Charge current, noise and thermodynamic uncertainty relations in a double quantum dot

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Abstract

The objective of this thesis is to study theoretically the violations of a thermodynamic uncertainty relation, denoted as the TUR, in a double quantum dot. The current and the noise in the system are calculated, while the system is in contact with two fermionic baths. These quantities are used for studying the violations of the TUR, which is found to be violated. This is the key finding of this thesis and the violations happen possibly due to the quantum mechanical properties of the system. A reduced model, which includes only the states in the double dot, is introduced in order to study the system's time-evolution analytically. A master equation and the suitable Hamiltonian are introduced. The theoretical tools used in the calculations are the second-quantization formalism and the density matrix formalism.

Acknowledgements

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Abbreviations

LB Left bath

RB Right bath

LQD Left quantum dot

RQD Right quantum dot

TUR Thermodynamic uncertainty relation

Contents

1 Introduction

In the recent years there has been an interest in studying nano scale systems, which exhibit significant quantum mechanical behaviour. The aim of this thesis is to calculate analytically the current and the noise in a double quantum dot connected to two heat baths and study the violations of a thermodynamic uncertainty relation, denoted as a TUR. The current and the noise are calculated with the second quantization formalism and the density matrix formalism, which are commonly used for studying quantum many-body systems[1, 2].

The fluctuations are an important quantity to study, when it comes to nanoscale quantum systems. In this system, quantum mechanical properties of the electrons determine the electric current through the system, thus originate stochastic processes. The stochastic processes are known to give a high-level of noise[3]. In general, a high-level of noise is common to small systems. However, significant quantum mechanical properties in a small system can cause the type of noise which is typical for the quantum systems. Therefore, noise is something that could distinguish such systems from systems, which do not possess significant quantum mechanical behaviour.

Thermodynamics is traditionally studied in larger scale systems, where the quantum mechanical properties are neglected, as they are not significant. However, in nanoscale systems quantum mechanics can be behind the thermodynamical properties[4]. To study further the quantum mechanical properties of the system the TUR is studied. Violations of the TURs provide a way to see the quantum mechanical behaviour in the thermodynamical systems as the TURs are bounds derived for the classical systems. When these bounds are violated, a quantum mechanical behaviour can be assumed[5].

A serial double-quantum dot connected to fermionic baths has potential applications in consistency in nanoscale and in nanotechnology. In the recent years the electronic devices have become smaller and building nanoscale electronics is common. In spite of the smaller size, a nanoscale electronics is not immune to an unwanted over-heating of the components. A double-quantum dot connected to two thermal bath could provide a sustainable solution to this problem by turning the excess heat into work. This application of the system has been studied experimentally[6, 7]. In principle, a double-quantum dot can operate as a heat-engine if the temperatures of the baths are set unequal. The heat-engines are machines which are used for turning heat into work, which is a useful form of energy. This system has also been studied experimentally by looking at a thermodynamic double-quantum dot heat-engine and focusing on the impact that a maximal entanglement of the electrons can have on extracting work from a heat-bath[8].

In this thesis the key theoretical concepts are introduced first which is followed by introducing the system and the main equations used in the calculations. The current and the fluctuations of the current in a double-quantum dot are calculated. These results are analyzed to understand how different parameters impact the values that the current and the noise take. The system is analyzed further by writing the TUR in terms of the calculated expressions for the current and noise. A conclusion and an out-look are provided at the end of this thesis.

2 Theory and the system

2.1 Double quantum dot connected to two fermionic baths

A double quantum dot system studied in this thesis consists of two quantum dots, which are serially connected to two fermionic baths. The quantum dots are called the left and the right quantum dot and are denoted as LQD and RQD. The left and the right bath are denoted as LB and RB. The quantum dots have a single level and a dot can be occupied by one electron at a time. The electrons in this system are assumed to be non-interacting. In order the current to flow through the system, the electrons can tunnel between the baths and the quantum dots as well as between the quantum dots. This is illustrated in the following figure.

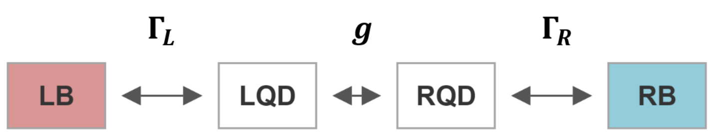


Figure 1: An illustration of the double quantum dot connected to the two fermionic bath. The tunneling rates of the electrons are denoted as Γ_L, g and Γ_R.

2.2 Second-quantization formalism

Particles which possess quantum mechanical properties do not have an exact position. Instead of being an one place at a given time, they are described by a wave-function of the possible positions that the particles could take[9].

The second-quantization formalism is commonly used when quantum many-body systems are studied. The second-quantization formalism can be expressed in a form of single particle states and the occupation of these states, instead of studying the positions of the particles in real space. In a fermionic system, a state can have an occupation number 0 or 1. This is due to the Pauli exclusion principle, which states that two identical fermions cannot occupy the same quantum state[1].

The occupation number of the state is obtained by applying a number operator on a ket, when the Dirac notation is used. The number operator $\hat{n}$ is defined as

$$\hat{n}|n\rangle = c_i^\dagger c_i|n\rangle = n|n\rangle \tag{1}$$

where $c_i^\dagger$ is a raising operator, c_i is a lowering operator and n is either 0 or 1 depending on the occupation of the state. The state $|n\rangle$ is an eigenstate of $\hat{n}$ and the index i denotes the quantum state the operator acts on.

The fermion operators obey the following anti-commutation relations.

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \tag{2}$$
$$\{c_i, c_j^\dagger\} = \delta_{ij} \tag{3}$$

2.3 Density matrix

In a density matrix, the state is expressed as an operator state, $|\psi\rangle\langle\psi|$, which is an outer product state of a ket and a bra. The states follow a probability density, meaning that there is a probability that a given state occurs. A density matrix can be written as a weighted sum of the states, where the weight, p_i, is the probability of the respective state occurring.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{4}$$

The sum of the probabilities adds up to 1. Thus, it holds that

$$\sum_i p_i = 1 \tag{5}$$

Essentially, the density matrix provides a description of a quantum many-body system[2]. The density matrix, written as a sum in Eq. (4), can be expressed in a matrix form. The diagonal elements are probabilities of the respective states occurring in the system.

It is common that the density matrix describes systems that are a statistical mixture of states, for example the double-quantum dot system, as the dots can have different occupations. Hence, the density matrix can consist of either a pure state or a mixed state. The difference between these states is described as follows. A pure state is a state that is completely known. This means that one probability in the sum given in Eq. (4) is equal to one and the rest are zero. A density matrix that describes a mixed state will have instead more than one non-zero term.

Using the double quantum dot connected to the two baths as an example, the density matrix, which includes only the states of the quantum dot, would look like the following matrix, when expressed with the matrix representation:

$$\rho = \begin{bmatrix} \rho_{00} & 0 & 0 & 0 \\ 0 & \rho_{10} & \alpha & 0 \\ 0 & \alpha* & \rho_{01} & 0 \\ 0 & 0 & 0 & \rho_{11} \end{bmatrix} \tag{6}$$

where ρ_{ij} is the probability of the state ij. The index i gives an occupation of the left dot and the index j gives the occupation of the right dot. The occupation of the dot can be 0 or 1.

When the tunneling between the dots can happen, the density matrix given in Eq.(6) will have non-zero off-diagonal elements, α and α^*, which are called coherences. In this system, the coherences will occur only in the matrix elements (2,3) and (3,2). This happens because the tunneling between the dots can only happen when there is an electron in one of the dots and the other one is empty.

When considering the density matrix representation of the system, the tunneling between the dots can be understood as the electron being in a superposition in the both quantum dots. The superposition of the electron is described in the the density matrix formalism as the coherences in the density matrix, since the electron is on the left and the right dot at the same time. If this system was described in terms of the wave-functions, the tunneling would be understood as the wave-function of an electron extending to the both dots.

The density matrix is a Hermitian matrix. A Hermitian matrix has diagonal elements, which are real. The off-diagonal elements obey the relation $a_{ij} = a^*_{ji}$.

2.4 Quantum mechanical description of the system

The double quantum dot connected to two fermionic baths can be described mathematically with a reduced model where only the states of the quantum dot are considered, instead of looking at all the possible states, which would include the baths as well. Therefore, the density matrix used in the calculations is a reduced one. It does not take into account the states that the baths have and the probabilities of these states. The reduced density matrix is given in Eq. (6).

The Hamiltonian of the system is

$$H = \epsilon_L c_L^\dagger c_L + \epsilon_R c_R^\dagger c_R + g(c_L^\dagger c_R + c_R^\dagger c_L) \tag{7}$$

Equation (7) contains terms which give the energies of the electrons in LQD and RQD. The third term in the Hamiltonian gives the coupling between the quantum dots. Because the model is a reduced one, the Hamiltonian does not have terms describing the baths or the connections between the baths and the system[10]. The tunneling strength between the quantum dots is denoted as g, which has the dimension of $[1/s]$, as $\hbar$ is set to 1.

