Impact of an interest rate coverage in a life insurance company

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Abstract

This master thesis proposes different methods to price interest rate vanilla derivatives. Those derivatives are priced in a negative interest rate environment, where the standard models are not applicable anymore. Those methods are tested on interest rate derivatives that were in portfolio at BNP Paribas Cardif as of december 31st 2018, mainly Cross Currency Swaps, CAPs and Swaptions. The prices obtained through the different methods will be compared with prices given by a third party.

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Introduction

This document is the report of my degree project at LTH Faculty of Engineering of Lund University. This master thesis is the final assignment of a two-year Master's program in Engineering Physics, worth 30 credits over 120 for the whole program. My specialization in this program is Financial Modelling, and as a consequence, the field concerned by this study is Financial Mathematics. The work for this project has been performed during my six months internship within BNP Paribas Cardif, from july 2019 to january 2020.

The purpose of this study is to find new methods to price (mainly) interest rate derivatives in a negative rate environment. In that regard, several methods have been suggested, implemented in VBA, and challenged on a benchmark of real derivatives from a BNP Paribas Cardif's portfolio.

The first chapter introduces the economical and financial background of BNP Paribas Cardif, the company I worked in. It then describes in details the goals and the constraints of the market risk team to better understand their need of a good estimation of the derivative risk coverage.

Chapter 2 explains the selected mathematical models. It details the mathematical justification which leads to the price formula.

Chapter 3 describes the different steps composing the code. It gives details on how the mathematical models from chapter 2 are in practice implemented.

Chapter 4, in the end, presents, analyzes, and compares the results obtained through each method.

Chapter 1

Background

1.1 BNP Paribas Cardif

BNP Paribas is a French international banking group. Operating in 77 countries on all continents, it is the largest bank in the Eurozone and the 8th worldwide by total assets. The group is split into three business streams, the retail banking activities, which represents 70% of the revenues, the Corporate and Institutional banking (Global investment banking) and the Investment solutions unit, which includes asset management, real estate, insurance business (insurance and wealth management).

BNP Paribas Cardif is also an international player which represents 13% of the group total pre tax income (1 479 million of euros). Figure 1 shows the position of BNP Paribas Cardif in the group.

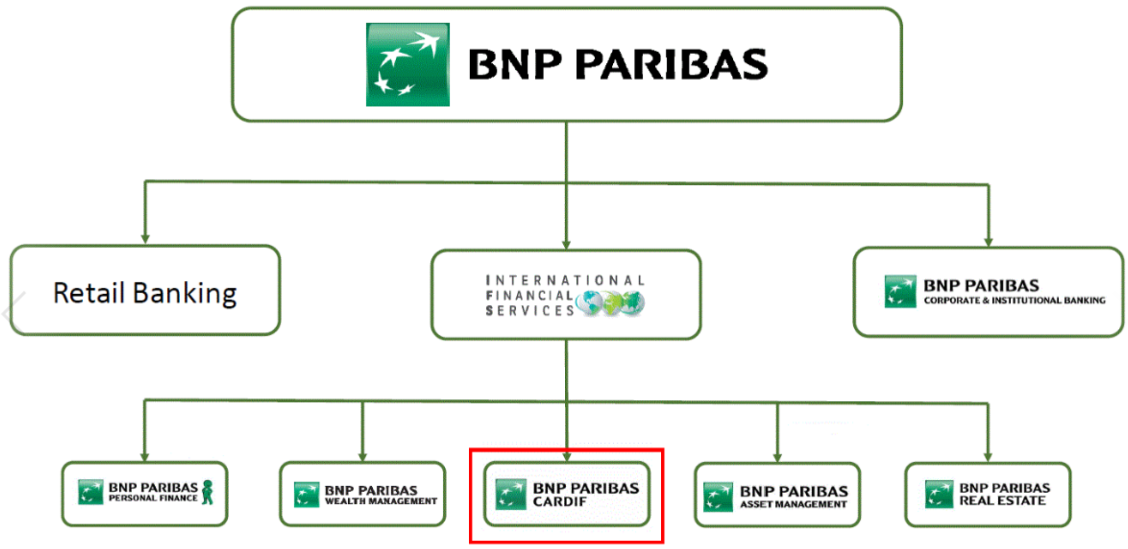

Figure 1 : Position of BNP Paribas Cardif in the group

Insurance activities can be split into two families:

- Damage insurance which corresponds to traditional insurance activities like property or health coverage. The client pays a premium and get an insurance against the risk he wished to be protected.
- Life insurance which corresponds to savings. The premium is invested on the market and parts of the dividends are paid to the client. Life insurance activities represent 95% of the total asset under management of AuM (227.4 billion out of 239.2).

Most life insurance assets come from French businesses. In France there exists two types of contracts:

- General funds. Every year, the client is given a credited rate depending of the return generated by the assets with a 0% guaranteed rate, meaning that the credited rate (net of fees) cannot fall below zero.
- Unit linked. In these contracts, the market risk is taken by the client.

The risk is covered by the insurer only in the case of general funds. It becomes quite constraining when 239 billion euros are under management in an environment where rates are close to zero or even negative. In order to control the risk and therefore the solvency of European's insurance companies and to harmonize the regulation at a European level, the European commission voted new regulations in 2009 called Solvency II. Cardif falls under these new regulations. Solvency II came into application on January 1st 2016.

1.2 Solvency II

Solvency II directives require setting aside a certain amount of capital to meet exceptional losses that could arise from insurance activities and that could threaten the solvency of the insurer. In a low (or even negative) interest rate environment, it forces European insurers to rethink their business model. They are based on three pillars:

- The first pillar consists of quantitative requirements. The solvency of an insurance company is determined by the SCR. The SCR, for Solvency Capital Requirement, is the amount of own fund an insurance company must hold to ensure its solvency during one year when facing events with a return period inferior to 200 years.
- A second pillar sets out requirements for the governance and risk management of insurers, as well as for the effective supervision of insurers.
- A third pillar focuses on disclosure and transparency requirements.

The SCR depends of four main types of risk:

- **Underwriting risk** covers the risk arising from the underwriting of insurance, associated with both the perils covered and the processes followed in the conduct of the business (definition from EIOPA). It can be separated into life insurance underwriting, non-life insurance underwriting and health underwriting.
- **Market risk** arises from the volatility of market prices of financial instruments. Exposure to market risk is measured by the impact of movements in the level of financial variables such as equity prices, interest rates, real estate prices and exchange rates (definition from EIOPA).

- **Counterparty risk:** reflects possible losses due to unexpected default of the counterparties and debtors of undertakings over the forthcoming twelve months. The scope of the counterparty default risk module includes risk-mitigating contracts, such as reinsurance arrangements, securisations and derivatives (definition from EIOPA).
- **Operational risk** is the risk of loss arising from inadequate or failed internal processes, or from personnel and systems, or from external events.

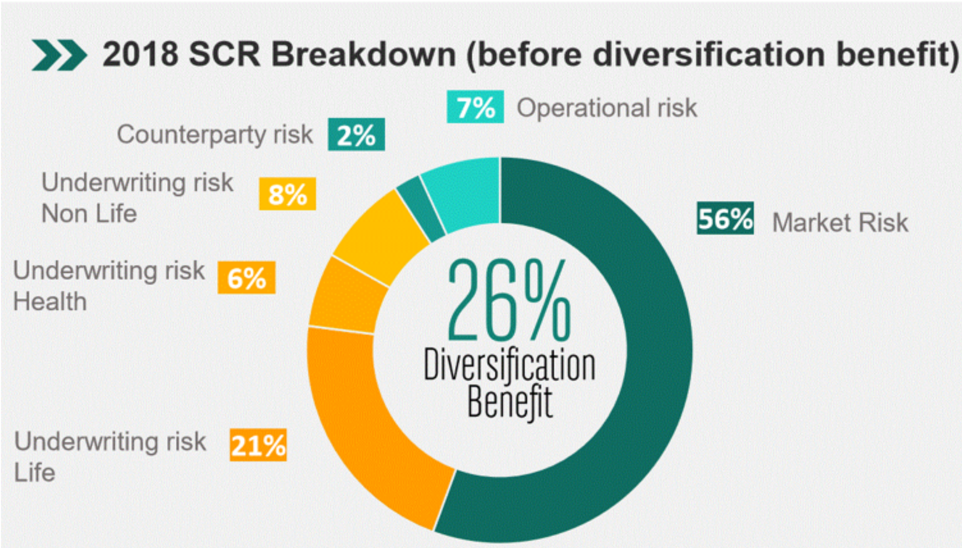

Figure 2 : SCR composition of BNP Paribas Cardif in 2018

For a life insurance company such as BNP Paribas Cardif, the most important risk faced arise from market fluctuations. There exists different types of market risk:

- Interest rate risk (free interest rate risk here)
- Equity risk
- Property risk
- Spread risk (can also be part of interest rate risk)
- Currency risk (FX)

To reduce its capital requirement and therefore its (market) SCR, Solvency II allows the insurer to use hedging tools and in particular derivatives. The composition of the French portfolio in 2018 is shown in figure 3.

