The Uniformization Theorem

Nikita Zozoulenko

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Abstract

In this bachelor's thesis we provide a self contained proof of the Uniformization Theorem. It states that a simply connected Riemann surface is conformally equivalent to either the complex plane $\mathbb{C}$, the unit disk $\mathbb{D}$, or the Riemann sphere $\mathbb{C}^*$. The proof given in this thesis is purely analytic and is based on the construction of Green's function and bipolar Green's function of a Riemann surface through the Perron method.

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Populärvetenskaplig Sammanfattning

En myra som lever på en sfär kan förflytta sig i två dimensioner, framlänges och i sidled. Utan några kunskaper om den globala topologin av myrans värld, så kan den inte urskilja om den lever på en sfär eller i det tvådimensionella talplanet. På ett liknande sätt kan man inom matematiken formalisera en abstrakt topologisk yta som lokalt har samma egenskaper som det komplexa talplanet: En så kallad Riemannyta. Dessa ytor är viktiga objekt inom komplex analys och de används för att studera analytiska funktioner. I detta verk presenteras ett bevis för en känd sats som i stora drag säger att en Riemannyta utan "hål" antingen måste vara det komplexa talplanet, en cirkular skiva eller en sfär.

Contents

1 Introduction

The purpose of this bachelor's thesis is to introduce the notion of a Riemann surface and to prove the Uniformization Theorem. It is a generalization of the classical Riemann Mapping Theorem lifted to the setting of a Riemann surface. This generalization was first proven by Poincaré and Koebe in 1907 [5], and it states the following:

Theorem 1.1 (**The Uniformization Theorem**). *A simply connected Riemann surface is conformally equivalent to either the complex plane $\mathbb{C}$, the unit disk $\mathbb{D}$, or the Riemann sphere $\mathbb{C}^*$.*

The proof presented in this thesis is self contained, and is based on the outline presented in [4] and [7] using analytic methods. For an extensive reference regarding the topic of Riemann surfaces one can look at [1] or [3]. As for the contents of this thesis, it is assumed that the reader has general knowledge of point set topology, real analysis, complex analysis, and the basics of covering spaces as stated in the preliminaries in Section 2.

We begin Section 3 by defining a Riemann surface R, subharmonic functions on R, and analytic functions between Riemann surfaces. We prove three maximum principles for subharmonic functions, Harnack's Inequality, Harnack's Theorem, and the Identity Theorem on R. Next, we introduce Perron families and a special case of the Dirichlet problem for exterior domains on R.

In section 4 we define Green's function of a Riemann surface R and develop the tools needed to prove the Uniformization Theorem. We begin by proving some basic properties of Green's function, and then proceed to prove a number of theorems concerning the existence of Green's function. Finally, we relate Green's function of R to the Green's function of a covering surface.

Section 5 begins by proving the Uniformization Theorem in the case where Green's function exists. We use this result to show that all Riemann surfaces are second countable, and that Green's function is symmetric. With this knowledge we proceed to define bipolar Green's function and prove that it always exists. This is then used to prove the Uniformization Theorem in the case where Green's function does not exist.

2 Preliminaries

In this section we state a number of key theorems from complex analysis and the theory of covering spaces without proof. The reader is assumed to have general knowledge of point set topology, real analysis, and complex analysis in addition to the theorems presented in this section. For reference material, see for example [4] regarding complex analysis and [1] for the theory of covering spaces.

2.1 Complex Analysis

Definition 2.1. A continuous function $u : U \to \mathbb{R}$ is said to be **harmonic** in an open set $U \subset \mathbb{C}$ if for all $z \in U$ there is an $\epsilon > 0$ such that u satisfies the mean value property

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta, \qquad 0 < r < \epsilon$$

Theorem 2.2 (**Harnack's Inequality**). *If u is a positive harmonic function in the open disk $\{z;\ |z - z_0| < R\}$ of radius R and center z_0, then*

$$\frac{R-r}{R+r} \leq \frac{u(z)}{u(z_0)} \leq \frac{R+r}{R-r}, \qquad |z - z_0| \leq r < R$$

Theorem 2.3 (**Harnack's Theorem**). *Let $\{u_n\}_{n=1}^{\infty}$ be an increasing sequence of harmonic functions in an open connected set D. Then u_n converges uniformly on compact subsets of D to a harmonic function or to $+\infty$.*

Definition 2.4. A continuous function $h : U \to [-\infty, +\infty)$ is said to be **subharmonic** in an open set $U \subset \mathbb{C}$ if for all $z \in U$ there is an $\epsilon > 0$ such that u satisfies the mean value inequality

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta, \qquad 0 < r < \epsilon$$

Definition 2.5. Let $D \subset \mathbb{C}$ be a bounded open set. The **Dirichlet problem** for a given continuous function $f : \partial D \to \mathbb{R}$ is to find a harmonic function $u : D \to \mathbb{R}$ such that u extends to coincide with f on ∂D.

Definition 2.6. Let $D \subset \mathbb{C}$ be a bounded open set and $f : \partial D \to \mathbb{R}$ a continuous function. We define the **Perron family of subsolutions** corresponding to f, denoted by $\mathcal{F}_f$, to be the family of all subharmonic functions u on D such that

$$\limsup_{z \to \zeta} u(z) \leq f(\zeta), \qquad \forall \zeta \in \partial D$$

We define the **Perron solution** to the Dirichlet problem with boundary data f on ∂D to be the upper envelope $\widetilde{f}$ of the family $\mathcal{F}_f$ defined by

$$\widetilde{f}(z) = \sup\{u(z);\ u \in \mathcal{F}_f\}, \qquad z \in D$$

Theorem 2.7. *The Perron solution $\widetilde{f}$ to the Dirichlet problem with boundary data f on ∂D is harmonic in D.*

Definition 2.8. Let $D \subset \mathbb{C}$ be a bounded open set. We define a **subharmonic barrier** at $\zeta_0 \in \partial D$ to be a subharmonic function $w : D \cap B_\delta(\zeta_0) \to [-\infty, \infty)$ for some open ball $B_\delta(\zeta_0)$ satisfying

(i) $w(z) < 0$ for $z \in D \cap B_\delta(\zeta_0)$

(ii) $w(z) \to 0$ as $z \to \zeta_0$

(iii) $\limsup_{z \to \zeta} w(z) < 0$ for all $\zeta \in \partial D$, $0 < |\zeta - \zeta_0| < \delta$

Theorem 2.9. *If there exists subharmonic barriers at every point of ∂D, then the Perron solution is the solution to the Dirichlet problem.*

Definition 2.10. Let f_0 be an analytic function defined in an open neighbourhood of z_0, and let $\gamma : [0, 1] \to \mathbb{C}$ be a path starting at z_0. We say that f_0 can be **continued analytically along the path** γ if for each $t \in [0, 1]$ there is an analytic function f_t defined in an open neighbourhood of $\gamma(t)$ and if r is near t, then $f_t(z) = f_r(z)$ for z in the intersection of their disks of convergence.

Theorem 2.11 (**The Monodromy Theorem**). *Let D be simply connected and assume that an analytic function f can be analytically continued along any path in D. Then f defines a unique analytic function for all of D.*

The Monodromy Theorem can be extended to meromorphic functions as well, by viewing them as analytic functions into the extended complex plane.

Theorem 2.12 (**Riemann Mapping Theorem**). *Let D be a simply connected, open, and connected set different from $\mathbb{C}$ with $a \in D$. Then there is a unique bijective analytic function $\varphi : D \to \mathbb{D}$ such that $\varphi(a) = 0$ and $\varphi'(a) > 0$.*

Theorem 2.13 (**Identity Theorem**). *If f and g are analytic functions in an open connected set D, and f and g coincide on a set $A \subset D$ where A has a limit point, then $f = g$ on D.*

Theorem 2.14. *All bijective analytic functions $\tau : \mathbb{D} \to \mathbb{D}$ are given by the particular Möbius transformations*

$$\tau(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z}$$

where $a \in \mathbb{D}$ and $0 \leq \theta < 2\pi$

2.2 Covering Spaces

Definition 2.15. Let X be a topological space. A **covering space** of X is a topological space C with a surjective function $\pi : C \to X$ such that for all $x \in X$ there is an open neighbourhood U of x such that $\pi^{-1}(U)$ is a disjoint union of sets, all of which are mapped homeomorphically onto U by π.

Definition 2.16. A **universal cover** of a connected topological space X is a simply connected covering space.