The electrons can tunnel from the fermionic baths to the quantum dots with given rates. The tunneling rate between LB to LQD is denoted as Γ_L. Similarly, the tunneling rate between RQD and RB is denoted as Γ_R. Γ_L and Γ_R have a dimension of $[1/s]$.

The Fermi-function contributes to the probability of the electrons entering the dots from the baths. It occurs in the calculations together with the tunneling rates Γ_L and Γ_R to describe how the system evolves. The Fermi-function is

$$f_i(\epsilon_i) = \frac{1}{e^{(\epsilon_i - \mu_i)/k_B T_i} + 1} \tag{8}$$

where ϵ_i is the energy of the quantum dot, T_i is the temperature of the bath, μ_i is the chemical potential level and k_B is the Boltzmann constant. The index i assigns the position left or right, denoted as L and R, to the given quantities.

2.5 Master equation of the double quantum dot connected to the fermionic baths

Master equations are the first-order differential equations that describe the time evolution of states, which follow some statistical distribution. The master equation is written as

$$\frac{\mathrm{d}P_k}{\mathrm{d}t} = \sum_l \left[T_{kl}P_l - T_{lk}P_k\right] \tag{9}$$

where P_k is the probability of the particle being in the state k and T_{kl} is the transition rate from the state l to the state k. The master equation can be expressed as a matrix equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{P} = A\vec{P}$, where A is a transition matrix consisting of the coefficients T_{lk} and T_{kl}. The transition matrix A gives the connections between the statistical states as the time evolves[11].

The master equation given in Eq. (9) does not take into an account that there is coherent tunneling between the dots. Therefore, a master equation for a quantum mechanical system is formulated with the second quantization. The time evolution of the density matrix of the double quantum dot connected to the fermionic baths has the master equation

$$\partial_t\rho = -i[H,\rho] + \mathscr{L}_L\rho + \mathscr{L}_R\rho \tag{10}$$

where $\mathscr{L}_i$ is given by,

$$\mathscr{L}_i = \Gamma_i f_i(\epsilon_i)\mathscr{D}[c_i^\dagger] + \Gamma_i[1 - f_i(\epsilon_i)]\mathscr{D}[c_i] \tag{11}$$

and $\mathscr{D}$ in Eq. (11) is defined as

$$\mathscr{D}[\hat{A}]\rho = \hat{A}\rho\hat{A}^\dagger - \frac{1}{2}\{\hat{A}^\dagger\hat{A},\rho\} \tag{12}$$

The first term in Eq. (10) is a commutator of the Hamiltonian and the density matrix. The second and the third term give a contribution from the baths into the time evolution of the density matrix.

3 Method and Calculations

3.1 Current

A current through the system is defined as the ratio of the amount of charge Q flowing through the system in a given time t and the time t, $I = Q/t$. In the long-time limit, the double quantum dot system will reach a steady state, meaning that the current over the system will be constant. Physically this means that as enough time has passed, the current between LB and LQD, denoted as $I_{LB,LQD}$, and the current between RQD and LQD, denoted as $I_{RQD,LQD}$ will have equal magnitudes but opposite signs, giving that

$$I_{LB,LQD} + I_{RQD,LQD} = 0 \tag{13}$$

The amount of electrons in the quantum dot corresponds to the average occupation of the quantum dot. Thus, $\langle Q_L \rangle = \langle c_L^\dagger c_L \rangle e$ is an expression for the average charge inside LQD, where e is the electron charge. For simplicity, e will be set to 1.

$\partial_t \langle c_L^\dagger c_L \rangle$ gives the rate that the occupation of LQD changes. In the long time limit, this rate will be zero, as the current over LQD will be constant. Same reasoning holds for $\partial_t \langle c_R^\dagger c_R \rangle$ and $\partial_t \langle c_L^\dagger c_R \rangle$.

Derivatives of the expectation values $\partial_t \langle c_i^\dagger c_j \rangle$, are calculated by using that $\partial_t \langle c_i^\dagger c_j \rangle = Tr\{c_i^\dagger c_j \, \partial_t \rho\}$, where $\partial_t \rho$ is determined by the master equation given in Eq. (10). It is set that $\epsilon = \epsilon_R = \epsilon_L$, which physically means that LQD and RQD are identical. The fermionic anti-commutation relations, given in Eq. (2)-(3), are used to simplify the expressions.

The calculations give a linear system of equations.

$$\partial_t \langle c_L^\dagger c_L \rangle = -ig \langle c_L^\dagger c_R \rangle + ig \langle c_R^\dagger c_L \rangle + \Gamma_L f_L(\epsilon) - \Gamma_L \langle c_L^\dagger c_L \rangle \tag{14}$$

$$\partial_t \langle c_R^\dagger c_R \rangle = -ig \langle c_R^\dagger c_L \rangle + ig \langle c_L^\dagger c_R \rangle + \Gamma_R f_R(\epsilon) - \Gamma_R \langle c_R^\dagger c_R \rangle \tag{15}$$

$$\partial_t \langle c_L^\dagger c_R \rangle = -ig \langle c_L^\dagger c_L \rangle + ig \langle c_R^\dagger c_R \rangle - \frac{1}{2}(\Gamma_L + \Gamma_R) \langle c_L^\dagger c_R \rangle \tag{16}$$

$$\partial_t \langle c_R^\dagger c_L \rangle = -ig \langle c_R^\dagger c_R \rangle + ig \langle c_L^\dagger c_L \rangle - \frac{1}{2}(\Gamma_L + \Gamma_R) \langle c_R^\dagger c_L \rangle \tag{17}$$

The current through the system is a stochastic process since the electrons can attempt to enter the system from both baths with given probabilities. It follows that despite the majority of the electrons are going in one direction, there will be electrons moving in the opposite direction. This can be understood by studying the terms of Eq. (14) and noticing that the terms have different signs.

Each term in Eq. (14) can be interpreted as the electrons moving in the system in a given direction with a certain rate, which is by definition a current. The first two terms of Eq. (14) constitute $I_{RQD,LQD}$. The first term is the current from RQD to LQD and the second term is the current from LQD to RQD. $I_{RQD,LQD}$ is defined as the expectation value of the current operator, giving that

$$\langle \hat{I} \rangle = -ig\langle c_L^\dagger c_R \rangle + ig\langle c_R^\dagger c_L \rangle \tag{18}$$

Similarly, the third and the fourth term constitute $I_{LB,LQD}$. The third term is the current from LB to LQD, which is simply the tunneling rate multiplied by the Fermi-function of the left bath. The fourth term is the current from LQD to LB.

Equation (16) consists of three terms. The first two terms are purely imaginary, since $\langle c_L^\dagger c_L \rangle$ and $\langle c_R^\dagger c_R \rangle$ are real. Multiplying them by i gives a purely imaginary expression.

Equation (16) is re-written by separating the third term of Eq. (16) into a real and an imaginary part. Thus, Eq. (16) becomes

$$ig\langle c_R^\dagger c_R \rangle - ig\langle c_L^\dagger c_L \rangle + \frac{1}{2}(\Gamma_L + \Gamma_R)\langle c_L^\dagger c_R \rangle = ig\langle c_R^\dagger c_R \rangle - ig\langle c_L^\dagger c_L \rangle + \frac{1}{2}(\Gamma_L + \Gamma_R)(\mathrm{Re}\langle c_L^\dagger c_R \rangle + i\mathrm{Im}\langle c_L^\dagger c_R \rangle) \tag{19}$$

The first, the second and the fourth term of Eq. (19) are imaginary. Therefore, for the terms to add up to zero, it has to hold that

$$\mathrm{Re}\langle c_L^\dagger c_R \rangle = 0 \tag{20}$$

The density matrix is Hermitian, thus the expectation values $\langle c_R^\dagger c_L \rangle$ and $\langle c_L^\dagger c_R \rangle$ obey the relation

$$\langle c_L^\dagger c_R \rangle = \langle c_R^\dagger c_L \rangle^* \tag{21}$$

The imaginary and the real part of the complex number given in Eq. (21) are separated.

$$\mathrm{Re}\langle c_L^\dagger c_R \rangle + i\mathrm{Im}\langle c_L^\dagger, c_R \rangle = (\mathrm{Re}\langle c_R^\dagger c_L \rangle + i\mathrm{Im}\langle c_R^\dagger c_L \rangle)^* \tag{22}$$

$$= \mathrm{Re}\langle c_R^\dagger c_L \rangle - i\mathrm{Im}\langle c_R^\dagger c_L \rangle \tag{23}$$

The real parts of Eq. (23) are zero because of the result given in Eq. (20). Thus, Eq. (23) becomes

$$\begin{aligned} i\mathrm{Im}\langle c_L^\dagger c_R \rangle &= -i\mathrm{Im}\langle c_R^\dagger c_L \rangle \\ \langle c_L^\dagger c_R \rangle &= -\langle c_R^\dagger c_L \rangle \end{aligned} \tag{24}$$

