (2 *log (2))/(log (10/9))
envelopesNotForSwitching <- which(allocatedValue.is.even & (envelopes$allocatedValues >= switchingStrategyCutoffValue))
#envelopesNotForSwitching#this line, if uncommented, gives the rows where switching should not be done

#valuesAfterSwitchingStrategy[!c(envelopesNotForSwitching)] <- shuffledEnvelopes[!c(envelopesNotForSwitching),2]
envelopes$valuesAfterSwitchingStrategy <- shuffledEnvelopes[,2]
envelopes$valuesAfterSwitchingStrategy[c(envelopesNotForSwitching)] <- envelopes$allocatedValues[c(envelopesNotForSwitching)]
#envelopes

#get the sums of the non-switch strategy and the switching strategy
envelopes$valuesAfterSwitchingStrategy <- {as.numeric(envelopes$valuesAfterSwitchingStrategy)}
print('With the Switching Strategy, the total value of all the envelopes is ')
sum(as.vector(envelopes$valuesAfterSwitchingStrategy))
envelopes$allocatedValues <- {as.numeric(envelopes$allocatedValues)}
print('With the Never Switching Strategy, the total value of all the envelopes is ')
sum(envelopes$allocatedValues)


print('With the Switching Strategy, the number of times the value after switching was greater than the allocated value is')
length(which(envelopes$valuesAfterSwitchingStrategy > envelopes$allocatedValues))

print('With the Switching Strategy, the number of times the value after switching is less than the allocated value is')
length(which(envelopes$allocatedValues > envelopes$valuesAfterSwitchingStrategy))

print('With the Switching Strategy, the number of times the value after switching is equal to the allocated value is')
length(which(envelopes$allocatedValues == envelopes$valuesAfterSwitchingStrategy))

# vector whose ith entry is the the difference of the cumulative winnings of the switchingStrategy minus
# the cumulative winnings of the never-switching strategy at the ith iteration
differenceOfCumulativeWinnings <- cumsum(envelopes$valuesAfterSwitchingStrategy) - cumsum(envelopes$allocatedValues)
```

```
#specify width and height of plot
options(repr.plot.width=18, repr.plot.height=15)



par(cex.main=2.4,cex.lab=1.7)
plot(1:numberOfGames,differenceOfCumulativeWinnings, lwd=3, col="green", ann=FALSE, las=2,type='l')
 title("Discrete Distribution P(n)=(9^(n-1))/(10^n) \n Switching Strategy cumulative winnings minus Never Switching cumulative winnings",
 xlab="Number of Games", ylab="Winnings")
 axis(4)

#we plot the average gain over the number of iterations
games = seq(1, numberOfGames, by=1)
movingAverageWinningsSwitchingStrategy <- cumsum(envelopes$valuesAfterSwitchingStrategy) / games
movingAverageWinningsNeverSwitching <- cumsum(envelopes$allocatedValues) / games
rangeOfAverageWinnings <- range(c(movingAverageWinningsSwitchingStrategy,movingAverageWinningsNeverSwitching))

plot(range(games),range(rangeOfAverageWinnings),type='n', ann=FALSE, las=2)
    lines(1:numberOfGames,movingAverageWinningsSwitchingStrategy, type='l', col="red")
    lines(1:numberOfGames,movingAverageWinningsNeverSwitching, col="blue", type='l')
    title("Average Winnings vs Number of iterations\n Switching Strategy (red), Never Switching (blue)",xlab="Number of Games", ylab="Average Winnings'
     axis(4)


#lets make 100 samples, where each sample has 10000 iterations
samples = 100;
sampleOfSwitchingStrategyWinnings <- numeric(samples)
sampleOfNeverSwitchingWinnings <- numeric(samples)
for (i in 1:samples) {
    envelopes <- data.frame(matrix(ncol = 0, nrow = numberOfGames))
    sampleFromDiscreteDistribution <- as.matrix(sampleDist(numberOfGames))
    envelopes <- as.data.frame(cbind(sampleFromDiscreteDistribution,2*sampleFromDiscreteDistribution))
    perms<- t(replicate(nrow(envelopes), sample(ncol(envelopes))))
    shuffledEnvelopes <- t(sapply(seq_len(nrow(perms)),
               function(i, perms, mat) mat[i, perms[i,]],
               mat = envelopes, perms = perms))
    envelopes$allocatedValues <- as.numeric(shuffledEnvelopes[,c(1)])
    #envelopes$complementaryValues <- as.numeric(shuffledEnvelopes[,c(2)])
    allocatedValue.is.even <- sapply(envelopes$allocatedValues, function(i) i %% 2 == 0)
    #switchingStrategyCutoffValue = (2 *log (2))/(log (10/9))
    envelopesNotForSwitching <- which(allocatedValue.is.even & (envelopes$allocatedValues >= switchingStrategyCutoffValue))
    envelopes$valuesAfterSwitchingStrategy <- shuffledEnvelopes[,2]
    envelopes$valuesAfterSwitchingStrategy[c(envelopesNotForSwitching)] <- envelopes$allocatedValues[c(envelopesNotForSwitching)]

    #append the vector of winnings for the Switching Strategy
    envelopes$valuesAfterSwitchingStrategy <- {as.numeric(envelopes$valuesAfterSwitchingStrategy)}
    sampleOfSwitchingStrategyWinnings[i] <- sum(as.vector(envelopes$valuesAfterSwitchingStrategy))

    #append the vector of winnings for the Never Switching Strategy
    envelopes$allocatedValues <- {as.numeric(envelopes$allocatedValues)}
    sampleOfNeverSwitchingWinnings[i] <- sum(envelopes$allocatedValues)

}


#sampleOfSwitchingStrategyWinnings #uncomment this to reveal the contents of the vector
#sampleOfNeverSwitchingWinnings #uncomment this to reveal the contents of the vector




par(mfrow=c(1,2))

boxplot(sampleOfSwitchingStrategyWinnings, main = paste("Switching strategy winnings for \n 100 samples, each has ", numberOfGames," iterations"))
boxplot(sampleOfNeverSwitchingWinnings, main = paste("Never Switching strategy for \n 100 samples, each has ", numberOfGames," iterations"))

hist(sampleOfSwitchingStrategyWinnings, breaks = 19, main = paste("Switching Strategy winnings for \n 100 samples, each sample has", numberOfGames,"
iterations"))

hist(sampleOfNeverSwitchingWinnings, breaks = 19, main = paste("Never Switching winnings for \n 100 samples, each has ", numberOfGames," iterations"))
```