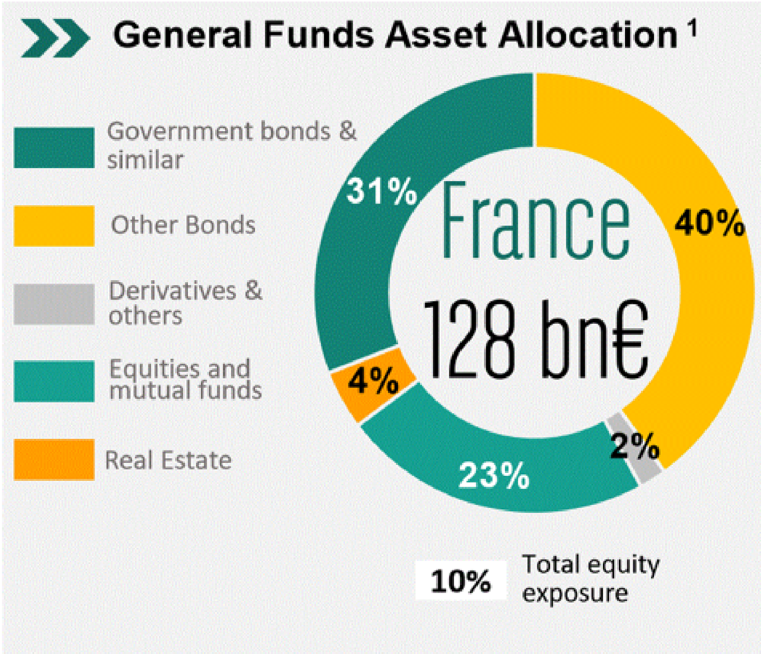

Figure 3 : French portfolio composition in 2018 (Roughly half of asset under management)

We see that more than 70% of the portfolio is composed of bonds (government + other). This means that market risk mainly arises from interest rate (and spread) variations.

My internship was done in the market risk team which is itself part of the risk department. The risk department, as stated in the solvency II directive (second pillar) acts as a second line of defense. It is there to control what the IAM department (Investment Asset Management) does on the market.

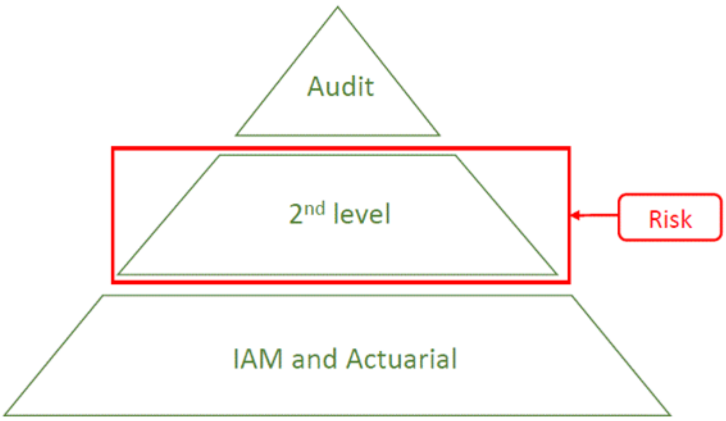

Figure 4 : position of the risk department in the company

Each year, the IAM gets directives from:

- The ALM department (part of the actuarial department) gives strategic constraints. They give an upper bound for the modified duration and the beta, and gives constraints on real estate allocations.
- Every department (such as risk, ALM or actuarial) can give additional constraints related to their fields. It is here for example that upper bounds for bonds with certain credit notations are fixed, and that a budget for derivatives is fixed.

As of today, a third party under the responsibility of the IAM department prices derivatives in Cardif. Risk also needs to price the derivatives to challenge these valorizations. The goal of my internship was to suggest different methods to price interest rates derivatives and then implement them in VBA (and Python).

BNP Cardif uses several derivatives to protect itself against interest rate variations:

- Cross Currency Swaps (CCS): This derivative is used to hedge against change rate variations. It is used as a tactical tool (for specific and identified bonds) to secure on the yield of a bond in a foreign currency by locking the change rate between euro and this currency.
- Interest rate swaps: It is used as a tactical tool to secure the coupon rate of a bond with variable coupons.
- CAP: It is used as a global tool to protect the insurer against interest rate variations.
- Swaptions: It is used as a global tool to protect the insurer against interest rate variations.

The pricing of derivatives must be included in a negative interest rate environment. The market standard model, the Black-76, is only applicable for positive interest rates. This text will present three models, all applicable in a negative rate environment. Each model needs a volatility as an input. This volatility depends of the maturity and the strike of the option considered.

Chapter 2

Change and Interest rate theory

In this section, the different derivatives that have been priced are introduced, as well as the methods used to get their value. This section focuses on three types of derivatives, Cross Currency Swaps, CAPs and Swaptions. Two more types of derivatives (interest rate swaps and bond forwards) have also been priced but they are considered as more standard and simple. The methods used to price them are described in appendix A.

2.1 Cross Currency Swap

Due to the situation of the bond market (negative rates), investors (and therefore insurers) look for better yields outside of the Eurozone. In order to secure this yield, it becomes necessary to fix the exchange rate. The cross currency swap (CCS) is a good way to do it. It corresponds to an exchange of a bond in a foreign currency with a bond in euro. Figure 5 shows how to hedge itself using a CCS (for a bond in US dollar).

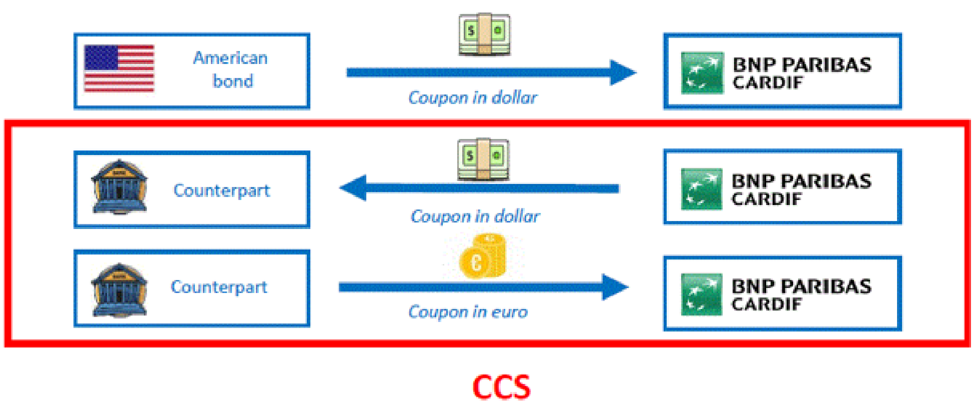

Figure 1 : *Hedge of a US dollar bond using a Cross Currency Swap*

The insurer secures its yield by paying coupons in the foreign currency and receiving coupons in euro. In the CCS considered in this study, both bonds have fixed coupon rates specified in the term sheet.

The market convention is to convert the future foreign currency cash flows in euro using the forward change rate. It is the present value, in absence of arbitrage, of the change rate at a time $t > 0$. We denote it by $\tau_X(t)$.

Let us denote by $\tau_X(0)$ the spot exchange rate (known at time 0, fixed by the market). Consider now the two following strategies:

- Convert one euro into the foreign currency, invest it at the LIBOR rate until time t, and then convert it back at time t in euro.
- Invest one euro until time t at the EURIBOR rate.

In absence of arbitrage, both strategies have the same payoff i.e:

$$\frac{1}{\tau_X(0)}(1 + Z_{LIBOR}(t))^t \ \tau_X(t) = (1 + Z_{EURIBOR}(t))^t \tag{2.1}$$

Here, $Z(t)$ is the zero coupon rate with tenor t.

Therefore:

$$\tau_X(t) = \tau_X(0) \ \left(\frac{1 + Z_{EURIBOR}(t)}{1 + Z_{LIBOR}(t)}\right)^t \tag{2.2}$$

In this work, the foreign currencies that have been used are the US dollar, the Japanese yen and the British pound.

In practice, formula (2.2) is not correct. In fact, it would if the US 3M LIBOR was equivalent to the 3M EURIBOR. To explain this, let us define basis swaps.

Definition: A basis swap is a swap where both legs have a floating rate.

This definition is very general and a basis swap can have both legs in the same currency (6M EURIBOR vs Euro treasure bills for example). Let us consider as an example the 3M US LIBOR vs 3M JPY LIBOR swap market. This market reflects the global demand for swapping from US dollar to Japanese yen. The swap is in practice not exactly 3M US LIBOR vs 3M JPY LIBOR because there is a spread added to it. In our example, in order to enter a swap where one would pay 3M US LIBOR, one would require to receive 3M JPY LIBOR plus a spread. This difference can be explained by the fact that US financial institutions may have better credit and liquidity guarantees than their Japanese counterparts and are therefore more attractive. Hence, the spread is here to reward the investor receiving yen for getting a less valuable currency. This spread is called the basis spread. If we denote by s(t) the forward basis spread at time t from the foreign currency to euro (obtained directly from Bloomberg), formula (2.2) becomes:

$$\tau_X(t) = \tau_X(0) \left(\frac{1 + Z_{EURIBOR}(t)}{1 + Z_{LIBOR}(t) + s(t)} \right)^t \tag{2.3}$$

If the foreign currency is more attractive than the euro, s(t) can be negative. This basis spread is particularly important for the Japanese yen and the British pound (it is smaller for the US dollar but still noticeable).

Once we can the forward change rate at any maturity, we can compute the price of the Cross Currency swap. To do that, we use the price of a bond with nominal N, coupon rate i and maturity T:

$$P_{BOND} = \sum_{l=1}^{T} \frac{i.N}{(1 + Z_{act}(l))^l} + \frac{N}{(1 + Z_{act}(T))^T} \tag{2.4}$$

Since the cross currency swap is the difference of two bonds with different currencies, we obtain its price by the following formula:

$$\Pi = P_{BOND\ EUR} - P_{BOND\ USD} \tag{2.5}$$

$$\Pi = \sum_{l=1}^{T} \frac{i_{EUR}.N_{EUR} - \tau_X(l).i_{USD}.N_{USD}}{(1 + Z_{act}(l))^l} + \frac{N_{EUR} - \tau_X(T).N_{USD}}{(1 + Z_{act}(T))^T} \tag{2.6}$$

2.2 CAP, Caplet and Swaption:

Interest rate options with a simple payoff are known as vanillas. We will start by defining the simplest option.

Definition: A (put) call option gives its holder the right without obligation to (sell) buy an underlying asset from the writer at a predetermined strike price K on or before an agreed future maturity date T.

If the right can be exercised before the expiry date, it is an American option. Otherwise, it is a European option.