Finally, Eq. (14) and (15) are rewritten by using Eq. (24) and setting the expressions to zero due to the steady state. The system of equations becomes

$$\partial_t \langle c_L^\dagger c_L \rangle = -2ig\langle c_L^\dagger c_R \rangle + \Gamma_L f_L(\epsilon) - \Gamma_L \langle c_L^\dagger c_L \rangle = 0 \tag{25}$$

$$\partial_t \langle c_R^\dagger c_R \rangle = 2ig\langle c_L^\dagger c_R \rangle + \Gamma_R f_R(\epsilon) - \Gamma_R \langle c_R^\dagger c_R \rangle = 0 \tag{26}$$

$$\partial_t \langle c_L^\dagger c_R \rangle = -ig\langle c_L^\dagger c_L \rangle + ig\langle c_R^\dagger c_R \rangle - \frac{1}{2}(\Gamma_L + \Gamma_R)\langle c_L^\dagger c_R \rangle = 0 \tag{27}$$

Solving the system of equations above gives the result

$$\langle c_L^\dagger c_L \rangle = \frac{-4g^2\Gamma_R(f_L - f_R)}{(4g^2 + \Gamma_L\Gamma_R)(\Gamma_L + \Gamma_R)} + f_L \tag{28}$$

$$\langle c_R^\dagger c_R \rangle = \frac{4g^2\Gamma_L(f_L - f_R)}{(4g^2 + \Gamma_L\Gamma_R)(\Gamma_L + \Gamma_R)} + f_R \tag{29}$$

$$\langle c_L^\dagger c_R \rangle = -\frac{i2g\Gamma_R\Gamma_L(f_L - f_R)}{(4g^2 + \Gamma_L\Gamma_R)(\Gamma_L + \Gamma_R)} \tag{30}$$

In the steady state, the current flowing through the system is the same everywhere. Therefore, the current I is defined as $I = I_{LB,LQD}$. I has the following expression when the expectation value $\langle c_L^\dagger c_L \rangle$ given in Eq. (28) is substituted in

$$I = \Gamma_L f_L - \Gamma_L \langle c_L^\dagger c_L \rangle \tag{31}$$

$$= \frac{4g^2\Gamma_R\Gamma_L(f_L - f_R)}{(4g^2 + \Gamma_L\Gamma_R)(\Gamma_L + \Gamma_R)} \tag{32}$$

The equations of motion for $\langle \hat{I} \rangle$ $\langle c_L^\dagger c_L \rangle$ and $\langle c_R^\dagger c_R \rangle$ can be found by re-writing Eq. (14)-(17). These expressions will be used in the further calculations.

The time derivative of the expectation value of the current operator becomes

$$\partial_t \langle \hat{I} \rangle = -ig\partial_t \langle c_L^\dagger c_R \rangle + ig\partial_t \langle c_R^\dagger c_L \rangle \tag{33}$$

$\partial_t \langle c_L^\dagger c_R \rangle$ and $\partial_t \langle c_R^\dagger c_L \rangle$ are given in Eq. (16)-(17). These expressions are substituted into Eq. (33) giving

$$\partial_t \langle \hat{I} \rangle = 2g^2 \langle c_R^\dagger c_R \rangle - 2g^2 \langle c_L^\dagger c_L \rangle + ig\frac{1}{2}\Big(\Gamma_L + \Gamma_R\Big)\langle c_L^\dagger c_R \rangle - ig\frac{1}{2}\Big(\Gamma_L + \Gamma_R\Big)\langle c_R^\dagger c_L \rangle \tag{34}$$

This expression can be simplified. $\langle c_L^\dagger c_R \rangle$ and $\langle c_R^\dagger c_L \rangle$ are eliminated from Eq. (34) by using the definition of $\langle \hat{I} \rangle$, which is given in Eq. (18).

Similarly, $\partial_t\langle c_L^\dagger c_L\rangle$ and $\partial_t\langle c_R^\dagger c_R\rangle$, given in Eq. (14)-(15), are expressed with $\langle\hat{I}\rangle$. This gives the equations of motion for $\langle\hat{I}\rangle$, $\langle c_L^\dagger c_L\rangle$ and $\langle c_R^\dagger c_R\rangle$.

$$\partial_t\langle\hat{I}\rangle = -\frac{1}{2}\Big(\Gamma_L+\Gamma_R\Big)\langle\hat{I}\rangle + 2g^2\langle c_R^\dagger c_R\rangle - 2g^2\langle c_L^\dagger c_L\rangle \tag{35}$$

$$\partial_t\langle c_L^\dagger c_L\rangle = \langle\hat{I}\rangle - \Gamma_L\langle c_L^\dagger c_L\rangle + \Gamma_L f_L(\epsilon) \tag{36}$$

$$\partial_t\langle c_R^\dagger c_R\rangle = -\langle\hat{I}\rangle - \Gamma_R\langle c_R^\dagger c_R\rangle + \Gamma_R f_R(\epsilon) \tag{37}$$

We now define the vector $\vec{B}$ as

$$\begin{aligned}\vec{B} &= (\langle B_1\rangle, \langle B_2\rangle, \langle B_3\rangle, \langle B_4\rangle)\\ &\equiv (\langle\hat{I}\rangle, \langle c_L^\dagger c_L\rangle, \langle c_R^\dagger c_R\rangle, \langle B_I\rangle)\end{aligned} \tag{38}$$

where B_I is an identity operator.

The time-derivative of $\vec{B}$ is

$$\dot{\vec{B}} = (\langle\dot{B}_1\rangle, \langle\dot{B}_2\rangle, \langle\dot{B}_3\rangle, \langle\dot{B}_4\rangle) \tag{39}$$

where $\langle\dot{B}_4\rangle = 0$. Then we set that $\dot{\vec{B}}_L = (\langle\dot{B}_1\rangle, \langle\dot{B}_2\rangle, \langle\dot{B}_3\rangle)$ so that $\dot{\vec{B}}_L$ gives the left-hand sides of Eq. (35)-(37).

The matrix G with coefficients G_{ij} is defined as

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14}\\ G_{21} & G_{22} & G_{23} & G_{24}\\ G_{31} & G_{32} & G_{33} & G_{34}\end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\Big(\Gamma_L+\Gamma_R\Big), & -2g^2, & 2g^2, & 0\\ 1, & -\Gamma_L, & 0, & \Gamma_L f_L(\epsilon)\\ -1, & 0, & -\Gamma_R, & \Gamma_R f_R(\epsilon)\end{bmatrix} \tag{40}$$

Finally, the equations of motion of $\langle\hat{I}\rangle$, $\langle c_L^\dagger c_L\rangle$ and $\langle c_R^\dagger c_R\rangle$ are written into the matrix equation

$$\dot{\vec{B}}_L = G\vec{B} \tag{41}$$

3.2 Fluctuations of the current

The fluctuations of the current, also called the noise, are a measure of the time dependent current differing from the average current. The noise is prominent in the system, as the current is originated by a stochastic process and it gives information about the microscopic processes in the system.

The noise is defined as the following integral of the two-point correlator $\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle$.

$$S = \int_{-\infty}^{\infty} \mathrm{d}\tau \big\langle\Delta\hat{I}(\tau)\Delta\hat{I}\big\rangle \tag{42}$$

where the operator $\Delta\hat{I}$, with the time dependence given by the variable τ, is defined as

$$\Delta\hat{I}(\tau) = \hat{I}(\tau) - \langle\hat{I}\rangle \tag{43}$$

The two-point correlators have an identity $\langle B_i(-\tau)B_l\rangle = \langle B_i(\tau)B_l\rangle^*$[12]. It follows from the identity that the real part of a two-point correlator is an even function and the imaginary part is an odd function. Therefore, the noise S becomes

$$S = 2\mathrm{Re}\int_0^\infty \mathrm{d}\tau \big\langle\Delta\hat{I}(\tau)\Delta\hat{I}\big\rangle \tag{44}$$

In order to calculate the integral we introduce the quantum regression theorem[11]. The quantum regression theorem, which holds for some systems, states that operators, which can be written as $\langle\dot{B}_i\rangle = \sum_j G_{ij}\langle B_j\rangle$, have a two-point correlation function which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle B_i(t+\tau)B_k(t)\rangle = \sum_j G_{ij}\langle B_j(t+\tau)B_k(t)\rangle \tag{45}$$

As before, the operators, such as the current operator, are expressed as averages since they are used for calculating the quantities in this system which exhibits a quantum mechanical behaviour. The quantum mechanical behaviour in the system makes it possible to only study these quantities as average values, because the behaviour of the electrons in this system is formulated in a probabilistic manner.

The quantum regression theorem is applied to the matrix equation given in Eq. (41). Essentially, the quantum regression theorem connects the two-point correlation functions to the equations of motion of $\langle\hat{I}\rangle$, $\langle c_L^\dagger c_L\rangle$ and $\langle c_R^\dagger c_R\rangle$. It is set that $B_k = \hat{I}$ to obtain $\langle\hat{I}(\tau+t)\hat{I}(t)\rangle$. B_I is independent of t and τ. Therefore, $\langle B_I(t+\tau)\hat{I}(t)\rangle = \langle\hat{I}(t)\rangle$.