Definition 2.17. Let X be a topological space. We say that X is **locally path-connected** if for all $x \in X$ and all open neighbourhoods V of x there exists a smaller open neighbourhood U of x that is path-connected.

Definition 2.18. The topological space X is said to be **semi-locally simply connected** if for all $x \in X$ there exists a neighbourhood V of x such that any closed curve in V can be contracted to a point in X.

Theorem 2.19. *A connected and locally path-connected topological space X has a universal cover iff it is semi-locally simply connected.*

Definition 2.20. Let C be a covering space of X with covering map $\pi : C \to X$. A **deck transformation** of π is a homeomorphism $\tau : C \to C$ such that $\pi \circ \tau = \pi$. The set of all deck transformations of π form a group under function composition, denoted by $\mathrm{Aut}(\pi)$.

Theorem 2.21. *If S is a universal cover of X with covering map $\pi : S \to X$, then $\pi(x) = \pi(y)$ iff there exists a unique $\tau \in \mathrm{Aut}(\pi)$ satisfying $\tau(x) = y$.*

3 Riemann Surfaces

In this section we define the notion of a Riemann surface R, subharmonic functions on R, and analytic functions between Riemann surfaces. We prove three maximum principles for subharmonic functions, Harnack's Inequality, Harnack's Theorem, and the Identity Theorem on R. Next, we introduce Perron families and a special case of the Dirichlet problem for exterior domains on R. For a complete reference on the theory of Riemann surfaces, see [3] which this section is based on.

3.1 Definition and Functions on Riemann Surfaces

Definition 3.1. A **Riemann surface** is a connected Hausdorff topological space R with a collection of open subsets $\{U_\alpha \subset R;\ \alpha \in I\}$ and functions $\{z_\alpha : U_\alpha \to \mathbb{C};\ \alpha \in I\}$ with the following properties:

(i) $R = \bigcup_{\alpha \in I} U_\alpha$

(ii) $z_\alpha : U_\alpha \to z_\alpha(U_\alpha)$ is a homeomorphism

(iii) the compositions $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ are analytic

The set $\mathcal{A} = \{(U_\alpha, z_\alpha);\ \alpha \in I\}$ is said to be an atlas for R, where U_α and z_α are called coordinate disks and charts respectively.

Remark. *A Riemann surface is an example of a one-dimensional complex manifold. For higher-dimensional manifolds the topological space is also required to be second countable. This is not needed for Riemann surfaces, and we will prove in Section 5 that all Riemann surfaces are second countable.*

We will assume that the charts z_α map the coordinate disks U_α homeomorphically onto the open unit disk $\mathbb{D}$ by the following construction: Fix a point $p \in U_\alpha$ and consider a small open ball $B_\epsilon(q)$ of radius ϵ and center $q = z_\alpha(p)$. Set $V_\alpha = z_\alpha^{-1}(B_\epsilon(q))$. Since z_α is a homeomorphism the restriction $z_\alpha|_{V_\alpha} : V_\alpha \to B_\epsilon(q)$ is also a homeomorphism. We now map our new coordinate disk V_α to the unit disk using the chart $f = h_1 \circ z_\alpha$ where $h_1(z) = \frac{z-q}{\epsilon}$. If $g = h_2 \circ z_\beta$ is another similarly constructed chart we see that $f \circ g^{-1} = h_1 \circ z_\alpha \circ z_\beta^{-1} \circ h_2^{-1}$ which is analytic.

Note that by restricting the charts in this manner, we may assume that the closure of a coordinate disk is compact in R. Additionally, by composing our charts with suitable Möbius transformations we may freely choose which point the charts map to 0.

Definition 3.2. Let $z : U \to \mathbb{D}$ be a chart of a Riemann surface R and $0 < r < 1$ a real number. We define rU to be the set $rU = \{p \in R;\ z(p) < r\}$.

We now give a few basic examples of Riemann surfaces.

Example 3.3. The complex plane $\mathbb{C}$ is a Riemann surface by considering a single coordinate chart, the identity on $\mathbb{C}$.

Note that any connected open subset $U \subset R$ of a Riemann surface R can be made into a Riemann surface by restricting the charts on R to U. In particular, any domain in $\mathbb{C}$ is a Riemann surface.

Example 3.4. The Riemann sphere $R = \{\overline{x} \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = 1\}$ equipped with the subspace topology is a Riemann surface. We let the coordinate disks be $R \setminus \{(0,0,1)\}$ and $R \setminus \{(0,0,-1)\}$ with charts $f(x,y,z) = \frac{x+iy}{1-z}$ and $g(x,y,z) = \frac{x-iy}{1+z}$ respectively. These maps are homeomorphisms, and $f \circ g^{-1}(w) = \frac{1}{w}$ which is analytic in the domain of definition. We identify the Riemann sphere with the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ using these charts.

Riemann surfaces originally arose in the study of multi-valued analytic functions such as the complex logarithm [8]. The idea is encapsulated in the following example.

Example 3.5. Consider the complex square root defined by $\sqrt{z} = w$ whenever $z^2 = w$. This is not a function in the classical sense since it is multi-valued. To make it into a well-defined function, consider instead the graph

$$S = \{(z,w) \in \mathbb{C}^2;\ w^2 = z\}$$

which will be given the structure of a Riemann surface. In this particular example it is sufficient to have a single chart $f : S \to \mathbb{C}$ defined by $f(z,w) = w$ which is a homeomorphism. The square root is then just the projection of the graph into the w-plane. We can now study the map as a function on a Riemann surface instead of a multi-valued function on $\mathbb{C}$. Analogously one can associate any multi-valued analytic function $w(z)$ which satisfies $P(z, w(z)) = 0$ for some irreducible polynomial $P(z,w)$ with a Riemann surface [6] obtained from the graph G defined by

$$G = \{(z,w) \in \mathbb{C}^2;\ P(z,w) = 0,\ \partial_w P(z,w) \neq 0\}$$

One can also study the complex logarithm using the same method by considering the set of all points in $\mathbb{C}^2$ which satisfy $e^w = z$.

For the remaining parts of this thesis we will only discuss abstract Riemann surfaces. We proceed by defining analytic functions on a Riemann surface into the complex plane, and analytic functions between Riemann surfaces.

Definition 3.6. Let R be a Riemann surface. A function $f : R \to \mathbb{C}$ is **analytic** if for all charts z_α, the function $f \circ z_\alpha^{-1} : \mathbb{D} \to \mathbb{C}$ is analytic. We define meromorphic functions similarly.

The definition of an analytic function is independent of the choice of chart. If f is analytic near a point $p \in R$, and $p \in U_\alpha \cap U_\beta$ for some coordinate disks U_α and U_β, then $f \circ z_\beta^{-1} = (f \circ z_\alpha^{-1}) \circ (z_\alpha \circ z_\beta^{-1})$ is analytic if $f \circ z_\alpha^{-1}$ is.

Definition 3.7. Let R and W be Riemann surfaces. A continuous function $f : R \to W$ is **analytic** if for all pairs of charts $z : U_R \to \mathbb{D}$ on R and $w : U_W \to \mathbb{D}$ on W with $f(U_R) \subset U_W$, the function

$$w \circ f \circ z^{-1} : \mathbb{D} \to \mathbb{D}$$

is analytic. Meromorphic functions are defined similarly.

The function $f : R \to W$ is said to be **conformal** if it is bijective and analytic. In that case f^{-1} is analytic, and we say that R and W are **conformally equivalent**.

Theorem 3.8 (**Identity Theorem**). *If R and W are Riemann surfaces and $f_1, f_2 : R \to W$ are two analytic functions that coincide on a set $A \subset R$ where A has a limit point, then $f_1 = f_2$ in R.*

Proof. Let X be the set of all the points $p \in R$ that have an open neighbourhood V of p such that $f_1|_V = f_2|_V$. This set is open by definition. To show that X is closed we prove that all limit points q of X belong to X. Let $z : U \to \mathbb{D}$ be a chart on R with $z(q) = 0$. For each $n \in \mathbb{N}$ the set $\frac{1}{n}U = \{p \in R;\ |z(p)| < \frac{1}{n}\}$ is an open neighbourhood of q which intersects X. We may therefore choose a sequence $q_n \in X$ such that $q_n \to q$. Since $f_1(q_n) = f_2(q_n)$ for all $n \in \mathbb{N}$, we have that $f_1(q) = f_2(q)$ by continuity. Let $w : V \to \mathbb{D}$ be a chart on W with $w(f_i(q)) = 0$ and choose an integer N such that $\frac{1}{N}U \subset f_i^{-1}(V)$. Then the function

$$g_i = w \circ f_i \circ z^{-1} : \{x \in \mathbb{C};\ |x| < \frac{1}{N}\} \to \mathbb{D}$$

is analytic such that g_1 and g_2 coincide on the set $\{z(q_n) \in \mathbb{D};\ n > N\}$ which has a limit point $z(q)$. The Identity Theorem in the complex plane implies that $g_1 = g_2$, so $f_1|_{\frac{1}{N}U} = f_2|_{\frac{1}{N}U}$ and consequently $q \in X$, which shows that X is closed.