The payoff of a European call option is:

$$P = \max(S_T - K, 0) \tag{2.7}$$

Where S_T is the price of the underlying at time T.

Similarly, the payoff of a European put option is:

$$P = \max(K - S_T, 0) \tag{2.8}$$

We will now define the vanillas we will price. The first one is called the CAP. Consider first a set of increasing times $T_0, T_1, \ldots, T_n$ with $T_1 > 0$.

Definition: A **CAP** with strike price K and notional N gives its holder at each date T_k $k = 1, \ldots, n$ the payoff:

$$P = N \; (T_k - T_{k-1}) \; \max(F_{T_k} - K, 0) \tag{2.9}$$

Where F_t is the underlying forward rate at time t of the CAP. Usually CAPs have a monetary underlying rate like the 6M EURIBOR or 3M EURIBOR. Yet, in our case, the underlying is a swap rate, the 10Y vs 6M EURIBOR.

The CAP can be seen as a portfolio of European call options exercised at each date T_k. Each call option is called a caplet. Similarly, a floor is a portfolio of put European options exercised at each date T_k. The CAP is used to reduce the exposure to a specific rate. If the underlying rate rises, the CAP's present value increases to compensate a possible loss on bonds.

The last type of derivatives priced in this text is Swaptions. Swaptions, as considered here, are different from the standard definition.

Usually, a swaption gives its holder the right but not the obligation to enter into a swap with maturity T and strike price K. All the characteristics of the underlying swap are determined at the signature of the swaption. It is a combination of a swap and an option.

In this text, the considered swaptions are defined as follow:

Definition: Consider a set of increasing times $T_0, T_1, \ldots, T_n$ with $T_0 \geq 0$. A **swaption** with strike price K and notional N gives its holder at each date T_k, $k = 1, \ldots, n$ the payoff:

$$P = N \; \frac{D_k}{D_0} (T_k - T_{k-1}) \; \max(F_{T_k} - K, 0) \tag{2.10}$$

Where F_t is the underlying forward rate at time t of the swaption and, if we denote by τ the tenor of the underlying rate (for the 10Y swap rate, $\tau = 10$) :

$$D_k = \sum_{l=1}^{\tau} \frac{1}{(1 + F_k)^l} \tag{2.11}$$

and:

$$D_0 = \sum_{l=1}^{\tau} \frac{1}{(1+\tau_0)^l} \tag{2.12}$$

τ_0 is specified by the term sheet and is agreed on when the contract is signed.

Swaptions, as considered in this text, have therefore a similar structure to CAPs, the only difference being that cash flows are weighted by duration ratios. The underlying rate considered, like for caps, is the 10Y vs 6M EURIBOR swap rate. Swaptions also starts at a future date while CAP are always beginning few days after the contract is signed.

2.3 Black-76 model and shifted Black model

This section presents the standard model that was used to price caplets when rates were positive. We will assume at first that forward rates cannot turn negative and then slightly change the model to make it compatible with negative interest rates. The proof of the formula is available in appendix B.

In the Black model, the forward underlying rate F_t is supposed to have, under its forward measure Q, the following dynamics:

$$dF_t = \sigma F_t \; dW_t^Q \tag{2.13}$$

The forward rate is therefore a geometrical Brownian motion (which is why it cannot turn negative).

Using the risk neutral valuation formula, we find that:

$$\Pi = E^Q \left[DF(T) \; \max(F_T - K, 0) \right] \tag{2.14}$$

Computing this expectation (see appendix B) we find the Black formula:

$$\Pi = DF(T) \left[F_0 \; N(d_1) - K \; N(d_2) \right] \tag{2.15}$$

with:

$$d_1 = \frac{\ln\left(\frac{F_0}{K}\right) + \frac{1}{2}\sigma^2 \; T}{\sigma\sqrt{T}} \tag{2.16}$$

$$d_2 = \frac{\ln\left(\frac{F_0}{K}\right) - \frac{1}{2}\sigma^2 \; T}{\sigma\sqrt{T}} \tag{2.17}$$

and N is the cumulative function of the standard Gaussian distribution.

The formula is similar to the Black Scholes formula which price a European call (or put) option on a stock. Market standard when rates were positive was to price caplets using the Black model.

The Black formula gives a link between the volatility and the price. If one knows the volatility of the forward rate, one can price a caplet. It is also possible to do the opposite. Knowing the price, one can deduce the implied volatility. In practice, on Bloomberg, prices are presented by their implicit volatility.

Since interest rates have turned negative in the Eurozone (the swap curve went negative for tenors up to 10Y in summer 2019), the Black model is no longer usable. One way to allow negative rates is to shift by an arbitrary number the underlying rate $F_t^{shift} = F_t + s$. In order to keep the same payoff, the strike is shifted by the same amount. The shifted rate will also be a geometrical Brownian motion and the Black formula still holds with the new strike and forward rate. This model will be referred to as the **shifted Black model**.

2.4 Bachelier model

The Bachelier model is an alternative to the Black model and is compatible with negative rates. The forward rate is supposed to have the following variations under its forward measure:

$$dF_t = \sigma \ dW_t^Q \tag{2.18}$$

And therefore:

$$F_t = F_0 + \sigma W_t^Q \tag{2.19}$$

Since the forward rate is normally distributed, Bachelier model is also called the normal model. The price of a caplet under this model is obtained by the Bachelier formula (see appendix B for the proof):

$$\Pi = DF(T) \ [(F_0 - K) \ N(d) + \sigma\sqrt{T} \ N'(d)] \tag{2.20}$$

with

$$d = \frac{F_0 - K}{\sigma\sqrt{T}} \tag{2.21}$$

and N and N' are respectively the cumulative function and the probability density of the standard Gaussian distribution. Bachelier model has become more important since rates turned negatives. For swap rates as underlying, the volatilities obtained on Bloomberg (VCUB) are normal volatilities.

2.5 SABR model

The SABR model is an expansion of both Black and Bachelier models. It has been developed by Hagan, Kumar, Lesniewski and Woodward in 2002. This model describes variations for both the forward rate and its volatility. In the SABR model, the forward rate F_t has the following dynamics:

$$dF_t = V_t\ F_t^{\beta}\ dW_t \tag{2.22}$$

$$dV_t = \nu\ V_t\ dZ_t \tag{2.23}$$

$$V_0 = \alpha \tag{2.24}$$

$$dW_t\ dZ_t = \rho\ dt \tag{2.25}$$

where W_t and Z_t are Wiener processes under the forward measure, $\beta \in [0; 1]$, $\rho \in [-1; 1]$ and α and ν are positive constants.

The SABR model has become a market standard for its simplicity and because it is able to give a close form formula for the implicit volatility. For a given maturity T, the volatility is a function of the strike K and the curve, due to the form it takes, is called a volatility smile.

A notable result is that under the SABR model, and for any beta, the price of a caplet is given by a Black-type formula, i.e. :

$$\Pi = DF(T)\ [F_0\ N(d_1) - K\ N(d_2)] \tag{2.26}$$

with:

$$d_1 = \frac{\ln\left(\frac{F_0}{K}\right) + \frac{1}{2}\sigma_{SABR}^2\ T}{\sigma_{SABR}\sqrt{T}} \tag{2.27}$$

$$d_2 = \frac{\ln\left(\frac{F_0}{K}\right) - \frac{1}{2}\sigma_{SABR}^2\ T}{\sigma_{SABR}\sqrt{T}} \tag{2.28}$$

The implied volatility σ_{SABR} is a function of the maturity T and the strike K. The SABR model, like the Black model, requires the forwards underlying rates to be positive. In a negative environment, the rates can also be shifted, to obtain the shifted SABR model. The shifted forward rate is supposed to have the following variations:

$$F_t^* = F_t + s \tag{2.29}$$

$$dF_t^* = V_t \; F_t^{*\beta} \; dW_t^* \tag{2.30}$$

$$dV_t = \nu \; V_t \; dZ_t^* \tag{2.31}$$

$$V_0 = \alpha \tag{2.32}$$

$$dW_t^* \; dZ_t^* = \rho \; dt \tag{2.33}$$

Here again, the price of a caplet is given by a Black type formula:

$$\Pi = DF(T) \; [F_0^* \; N(d_1) - K^* \; N(d_2)] \tag{2.34}$$

with:

$$d_1 = \frac{\ln\left(\frac{F_0^*}{K^*}\right) + \frac{1}{2}\sigma_{SABR}^2 \; T}{\sigma_{SABR}\sqrt{T}} \tag{2.35}$$

$$d_2 = \frac{\ln\left(\frac{F_0^*}{K^*}\right) - \frac{1}{2}\sigma_{SABR}^2 \; T}{\sigma_{SABR}\sqrt{T}} \tag{2.36}$$

$$K^* = K + s \tag{2.37}$$

The approximated Black volatility is given by the following formula:

$$\begin{aligned} \sigma_{SABR}(F^*, K^*) = & \frac{\alpha \; \log\left(\frac{F^*}{K^*}\right)}{\int_{K^*}^{F^*} \frac{du}{C(u)}} \; \frac{\xi}{\hat{x}(\xi)} \cdot \\ & \left(1 + \left[\frac{2 \; \gamma_2 - \gamma_1^2 + \frac{1}{x_{av}^2}}{24} \; \alpha^2 \; C(x_{av})^2 \right.\right. \\ & \left.\left. + \frac{1}{4} \; \rho \; \alpha \; \nu \; \gamma_1 C(x_{av}) + \frac{2 - 3\rho^2}{24} \; \nu^2 \right] \; T + ... \right) \end{aligned} \tag{2.38}$$

With:

$$x_{av} = \sqrt{F^* \; K^*} \tag{2.39}$$

$$C(x) = x^\beta \tag{2.40}$$

$$\gamma_1 = \frac{\beta}{x_{av}} \tag{2.41}$$

$$\gamma_2 = \frac{\beta\ (\beta - 1)}{x_{av}^2} \tag{2.42}$$

$$\xi = \frac{\nu}{\alpha}\ \frac{F^* - K^*}{C(x_{av})} \tag{2.43}$$

$$\hat{x}(\xi) = \log\left(\frac{\sqrt{1 - 2\ \rho\ \xi + \rho^2} - \rho + \xi}{1 - \rho}\right) \tag{2.44}$$

The goal is to find the parameter α, β, ρ and ν which match the best the market prices. This can be done by minimizing for example the sum of square differences between the market implied volatilities and the SABR volatilities. The influence of each parameters on the curve $\sigma_{SABR} = f(K)$ at a fixed maturity, is described in table 2.1.

