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On the $(\frac{1}{2}, \frac{1}{2})$ Representation of the Lorentz Group and the Discrete CPT Symmetries

Nikolaj Skog Pirinen

Department of Astronomy and Theoretical Physics, Lund University

Bachelor thesis supervised by Malin Sjödahl

Abstract

This thesis derives the explicit form of the elements of the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group, by actually performing a direct product of the chiral $(\frac{1}{2}, 0)$- and $(0, \frac{1}{2})$-representations. The Lorentz transformations of fourvectors are thereafter recovered from this direct-product representation, allowing the derivation of a transformation matrix between the fourvector- and direct product basis. This matrix is then used to explore the discrete $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations in both bases.

Populärvetenskaplig beskrivning

Början av 1900-talet såg en oerhörd utveckling av nya forskningsområden inom fysiken. 1905 lade Einstein grunden för den speciella relativitetsteorin, som revolutionerade hur fysiker ser på själva tiden och rummet genom att föreslå att vi kan behandla de båda på samma sätt. Samtidigt växte kvantfysikens värld fram i jakten på en beskrivning av vår verklighets allra minsta beståndsdelar.

Kvantfysikens historia anses ofta börja med Max Planck, som insåg att verkligheten verkar kräva att energi endast kan färdas i små paket, så kallade kvanta. Därefter fylldes kvantfysikens unga rymder av stjärnnamn som Werner Heisenberg, Erwin Schrödinger, Niels Bohr och Wolfgang Pauli, som alla bidrog till att förändra fysikens världsbild i grunden. Bland dessa namn sticker dock ett ut som kanske det mest betydelsefulla: Paul Dirac.

Bland hans många resultat var en av de mest revolutionerande den numera så kallade *Diracekvationen*, som av många fortfarande anses vara en av den teoretiska fysikens främsta bedrifter. Anledningarna är flera. Till exempel lyckas ekvationen beskriva elektronen (och därmed atomen) mer exakt än någon annan tidigare teori kunnat göra, genom att sammanföra kvantfysikens regelverk med den speciella relativitetsteorin. Förutom detta bevisade den även att alla partiklar måste vara ständigt magnetiska, ett fenomen som fysiker kallar spinn, samt förutsade antimaterians existens flera år före dess upptäckt.

Diracekvationen beskriver likt Schrödingerekvationen alla partiklar som vågor. Eftersom den förstnämnda inte bara måste passa in med kvantfysiken, utan även relativitetsteorin, måste ekvationens vågor breda ut sig i rummet såväl som i tiden. Vågorna måste även av samma orsak lyda under en universell grundlag från den speciella relativitetsteorin känd som *Relativitetsprincipen*.

Relativitetsprincipen uttrycker att oavsett var ett objekt befinner sig i rumstiden och hur objektet där rör sig, så måste det lyda under samma fysikaliska lagar. Den här egenskapen kallar fysiker för *symmetri* i rumstiden, och att ett objekt skall uppfylla den egenskapen brukar ses som ett grundläggande krav för att det överhuvudtaget ska kunna existera i universum.

Dessa vågor som Diracekvationen tillskriver alla partiklar måste alltså även vara symmetriska i rumstiden. Vågorna beskrivs formellt av objekt kallade spinorer, och de står i kontrast till vektorer, vilka beskriver partiklar inom den klassiska speciella relativitetsteorin.

Trots att vektorer och spinorer ser matematiskt väldigt annordlunda ut, måste de stämma överens med varandra eftersom de i slutändan beskriver samma universum. Syftet med denna uppsats är att härleda en slags översättningsmetod mellan de två objekten genom att uttnytja deras rumstidssymmetri, och därmed bygga en bro mellan Albert Einsteins och Paul Diracs arbeten.

Contents

1 Introduction

The theory of special relativity was published in 1905 [1], and has since then formed the foundation for many modern branches of physics, such as particle physics and cosmology. Informally, it is a framework for describing what changes in a system when considering different frames of reference. The certain kind of reference frame transformations that the theory concerns are known as Lorentz transformations, commonly described by the property that they keep the speed of light in vacuum constant.

Lorentz transformations are an example of a notion called symmetry transformation, generally defined as any transformation which leaves the dynamics of a system unchanged. Lorentz transformations cannot change the dynamics of a system consistent with special relativity, and are therefore said to be a symmetry of spacetime.