Using the same argument as above on the set A with limit point $a \in R$ shows that $a \in X$. Since R is connected, and X is both open and closed, we conclude that $X = R$. □

Definition 3.9. A continuous function $u : R \to [-\infty, +\infty)$ is said to be **subharmonic** on a Riemann surface R if for all charts $z : U \to \mathbb{D}$ the function $u \circ z^{-1} : \mathbb{D} \to [-\infty, +\infty)$ is subharmonic. We again find that subharmonicity is independent of the choice of chart. Harmonic functions are treated similarly.

Note that by the definition of a Riemann surface we know that $z_\alpha \circ z_\beta^{-1}$ is an injective analytic function. This means that theorems in the plane requiring only local properties that are invariant under analytic change of variables are also valid on Riemann surfaces. The Monodromy Theorem is one such theorem, and it is the only major result which we will not provide a proof of.

Theorem 3.10 (**Maximum principle**). *Let $u : R \to [-\infty, \infty)$ a subharmonic function on a Riemann surface R. If u attains its maximum value M at some point of R, then u is constant on R.*

Proof. We want to show that the set $E = \{p \in R;\ u(p) = M\}$ is open. Let $p \in E$, and let z be a chart with $z(p) = q$. By the definition of a subharmonic function there is an $\epsilon > 0$ such that for all $r < \epsilon$

$$0 \geq \int_0^{2\pi} \Big(u\big(z^{-1}(q)\big) - u\big(z^{-1}(q + re^{i\theta})\big) \Big) d\theta$$

The integrand is nonnegative by the maximality assumption, hence we have that $u\big(z^{-1}(w)\big) = M$ for all $w \in B_\epsilon(q)$ by the continuity of u. Since z is a homeomorphism $z^{-1}\big(B_\epsilon(q)\big)$ is open, which proves that E is open. By continuity $R \setminus E$ is also open, and thus $R = E$ by connectedness. □

Theorem 3.11 (**Maximum principle 2**). *Let R be a Riemann surface and $u : R \to [-\infty, \infty)$ a subharmonic function. If $u(p) \leq M$ for all $p \in R \setminus K$ where K is a compact subset of R, then $u(p) \leq M$ for all $p \in R$.*

Proof. Since u is continuous it attains a maximum value C on the compact set K. If $M \leq C$ then u is constant on R by the maximum principle, otherwise $u(p) \leq M$ for $p \in K$ as well. □

Theorem 3.12 (**Maximum principle 3**). *Let R be a Riemann surface, W an open subset of R such that $\overline{W}$ is compact, and $u : W \to [-\infty, \infty)$ a subharmonic function such that u extends continuously to ∂W. If $u(p) \leq M$ for all $p \in \partial W$, then $u(p) \leq M$ for all $p \in \overline{W}$.*

Proof. Since u is continuous, it attains its maximum value at some point $q \in \overline{W}$. If $q \in \partial W$ then we are done. Otherwise $q \in W$ and u is constant by the maximum principle. □

All three versions of the maximum principle will unanimously be referred to as the maximum principle and they will all be used to prove key results leading up to the Uniformization Theorem.

Theorem 3.13 (**Harnack's Inequality**). *Let R be a Riemann surface and $u : R \to \mathbb{R}$ a positive harmonic function. Then for every compact set $K \subset R$ there exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq \frac{u(p)}{u(q)} \leq C \qquad \forall p, q \in K$$

Proof. Since K is compact we can cover it with a finite number of open sets $V_i = z_i^{-1}(\frac{1}{3}\mathbb{D})$, where z_i are charts with $z_i(p_i) = 0$, and $0 < i < n$. Now let $p, q \in K$ be two arbitrary points. If p and q belong to the same coordinate disk, say V_1, we use the following estimate: Let $z_1(p) = x$ and $z_1(q) = y$. By

Harnack's inequality in the plane there is a constant $D > 0$ not dependent on p or p_1 such that

$$\frac{1}{D}u(p) \leq u(p_1) \leq Du(p)$$

A repeated use of the above inequality for the points p and q yields that

$$\frac{1}{D^2} \leq \frac{u(p)}{u(q)} \leq D^2$$

If p and q belong to different coordinate disks, we may find a finite sequence of open covering disks $\{V_{k_j}\}_{j=1}^m$ such that $V_{k_j} \cap V_{k_{j+1}} \neq \emptyset$ for each connected component. By creating a chain of inequalities using points along the intersections we get that

$$\frac{1}{D^{2m}} \leq \frac{u(p)}{u(q)} \leq D^{2m}$$

To get the same constant C for all points in K, we set $C = D^{2n}$ and consider the estimate for the finitely many connected components. □

Theorem 3.14 (Harnack's Theorem). *Let $\{u_n\}_{n=1}^{\infty}$ be an increasing sequence of harmonic functions on a Riemann surface R. Then u_n either converges to a harmonic function u or to $+\infty$.*

Proof. We want to show that the set $E = \{p \in R;\ u(p) < +\infty\}$ is both open and closed, and that u is harmonic in E. If $p \in E$ and $z : U \to \mathbb{D}$ is a chart with $z(p) = 0$, then $u_n \circ z^{-1}$ is an increasing sequence of harmonic functions in $\mathbb{D}$ such that $u_n \circ z^{-1}(0) \to u(p) < +\infty$. Hence by Harnack's Theorem in the plane $u \circ z^{-1}$ is harmonic in $\mathbb{D}$ and $U \subset E$, which shows that E is open. To show that E is closed we need to show that all limit points q of E belong to E. If $w : V \to \mathbb{D}$ is a chart where V is an open neighbourhood of q, then V intersects E, so by the same argument as above $V \subset E$, and in particular $q \in E$. We conclude that $R = E$ or $R = \emptyset$ by connectedness. □

3.2 Perron Families of Subharmonic Functions

Definition 3.15. Let R be a Riemann surface and $W \subset R$ an open subset. A nonempty family $\mathcal{F}$ of subharmonic functions on W is called a **Perron family** if it satisfies

(i) if $u, v \in \mathcal{F}$, then $\max(u, v) \in \mathcal{F}$

(ii) if $u \in \mathcal{F}$ and U is a coordinate disk such that u is finite on ∂U, then the function defined to be equal to u on $W \setminus U$ and the harmonic extension of $u|_{\partial U}$ on U is in $\mathcal{F}$

Theorem 3.16. *Let R be a Riemann surface, $W \subset R$ an open subset, and $\mathcal{F}$ a Perron family on W. Then the upper envelope ω of the family $\mathcal{F}$*

$$\omega(p) = \sup\{u(p);\ u \in \mathcal{F}\}, \qquad p \in W$$

is harmonic in W, or $\omega(p) = +\infty$ for all $p \in W$

Proof. The proof follows the outline in [2]. We want to show that the set $E = \{p \in R;\ \omega(p) < \infty\}$ is open and closed, as well as ω being harmonic in E. Let $q \in R$ and choose a coordinate disk U containing q whose closure is compactly contained in R. By the definition of ω there exists a sequence of subharmonic functions $a_n \in \mathcal{F}$ such that $a_n(q) \leq a_{n+1}(q)$ for all n and $a_n(q) \to \omega(q)$. Set $u_n = \max\{a_1, a_2, ..., a_n\}$, which is an increasing sequence in $\mathcal{F}$ such that $u_n(q) \to \omega(q)$. Now let h_n be the harmonic extension of u_n in U. Then h_n is an increasing sequence in $\mathcal{F}$ such that $u_n \leq h_n$ and $h_n(q) \to \omega(q)$. If $h = lim_{n\to\infty} h_n$, then by Harnack's Theorem $h = +\infty$ or h is harmonic in U. If $h(q) = +\infty$, then since $h \leq \omega$ we get that $\omega(p) = +\infty$ for all $p \in U$, which proves that E is closed.