Parameter	**Curve Property**	**Direction**
Alpha α	Level	The curve shift upwards when alpah increases
Beta β	Slope	The curve steepens when beta decreases
Rho ρ	Slope	The curve steepens when rho decreases
Nu ν	Curvature	The curvature increases when nu increases

Table 2.1: Impact of each parameter of the SABR model on the smile

Based on table 2.1, β and ρ have the same impact on the smile. It is then easier to fix β and optimize on α, ρ and ν. In their article, Hagan, Kumar, Lesniewski and Woodward state that the choice of β does not affect the fitting of the three other parameters. In the program, $\beta = 0.2$ was arbitrarily chosen.

Chapter 3

Methodology

The different methods to price each derivative are described here. The pricer have been implemented in VBA. It can be decomposed into several module classes, which represent each type of derivative. In this section, the methods used to price caplets and CCS will be described and the choices of market data will be justified.

In a negative interest rate environment, it is difficult to have access to shifted Black volatilities for caplets with a swap rate as underlying. The data chosen as input are therefore all normal volatilities. They are taken from Bloomberg and the chosen valuation date is the 31/12/2018. This date was specifically chosen because there are prices from the third party to compare with. Each volatility is expressed in basis point scale. This means that it has to be divided by 10 000 before putting it in the Bachelier formula.

Each method is tested on a panel of 23 swaptions with different strikes and different maturities. All these swaptions were in portfolio on 31/12/2018 and their parameters (strike, day count method, maturity, notional amount…) were directly taken from the corresponding term sheets. The third party price of each of these swaptions is taken to compare with the obtained value. For each method, the gap between each swaption's price and the third party value is analyzed (in percentage of the notional). However, a smaller gap does not imply that one method is better than another. The gap is here to see if each method give prices with a good magnitude. Since the goal is to challenge the already given prices, one cannot give too much credit to them. Moreover, the third party prices also use one method and the similar method could give a smaller gap in the end. The main goal here is to have a good understanding of the derivatives and of the methods in order to have a benchmark of prices that gives the team more trust in the obtained values.

3.1 Bachelier

Since the input data are all normal volatilities, the easiest pricing method to implement is therefore the Bachelier model. The method considered works as follow:

1. Import the normal volatilities for different strikes and maturities and regroup them by strike.

2. The volatility is considered constant between two known maturities and a linear interpolation is made between two strikes.

3. Compute the forward swap rate F_t for different maturities. A linear interpolation is made to obtain the forward rate for any maturity.

4. Price each caplet of the CAP (or swaption) using the Bachelier formula. Every caplet is discounted at the OIS (EONIA) rate.

This method is the simplest method because it almost does not require any volatility transformation. It is useful to have a rough but robust approximation of the price to check the order of magnitude of the third party valorization and to check if the forward rates are calculated correctly.

This method does not take into account at the money (ATM) caplets because the ATM strike changes for each maturity and it is therefore more difficult to implement in the code. This is not a big deal because the methods are made to price caplets with any strike (not necessarily ATM caplets) and because some derivative in the portfolio are far from the money.

3.2 Shifted Black

Black model is less simplistic than the normal one and may catch better the forward rate variations. Yet, as explained before, it is difficult to get access to shifted Black volatilities for CAPs or swaptions with a swap rate as underlying. To use the Black model, one has first to transform Bachelier volatilities into shifted Black ones. The method to price caplets using the shifted Black model is described as follow:

1. Import the normal volatilities for different strikes and maturities and regroup them by strike.

2. Compute the market prices using the Bachelier formula.

3. Find the implied Black volatilities by optimizing the shifted Black formula to the market prices using 1-D root finding. The shift was chosen arbitrarily and is of 3

4. Interpolate linearly between two strikes to get the Black volatility for any strike.

5. Compute the forward swap rate F_t for different maturities. A linear interpolation is made to obtain the forward rate for an unknwon maturity.

6. Price each caplet of the CAP (or swaption) using shifted Black formula with the same shift. Every caplet is discounted at the OIS (EONIA) rate.

The whole program was implemented in vba. The 1-D root finding algorithm was implemented by hand using a Newton type method on the squared difference. The Newton method requires vega which, in the shifted Black model is given by:

$$\nu = F_0\sqrt{T}N'(d_1) \tag{3.1}$$

with again:

$$d_1 = \frac{\ln\left(\frac{F_0^*}{K^*}\right) + \frac{1}{2}\sigma^2\,T}{\sigma\sqrt{T}} \tag{3.2}$$

The volatility is supposed to be piecewise constant between two known maturities. It is illustrated in figure 6.

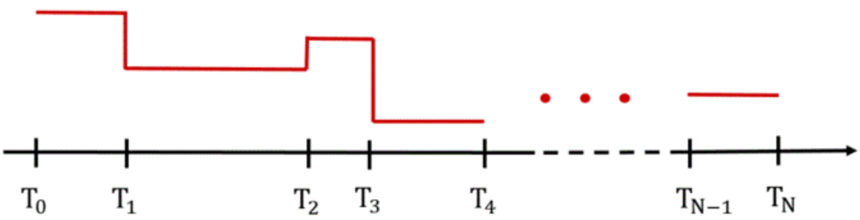

Figure 6 : *Caplet volatility as a function of maturity (with a random fixed strike)*

This method again does not calculate specifically ATM caplet volatilities but like for the Bachelier model it is not a big deal since some derivatives are far from the money.

3.3 SABR model

The third method is here to improve the modelling of the strike dependence of the caplet volatility. Instead of doing a basic linear interpolation, a SABR model is used to fit the volatility smile for each maturity. The method used can be summarized as follow:

1. Import the normal volatilities for different strikes and maturities and regroup them by strike.
2. Compute the market prices using the Bachelier formula.
3. Find the implied Black volatilities by optimizing the shifted Black formula to the market prices using 1-D root finding. The shift was chosen arbitrarily and is of 3
4. Regroup Black volatilities by maturity. For each tenor, fit a SABR model on the volatility smile. The parameters are obtained by minimizing the squared difference. For each smile $\beta = 0.2$ is chosen arbitrarily.
5. Compute the forward swap rate F_t for different maturities. A linear interpolation is made to obtain the forward rate for any maturity.
6. Price each caplet of the CAP (or swaption) using shifted Black formula with the same shift. Every caplet is discounted at the OIS (EONIA) rate.

SABR fitting requires an optimization on three parameters. Yet, while 1-D root finding was possible to implement by hand in VBA, optimization on three parameters are more difficult to do. This is why the optimization was done in Python using the `scipy.optimize` library and the minimize function. Since the main program remains implemented in vba, the xlwings module is used to make the junction between VBA and Python. Black volatilities, strikes and forward rates are given as input to the Python script and the algorithm gives as output the SABR parameters for each maturity. The method chosen for the optimization in the minimize function is the Broyden-Fletcher-Goldfarb-Shanno algorithm (BFGS). Several methods were tested and BFGS gave the best results.

Chapter 4

Results

4.1 Bachelier

The Bachelier volatilities are directly taken from Bloomberg. Figure 7 shows for different strikes the volatility function.

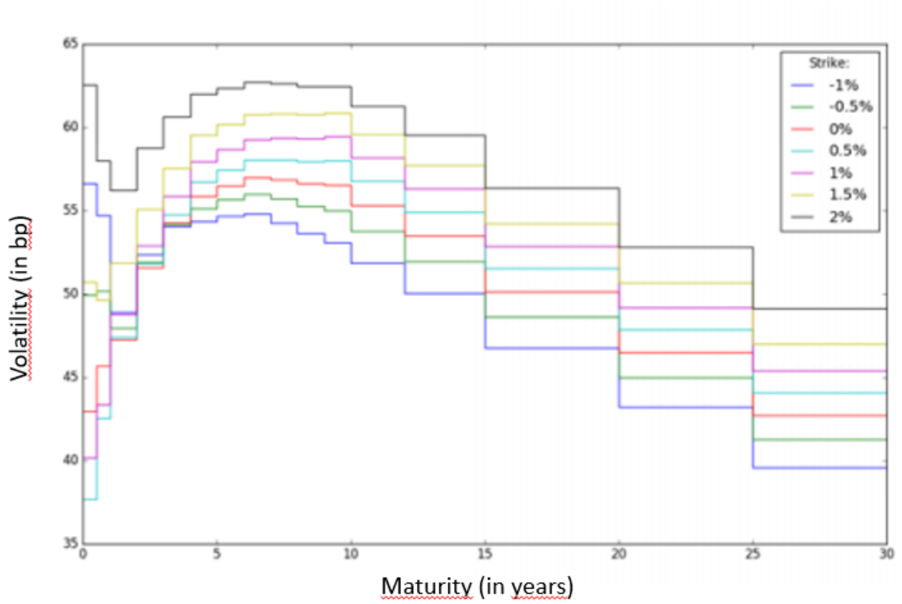

Figure 7 : Plot of the normal volatility as a function of the maturity for different strikes. The volatility is in basis point and the maturity in years

The different errors between the third party market values and the Bachelier values can be seen on figure 8.