Quantum regression theorem gives the set of equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\hat{I}(t+\tau)\hat{I}(t)\rangle = -\frac{1}{2}\Big(\Gamma_L+\Gamma_R\Big)\langle\hat{I}(t+\tau)\hat{I}(t)\rangle - 2g^2\langle c_L^\dagger c_L(t+\tau)\hat{I}(t)\rangle + 2g^2\langle c_R^\dagger c_R(t+\tau)\hat{I}(t)\rangle \tag{46}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle c_L^\dagger c_L(t+\tau)\hat{I}(t)\rangle = \langle\hat{I}(t+\tau)\hat{I}\rangle - \Gamma_L\langle c_L^\dagger c_L(t+\tau)\hat{I}(t)\rangle + \Gamma_L f_L(\epsilon)\langle\hat{I}(t)\rangle \tag{47}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle c_R^\dagger c_R(t+\tau)\hat{I}(t)\rangle = -\langle\hat{I}(t+\tau)\hat{I}(t)\rangle - \Gamma_R\langle c_R^\dagger c_R(t+\tau)\hat{I}(t)\rangle + \Gamma_R f_R(\epsilon)\langle\hat{I}(t)\rangle \tag{48}$$

The expressions above are considered to be in the steady state, thus they are independent of t and only depend on τ. Therefore, the choice of t is has no impact on the expression. For convenience, it is chosen that $t = 0$.

Analogous to the operator $\Delta\hat{I}(t)$, given in Eq. (43), the operators $\Delta\hat{n}_L(t)$ and $\Delta\hat{n}_R(t)$ are introduced.

$$\Delta\hat{n}_L(t) = \hat{n}_L(t) - \langle\hat{n}_L\rangle \tag{49}$$
$$\Delta\hat{n}_R(t) = \hat{n}_R(t) - \langle\hat{n}_R\rangle \tag{50}$$

where $\hat{n}_L = c_L^\dagger c_L$ and $\hat{n}_R = c_R^\dagger c_R$. The operators $\Delta\hat{I}(t)$,$\Delta\hat{n}_L(t)$ and $\Delta\hat{n}_R(t)$ give the difference between the t dependent operator and the expectation value of that operator. Thus, they have a zero expectation value, as the steady state is assumed for all t.

The equations of motion of the two-point correlators $\langle\hat{I}(t+\tau)\hat{I}(t)\rangle$, $\langle c_L^\dagger c_L(t+\tau)\hat{I}(t)\rangle$ and $\langle c_R^\dagger c_R(t+\tau)\hat{I}(t)\rangle$ will now be expressed in terms of $\Delta\hat{I}(t)$, $\Delta\hat{n}_L(t)$ and $\Delta\hat{n}_R(t)$.

The two-point correlator $\langle\hat{I}(\tau)\hat{I}\rangle$ is expressed by using $\Delta\hat{I}(\tau)$. It is noted that $\hat{I}(0) = \hat{I}$.

$$\begin{aligned}
\langle\hat{I}(\tau)\hat{I}\rangle &= \big\langle\big(\Delta\hat{I}(\tau) + \langle\hat{I}\rangle\big)\big(\Delta\hat{I} + \langle\hat{I}\rangle\big)\big\rangle \\
&= \big\langle\Delta\hat{I}(\tau)\Delta\hat{I} + \Delta\hat{I}(\tau)\langle\hat{I}\rangle + \Delta\hat{I}\langle\hat{I}\rangle + \langle\hat{I}\rangle^2\big\rangle \\
&= \langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle + \langle\Delta\hat{I}(\tau)\rangle\langle\hat{I}\rangle + \langle\Delta\hat{I}\rangle\langle\hat{I}\rangle + \langle\hat{I}\rangle^2 \\
&= \langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle^2
\end{aligned} \tag{51}$$

where the expression is simplified by using that $\langle\Delta\hat{I}(t)\rangle = 0$ for all t.

Similarly, the two-point correlators $\langle c_L^\dagger c_L(\tau)\hat{I}\rangle$ and $\langle c_R^\dagger c_R(\tau)\hat{I}\rangle$ are expressed with the operators $\Delta\hat{n}_L(t)$, $\Delta\hat{n}_R(t)$ and $\Delta\hat{I}(t)$. This gives

$$\begin{aligned}
\langle c_L^\dagger c_L(\tau)\hat{I}\rangle &= \big\langle\big(\Delta\hat{n}_L(\tau) + \langle\hat{n}_L\rangle\big)\big(\Delta\hat{I} + \langle\hat{I}\rangle\big)\big\rangle \\
&= \langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle\langle\hat{n}_L\rangle
\end{aligned} \tag{52}$$

$$\begin{aligned}
\langle c_R^\dagger c_R(\tau)\hat{I}\rangle &= \big\langle\big(\Delta\hat{n}_R(\tau) + \langle\hat{n}_R\rangle\big)\big(\Delta\hat{I} + \langle\hat{I}\rangle\big)\big\rangle \\
&= \langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle\langle\hat{n}_R\rangle
\end{aligned} \tag{53}$$

Equation (51) is substituted into the left-hand side of Eq. (46) in order to express it with $\Delta\hat{I}$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big\langle\big(\Delta\hat{I}(\tau) + \langle\hat{I}\rangle\big)\big(\Delta\hat{I} + \langle\hat{I}\rangle\big)\big\rangle = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle^2\big) \tag{54}$$
$$= \frac{\mathrm{d}}{\mathrm{d}\tau}\big\langle\Delta\hat{I}(\tau)\Delta\hat{I}\big\rangle \tag{55}$$

The second term in Eq. (54) becomes zero when the derivative is performed, as $\langle\hat{I}\rangle^2$ is not a function of τ.

Similarly, Eq. (52)- (53) are substituted into the left-hand sides of Eq. (47)-(48).

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\big(\Delta\hat{n}_L(\tau)+\langle\hat{n}_L\rangle\big)\big(\Delta\hat{I}+\langle\hat{I}\rangle\big)\Big\rangle = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\Big\rangle \tag{56}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\big(\Delta\hat{n}_L(\tau)+\langle\hat{n}_L\rangle\big)\big(\Delta\hat{I}+\langle\hat{I}\rangle\big)\Big\rangle = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\Big\rangle \tag{57}$$

The two-point correlation function given in Eq. (46) is equated to the left-hand side of Eq. (55). This gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\Delta\hat{I}(\tau)\Delta\hat{I}\Big\rangle = -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\langle\hat{I}(\tau)\hat{I}\rangle - 2g^2\langle c_L^\dagger c_L(\tau)\hat{I}\rangle + 2g^2\langle c_R^\dagger c_R(\tau)\hat{I}\rangle \tag{58}$$

The two-point correlators $\langle\hat{I}(\tau)\hat{I}\rangle$, $\langle c_L^\dagger c_L(\tau)\hat{I}\rangle$ and $\langle c_R^\dagger c_R(\tau)\hat{I}\rangle$ appearing in the right-hand side of Eq. (58) are replaced with $\langle\Delta\hat{I}(\tau)\hat{I}\Delta\rangle$, $\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle$ and $\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle$ by substituting in Eq. (51)-(53).

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\Delta\hat{I}(\tau)\Delta\hat{I}\Big\rangle =& -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\langle\hat{I}(\tau)\hat{I}\rangle - 2g^2\langle c_L^\dagger c_L(\tau)\hat{I}\rangle + 2g^2\langle c_R^\dagger c_R(\tau)\hat{I}\rangle \\ =& -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\Big(\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle^2\Big) \\ & - 2g^2\Big(\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle\langle\hat{n}_L\rangle\Big) \\ & + 2g^2\Big(\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle\langle\hat{n}_R\rangle\Big)\end{aligned} \tag{59}$$

Similar step is performed to Eq. (56)-(57) giving

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\Big\rangle =& \langle\hat{I}(\tau)\hat{I}\rangle - \Gamma_L\langle c_L^\dagger c_L(\tau)\hat{I}\rangle + \Gamma_L f_L(\epsilon)\langle\hat{I}\rangle \\ =& \Big(\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle^2\Big) - \Gamma_L\Big(\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle\langle\hat{n}_L\rangle\Big) + \Gamma_L f_L(\epsilon)\langle\hat{I}\rangle\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle\Big\rangle =& -\langle\hat{I}(\tau)\hat{I}\rangle - \Gamma_R\langle c_R^\dagger c_R(\tau)\hat{I}\rangle + \Gamma_L f_L(\epsilon)\langle\hat{I}\rangle \\ =& -\Big(\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle^2\Big) - \Gamma_R\Big(\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle + \langle\hat{I}\rangle\langle\hat{n}_R\rangle\Big) + \Gamma_R f_R(\epsilon)\langle\hat{I}\rangle\end{aligned} \tag{61}$$

Now Eq. (59)-(61) are expressed in terms of the operators $\Delta\hat{I}$, $\Delta\hat{n}_L$ and $\Delta\hat{n}_R$.

These equations are simplified further by using the equations of motion for $\langle\hat{I}\rangle$, $\langle c_L^\dagger c_L\rangle$ and $\langle c_R^\dagger c_R\rangle$, given in Eq. (35)-(37). The equations of motions take the zero value in the steady state.