In the case $\omega(q) < +\infty$, we want to show that $\omega = h$ in U, which would show that E is open. Assume by contradiction that $h(q_0) < \omega(q_0)$ for some $q_0 \in U$. We can then find a sequence $b_n \in \mathcal{F}$ such that $b_n(q_0) \to \omega(q_0)$. Set $\widetilde{a}_n = \max(a_n, b_n)$ and define the functions $\widetilde{u}_n = \max\{\widetilde{a}_1, \widetilde{a}_2, \widetilde{...}, \widetilde{a}_n\}$, $\widetilde{h_n}$, and $\widetilde{h}$ as in the previous paragraph by replacing a_n with $\widetilde{a_n}$. Then $\widetilde{h}$ is harmonic in U with $h \leq \widetilde{h}$ and satisfies

$$\widetilde{h}(q) = \omega(q) = h(q)$$

$$\widetilde{h}(q_0) = \omega(q_0) > h(q_0)$$

which shows that the nonconstant harmonic function $\widetilde{h} - h$ attains its minimum at $q \in U$, contradicting the maximum principle. We conclude that $E = R$ or $E = \emptyset$ by connectedness. $\square$

In the following lemma we will consider solving the Dirichlet problem on a domain obtained by deleting from a given Riemann surface a finite disjoint union of small coordinate disks as defined in Definition 3.2.

Lemma 3.17. *Let R be a Riemann surface. For each $k = 1, 2, ..., n$ let $z_k : U_k \to \mathbb{D}$ be a chart where $\overline{U_k}$ is compactly contained in R with $\overline{U_i} \cap \overline{U_j} = \emptyset$, and $0 < r_k < 1$ a real number. Let $f_k : \partial r_k U_k \to \mathbb{R}$ be a non-negative continuous function and set $W = R \setminus \bigcup_{k=1}^{n} \overline{r_k U_k}$. Define the family $\mathcal{F}$ as the set of all subharmonic functions $u : W \to [-\infty, +\infty)$ such that $u = 0$ off a compact subset of R, and that the following limit exists and satisfies*

$$\lim_{p \to \zeta} u(p) \leq f_k(\zeta), \qquad \forall \zeta \in \partial r_k U_k,\ k = 1, ..., n$$

If $f_k(p) \leq M$ for all $p \in \partial r_k U_k$ and all $k = 1, 2, ..., n$, then the upper envelope ω of the family $\mathcal{F}$ is harmonic on W and satisfies

$$\lim_{p \to \zeta} \omega(p) = f_k(\zeta), \qquad \forall \zeta \in \partial r_k U_k,\ k = 1, ..., n \tag{1}$$

$$0 \leq \omega(p) \leq M, \qquad \forall p \in W \tag{2}$$

Proof. The family $\mathcal{F}$ is a Perron family and is nonempty since $0 \in \mathcal{F}$. We start by checking (2). The inequality $0 \leq \omega(p)$ follows trivially since $0 \in \mathcal{F}$. Consider any arbitrary $u \in \mathcal{F}$. Since $u = 0$ off a compact set $K \subset R$, the maximum principle applied to $K \setminus \bigcup_{k=1}^{n} r_k U_k$ says that

$$u(p) \leq \max(0, M) = M, \qquad \forall p \in W$$

and we obtain (2) by taking the supremum over all $u \in \mathcal{F}$. Additionally, since $\mathcal{F}$ is bounded, its upper envelope w is harmonic by Theorem 3.16.

We now prove condition (1). For each k we can find a real number s_k such that $r_k < s_k < 1$. Consider the annulus $\{z \in \mathbb{C};\ r_k < z < s_k\}$ where for each boundary point we can find a subharmonic barrier. Let v_k be the solution of the Dirichlet problem on the annulus with data $f_k \circ z_k^{-1}$ for $|z| = r_k$ and data 0 for $|z| = s_k$. Then the function $u_k : W \to [-\infty, +\infty)$ defined by

$$u_k(p) = \begin{cases} v_k \circ z_k(p) & \text{if } r_k < |z_k(p)| < s_k \\ 0 & \text{otherwise} \end{cases}$$

is in $\mathcal{F}$ and satisfies $\lim_{z \to \zeta} u_k(p) = f_k(\zeta)$ for all $\zeta \in \partial r_k U_k$. Since $u_k \leq \omega$, we have that $f_k(\zeta) \leq \liminf_{p \to \zeta} \omega(p)$. To show the reverse inequality, consider for any $u \in \mathcal{F}$ the subharmonic function $h : s_k U_k \setminus \overline{r_k U_k} \to [-\infty, +\infty)$ defined by

$$h(p) = u(p) - u_k(p) - M \frac{\log |z_k(p)| - \log r_k}{\log s_k - \log r_k}$$

Then $\lim_{p \to q} h(p) \leq 0$ for all $q \in \partial\big(s_k U_k \setminus \overline{r_k U_k}\big)$, so by the maximum principle $h(p) \leq 0$ for all $p \in s_k U_k \setminus \overline{r_k U_k}$. Since u was arbitrary, taking the supremum over the family $\mathcal{F}$ gives us that

$$\omega(p) \leq u_k(p) + M \frac{\log |z_k(p)| - \log r_k}{\log s_k - \log r_k}, \qquad \forall p \in s_k U_k \setminus \overline{r_k U_k}$$

In particular we have that $\limsup_{p \to \zeta} \omega(p) \leq f_k(\zeta)$ for all $\zeta \in \partial r_k U_k$. We conclude that

$$\limsup_{p \to \zeta} \omega(p) \leq f_k(\zeta) \leq \liminf_{p \to \zeta} \omega(p)$$

which shows that the limit exists and satisfies (1). Since k was arbitrary, we are done.

$\square$

4 Green's Function of a Riemann Surface

In this section we define Green's function of a Riemann surface R and develop the tools needed to prove the Uniformization Theorem. We begin by proving some basic properties of Green's function, and then proceed to prove a number of theorems concerning the existence of Green's function. Finally, we are going to relate Green's function of R to the Green's function of a covering surface. We follow the outline presented in [4] and [7].

4.1 Definition and Basic Properties

Definition 4.1. Let R be a Riemann surface, $q \in R$ a point, and z a chart such that $z(q) = 0$. We define $\mathcal{F}_q$ to be the set of all subharmonic functions $u : R \setminus \{q\} \to [-\infty, +\infty)$ such that $u = 0$ off a compact subset of R, and $u(p) + \log|z(p)|$ extending to a subharmonic on the coordinate disk. We define Green's function $g(p, q)$ of R with pole at q to be the upper envelope of the family $\mathcal{F}_q$.

$$g(p, q) = \sup\{u(p) : u \in \mathcal{F}_q\}$$

Since $\mathcal{F}_q$ is a Perron family, Theorem 3.16 says that $g(p, q)$ is either a harmonic function on $R \setminus \{q\}$, or $g(p, q) = \infty$. In the former case we say Green's function with pole at q exists, and in the latter case we say that the function does not exist.

Theorem 4.2. *Let R be a Riemann surface and assume Green's function $g(p, q)$ with pole at q exists, and let $z : U \to \mathbb{D}$ be a chart with $z(q) = 0$. Then $g(p, q)$ is a positive harmonic function for $p \in R \setminus \{q\}$, and $g(p, q) + \log|z(p)|$ extends to be harmonic at q.*

Proof. Since $g(p, q)$ exists it is harmonic on $R \setminus \{q\}$. The function 0 is in the Perron family $\mathcal{F}_q$ defining $g(p, q)$, hence $g(p, q) \geq 0$. Let $0 < r < 1$ and define the function $v : R \to (-\infty, +\infty]$ by

$$v(p) = \begin{cases} \log r - \log|z(p)| & \text{if } z(p) \leq r \\ 0 & \text{otherwise} \end{cases}$$

Then v is subharmonic on $R \setminus \{q\}$ and is equal to 0 off a compact subset of R, so $v \in \mathcal{F}_q$. Therefore $g(p, q) \geq \log r - \log|z(p)|$ for all $p \in R \setminus \{q\}$. Taking the limit as $p \to q$ we obtain that $g(p, q) \to \infty$. If $g(p, q) = 0$ for some p, then g attains its minimum and is constant by the maximum principle, contradicting the previous statement. Thus $g(p, q)$ is strictly positive.