	0,6523%	0,6537%	0,6603%	0,4810%	0,9392%
0,7393%	0,6640%	0,8500%	0,5226%	0,1877%	0,4393%
0,3032%	0,5708%	0,3048%	0,4035%	0,4090%	0,4047%
0,4051%	0,4706%	0,4179%	0,8431%	0,5609%	0,7830%

Relative global error	0,5506%
Quadratic error	0,1239%

Figure 8: Market value gaps between the Bachelier method and the third party method. The first table shows the gap for each of the 23 swaptions in portfolio (in percentage of notional). The second table gives the global relative and quadratic error.

The order of magnitude of each price is good since the gap is inferior to 1% of the notional for each swaption.

4.2 Black

After optimizing the Bachelier prices, the shifted Black volatilities (with a shift of 3%) can be seen on figure 9.

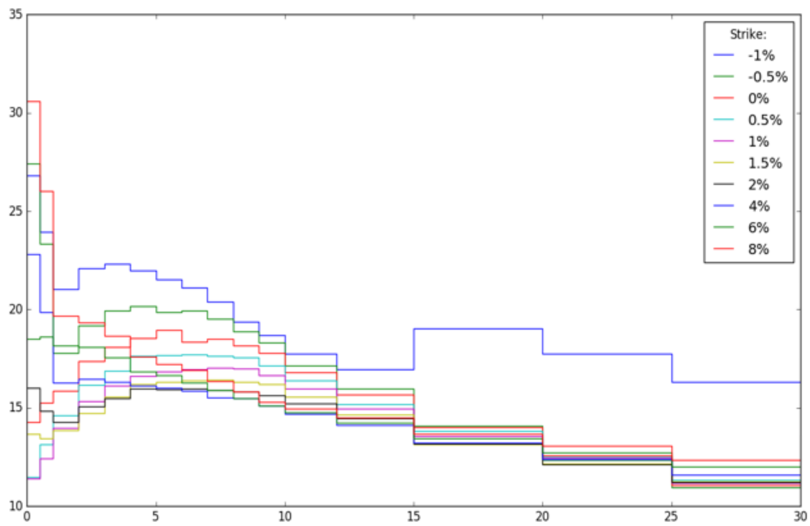

Figure 9 : Plot of the shifted Black volatility as a function of the maturity for different strikes. The volatility is in percentage and the maturity in years

The relative errors with the third party values are on figure 10.

	1,2778%	0,9545%	0,9552%	0,9579%	0,9045%
1,2866%	1,0428%	0,9550%	1,2286%	1,0953%	0,1687%
0,7629%	0,6647%	1,1028%	0,6651%	0,7817%	0,6647%
0,6543%	0,6543%	0,7200%	0,8601%	1,2353%	0,8257%

Relative global error	0,8862%
Quadratic error	0,1955%

Figure 10 : Market value gaps between the shifted Black method and the third party method. The first table shows the gap for each of the 23 swaptions in portfolio (in percentage of notional). The second table gives the global relative and quadratic error. Gaps superior to 1% of the notional are colored in yellow

The Gaps are on average higher than for the Bachelier method but stay inferior to 1.3% of the notional amount. The order of magnitude therefore stays acceptable.

4.3 SABR

The volatility obtained in the Black section (see figure 9 above) are regrouped by maturities and, for each tenor, a SABR model is fitted. The figures below show the volatility smile for different maturities.

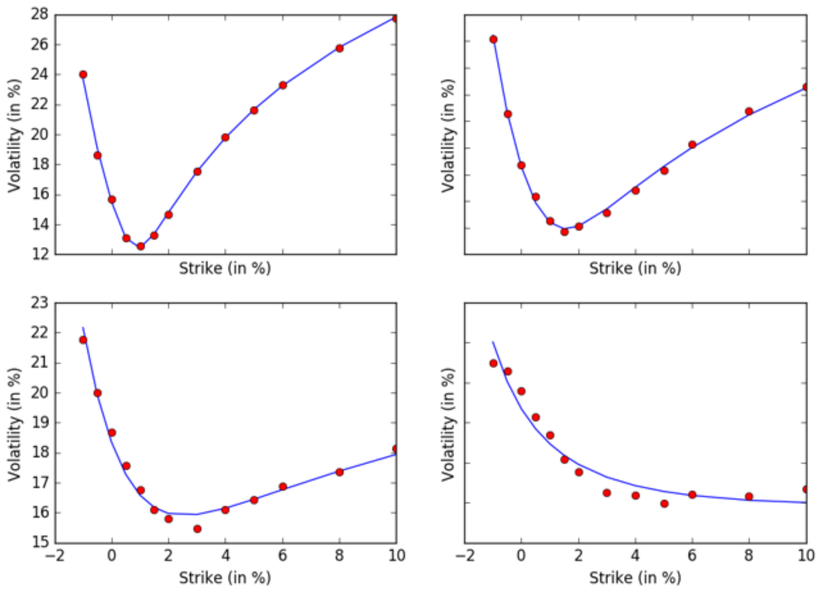

Figure 11 : Volatility smile for different maturities using a SABR model. The red dots correspond to the Black volatilities obtained in the Black section and the blue curves are the SABR functions. The maturities considered for each plot are: top left: 1 year; top right: 3 years; bottom left: 5 years; bottom right: 10 years

The fit of the SABR model is better for short maturities, which is consistent with the fact that the SABR volatilities formula is an approximation for short maturities.

The SABR coefficient for each maturity can be seen on figure 12 and 13.

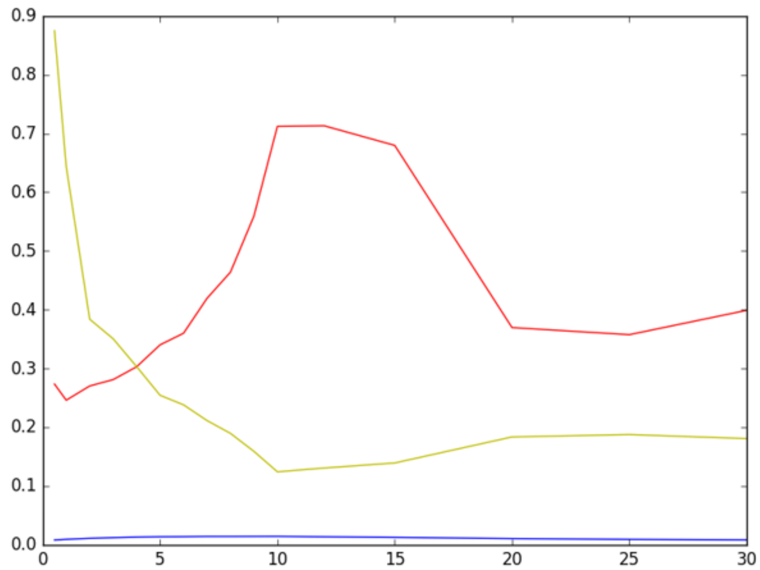

Figure 12 : plot of the different SABR coefficient as a function of the maturity. Alpha corresponds to the blue curve, rho to the red curve and nu to the yellow curve. Maturities are expressed in years

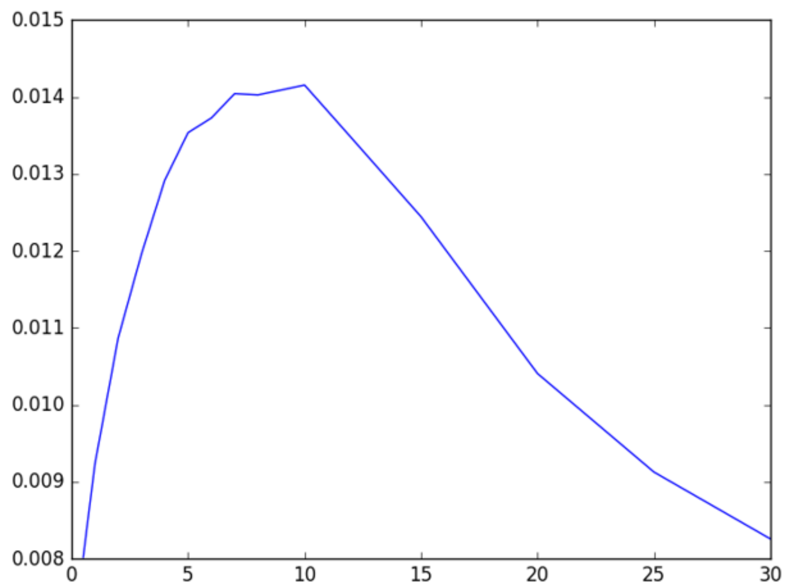

Figure 13 : Plot of the alpha coefficient as a function of the maturity. Tenors are expressed in years.

The gaps with the third party values are shown on figure 14:

	1,1574%	0,6465%	0,6489%	0,6617%	0,9359%
0,9761%	0,7238%	0,6312%	0,9090%	1,0455%	0,0533%
0,7146%	0,5842%	1,0624%	0,5863%	0,7247%	0,5686%
0,5287%	0,5295%	0,7065%	0,7869%	1,1668%	0,8108%

Relative global error	0,7364%
Quadratic error	0,1615%

Figure 14 : Market value gaps between the SABR method and the third party method. The first table shows the gap for each of the 23 swaptions in portfolio (in percentage of notional). The second table gives the global relative and quadratic error. Gaps superior to 1% of the notional are colored in yellow

4.4 CCS

The CCS pricing method was tested on 2 JPY swaps, 8 USD swaps and 1 GBP swap. The relative error (in percentage of the notional) are displayed in table 2.

USD	**JPY**	**GBP**
0,850%	2,249%	1,252%
0,590%	3,649%	
0,021%		
1,231%		
1,255%		
1,220%		
1,355%		
1,321%		

Table 2: Relative error of all CCS (in percentage of the notional).

The relative error are bigger than for the interest rate products. JPY CCS in particular, have an error bigger than 2% of the notional. One explication of this gap is the sensitivity of the pricing formula to the choice of market data. The results were still considered acceptable after a meeting with the third party to validate our model.