Symmetry itself may be mathematically described through a certain algebraic structure called a group. Groups are sets of elements which can be combined in a specific way such that they produce another element within the set. Since any amount of Lorentz transformations performed on spacetime will yield another Lorentz transformation, the set of all such transformations form a group known as the Lorentz group.

In practice, one way to express the elements of the Lorentz group is through matrices applied onto vectors or similar objects. The structure of the vectors in question require the matrices of the Lorentz group to have a certain form, the choice of which is known as a representation of the group. In order to differentiate between different Lorentz group representations, they are commonly labeled by a pair of integer- or half-integer values.

For example, the ordinary kinematic models of special relativity in Minkowski spacetime applies Lorentz transformations to fourvectors. This induces linear transformations of the spacetime coordinates of the fourvectors, which may be expressed as a matrix. These matrices are a Lorentz group representation, and are labeled by the numbers $(\frac{1}{2}, \frac{1}{2})$.

Other representations of the Lorentz group are gained when considering Lorentz transformations on something else than fourvectors. For example within quantum field theory, when fermions are described as solutions to the Dirac equation. The objects that solve this equation are known as Dirac spinors, which have four complex components, and must transform in a certain way under the Lorentz group, like the fourvectors of special relativity. However, since the mathematical structure of the Dirac spinors is different to that of the fourvectors, they must transform according to a different representation of the Lorentz group.

Specifically, Dirac spinors transform according to a direct sum of two two-dimensional representations of the Lorentz group labeled by $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, called the left- and right-chiral representations respectively. The transformations have this structure because the Dirac spinor itself consists of the direct sum of a pair of two-component objects called Weyl spinors. These objects appear individually as the solutions to the Dirac equation in the limit of a particle without mass, known as the Weyl equations. The Weyl spinors transform according to the two chiral representations, so the Lorentz transformations of the Dirac spinor is described by the direct sum of both of these operations.

On the other hand, if one was to consider the direct *product* of the chiral representations instead of the sum, one gains a $(\frac{1}{2}, \frac{1}{2})$ representation, equivalent to that of the fourvectors. However, in many pedagogical sources, the direct product of the chiral representations is not explicitly performed in discussions of the $(\frac{1}{2}, \frac{1}{2})$ representation. Instead, one typically considers the action of both chiral transformations acting on a basis of hermitian (2×2) matrices. By doing this, it is possible to derive the standard form of all Lorentz transformations from the complex components of the matrices. The purpose of this thesis is not to consider the hermitian matrix basis however, but instead calculate the direct product of the left- and right chiral representations in order to gain the $(\frac{1}{2}, \frac{1}{2})$ representation in a direct-product basis, and thereafter explore its consequences.

This thesis will first introduce the theoretical background necessary for the later derivations in section 2. This section begins by introducing some fundamental group theory, which is followed by the representation theory of the fourvector representation of the Lorentz group. The section then continues with the theory of the Dirac- and Weyl equations, along with the theory of the chiral representations of the Lorentz group, and ends with a description of the $(\frac{1}{2}, \frac{1}{2})$ representation. The subsequent section 3 consists of the results of the derivation of the direct product representation, along with an exploration of its consequences. The transformation matrices of the representation are derived first. Thereafter, it is found that each transformation has invariant eigenvectors, which are used to identify the space and time axes of the fourvector representation. This then allows the derivation of a linear transformation matrix between the direct product- and fourvector basis. Finally, the form of the fourvector $\mathcal{P}$ and $\mathcal{T}$ transformations are explored using the found matrix in the direct product basis, and likewise the effects of one Weyl spinor charge conjugation operation is explored in the fourvector basis.

2 Theoretical Background

What follows in this section is an review of the necessary theoretical background required to understand the process done in section 3. First, some general matrix group representation theory is reviewed, whereafter the Lorentz group and its representations are considered, along with its connection to the Dirac equation and the Weyl equations.

2.1 Fundamental Matrix Group Theory

In many applications within physics, one considers groups whose elements are matrices [4]. Many of the common groups are subgroups of the so-called general linear group, denoted $GL(n, \mathbb{F})$, defined as the set of all invertible $(n \times n)$ matrices over field $\mathbb{F}$ under the operation of matrix multiplication.