For the second part of the theorem, let $M = \max\{g(p, q);\ p \in \partial rU\}$ and take $u \in \mathcal{F}_q$. Since $u(p) + \log|z(p)|$ is subharmonic in rU, the maximum principle implies that

$$u(p) + \log|z(p)| \leq M + \log r \qquad \forall p \in \overline{rU}$$

Taking the supremum over all $u \in \mathcal{F}_q$ and combining results we obtain that

$$\log r \leq g(p,q) + \log|z(p)| \leq M + \log r \qquad \forall p \in \overline{rU} \setminus \{q\}$$

By Riemann's Theorem on removable singularities, $g(p,q) + \log|z(p)|$ extends to be harmonic at q which completes the proof. $\square$

Theorem 4.3. *Let R be a Riemann surface where Green's function $g(p,q)$ with pole at q exists, and let $z : U \to \mathbb{D}$ be a chart with $z(q) = 0$. If $h : R \setminus \{q\} \to \mathbb{R}$ is a positive harmonic function such that $h(p) + \log|z(p)|$ is harmonic in U, then $h(p) \geq g(p,q)$ for all $p \in R \setminus \{q\}$.*

Proof. Let $\mathcal{F}_q$ be the Perron family defining $g(p,q)$. If $u \in \mathcal{F}_q$, then for all $p \in U$ we have that the function

$$u(p) - h(p) = \big(u(p) + \log|z(p)|\big) - \big(h(p) + \log|z(p)|\big)$$

is subharmonic in U. Hence $u - h$ is subharmonic on all of R, and $u - h < 0$ off a compact subset of R. By the maximum principle $u - h \leq 0$ on R. Taking the supremum over all $u \in \mathcal{F}_q$ we obtain that $g(p,q) - h(p) \leq 0$ for all $p \in R \setminus \{q\}$. $\square$

4.2 Existence Theorems

Lemma 4.4. *Let R be a Riemann surface, $z : U \to \mathbb{D}$ a chart, and $0 < r < 1$ a real number. If Green's function $g(p,q_0)$ of R exists for some $q_0 \in rU$, then it exists for for all $q \in rU$.*

Proof. If $z(q_0) \neq 0$, replace z with a composition of itself with a suitable Möbius transformation so that $z(q_0) = 0$. Let ω be the solution to the Dirichlet problem on $R \backslash \overline{rU}$ with data 1 on ∂rU as in Lemma 3.17, defined by the Perron family $\mathcal{F}$. By the same lemma we have that $0 \leq \omega \leq 1$. There are three possible cases: Either $\omega = 0$, $0 < \omega < 1$, or $\omega = 1$ on $R \setminus \overline{rU}$ by the maximum principle. The first case is impossible since $\omega(p) \to 1$ as p approaches ∂rU. We now prove that the third case is also impossible:

Since $g(p,q_0)$ exists, let $c = \min\{g(p,q_0);\ p \in \partial rU\}$. For any $u \in \mathcal{F}$ we have that $u \leq 1$, and $u = 0$ off a compact subset of R. By applying the maximum principle we obtain that

$$u(p) - \frac{g(p,q_0)}{c} \leq 0 \qquad \forall p \in R \setminus \overline{rU}$$

Taking the supremum over all $u \in \mathcal{F}$ yields

$$\omega(p) \leq \frac{g(p,q_0)}{c} \qquad \forall p \in R \setminus \overline{rU} \tag{3}$$

The infimum of $g(p,q_0)$ over $R\setminus\{q_0\}$ is 0, for if it was $a>0$ then $g-a$ would be harmonic on $R\setminus\{q_0\}$ with logarithmic pole at q_0, and thus satisfies $g-a\geq g$ by Theorem 4.3. This contradicts that $a>0$. Now the infimum of $g(p,q_0)$ over $R\setminus\overline{rU}$ also has to be zero, thus by equation (3) the infimum of ω is also zero, which proves that $0<\omega<1$.

Continuing the proof, let $q\in rU$ be arbitrary and choose s such that $r<s<1$ and define

$$\begin{aligned} K &= \max\{\omega(p);\ p\in\partial sU\} \\ C_s &= \min\{|\log|z(p)-z(q)||;\ p\in\partial sU\} \\ C_r &= \min\{|\log|z(p)-z(q)||;\ p\in\partial rU\} \end{aligned}$$

Let $u\in\mathcal{F}_q$, the Perron family defining Green's function, and set

$$M=\max\{u(p);\ p\in\partial sU\}$$

Using the maximum principle we obtain that

$$u(p)+\log|z(p)-z(q)|\leq M+C_s \qquad \forall p\in\overline{sU}$$

This estimate holds for $p\in\partial rU$ in particular, so since $u=0$ off a compact subset of R and $\omega(p)=1$ for $p\in\partial rU$, the maximum principle gives that

$$u(p)\leq\big(M+C_s+C_r\big)\omega(p) \qquad \forall p\in R\setminus\overline{rU}$$

If we take the maximum over all $p\in\partial sU$ we get that

$$M\leq\big(M+C_s+C_r\big)K \iff M\leq\frac{(C_r+C_s)K}{1-K}$$

Therefore, we have that $u(p)\leq M\leq\frac{(C_r+C_s)K}{1-K}$ for $p\in\partial sU$ independent of $u\in\mathcal{F}_q$, which proves that Green's function with pole at q exists. $\square$

Theorem 4.5. *Green's function $g(p,q)$ of a Riemann surface R exists for all $q\in R$, or for none.*

Proof. Let X be the set of points where Green's function exists. Lemma 4.4 implies that X is open. To show that X is closed, we prove that all limit points q of X are contained in X. Let $z:U\to\mathbb{D}$ be a chart with $z(q)=0$. Then $\frac{1}{2}U$ is an open neighbourhood of q, so it intersects X, and hence there is an $x\in\frac{1}{2}U$ such that $g(p,x)$ exists. Since $q,x\in\frac{1}{2}U$, Lemma 4.4 implies that $g(p,q)$ exists. This shows that $q\in X$, so X is closed. Since R is connected we have that $R=X$ or $R=\emptyset$. $\square$

The following theorem provides a wide class of Riemann surfaces where Green's function exists. This theorem will be necessary to deal with the Uniformization Theorem in the case where Green's function does not exist.

Theorem 4.6. *Let R be a Riemann surface, U_0 a coordinate disk, $0 < t < 1$ a real number, and set $W = R \setminus \overline{tU_0}$. Then Green's function $g(p, q)$ of W exists for all $q \in W$.*

Proof. Fix $q \in W$, let $0 < r < 1$ be a real number, and $z : U \to \mathbb{D}$ be a chart where $\overline{U}$ is compactly contained in W with $z(q) = 0$. Let ω be the Perron solution to the Dirichlet problem on $W \setminus \overline{rU}$ with data 1 on ∂rU and 0 on ∂tU_0, as in Lemma 3.17. By the same lemma ω satisfies $0 \leq \omega \leq 1$. The function is not constant by definion, hence it cannot attain its minimum or maxmimum inside this domain by the maximum principle. Hence we have that $0 < \omega < 1$. Let $u \in \mathcal{F}_q$, the Perron family defining Green's function of W, and set

$$\begin{aligned} C &= \max\{\omega(p);\ p \in \partial U\} \\ K &= \max\{u(p);\ p \in \partial U\} \\ M &= \max\{u(p);\ p \in \partial rU\} \end{aligned}$$

Since $u(p) + \log |z(p)|$ is subharmonic in U, the maximum principle gives that

$$u(p) + \log |z(p)| \leq K \qquad \forall p \in \overline{U}$$

$$u(p) + \log |z(p)| \leq M + \log r \leq K \qquad \forall p \in \overline{rU} \tag{4}$$

Since $u = 0$ off a compact subset of R and $0 < \omega < 1$, the maximum principle says that

$$u(p) \leq M\omega(p) \qquad \forall p \in W \setminus \overline{rU}$$

Taking the maximum over $p \in \partial U$ we obtain that

$$K \leq MC \tag{5}$$

By adding inequalities (4) and (5) we finally obtain that

$$M + \log r + K \leq K + MC \iff M \leq \frac{\log r}{C - 1}$$

Since r and C don't depend on $u \in \mathcal{F}_q$, and M was the maximum of u over ∂rU, we see that $g(p, q)$ has to be bounded for $p \in \partial rU$ and therefore exists. $\square$