Chapter 5

Conclusion

This study presents and challenges different methods to price interest rate derivative in a negative environment. Starting from scratch, mathematical formulas have been justified, implemented in VBA and challenged using real derivatives from BNP Paribas Cardif's portfolio.

In practice, one has to conciliate a theoretical mathematical model with the limitations of the tools available. How good can a optimisation be using vba, and other tools that employees can easily install on their computer and fully master? How good are the market data available to you? It is always possible to make a more precise model, have a better optimisation method, have a better interpolation method, but as a risk analyst, the important question is, for what I want and with the tools I have, are my results satisfactory?

In the end, especially in the risk department, more than the number, it is the method which matters. Being able to fully justify, all the market data choices, and to understand each step which lead to the end price is more important that the final price itself. In that regard, my personal value to the project was to give mathematical tools and methods, and to justify them, in order to get a good estimation of the fair value of the derivative products.

Today, the pricer has been fully coded in Python using my VBA program, and is now daily used by the team.

Chapter 6

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Appendices

Appendix A

Interest rate swaps and bond forwards

A.1 Interest rate swap

These swaps are used as tactical hedging for specific bonds. The payer leg is a bond with a floating rate while the receiver leg is a bond with a fixed rate. Figure 1 shows how an interest rate swap is used to hedge a bond.

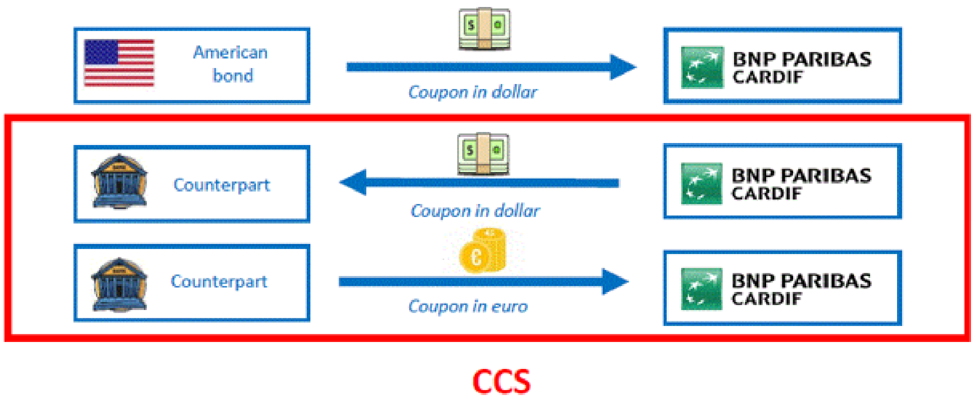

Figure A.1 : Hedge of a bond using an Interest Rate Swap

By using an interest rate swap, an insurer can freeze the interest rate of a floating rate bond. The interest risk relative to the bond is therefore nullified.

The price of an interest rate swap is equal to the difference of the value of both legs. To simply price it one can use the simple price formula of a bond with nominal N, coupon rate i and maturity T:

$$P_{BOND} = \sum_{l=1}^{T} \frac{i.N}{(1 + Z_{act}(l))^l} + \frac{N}{(1 + Z_{act}(T))^T} \tag{A.1}$$

Since the interest rate swap is the difference of two bonds, we obtain its price by the following formula:

$$\Pi = P_{BOND\ fixed} - P_{BOND\ float} \tag{A.2}$$

$$\Pi = \sum_{l=1}^{T} \frac{(i_{fixed} - i_{float}).N}{(1 + Z_{act}(l))^l} \tag{A.3}$$

Since both bonds have the same principal N, both parties pays back the same amount of money at maturity. There is therefore no pay back term in the price formula.

$i_{float}(t)$ corresponds to the predicted value of the floating rate at time t in absence of arbitrage and is called the forward rate.

Consider for example that the floating rate is the 6 months EURIBOR rate. In absence of arbitrage:

$$(1 + Z_{EURIBOR}(t))^t \ (1 + i_{float}(t))^{1/2} = (1 + Z_{EURIBOR}(t + 1/2))^{t+1/2} \tag{A.4}$$

The left term of the equation corresponds to the value of a strategy where the insurer invests one euro at the EURIBOR return rate until time t and then reinvest it again at the EURIBOR rate during 6 months. The right term corresponds to the value of a strategy where the insurer invests one euro at the EURIBOR return rate until t plus 6 months. Since both strategies have the same payoff, in absence of arbitrage they must have at any time the same value.

In the end:

$$i_{float}(t) = \left(\frac{1 + Z_{EURIBOR}(t + 1/2)}{1 + Z_{EURIBOR}(t)} \right)^2 - 1 \tag{A.5}$$

Remark: we have $i_{float}(0) = Z_{EURIBOR}(1/2)$ where $i_{float}(0)$ is the spot rate.

A.2 Bond forward

The simplest type of OTC derivative contract is the forward contract. It is defined as follow.

Definition: A forward contract is a contract between two parties to buy or sell an asset at a specified future time T (the maturity) at a price K (the strike) agreed on at the time of conclusion of the contract.

The value at time t of a forward contract is: $P(t) = S_t - K$ where S_t is the price of the underlying at maturity.

One type of derivatives priced in this text is the bond forward. It is, as its name suggests, a forward contract with a bond as underlying. The bond chosen is specified by the term sheet of the contract and usually quoted on Bloomberg. Bond forward can be used as a hedge tool but also to anticipate the purchase of a bond by locking up its price (it is useful if rates are expected to fall).

The main problem when pricing a bond forward is to price correctly the underlying. The spread of a bond is often difficult to obtain. In order to reduce its influence, it is usually more interesting to take the present value of the bond and then to subtract the cash flows happening before the maturity of the contract. The removed cash flow are computed with a spread equal to zero. By doing that, less cash flows have to be estimated, reducing therefore model errors.

Appendix B

Black and Bachelier formulas

B.1 Useful lemma

Lemma: Let f be the probability density function of a normal law with expectation μ and volatility σ. Let $\alpha \in \mathbb{R}$.
Then:

$$\int_{\alpha}^{\infty} f(u)\ du = N\left(\frac{\mu - \alpha}{\sigma}\right) \tag{B.1}$$

Where N is the repartition function of the standard normal law.

This lemma is an immediate consequence of the symmetry of the standard normal distribution. For all real number x we have:

$$1 - N(x) = N(-x) \tag{B.2}$$

In our case, $x = \frac{\alpha - \mu}{\sigma}$.

B.2 Black formula

In the Black model, we suppose that the forward rate F_t has the following dynamics under its forward measure $\mathbb{Q}$:

$$dF_t = \sigma F_t\ dW_t^{\mathbb{Q}} \tag{B.3}$$

The volatility σ is supposed constant. The forward rate is therefore a geometric Brownian motion and:

$$F_T = F_0\ \exp\left(-\frac{1}{2}\sigma^2 T + \sigma W_T^{\mathbb{Q}}\right) \tag{B.4}$$

Remember here that F_0 is not the spot value of the underlying rate but the value of the forward rate at time 0.

Let us now define Z by:

$$Z = \ln\left(\frac{F_T}{F_0}\right) \tag{B.5}$$

Z follows a normal law with mean value $-\frac{1}{2}\sigma^2 T$ and standard deviation $\sigma\sqrt{T}$ i.e

$$Z \ \sim \ N\left(\frac{1}{2}\sigma^2 T,\ \sigma\sqrt{T}\right) \tag{B.6}$$

Using the risk neutral formula, the present value of the caplet is given by:

$$\Pi = DF(T)\int_{-\infty}^{\infty} \max(F_0\ e^z - K,\ 0)\ f(z)\ dz \tag{B.7}$$

Where f is the probability density of the normal law followed by Z. We have therefore:

$$\Pi = DF(T)\left(F_0\ \int_{\ln\left(\frac{F_0}{K}\right)}^{\infty} e^z\ f(z)\ dz - K\ \int_{\ln\left(\frac{F_0}{K}\right)}^{\infty} f(z)\ dz\right) \tag{B.8}$$

Using the lemma, we have,

$$\Pi = DF(T)\left(F_0\ \int_{\ln\left(\frac{F_0}{K}\right)}^{\infty} e^z\ f(z)\ dz - K\ N(d_2)\right) \tag{B.9}$$

with:

$$d_2 = \frac{\ln\left(\frac{F_0}{K}\right) - \frac{1}{2}\sigma^2\ T}{\sigma\sqrt{T}} \tag{B.10}$$

Let $z \in \mathbb{R}$. We have:

$$e^Z\ f(z) = \frac{1}{\sqrt{2\pi T}\sigma}\ \exp\left(z - \frac{\left(z + \frac{1}{2}\sigma^2 T\right)^2}{2\ \sigma^2 T}\right) \tag{B.11}$$

then:

$$e^Z\ f(z) = \frac{1}{\sqrt{2\pi T}\sigma}\ \exp\left(z - \frac{z}{2} - \frac{z^2 + \frac{1}{4}\ \sigma^4\ T^2}{2\ \sigma^2 T}\right) \tag{B.12}$$

then:

$$e^Z\ f(z) = \frac{1}{\sqrt{2\pi T}\sigma}\ \exp\left(\frac{\left(z - \frac{1}{2}\sigma^2 T\right)^2}{2\ \sigma^2 T}\right) \tag{B.13}$$

in the end:

$$e^Z \ f(z) = g(z) \tag{B.14}$$

Where g is the probability density of a normal law with expectation $\frac{1}{2}\sigma^2 T$ and volatility $\sigma\sqrt{T}$.