Equation (59) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle =& -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\Big(\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle^2\Big) \\
& -2g^2\Big(\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle\langle n_L\rangle\Big) \\
& +2g^2\Big(\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle\langle n_R\rangle\Big) \\
=& -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle-2g^2\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle+2g^2\langle\Delta n_R(\tau)\Delta\hat{I}\rangle \\
& +\langle\hat{I}\rangle\big(-\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\langle\hat{I}\rangle-2g^2\langle n_L\rangle+2g^2\langle n_R\rangle\big) \\
=& -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle-2g^2\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle+2g^2\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle
\end{aligned} \tag{62}$$

Equation (60) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle =& \Big(\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle^2\Big)-\Gamma_L\Big(\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle\langle\hat{n}_L\rangle\Big)+\Gamma_L f_L(\epsilon)\langle\hat{I}\rangle \\
=& \langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle-\Gamma_L\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle\Big(\langle\hat{I}\rangle-\Gamma_L\langle\hat{n}_L\rangle+\Gamma_L f_L(\epsilon)\Big) \\
=& \langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle-\Gamma_L\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle
\end{aligned} \tag{63}$$

Equation (61) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle =& -\Big(\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle^2\Big)-\Gamma_R\Big(\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle\langle\hat{n}_R\rangle\Big)+\Gamma_R f_R(\epsilon)\langle\hat{I}\rangle \\
=& \langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle-\Gamma_R\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle+\langle\hat{I}\rangle\Big(-\langle\hat{I}\rangle-\Gamma_R\langle\hat{n}_R\rangle+\Gamma_R f_R(\epsilon)\Big) \\
=& -\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle-\Gamma_R\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle
\end{aligned} \tag{64}$$

Thus, the equations of the motion for the two-point correlators of $\Delta\hat{I}(\tau)$, $\Delta\hat{n}_L(\tau)$ and $\Delta\hat{n}_R(\tau)$

are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle = -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle - 2g^2\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle + 2g^2\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle \tag{65}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle = \langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle - \Gamma_L\langle\Delta\hat{n}_L(\tau)\Delta\hat{I}\rangle \tag{66}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle = -\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle - \Gamma_R\langle\Delta\hat{n}_R(\tau)\Delta\hat{I}\rangle \tag{67}$$

Next, the variables X, Y and Z are introduced.

$$X = \int_0^\infty \mathrm{d}\tau\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle \tag{68}$$

$$Y = \int_0^\infty \mathrm{d}\tau\langle\Delta n_L(\tau)\Delta\hat{I}\rangle \tag{69}$$

$$Z = \int_0^\infty \mathrm{d}\tau\langle\Delta n_R(\tau)\Delta\hat{I}\rangle \tag{70}$$

The equations of motions of the two-point correlators, given in Eq. (65)-(67) are expressed with the variables X, Y and Z and integrated giving

$$-\big\langle\Delta\hat{I}\Delta\hat{I}\big\rangle = -\frac{1}{2}\big(\Gamma_L+\Gamma_R\big)X - 2g^2Y + 2g^2Z \tag{71}$$

$$-\big\langle\Delta n_L\Delta\hat{I}\big\rangle = X - \Gamma_L Y \tag{72}$$

$$-\big\langle\Delta n_L\Delta\hat{I}\big\rangle = -X - \Gamma_R Z \tag{73}$$

where the two-point correlators evaluated at ∞ take the zero value. This holds since

$$\lim_{\tau\to\infty}\langle\Delta\hat{I}(\tau)\Delta\hat{I}\rangle = \langle\Delta\hat{I}\rangle^2 = 0 \tag{74}$$

The first equality in Eq. (74) holds as there are no correlations between $t=0$ and τ tending to infinity. The systems are statistically independent at $t=0$ when τ tends to infinity. Physically, this can be understood as the system having a limited memory. Eq. (74) being zero follows from the construction of the operators, as these operators have the expectation value zero.

Equations (43)-(50) with t=0, are substituted into the left-hand sides of Eq. (71)-(73) in order to express the expectation values with the second quantization operators. The expressions are simplified by using the fermionic anti-commutation relations.

The left-hand side of Eq. (72) becomes

$$
\begin{aligned}
-\big\langle \Delta\hat{n}_L \Delta\hat{I}\big\rangle &= -\Big\langle \big(\hat{n}_L - \langle\hat{n}_L\rangle\big)\big(\hat{I} - \langle\hat{I}\rangle\big)\Big\rangle \\
&= ig\langle c_L^\dagger c_R\rangle + \langle c_L^\dagger c_L\rangle\langle\hat{I}\rangle \\
&= ig\langle c_L^\dagger c_R\rangle + \langle c_L^\dagger c_L\rangle\big(-ig\langle c_L^\dagger c_R\rangle + ig\langle c_R^\dagger c_L\rangle\big) \\
&= ig\langle c_L^\dagger c_R\rangle - ig2\langle c_L^\dagger c_L\rangle\langle c_L^\dagger c_R\rangle \\
&= ig\langle c_L^\dagger c_R\rangle\big(1 - 2\langle c_L^\dagger c_L\rangle\big) \qquad (75)
\end{aligned}
$$

The left-hand side of Eq. (73) becomes

$$
\begin{aligned}
-\big\langle \Delta\hat{n}_R \Delta\hat{I}\big\rangle &= -\Big\langle \big(n_R - \langle\hat{n}_R\rangle\big)\big(\hat{I} - \langle\hat{I}\rangle\big)\Big\rangle \\
&= -ig\langle c_R^\dagger c_L\rangle + \langle c_R^\dagger c_R\rangle\langle\hat{I}\rangle \\
&= ig\langle c_L^\dagger c_R\rangle\big(1 - 2\langle c_R^\dagger c_R\rangle\big) \qquad (76)
\end{aligned}
$$

where the result given in Eq. (24) is used to simplify the expression.

Equation (71) becomes

$$
\begin{aligned}
-\big\langle \Delta\hat{I}^2\big\rangle &= -\Big\langle \big(\Delta\hat{I} - \langle\Delta\hat{I}\rangle\big)\big(\hat{I} - \langle\hat{I}\rangle\big)\Big\rangle \\
&= \langle\hat{I}\rangle^2 - \langle\hat{I}^2\rangle \qquad (77)
\end{aligned}
$$

where $\langle\hat{I}^2\rangle$ is

$$
\begin{aligned}
\langle\hat{I}^2\rangle &= \Big\langle \big(-igc_L^\dagger c_R + igc_R^\dagger c_L\big)^2\Big\rangle \\
&= g^2\langle c_L^\dagger c_R c_R^\dagger c_L + c_R^\dagger c_L c_L^\dagger c_R\rangle \qquad (78)
\end{aligned}
$$

Wick's theorem gives an identity which is applied to Eq. (78).

$$
\langle c_i^\dagger c_j c_k^\dagger c_r\rangle = \langle c_i^\dagger c_j\rangle\langle c_k^\dagger c_r\rangle + \langle c_i^\dagger c_r\rangle\langle c_j c_k^\dagger\rangle \qquad (79)
$$

Equation (78) becomes,

$$
\langle\hat{I}^2\rangle = g^2\big(2\langle c_R^\dagger c_L\rangle\langle c_L^\dagger c_R\rangle + \langle c_L^\dagger c_L\rangle\langle c_R c_R^\dagger\rangle + \langle c_R^\dagger c_R\rangle\langle c_L c_L^\dagger\rangle\big) \qquad (80)
$$

Equation (80) is simplified by using the result given in Eq. (24) and that $\langle c_i^\dagger c_i\rangle = -\langle c_i c_i^\dagger\rangle + 1$ for $i = L$, R. This holds due to the anti-commutation relation given in Eq. (3).

$$
\langle\hat{I}^2\rangle = g^2\big(-2\langle c_L^\dagger c_R\rangle^2 - 2\langle c_L^\dagger c_L\rangle\langle c_R^\dagger c_R\rangle + \langle c_L^\dagger c_L\rangle + \langle c_R^\dagger c_R\rangle\big) \qquad (81)
$$

The result given in Eq. (24) is used to simplify $\langle \hat{I} \rangle^2$. This gives

$$\begin{aligned}\langle \hat{I} \rangle^2 &= \big(- ig\langle c_L^\dagger c_R \rangle + ig\langle c_R^\dagger c_L \rangle \big)^2 \\ &= \big(- i2g\langle c_L^\dagger c_R \rangle \big)^2 \\ &= -4g^2 \langle c_L^\dagger c_R \rangle^2 \end{aligned} \tag{82}$$

Equations (75)-(77) and Eq. (71)-(73) give a linear system of equations.

$$g^2\big(2\langle c_L^\dagger c_L \rangle \langle c_R^\dagger c_R \rangle - \langle c_R^\dagger c_R \rangle - \langle c_L^\dagger c_L \rangle - 2\langle c_L^\dagger c_R \rangle^2\big) = -\frac{1}{2}\big(\Gamma_L + \Gamma_R\big)X - 2g^2 Y + 2g^2 Z \tag{83}$$

$$ig\langle c_L^\dagger c_R \rangle \big(1 - 2\langle c_L^\dagger c_L \rangle\big) = X - \Gamma_L Y \tag{84}$$

$$ig\langle c_L^\dagger c_R \rangle \big(1 - 2\langle c_R^\dagger c_R \rangle\big) = -X - \Gamma_R Z \tag{85}$$

The left-hand side of the linear system of equations is expressed in terms of $\langle c_L^\dagger c_L \rangle$, $\langle c_R^\dagger c_R \rangle$ and $\langle c_L^\dagger c_R \rangle$, given in Eq. (29)-(30), which are already solved for the steady state. Therefore, it is possible to notice that the left-hand sides of Eq. (83)-(85) are real values.