Lemma 4.7. *If there is a nonconstant bounded analytic function $\varphi : R \to \mathbb{C}$ on a Riemann surface R, then Green's function $g(p, q)$ exists for all $q \in R$.*

Proof. We may assume that $|\varphi(p)| < 1$ for all $p \in R$ by dividing by a suitable constant. Fix any arbitrary $q \in R$. By composing with a suitable Möbius transformation we may assume that $\varphi(q) = 0$. Let $u \in \mathcal{F}_q$, the Perron family defining $g(p, q)$. Then $u(p) + \log |\varphi(p)|$ is subharmonic in R. Since $u = 0$ off a compact subset of R, the maximum principle gives that

$$u(p) + \log |\varphi(p)| \leq 0 \qquad \forall p \in R$$

Taking the supremum over all $u \in \mathcal{F}_q$ shows that $g(p, q)$ is bounded by $-\log |\varphi(p)|$ and hence exists. $\square$

4.3 Green's Function of a Covering Space

Theorem 4.8. *Every connected covering space of a Riemann surface R is a Riemann surface, and its covering map is analytic.*

Proof. Let C be a covering space of R with covering map $\pi : C \to R$. We begin by constructing the atlas for C that makes π analytic. For each $x \in C$ and each chart $z : U \to \mathbb{D}$ with $\pi(x) \in U$, we construct a corresponding chart on C: By the definition of π there exists an open set V with $\pi(x) \in V$ such that $\pi^{-1}|_V$ is a disjoint union of subsets of C, each homeomorphic to V. Set $A = U \cap V$ which is open and nonempty, and homeomorphic to disjoint subsets of C. Finally, we let the chart be $\psi = z \circ \pi$, defined on the connected component of $\pi^{-1}(A)$ that contains x. If $\phi = w \circ \pi$ is another similarly constructed chart of C, then $\phi \circ \psi^{-1} = z \circ \pi \circ \pi^{-1} \circ w^{-1} = z \circ w^{-1}$ which is analytic. This, together with the fact that C is Hausdorff if R is, makes C a Riemann surface since it is connected.

Concerning the analycity of π, consider any pair of charts $\psi = z \circ \pi$ of C and w of R. Then $w \circ \pi \circ \psi^{-1} = w \circ z^{-1}$ which is analytic by construction. Hence π is analytic.

□

Theorem 4.9. *Let R be a Riemann surface where Green's function $g_R(p, q)$ with pole at $q \in R$ exists, and let C be a connected covering space of R with covering map $\pi : C \to R$. Then Green's function $g_C(p^*, q^*)$ of C exists for all $q^* \in \pi^{-1}(\{q\})$. Additionally, if C is second countable then the following identity holds*

$$g_R\big(\pi(p^*), q\big) = \sum_{q*:\, \pi(q^*)=q} g_C(p^*, q^*) \tag{6}$$

Proof. Theorem 4.8 says that C is a Riemann surface whose charts consist of the functions $z \circ \pi$, where z is a chart of R and π is analytic. Let $q_1^*, q_2^*, ..., q_n^*$ be distinct points in C such that $\pi(q_i^*) = q$ for all $i \leq n$. Then the function

$$g_R(\pi(p^*), q) + \log |z(\pi(p^*))|$$

is harmonic at q_i^* for all $i \leq n$. Let $u_i \in \mathcal{F}_{q_i^*}$, the Perron family defining Green's function, and define

$$f(p^*) = \left(\sum_{i=1}^{n} u_i(p^*) \right) - g_R(\pi(p^*), q)$$

which is subharmonic on C. Since each u_i is equal to 0 off some compact set $K_i \subset C$, we have that $f(p^*) \leq 0$ for $p^* \in C \setminus \cup_{i=1}^n K_i$, and thus for all $p^* \in C$ by the maximum principle. Taking the supremum over all $u_i \in \mathcal{F}_{q_i^*}$ for all $i \leq n$ we see that $g_C(p^*, q_i^*)$ exists and satisfies

$$\sum_{i=1}^{n} g_C(p^*, q_i^*) \leq g_R(\pi(p^*), q) \qquad \forall p^* \in C \setminus \cup_{i=1}^n \{q_i^*\}$$

Now assume that C is second countable. If $\pi^{-1}(\{q\})$ were uncountable, then it would have a limit point. This contradicts Theorem 3.8 since π is analytic and not constant. Hence we may assume that $\pi^{-1}(\{q\})$ is at most countable. Taking the limit as $n \to \infty$, we obtain a function $S(p^*)$ satisfying

$$S(p^*) = \sum_{q*:\, \pi(q^*)=q} g_C(p^*, q^*) \leq g_R(\pi(p^*), q)$$

As a limit of an increasing sequence of harmonic functions, $S(p^*)$ is harmonic for $p^* \notin \pi^{-1}(\{q\})$ by Harnack's Theorem. Additionally, $S(p^*) + \log|z(\pi(p^*))|$ is harmonic at each $q^* \in \pi^{-1}(\{q\})$ by the same theorem. This shows that if $u \in \mathcal{F}_q$, then $u(\pi(p^*)) - S(p^*)$ is subharmonic for all $p^* \in C$. Furthermore, we know that $u(p) = 0$ off some compact subset $K \subset R$. Since π is continuous and R is Hausdorff, $K^* = \pi^{-1}(K)$ is compact. Therefore $u(\pi(p^*)) = 0$ for $p^* \in C \setminus K^*$, and thus by the maximum principle we obtain that

$$u(\pi(p^*)) - S(p^*) \leq 0 \qquad \forall p^* \in C$$

Taking the supremum over all $u \in \mathcal{F}_q$ we get that

$$g_R(\pi(p^*), q) \leq S(p^*)$$

which completes the proof. □

Remark. *Since $g_R(p, q)$ and $g_C(p^*, q^*)$ exist for some $q \in R$ and $q^* \in C$, Theorem 4.5 implies that $g_R(p, q)$ and $g_C(p^*, q^*)$ exist for all $q \in R$ and all $q^* \in C$. Thus equation (6) holds for all $q \in R$.*

Remark. *The condition for C to be second countable is redundant, as we will prove that every Riemann surface is second countable in the next section.*

5 The Uniformization Theorem

This section begins with a proof of the Uniformization Theorem in the case where Green's function exists. We use this result to show that all Riemann surfaces are second countable, and that Green's function is symmetric. With this knowledge we proceed to define bipolar Green's function and prove that it always exists. This is then used to prove the Uniformization Theorem in the case where Green's function does not exist. The proof is based on [4] and [7].

5.1 The Case When Green's Function Exists

Theorem 5.1 (**Uniformization case 1**). *Let R be a simply connected Riemann surface. Then Green's function $g(p, q)$ exists for some $q \in R$ iff there exists a bijective analytic function $\varphi : R \to \mathbb{D}$.*

Proof. If there exists a bijective analytic function $\varphi : R \to \mathbb{D}$, then Green's function exists by Lemma 4.7.

For the converse, assume Green's function $g(p, q)$ exists for some $q \in R$. Let $z : U \to \mathbb{D}$ be a chart with $z(q) = 0$. Since $g(p, q) + \log |z(p)|$ is harmonic in U, we can write it as the real part $\mathrm{Re} f$ of an analytic function f defined in U. Hence the function

$$\varphi(p) = z(p)e^{-f(p)}$$

is analytic in U and satisfies $|\varphi(p)| = e^{-g(p,q)}$ there. The function only has one zero, and it is at q.

We now want to construct an analytic continuation of φ along any arbitrary path $\gamma : [0, 1] \to R$ with $\gamma(0) = q$. For any point $\gamma(t)$ where $0 < t \leq 1$, we can find a coordinate disk U_t centered at $\gamma(t)$ such that $q \notin U_t$. Since $g(p, q)$ is harmonic in U_t we can find an analytic function φ_t such that $|\varphi_t(p)| = e^{-g(p,q)}$, defined in U_t. If $\gamma(r)$, $0 < r \leq 1$ is another such point, and U_r intersects U_t, then we can choose φ_r such that $\varphi_t = \varphi_r$ since it is known that two harmonic conjugates differ by a constant. For t close to 0, we may restrict U_t such that $U_t \subset U$ by looking at the connected component containing $\gamma(t)$. We may pick the harmonic conjugate of $g(p, q)$ such that $|\varphi_t(p_0)| = |\varphi(p_0)|$ for some $p_0 \in U$. Since a non-zero analytic function with constant modulus on a connected open set is constant, we have that $\varphi(p) = \varphi_t(p)$ for all $p \in U_t$. Therefore, since R is simply connected, the function φ can be analytically continued to a unique function $\varphi : R \to \mathbb{C}$ by the Monodromy Theorem.