Using this we find:

$$\Pi = DF(T)\left(F_0 \int_{\ln\left(\frac{F_0}{K}\right)}^{\infty} g(z)\ dz - K\ N(d_2)\right) \tag{B.15}$$

Using the lemma, we prove the Black formula:

$$\Pi = DF(T)\left(F_0\ N(d_1) - K\ N(d_2)\right) \tag{B.16}$$

with:

$$d_1 = \frac{\ln\left(\frac{F_0}{K}\right) + \frac{1}{2}\sigma^2\ T}{\sigma\sqrt{T}} \tag{B.17}$$

$$d_2 = \frac{\ln\left(\frac{F_0}{K}\right) - \frac{1}{2}\sigma^2\ T}{\sigma\sqrt{T}} \tag{B.18}$$

B.3 Bachelier formula

In the Bachelier model, we suppose that the forward rate F_t has the following dynamics under its forward measure $\mathbb{Q}$:

$$dF_t = \sigma\ dW_t^{\mathbb{Q}} \tag{B.19}$$

The volatility σ is supposed constant. The forward rate can therefore be expressed as:

$$F_T = F_0 + \sigma W_T^{\mathbb{Q}} \tag{B.20}$$

F_T follows a normal law with mean value F_0 and standard deviation $\sigma\sqrt{T}$.

$$F_T \ \sim\ N\left(F_0,\ \sigma\sqrt{T}\right) \tag{B.21}$$

Using the risk neutral formula, the present value of the caplet is given by:

$$\Pi = DF(T)\int_{-\infty}^{\infty} \max(z - K,\ 0)\ f(z)\ dz \tag{B.22}$$

Where f is the probability density of the normal law followed by F_T. We have therefore:

$$\Pi = DF(T)\left(F_0 \int_K^{\infty} z\ f(z)\ dz - K \quad \int_K^{\infty} f(z)\ dz\right) \tag{B.23}$$

Using the lemma we have:

$$\Pi = DF(T)\left(F_0 \int_K^{\infty} z\ f(z)\ dz - K\ N(d)\right) \tag{B.24}$$

with:

$$d = \frac{F_0 - K}{\sigma\sqrt{T}} \tag{B.25}$$

Let $z \in \mathbb{R}$. We have:

$$\int_K^{\infty} z\ f(z)\ dz = \int_K^{\infty} \frac{z}{\sqrt{2\pi T}\sigma} \exp\left(-\frac{(z-F_0)^2}{2\sigma^2 T}\right) dz \tag{B.26}$$

then:

$$\int_K^{\infty} z\ f(z)\ dz = \int_K^{\infty} \frac{z-F_0}{\sqrt{2\pi T}\sigma} \exp\left(-\frac{(z-F_0)^2}{2\sigma^2 T}\right) dz + F_0 \int_K^{\infty} \frac{z}{\sqrt{2\pi T}\sigma} \exp\left(-\frac{(z-F_0)^2}{2\sigma^2 T}\right) dz \tag{B.27}$$

Using the lemma:

$$\int_K^{\infty} z\ f(z)\ dz = \left[-\frac{2\sigma^2 T}{2} N'\left(\frac{z-F_0}{\sigma\sqrt{T}}\right)\right]_K^{\infty} + F_0\ N(d) \tag{B.28}$$

If we denote by N' the probability density function of the standard Gaussian distribution, we have for all real number x,N'(-x)= N'(x) and therefore:

$$\Pi = (F_0 - K)N(d) + \sigma\sqrt{T}N'(d) \tag{B.29}$$

with:

$$d = \frac{F_0 - K}{\sigma\sqrt{T}} \tag{B.30}$$

Which concludes the proof of the Bachelier formula.

Appendix C

Market data

MAT	STK	-1	-0,5	0	0,5	1	1,5	2	3	4	5	6	8	10
0,5	0,92	56,62	49,93	42,95	37,66	40,16	50,69	62,52	85,74	107,84	129,04	149,55	189,05	227,05
1	1,03	54,7	50,18	45,66	42,53	43,32	49,62	57,98	75,85	93,45	110,52	127,11	159,1	189,89
2	1,23	48,87	47,94	47,25	47,4	48,77	51,83	56,21	66,44	77,38	88,39	99,29	120,56	141,16
3	1,41	52,34	51,9	51,56	51,78	52,88	55,06	58,76	67,73	77,71	87,96	98,22	118,41	138,05
4	1,56	54,03	54,15	54,27	54,75	55,82	57,54	60,6	68,26	77,01	86,19	95,48	113,97	132,07
5	1,7	54,34	55,13	55,83	56,7	57,92	59,53	61,97	68,41	75,76	83,58	91,61	107,76	123,73
6	1,78	54,65	55,64	56,47	57,42	58,64	60,17	62,34	68,35	75,2	82,54	90,11	105,43	120,65
7	1,84	54,78	55,98	56,97	58,03	59,27	60,76	62,72	68,35	74,71	81,55	88,64	103,07	117,47
8	1,88	54,27	55,69	56,86	58,04	59,33	60,79	62,6	67,88	73,76	80,07	86,63	100,04	113,49
9	1,91	53,61	55,26	56,62	57,93	59,29	60,74	62,42	67,39	72,79	78,56	84,56	96,88	109,29
10	1,93	53,07	54,97	56,53	57,98	59,41	60,86	62,45	67,11	72,05	77,26	82,68	93,84	105,13
12	1,86	51,84	53,74	55,31	56,74	58,16	59,59	61,26	65,9	70,8	75,98	81,34	92,37	103,52
15	1,75	50,04	51,93	53,48	54,9	56,29	57,69	59,51	64,08	68,92	74,02	79,29	90,1	101,01
20	1,52	46,75	48,62	50,13	51,51	52,85	54,2	56,32	60,81	65,56	70,55	75,69	86,19	96,75
25	1,41	43,22	45	46,47	47,82	49,16	50,65	52,81	57,31	62,07	67,03	72,1	82,39	92,69
30	1,37	39,57	41,27	42,7	44,05	45,39	46,96	49,13	53,62	58,33	63,19	68,13	78,07	87,97

Figure C.1 : Normal market volatilities. Volatilities are expressed in basis point, strikes in percentage and maturities in years. These data are obtained from VCUB on Bloomberg and are in date of 31/12/2018.

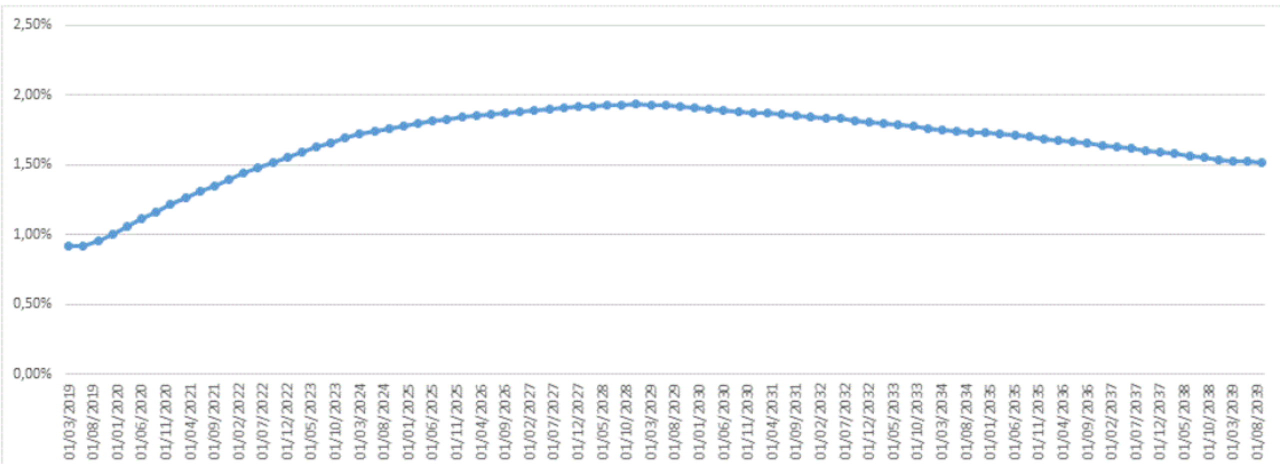

Figure C.2 : Plot of the10y vs 6M EURIBOR rate on the 31/12/2018. Calculations are made by the program with the 6M EURIBOR rate taken from Bloomberg

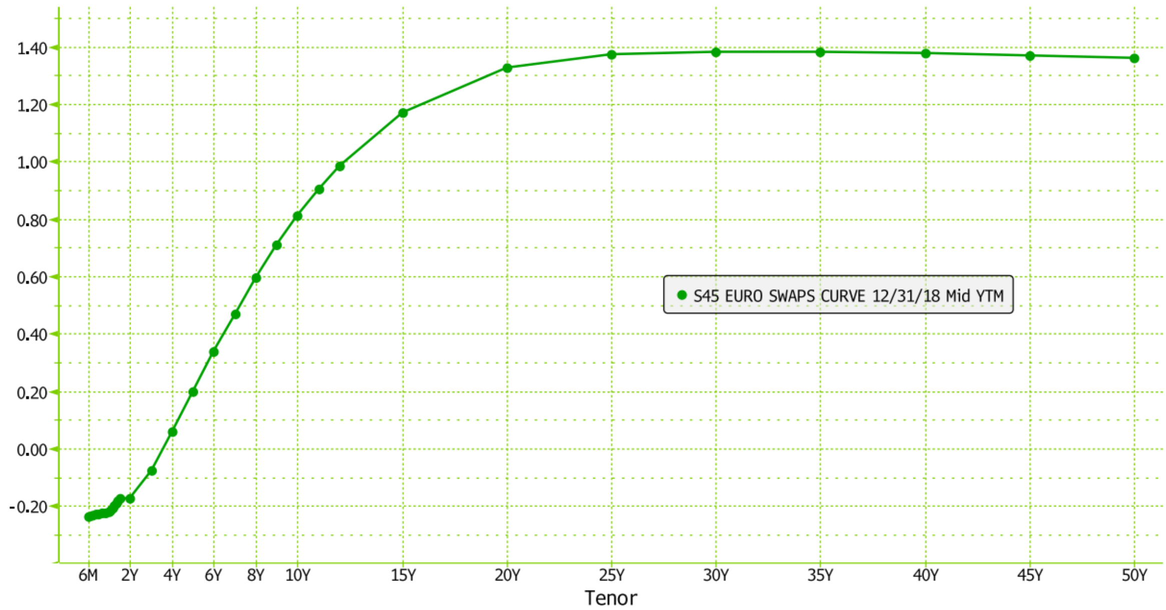

Figure C.3 : Plot of the Euro swap curve (S45) (from Bloomberg)