The linear system of equations is solved giving

$$X = \frac{A + B}{\big(\Gamma_L + \Gamma_R\big)\big(- \Gamma_L \Gamma_R / 2 - 2g^2\big)} \tag{86}$$

where A and B are

$$A = \Gamma_L \Gamma_R g^2 \big(2\langle c_L^\dagger c_L \rangle \langle c_R^\dagger c_R \rangle - \langle c_R^\dagger c_R \rangle - \langle c_L^\dagger c_L \rangle - 2\langle c_L^\dagger c_R \rangle^2\big) \tag{87}$$

$$B = 2g^2 \Big(- ig\langle c_L^\dagger c_R \rangle \big(1 - 2\langle c_L^\dagger c_L \rangle\big)\Gamma_R + ig\langle c_L^\dagger c_R \rangle \big(1 - 2\langle c_R^\dagger c_R \rangle\big)\Gamma_L \Big) \tag{88}$$

Equation (86) becomes the following when Eq. (28)-(30) are substituted in.

$$X = \frac{2g^2 \Gamma_L \Gamma_R (f_R(1 - f_L) + f_L(1 - f_R))}{(4g^2 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R)} - \frac{16(g^2 \Gamma_L \Gamma_R (f_L - f_R))^2 (\Gamma_L \Gamma_R + 4g^2 + (\Gamma_L + \Gamma_R)^2)}{((4g^2 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R))^3} \tag{89}$$

Thus, the noise in the system, given in Eq. (42) becomes the following when X is substituted into $S = 2\mathrm{Re}X$.

$$S = \frac{4g^2 \Gamma_L \Gamma_R (f_R(1 - f_L) + f_L(1 - f_R))}{(4g^2 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R)} - \frac{32(g^2 \Gamma_L \Gamma_R (f_L - f_R))^2 (\Gamma_L \Gamma_R + 4g^2 + (\Gamma_L + \Gamma_R)^2)}{((4g^2 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R))^3} \tag{90}$$

4 Results

4.1 Current

The expression of the current is analyzed. It is possible to determine from Eq. (18) what gives the direction to the current and some of the zero values that the current takes. The sign of the current I, which gives the direction of the flow, arises from the factor $f_L - f_R$. Thus, the higher value which f_L or f_R takes determines the direction of the current.

The temperature is the parameter of the Fermi-function, which determines the direction of the current when $\epsilon_L = \epsilon_R$, $\mu_L = \mu_R$ for $\epsilon > \mu$. It can be seen from Eq. (8) that when the temperature increases, the magnitude of the Fermi-function increases. Therefore, the current will flow from the hotter bath to the colder one. It is noted that the tunneling rates appear symmetrically in the expression for the current, meaning that interchanging the labels of the tunneling rates Γ_L and Γ_R has no impact on the current. Hence, the tunneling rates only change the magnitude of the current but do not contribute to the direction of the current.

The current I takes a zero value if $g = 0$. In this case, the Hamiltonian of the system does not have a term which would give a rise to the coherences in the density matrix. In other words, provide a coupling between the dots. If there is no coupling, the electrons will not tunnel between the dots.

Also, the current takes a zero value if $f_L = f_R$. The reason behind this is that the two baths and the two quantum dots are equal up to the tunneling rates Γ_L and Γ_R. Thus, there will be no favoured direction for the current to flow. It is noted that that by the current it is meant the average current in the steady state. When $f_L = f_R$ there will be on average an equal number of electrons passing through the system in the opposite directions.

4.1.1 Current's dependency on the parameters

The current is plotted as a function of Γ_L, Γ_R and g to analyze the behaviour of the current when the parameters are varied. It is studied what happens in the large limit of the parameters and whether there is a maximum value that the current can take.

The current is modified into a dimensionless equation because dimensionless plots give a more general description of the current's behaviour as no dimensions are fixed.

The steady state current given in Eq. (32) is divided by $g(f_L - f_R)$ to obtain a dimensionless current.

$$\begin{aligned}\frac{I}{g(f_L - f_R)} &= \frac{4g^2\Gamma_R\Gamma_L}{(4g^2 + \Gamma_L\Gamma_R)(\Gamma_L + \Gamma_R)} \cdot \frac{1/g^4}{1/g^3} \\ &= \frac{4(\Gamma_R/g)(\Gamma_L/g)}{\big(4 + (\Gamma_L/g)(\Gamma_R/g)\big)\big((\Gamma_L/g) + (\Gamma_R/g)\big)}\end{aligned} \tag{91}$$

The dimensionless current, $I/g(f_L - f_R)$, is plotted in a three-dimensional figure.

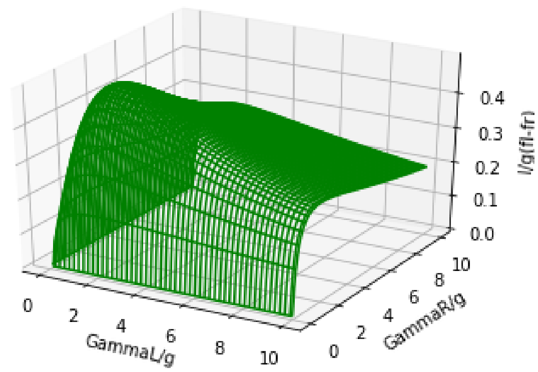


Figure 2: Dimensionless current, $I/g(f_L - f_R)$, as a function of Γ_L/g and Γ_R/g

Figure (2) shows that the current is a concave function. Therefore, it takes a visible global maximum value. This value is 1/2 and it occurs at the point $(\Gamma_L/g, \Gamma_R/g) = (2, 2)$.

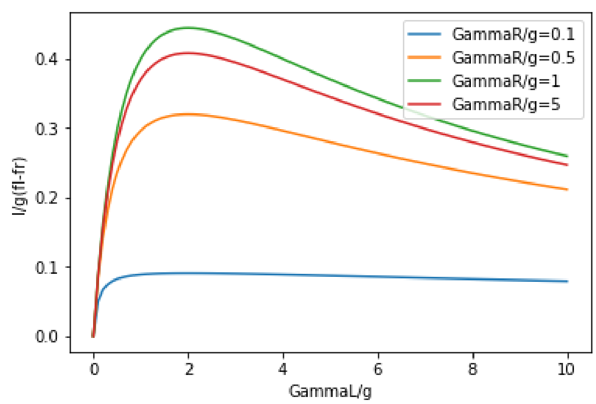


Figure 3: Plots of $I/g(f_L - f_R)$ as a function of Γ_L/g, while Γ_R/g takes the constant values. Γ_R/g is set to take values 0.1, 0.5, 1.0 and 5, which are labelled in the figure.

In Figure (3), the current's dependency of Γ_L is shown in a two-dimensional figure. $I/g(f_L - f_R)$ is plotted as a function of Γ_L/g while Γ_R/g is kept constant. The difference between the curves becomes more visible when several curves are plotted in the same figure. Keeping Γ_R/g constant is enough to see the current's dependency on either of the tunneling rates Γ_L and Γ_R, since Γ_L/g and Γ_R/g appear identically in Eq. (91). Keeping Γ_L/g constant and plotting $I/g(f_L - f_R)$ as a function of Γ_R/g would give the same group of curves.

The current tending to zero for large transition rates is verified analytically from Eq. (91).

$$\begin{aligned}\lim_{\Gamma_R/g\to\infty}\frac{I}{g(f_L-f_R)} &= \lim_{\Gamma_L/g\to\infty}\frac{I}{g(f_L-f_R)}\\ &= \lim_{\Gamma_L/g\to\infty}\frac{4(\Gamma_R/g)(\Gamma_L/g)}{\big(4+(\Gamma_L/g)(\Gamma_R/g)\big)\big((\Gamma_L/g)+(\Gamma_R/g)\big)}\\ &= 0 \end{aligned} \tag{92}$$

The quantum mechanical properties of the system are behind the physical reason why the current tends to zero in the large values of the tunneling rates Γ_L and Γ_R. The tunneling rate can be interpreted as an electron's attempt to enter a dot. During this attempt, it is evaluated if the dot is empty or not. Thus, the occupation of the dot is fully known. This can be understood as the bath conducting a measurement on the dot. The quantum mechanical phenomenon of an electron being in a superposition cannot take place when the occupation of the dot is fully known. Therefore, if the measurement on the dot is conducted repeatedly by the bath, it follows that the electron being in a superposition occurs less frequently. This means that the number of electrons tunneling between the dots will decrease, thus the current decreases. Eventually, the current tends to zero in the large limit of the tunneling rate. This phenomenon is called the quantum Zeno effect.