We want to show that φ is injective. Note that $|\varphi(p)| < 1$ for all $p \in R$. Fix $q_1 \in R$ and define

$$\Psi(p) = \frac{\varphi(p) - \varphi(q_1)}{1 - \overline{\varphi(q_1)}\varphi(p)}$$

which is analytic in R and satisfies $|\Psi(p)| < 1$. Let $u \in \mathcal{F}_{q_1}$, the Perron family defining Green's function. Then $u(p) + \log|\Psi(p)|$ is subharmonic on R and $u = 0$ off a compact subset of R, so the maximum principle implies that

$$u(p) + \log|\Psi(p)| \leq 0 \qquad \forall p \in R$$

Taking the supremum over all $u \in \mathcal{F}_{q_1}$ shows that $g(p, q_1)$ exists and that

$$g(p, q_1) + \log|\Psi(p)| \leq 0 \qquad \forall p \in R \setminus \{q_1\} \tag{7}$$

Substituting $p = q$ and a quick calculation shows that

$$g(q, q_1) \leq -\log|\Psi(q)| = -\log|\varphi(q_1)| = g(q_1, q)$$

Exchanging the roles of q and q_1 gives that $g(q, q_1) = g(q_1, q)$. In particular, equality holds in inequality (7) for $p = q$, and hence for all $p \in R \setminus \{q_1\}$ by the maximum principle. Thus $\Psi(p)$ has no zeros on $R \setminus \{q_1\}$, so if $\varphi(q_1) = \varphi(q_2)$, then by definition $\Psi(q_2) = 0$ and $q_1 = q_2$.

Since R is simply connected, $\varphi(R)$ is simply connected. By the Riemann mapping theorem $\varphi(R)$ is conformally equivalent with $\mathbb{D}$, and hence so is R. □

Remark. *In the above theorem we have also proved that if R is a simply connected Riemann surface, then $g(p, q)$ is symmetric, i.e. $g(p, q) = g(q, p)$ for all $p, q \in R$ with $p \neq q$.*

Corollary 5.2 (**Radó's Theorem**). *All Riemann surfaces are second countable.*

Proof. Let R be a Riemann surface with a chart $z : U \to \mathbb{D}$. Let $0 < t < 1$ be a real number and define $W = R \setminus \overline{tU}$. Theorem 4.6 implies that Green's function of W exists, so by Theorem 2.19, Theorem 4.9 and the Uniformization Theorem we may assume that $\mathbb{D}$ is the universal cover of W. Denote by $\mathcal{V}$ a countable basis for the topology on $\mathbb{D}$. Since the covering map $\pi : \mathbb{D} \to R$ is a local homeomorphism, it is also continuous and open. Define

$$\mathcal{B} = \{\pi(V);\ V \in \mathcal{V}\}$$

which is a countable collection of open subsets of W. We proceed by showing that $\mathcal{B}$ is a basis for the topology on W. For any open subset $A \subset W$ and point $p \in A$, we want to find a set $B \in \mathcal{B}$ with $p \in B \subset W$. Since π is surjective there is a point $x \in \mathbb{D}$ with $\pi(x) = p$. The set $\pi^{-1}(A)$ is an open neighbourhood of x, so we can choose a basis element $V \in \mathcal{V}$ with $x \in V \subset \pi^{-1}(A)$. Thus the set $\pi(V)$ satisfies $p \in \pi(V) \subset A$ and $\pi(V) \in \mathcal{B}$.

Since the coordinate disk U is second countable, we conclude that $R = U \cup W$ is second countable. □

Theorem 5.3. *Let R be a Riemann surface, not necessarily simply connected, where Green's function $g_R(p,q)$ exists for some $q \in R$. Then $g_R(p,q)$ is symmetric, i.e. $g_R(p,q) = g_R(q,p)$ for all $p, q \in R$ with $p \neq q$.*

Proof. Theorem 2.19 says that there exists a universal cover C of R. Since $g_R(p,q)$ exists for some $q \in R$, Theorem 4.5 implies that it exists for all $q \in R$, which by Theorem 4.9 implies that Green's function $g_C(p^*, q^*)$ of C exists for all $q^* \in C$. Hence we may assume that $C = \mathbb{D}$ by the Uniformization Theorem. Green's function $g_{\mathbb{D}}(p^*, q^*)$ of the unit disk $\mathbb{D}$ is given by

$$g_{\mathbb{D}}(p^*, q^*) = -\log \left| \frac{p^* - q^*}{1 - \overline{q^*} p^*} \right|$$

Let $\pi : \mathbb{D} \to R$ be the covering map, and $\textsc{Aut}(\pi)$ be the group of deck transformations. Theorem 2.21 states that $\pi \circ \tau = \pi$ for all $\tau \in \textsc{Aut}(\pi)$, and if $\pi(p^*) = \pi(q^*)$ then there is a unique $\tau \in \textsc{Aut}(\pi)$ such that $\tau(p^*) = q^*$. Since π is analytic by Theorem 4.8, each τ must be analytic and is thus a particular Möbius transformation which satisfies $g_{\mathbb{D}}(p^*, \tau(q^*)) = g_{\mathbb{D}}(\tau^{-1}(p^*), q^*)$ for all $p^* \neq q^*$. Using the result in Theorem 4.9 and the symmetry of $g_{\mathbb{D}}(p^*, q^*)$ we finally obtain that

$$\begin{aligned} g_R\big(\pi(p^*), q\big) &= \sum_{q*:\, \pi(q^*)=q} g_{\mathbb{D}}(p^*, q^*) \\ &= \sum_{\tau \in \textsc{Aut}(\pi)} g_{\mathbb{D}}(p^*, \tau(q^*)) \\ &= \sum_{\tau \in \textsc{Aut}(\pi)} g_{\mathbb{D}}(\tau^{-1}(p^*), q^*) \\ &= \sum_{\tau \in \textsc{Aut}(\pi)} g_{\mathbb{D}}(q^*, \tau^{-1}(p^*)) \\ &= g_R(q, \pi(p^*)) \end{aligned}$$

□

5.2 The Case When Green's Function Does Not Exist

Definition 5.4. Let q_1 and q_2 be distinct points on a Riemann surface R with charts $z_1 : U_1 \to \mathbb{D}$ and $z_2 : U_2 \to \mathbb{D}$ such that $z_1(q_1) = 0$ and $z_2(q_2) = 0$. A **bipolar Green's function** with poles at q_1 and q_2 is a harmonic function $G(p, q_1, q_2) : R \setminus \{q_1, q_2\} \to \mathbb{C}$ such that

(i) $G(p, q_1, q_2) + \log |z_1(p)|$ is harmonic at q_1

(ii) $G(p, q_1, q_2) - \log |z_2(p)|$ is harmonic at q_2

(iii) G is bounded on $R \setminus (U_1 \cup U_2)$

Theorem 5.5. *Let q_1 and q_2 be distinct points on a Riemann surface R. Then there always exists a bipolar Green's function $G(p, q_1, q_2)$.*

Proof. Fix some $q_0 \in R \setminus \{q_1, q_2\}$. Let z_0, z_1 and z_2 be charts on coordinate disks U_0, U_1 and U_2 such that $z_j(q_j) = 0$ for $j = 0, 1, 2$. By restricting the charts, assume that $\overline{U_i} \cap \overline{U_j} = \emptyset$ for all $i \neq j$ and that their closures are compact in R. Let $0 < t < 1$ and $0 < r < 1$ be real numbers and set $W_t = R \setminus tU_0$. Since $g_{W_t}(p, q_1)$ exists for all $p \in W_t$ with $p \neq q_1$ by Theorem 4.6, we can define

$$M(t, r) = \max\{g_{W_t}(p, q_1);\ p \in \partial rU_1\}$$

Using the maximum principle we obtain that

$$g_{W_t}(p, q_1) \leq M(t, r) \qquad \forall p \in W_t \setminus rU_1 \tag{8}$$

$$g_{W_t}(p, q_1) + \log |z(p)| \leq M(t, 1) \qquad \forall p \in \overline{U_1} \tag{9}$$