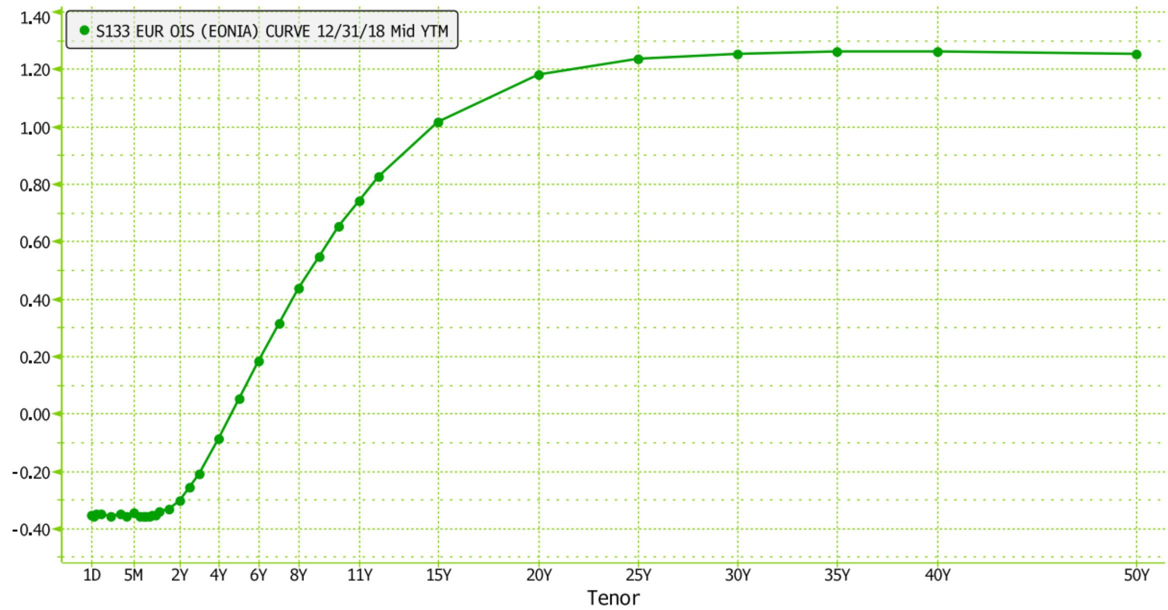

Figure C.4 : Plot of the Euro EONIA curve (S133) (from Bloomberg)

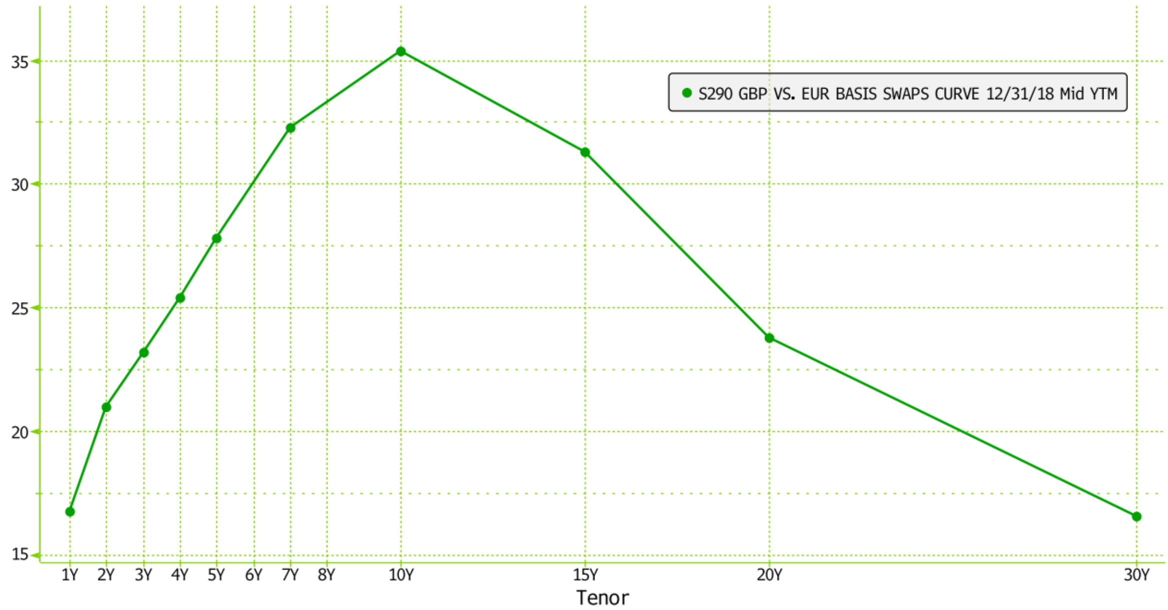

Figure C.5 : Plot of the GBP/EUR basis swap curve (S290) (from Bloomberg)

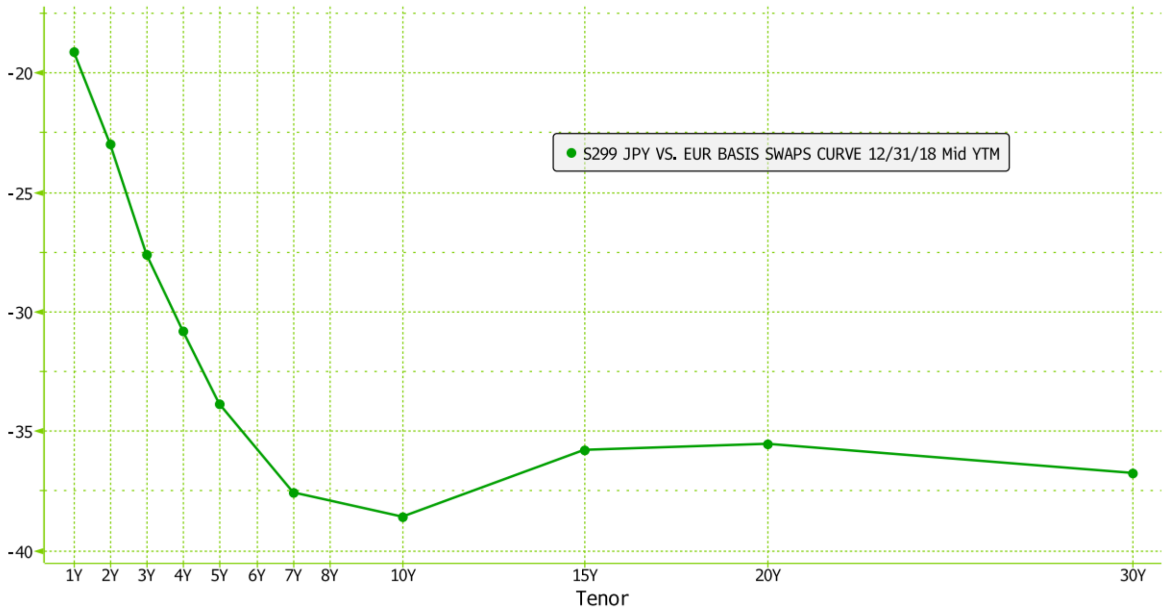

Figure C.6 : Plot of the JPY/EUR basis swap curve (S299) (from Bloomberg)

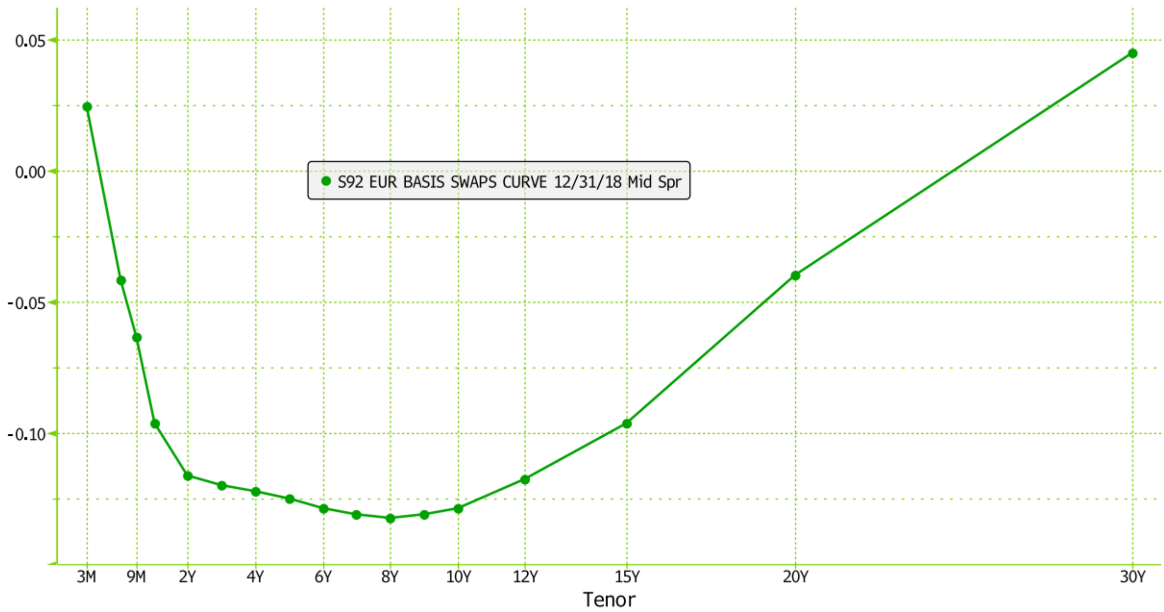

Figure C.7 : Plot of the EUR/USD basis swap curve (S92) (from Bloomberg)