The current, given in Eq. (32), is modified into a dimensionless quantity I/Γ to study the current's dependency of the tunneling strength g. Eq. (32) is divided by Γ and it is set that $\Gamma_L = \Gamma_R = \Gamma$ and $f_L - f_R = 1$.

$$\begin{aligned} I/\Gamma &= \frac{2g^2\Gamma^2}{4g^2\Gamma+\Gamma^3}\cdot\frac{1/\Gamma^4}{1/\Gamma^3}\\ &= \frac{2(g/\Gamma)^2}{4(g/\Gamma)^2+1} \end{aligned} \tag{93}$$

The current's dependency of g is illustrated by plotting Eq. (93) as a function of g/Γ.

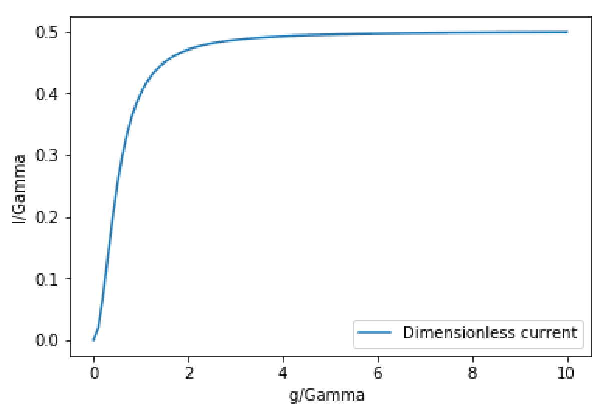


Figure 4: Dimensionless current I/Γ as a function of g/Γ with the parameters $f_L - f_R = 1$ and $\Gamma_L = \Gamma_R = \Gamma$

In Figure (4) the current seems to converge to 0.5 when the ratio g/Γ takes large values. This is verified analytically by taking the limit of Eq. (93).

$$\lim_{g\to\infty} I/\Gamma = \lim_{g\to\infty} \frac{2(g/\Gamma)^2}{4(g/\Gamma)^2+1} = 0.5 \tag{94}$$

When g increases, the tunneling between the quantum dots becomes more likely. Physically, the current tending to a constant value for large g can be understood as the current being limited by the process of the electrons entering and leaving the baths. The difference between magnitudes of the Fermi-functions, f_L and f_R, gives the limitations for the maximum value that the current can take for a given system, even when g tends to the infinity.

4.2 Fluctuations of the current

For simplicity, the noise is made dimensionless. It is done by setting that $\Gamma_L = \Gamma_R = \Gamma$ and dividing Eq. (90) by Γ. This gives

$$\begin{aligned} S/\Gamma &= \frac{4g^2\Gamma^2(f_L(1-f_R)+f_R(1-f_L))}{8g^2\Gamma+2\Gamma^3}\cdot\frac{1/\Gamma^4}{1/\Gamma^3} - \frac{32g^4\Gamma^4(f_L-f_R)^2(\Gamma^2+4g^2+4\Gamma^2)}{(8g^2\Gamma+2\Gamma^3)^3}\cdot\frac{1/\Gamma^{10}}{1/\Gamma^9} \\ &= \frac{2(g/\Gamma)^2(f_L(1-f_R)+f_R(1-f_L))}{4(g/\Gamma)^2+1} - \frac{32(g/\Gamma)^4(f_L-f_R)^2(5+4(g/\Gamma)^2)}{(8(g/\Gamma)^2+2)^3} \end{aligned} \tag{95}$$

The dimensionless noise and the dimensionless current are plotted as a function of g/Γ, with the parameters $f_L = 1$ and $f_R = 0$.

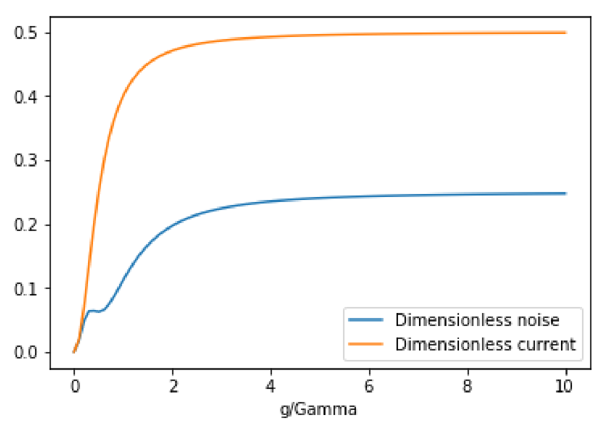


Figure 5: Dimensionless current and noise plotted as functions of g/Γ with the parameters $f_L = 1$ and $f_R = 0$ and $\Gamma_L = \Gamma_R = \Gamma$

From Figure (5) it can be seen that the noise increases as the current increases, except around $g/\Gamma = 0.7$. Around this value the noise exhibits a plateau.

The dimensionless noise, given in Eq. (95), is plotted as a function f_L and f_R while $g/\Gamma = 1$.

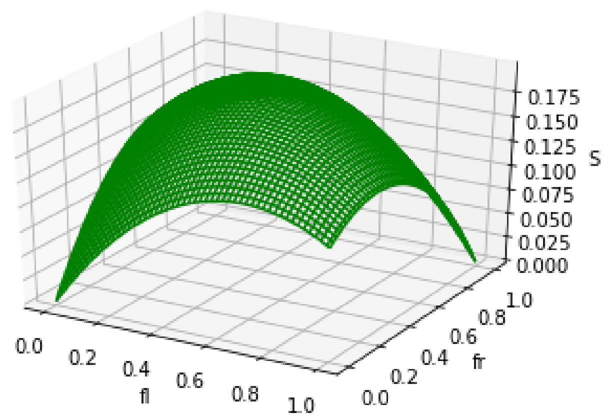


Figure 6: The dimensionless noise as a function of f_L and f_R with the parameter $g/\Gamma = 1$ and $\Gamma_L = \Gamma_R = \Gamma$.

In Figure (6), it can be seen that the noise is a concave function and takes a visible global maximum value. This value is equal to 1/5 and it occurs at the point $(f_L, f_R) = (1/2, 1/2)$. With these values of the Fermi-functions, the average current will be zero.

To study the connection between the noise and the current, the Fano factor is introduced. The Fano factor is defined as

$$F = S/\langle \hat{I} \rangle \tag{96}$$

The Fano factor is a useful quantity to plot since it makes it visible how the rates of the noise and current change when the parameters are varied. It also gives information about the particles that the system contains, based on what values the Fano factor will take. The Fano factor is plotted as a function of g/Γ. The noise and the current appearing in Eq. (96) are set dimensionless quantities to preserve the dimensionless. It is set that $f_L = 1$ and $f_R = 0$.

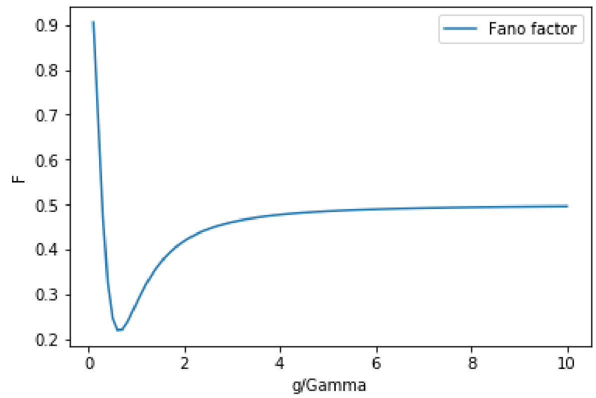


Figure 7: The Fano factor plotted as a function of g/Γ with the parameters $f_L = 1$ and $f_R = 0$ and $\Gamma_L = \Gamma_R = \Gamma$

From Figure (7), it can be seen that the Fano factor will be less than 1 for all values of g/Γ, which is a property of a fermionic system at low temperatures. This happens because only one electron can occupy a quantum dot at a time.