Of particular importance to this thesis is the subgroup of the general linear group known as the *pseudo-orthogonal groups*. These are defined through,

$$O(p,q) = \{g \in GL((p+q), \mathbb{R}) | \; g\,\mathbb{I}(p,q)\,g^T = \mathbb{I}(p,q)\}, \tag{2.1}$$

where $\mathbb{I}(p,q)$ is a diagonal matrix, with p positive ones, followed by q negative ones [4]. The group is only called orthogonal if $p \neq 0$ and $q = 0$. Furthermore, if the entries of the matrices g are complex, one instead considers the pseudo-unitary groups, similarly defined through

$$U(p,q) = \{g \in GL((p+q), \mathbb{C}) | \; g\,\mathbb{I}(p,q)\,g^\dagger = \mathbb{I}(p,q)\}. \tag{2.2}$$

If $\mathbb{I}(p,q)$ is purely positive, the notation is typically changed to $U(n)$, and the group is simply called unitary.

Moreover, if one enforces the constraint that the orthogonal and unitary matrices may only have unit determinant, the subgroups are called special, and are denoted $SO(p,q)$ and $SU(p,q)$ respectively [4].

2.1.1 Direct Product Representations

In this thesis, the concept of a direct product between two representations is of central importance. Between two matrix representations A and B, the direct product operation is defined such that each element of A is multiplied by the entirety of matrix B. For one $(m \times n)$ matrix A, and another B, of size $(p \times q)$, their direct product hence yields a $(mp \times nq)$ matrix $(A \times B)$ like for example [4]

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} A_{11} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} & A_{12} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \\ A_{21} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} & A_{22} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \end{pmatrix}. \tag{2.3}$$

Note how this differs from the direct sum, which appends the indices of one set to the end of another.

2.2 The Lorentz Group

The following thesis considers the group known as the Lorentz group, which is defined as the group of isometric transformations of spacetime that leave the Minkowski metric invariant [3].

2.2.1 The Fourvector Representation of the Lorentz Group

With the Minkowski metric defined as $\eta \equiv \mathrm{diag}[1, -1, -1, -1]$, the Lorentz group is identified as the pseudo-orthogonal group [4]

$$\mathrm{SO}(1,3) \equiv \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) |\, \Lambda^T \eta \Lambda = \eta\} \,. \tag{2.4}$$

This group may be divided into four components. To see this, note that for any element in the group,

$$\det(\Lambda^T \eta \Lambda) = \det(\eta) \tag{2.5}$$

Since $\det(\eta) = -1$, this grants the constraint on the Lorentz group elements that

$$\det(\Lambda) = \pm 1,$$

showing that Lorentz transformations are isometries in spacetime. Furthermore, definition (2.4) also provides the constraint

$$\Lambda_0{}^{\mu} \eta_{\mu\nu} \Lambda^{\nu}{}_0 = (\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^0_i)^2 = \eta_{00} = 1, \tag{2.6}$$

where Einstein-summation convention is utilized over μ and ν. Equation (2.6) then implies

$$\Lambda^0{}_0 \geq 1 \text{ or } \Lambda^0{}_0 \leq -1. \tag{2.7}$$

Any element of the Lorentz group can thereby be characterised with respect to the sign of its time component and the sign of its determinant. An element with positive determinant is known as *proper*, and one with positive time component is called *orthochronous*. The four possible combinations of these two properties defines the four components of the Lorentz group [4].

From this point on, the elements of the Lorentz group are assumed to belong to the proper orthochronous component, unless specified otherwise. Two notable exceptions to this are however the time reversal operator $\mathcal{T}$, defined as

$$\mathcal{T} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

and the parity inversion operator $\mathcal{P}$ defined as

$$\mathcal{P} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.9}$$

Neither of these are proper or orthochronous. However, they are still considered further on, because applying $\mathcal{T}$ to a Lorentz group element changes the element's orthochronous property, and likewise $\mathcal{P}$ changes its properness [4]. One can thus reach the other three components of the Lorentz group from the proper orthochronous component, by applying these two operators.

Furthermore, the elements of the proper orthochronous Lorentz group may themselves be divided into two categories of operations, namely spatial rotations and Lorentz boosts [4]. The form of a rotation around the x-axis in the fourvector representation may be expressed as,

$$R(\theta_1, \hat{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & 0 & \sin(\theta_1) & \cos(\theta_1) \end{pmatrix}, \tag{2.10}$$

while in the same representation, along the same axis, Lorentz boosts are represented by

$$\Lambda(\eta_1, \hat{x}) = \begin{pmatrix} \cosh(\eta_1) & -\sinh(\eta_1) & 0 & 0 \\ -\sinh(\eta_1) & \cosh(\eta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.11}$$

The matrices used for the other directions are shown in appendix A, and the objects which these operations are applied to are the fourvectors of special relativity, which the reader is assumed to be familiar with.