Hence by (8) the function

$$v_{t,r}(p) \equiv M(t, r) - g_{W_t}(p, q_1)$$

is a positive harmonic function in $W_t \setminus rU_1$. By taking the maximum over $p \in \partial rU_1$ in (9) we obtain that there exists a point $x \in \partial U_1$ such that $v_{t,r}(x) \leq \log \frac{1}{r}$. Let K be a compact subset of $W_1 \setminus rU_1$ containing $\{q_2\} \cup \partial rU_1$. Then by Harnack's inequality using the point x, there is a constant $C > 0$ depending only on K and r such that

$$0 \leq v_{t,r}(p) \leq C \qquad \forall p \in K$$

which gives us the inequality

$$|g_{W_t}(p, q_1) - g_{W_t}(q_2, q_1)| = |v_{t,r}(q_2) - v_{t,r}(p)| \leq 2C \qquad \forall p \in K$$

Similarly, if K' is a compact subset of $W_1 \setminus rU_2$ containing $\{q_1\} \cup \partial rU_2$, then there is a constant $C' > 0$ such that

$$|g_{W_t}(p, q_2) - g_{W_t}(q_1, q_2)| \leq 2C' \qquad \forall p \in K'$$

Set $D = 2C + 2C'$. Theorem 5.3 implies that $g_{W_t}(q_1, q_2) = g_{W_t}(q_2, q_1)$, so the function

$$\begin{aligned} G_t(p, q_1, q_2) &\equiv g_{W_t}(p, q_1) - g_{W_t}(p, q_2) \\ &= \big(g_{W_t}(p, q_1) - g_{W_t}(q_2, q_1)\big) - \big(g_{W_t}(p, q_2) - g_{W_t}(q_1, q_2)\big) \end{aligned}$$

is harmonic in $W_t \setminus \{q_1, q_2\}$ and satisfies

$$|G_t(p, q_1, q_2)| \leq D \qquad \forall p \in K \cap K',\ \forall t \in (0, 1)$$

where we may assume that $K \cap K'$ contains $\partial U_1 \cup \partial U_2$. Let $u \in \mathcal{F}_{q_1}$, the Perron family defining $g_{W_t}(p, q_1)$. Since $u = 0$ off a compact subset of R, the maximum principle gives that

$$\begin{aligned}
\sup_{p \in W_t \setminus U_1} \big(u(p) - g_{W_t}(p, q_2)\big) &\leq \max\left(0, \max_{p \in \partial U_1} \big(u(p) - g_{W_t}(p, q_2)\big)\right) \\
&\leq \max\left(0, \max_{p \in \partial U_1} G(p, q_1, q_2)\right) \\
&\leq D
\end{aligned}$$

Taking the supremum over all $u \in \mathcal{F}$ we obtain that

$$\sup_{p \in W_t \setminus U_1} G(p, q_1, q_2) \leq D$$

Using a similar argument for $W_t \setminus U_2$ we get that

$$\begin{aligned}
\inf_{p \in W_t \setminus U_2} G(p, q_1, q_2) &= - \sup_{p \in W_t \setminus U_2} -G(p, q_1, q_2) \\
&\geq -D
\end{aligned}$$

We have obtained the following bound:

$$|G_t(p, q_1, q_2)| \leq D \qquad \forall p \in W_t \setminus (U_1 \cup U_2)$$

Additionally, the functions $G_t + \log|z_1|$ and $G_t - \log|z_2|$ are harmonic in U_1 and U_2 respectively by the properties of Green's function, so by the maximum principle

$$\begin{aligned}
|G_t(p, q_1, q_2) + \log|z_1(p)|| &\leq D \qquad \forall p \in \overline{U_1} \\
|G_t(p, q_1, q_2) - \log|z_2(p)|| &\leq D \qquad \forall p \in \overline{U_2}
\end{aligned}$$

Since all Riemann surfaces are second countable by Radó's Theorem, we may find a sequence of increasing compact sets $K_1 \subset K_2 \subset ...$ such that

$$R \setminus \{q_0, q_1, q_2\} = \bigcup_{n=1}^{\infty} K_n$$

Given any decreasing sequence $t_n \to 0$, we may apply the Arzelà–Ascoli Theorem to find a subsequence $G_{t_{n_k}}$ which converges uniformly to a harmonic function $G(p, q_1, q_2)$ for $p \in K_1$. Continuing in this manner by taking subsequences of the previous subsequence, we may use a diagonalization argument to find a sequence G_{t_n} which converges uniformly to $G(p, q_1, q_2)$ for all $p \in R \setminus \{q_0, q_1, q_2\}$. This function satisfies (i), (ii), and (iii) in Definition 5.4 of the Bipolar Green's function, and extends to be harmonic at q_0 because it is bounded near that point. □

Theorem 5.6 (**Uniformization case 2**). *Let R be a simply connected Riemann surface. If Green's function $g(p,q)$ does not exist for some $q \in R$, then there is a bijective analytic function φ mapping R onto $\mathbb{C}$ or $\mathbb{C}^*$.*

Proof. Let $G(p,q_1,q_2)$ be a bipolar Green's function. Using the same construction as in Theorem 5.1 (Uniformization case 1), we may use the Monodromy theorem to find a meromorphic function $\varphi : R \to \mathbb{C}^*$ such that

$$|\varphi(p)| = e^{-G(p,q_1,q_2)}$$

with only one zero at q_1 and one pole at q_2. We want to show that φ is injective. Let $q_0 \in R \setminus \{q_1, q_2\}$ and define $\varphi_0 : \mathbb{R} \to \mathbb{C}^*$ to be a meromorphic function on R such that

$$|\varphi_0(p)| = e^{-G(p,q_0,q_2)}$$

and consider the function

$$\psi(p) = \frac{\varphi(p) - \varphi(q_0)}{\varphi_0(p)}$$

We have that ψ is bounded and analytic in R by the properties of bipolar Green's function and since the poles at q_2 and zeros at q_0 cancel out. If ψ were nonconstant then Green's function $g(p,q)$ would exist for all $q \in R$ by Lemma 4.7, hence

$$\psi(p) = \psi(q_1) = \frac{-\varphi(q_0)}{\varphi_0(q_1)}$$

is a finite nonzero constant function. If $\varphi(p) = \varphi(q_0)$, then $\varphi_0(p) = 0$ since otherwise $\psi(p)$ would be zero. Thus $p = q_0$ since φ_0 only has one zero, which proves injectivity.

This shows that φ is a bijective analytic function onto the simply connected region $\varphi(R) \subset \mathbb{C}^*$. If $\mathbb{C}^* \setminus \varphi(R)$ contains more than one point, then by the Riemann mapping theorem we can find a bijective analytic function mapping R onto $\mathbb{D}$, contradicting Theorem 5.1 since we assumed Green's function did not exist. Hence $\mathbb{C}^* \setminus \varphi(R)$ contains at most one point which we can move to ∞ with a Möbius transformation. □

Corollary 5.7. *The universal cover of a Riemann surface is conformally equivalent to either the complex plane $\mathbb{C}$, the unit disk $\mathbb{D}$, or the Riemann sphere $\mathbb{C}^*$.*

Proof. Theorem 4.8 implies that the universal cover is a Riemann surface. The claim now follows directly by the Uniformization Theorem. □

References

[1] L. Ahlfors and L. Sario. *Riemann Surfaces*. Princeton Math. Series. Princeton University Press, 1960.

[2] K. Carne. *Riemann surfaces and discrete groups*. Lecture notes. 2007. URL: www.dpmms.cam.ac.uk/tkc/.

[3] T. Forster and B. Gilligan. *Lectures on Riemann surfaces*. Graduate Texts in Mathematics. Springer Verlag, 1981.

[4] T. Gamelin. *Complex Analysis*. Undergraduate Texts in Mathematics. Springer New York, 2003.

[5] J. Gray. *On the history of the Riemann mapping theorem*. Rendiconti del Circolo Matematico di Palermo. Serie II, 1994.

[6] Y. Lekili. *Riemann Surfaces*. Lecture notes. URL: www.lekili.duckdns.org/teaching/curves/curves.pdf.

[7] D. Marshall. *The Uniformization Theorem*. Preprint. 2011. URL: www.math.washington.edu/~marshall/preprints/uniformizationII.pdf.

[8] C. Teleman. *Riemann Surfaces*. Lecture notes. 2003. URL: www.math.berkeley.edu/~teleman/math/Riemann.pdf.