4.2.1 Thermodynamic uncertainty relations

The system can be analyzed further by a TUR. The TURs are thermodynamic inequalities that provide bounds for thermodynamical quantities. These bounds are originally derived for classical systems. It is expected that in this quantum system a TUR could be violated due to the quantum coherence. [13] This can be assumed by looking at the plateau observed in Figure (5). A similar plateau was discovered earlier when coherent quantum tunneling was studied[14]. The TUR studied in this thesis is the uncertainty of the steady state current, which is bounded below by the entropy production in the system[5, 13]. The TUR is the following inequality

$$\frac{Var(I)}{\langle I \rangle^2} \geq \frac{2k_B}{\langle \dot{s} \rangle} \tag{97}$$

where $Var(I)$ is the variance of the current, $\langle I \rangle$ is the average current and $\langle \dot{s} \rangle$ is the entropy production. In the long time limit Eq. (97) is expressed with noise S.

$$\frac{S}{\langle I \rangle^2} \geq \frac{2k_B}{\langle \dot{s} \rangle} \tag{98}$$

The TUR given in Eq. (98) gives a bound for the systems in the long time limit. The temperatures of the baths are set equal, $T_L = T_R = T$, in order to study a simple system which does not require additional computations of heat currents in the system. This gives that the entropy production can be written as

$$\langle \dot{s} \rangle = \langle I \rangle (\mu_L - \mu_R)/T \tag{99}$$

Equation (99) is substituted into the TUR, given in Eq. (98). The expression is multiplied by $\langle I \rangle$ and the definition of the Fano factor is used. This gives the following expression for the TUR

$$F \geq \frac{2k_B T}{|eV|} \tag{100}$$

where $eV = \mu_L - \mu_R$. The left-hand side is simply the Fano factor and the right-hand side provides a classical bound. This difference, $D = F - 2k_b T/eV$, is plotted as a function of $eV/k_B T$ with parameters $\epsilon_L = \epsilon_R = 0$, $\mu_R = -\mu_L = eV/2$ and $g = 0.65\Gamma$. These parameters give a region where the TUR is violated[13].

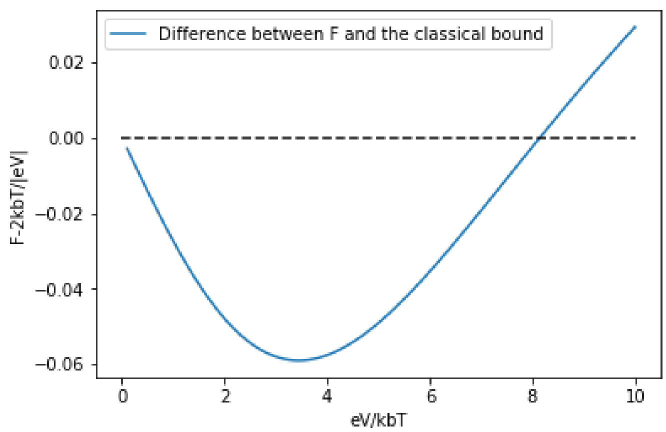
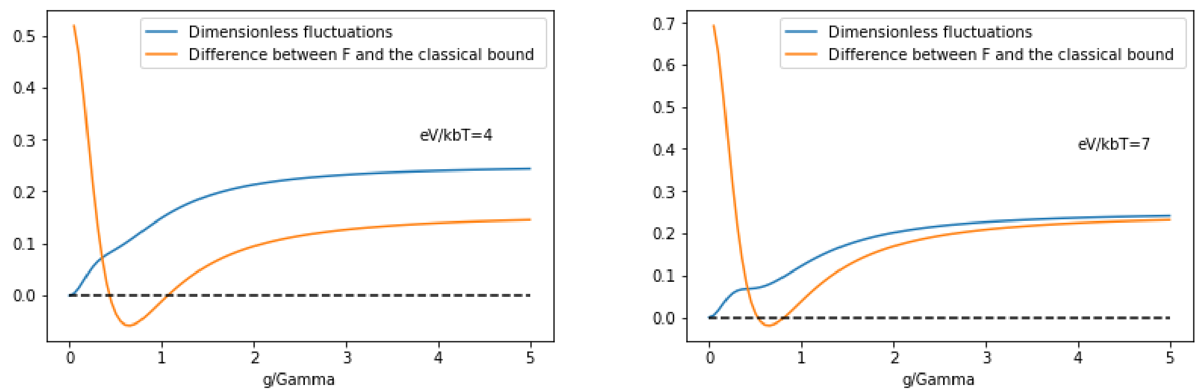


Figure 8: The difference between the Fano factor and the classical bound, $D = F - 2k_B T$, plotted as a function of $eV/k_B T$. The parameters used are $\epsilon_L = \epsilon_R = 0$, $\mu_R = -\mu_L = eV/2$ and $g = 0.65\Gamma$. The inequality is violated when the curve is below the dashed line.

As expected, because the Fano factor takes values near zero, the TUR is violated in the studied region, which can be seen in Figure (8). The violations happen when the curve is below the dashed line. Next, the connection between the plateau and the violations of the TUR is studied. This connection seems reasonable as the Fano factor takes smaller values when the fluctuations increase slower than the current. This happens in the plateau, where the derivative of the fluctuations is close to zero.

The fluctuations and the difference D are plotted as a function of g/Γ with the same set of parameters. The quantities are plotted for two different values of $eV/k_B T$ separately to observe what happens when $eV/k_B T$ is varied. First, a value of $eV/k_B T$ near the minimum of D is chosen, $eV/k_B T = 4$, and then a larger value away from the minimum, $eV/k_B T = 7$, is chosen.



(a) $eV/k_BT = 4$ (b) $eV/k_BT = 7$

Figure 9: The fluctuations S and the difference $D = F - 2k_BT/eV$ plotted as a function of g/Γ. The parameters used are $\epsilon_L = \epsilon_R = 0$, $\mu_R = -\mu_L = eV/2$ and $g = 0.65\Gamma$. The inequality is violated when the curve is below the dashed line.

Figure (9) shows that the violation of the TUR occurs near the plateau. However, it can be noticed from Figure (9a) that when $eV/k_BT = 4$ the plateau is not clearly visible, even though $eV/k_BT = 4$ gave a large violation of the TUR in Figure (8). The value $eV/k_BT = 4$, corresponds to the region with small difference between the chemical potentials, as $eV = \mu_L - \mu_R$, or a large value for the temperature.

When the value of eV/k_BT is increased, the plateau becomes more visible, which can be seen from Figure (9b). However, the violations of the TUR become smaller in that case, and eventually disappear when eV/k_BT is increased further. This region corresponds to a larger difference between the chemical potentials or a smaller value for the temperature, resulting a large entropy production, which can be seen in Eq. (99).

The physical reason behind the violations of the TURs in this system is quantum mechanics, which gives a rise to the current and therefore, to the significant fluctuations in the system. In the previously published paper [14], where the plateau was observed earlier, the effect of the coherences on the current was studied. This gives evidence that the coherences could possible be behind the plateau. Even though the connection is not that clear since the plateau smears out in the region of the higher violations of the TUR.

5 Conclusions

In this thesis, a study of a double quantum dot was conducted. The thesis began with presenting the theoretical concepts that are essential for understanding the content of this thesis. The central concepts are the second-quantization formalism and the density matrix formalism. The mathematical model to describe a system was introduced starting from the density matrix and advancing to the Hamiltonian and the master equation.

Once the preliminary knowledge was obtained, the current was calculated by using the second-quantization operators. The calculations of current involved two steps. First, an assumption of the steady state was made to obtain a linear system of equations. Secondly, the terms that constitute the current were identified to be able to define the current. The stochastic nature of the process was emphasized in order to understand how the flow of electrons gives a rise to the current through the system.

The noise was calculated by applying the quantum regression theorem to the equations of motions obtained from the calculations of the current. The time-integrals of the two-point correlatiors were evaluated to arrive to the expression for the noise.

The expression of the current was analyzed. It was found that the current divided by g takes a maximum value which is bound by the magnitudes of the Fermi-functions of the baths, regardless the magnitudes of the other parameters. When the noise was plotted as a function of g/Γ, it was found that there is a plateau in a region with small g/Γ. The plateau was studied by looking at violations of the TUR, which was expressed as a difference between the Fano factor and a classical bound. A possible connection between the Fano factor and the plateau was found. Even though the violations of the TUR are the strongest when the plateau becomes smeared out and when the plateau is the most visible the violations of the TUR became weaker, hence further studies would be needed to understand the physical phenomenon behind the existence of the plateau.

6 Outlook

Further studies on the double quantum dot system could be made. It could be studied when the coherences are the highest in the system and what are the parameters that affect the coherences the most. The connection between the magnitude of the current and the noise and the coherences could be particularly interesting.

In this sytem the energies of the double quantum dots were set equal. The double quantum dots with different energies could be an interesting subject of further studies. The TUR of such a system could be studied.

Appendices

A Trace

The trace is defined as a sum of diagonal elements of a matrix. The expectation value of an operator acting on a density matrix can be found by calculating the trace. The trace has the following computational rules.

The trace is linear.

$$\begin{aligned} \langle \hat{A} + C \rangle &= \mathrm{Tr}\left\{ \hat{A}\rho \right\} + \mathrm{Tr}\{C\rho\} \\ &= \mathrm{Tr}\left\{ \hat{A}\rho \right\} + C\,\mathrm{Tr}\{\rho\} \\ &= \mathrm{Tr}\left\{ \hat{A}\rho \right\} + C \end{aligned} \tag{101}$$

where $\hat{A}$ is an operator, C is a constant and ρ is the density matrix.

The trace is invariant under cyclic permutations.

$$\mathrm{Tr}\left\{ \hat{A}\hat{B}\rho \right\} = \mathrm{Tr}\left\{ \hat{B}\rho\hat{A} \right\} \tag{102}$$

where $\hat{A}$ and $\hat{B}$ are operators and ρ is the density matrix.

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