By linearizing the Lorentz group elements above with respect to the parameter close to the identity, one may gain the corresponding Lie-group generators. For a Lorentz boost along x-axis, the generator is [4]

$$K_1' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.12}$$

while for a rotation, one may use the generator,

$$L_1' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.13}$$

where as with the group elements, the form of the other directions are shown in appendix A.

By convention, these generators are now redefined to be hermitian, so that their eigenvalues can correspond to physical states [4]. The generators of the Lorentz group are therefore instead taken as

$$L_i \equiv iL_i', \;\; K_i \equiv iK_i'. \tag{2.14}$$

However, note that despite this redefinition, the new generators K_i' of the Lorentz boosts are still not hermitian, although the redefinition is still kept, such that the generators have the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \;\; [K_i, K_j] = -i\epsilon_{ijk}L_k, \;\; [K_i, L_j] = i\epsilon_{ijk}K_k, \tag{2.15}$$

where ϵ_{ijk} is the Levi-Civita tensor.

Any element of the fourvector representation of the Lorentz Group may thereby be generated by the exponential map,

$$\Lambda(\theta^i, \eta^i) = \exp[\,-i(\theta^i L_i + \eta^i K_i)\,], \tag{2.16}$$

for some rotation parameters θ_i and boost parameters η_i.

2.3 Spinors

The purpose of this thesis is to formalize a connection between the transformations of fourvectors and spinors, the second of which occurs within quantum field theory as solutions to the Dirac- and Weyl equations.

2.3.1 The Dirac and Weyl equation

The Dirac equation is a wave equation describing relativistic particles of half-integer spin, and is defined as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{2.17}$$

for the Dirac gamma matrices,

$$\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i \equiv \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.18}$$

where the σ^i are the Pauli matrices, shown in appendix A [2]. Note that the form of the gamma matrices is a matter of convention, where the form used here is known as the chiral- or Weyl basis. No matter the convention, the gamma matrices are, however, always four-dimensional. The solutions to the Dirac equation therefore has four components and constitute the objects known as Dirac spinors.

Considering such a spinor, equation (2.17) becomes

$$\begin{pmatrix} -m & i(\partial_0 - \sigma^i\partial_i) \\ i(\partial_0 + \sigma^i\partial_i) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0, \tag{2.19}$$

Where ψ_L, ψ_R are two-dimensional objects known as Weyl spinors [2]. These appear individually as a solution to the Dirac equation in the massless limit. Equation (2.19) is then decoupled into

$$i(\partial_0 - \sigma^i\partial_i)\psi_R = 0, \tag{2.20}$$

$$i(\partial_0 + \sigma^i \partial_i)\psi_L = 0, \tag{2.21}$$

which are known as the Weyl equations.

2.3.2 SU(2)

Before introducing the Lorentz group representations of spinors, some knowledge of the group $SU(2)$ is required [3]. $SU(2)$ is defined as the set of all (2×2) invertible unitary matrices of determinant 1, formally expressed as

$$\mathrm{SU}(2) \equiv \{g \in \mathrm{GL}(2,\mathbb{C})| \;\; g^\dagger \mathbb{I} g = \mathbb{I}, \det(g) = 1\}, \tag{2.22}$$

and any element within it may be generated by the Lie algebra spanned by the generators

$$J_1 = \frac{1}{2}\sigma^1, \;\; J_2 = \frac{1}{2}\sigma^2, \;\; J_3 = \frac{1}{2}\sigma^3. \tag{2.23}$$

The commutation relations between these generators are

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \tag{2.24}$$

where ϵ_{ijk} is the Levi-Civita tensor [4].

Furthermore, using these generators, one may define

$$J^2 \equiv J_1^2 + J_2^2 + J_3^2, \tag{2.25}$$

which is known as the Casimir operator of the group. It commutes with all generators of $SU(2)$ [4], and given a representation of $SU(2)$, this operator will have a single well-defined eigenvalue for all elements. This eigenvalue is typically denoted j, and is used to label different representations of $SU(2)$. As it turns out, this may also be used to label Lorentz group representations, as discussed in the following section.

2.3.3 Spinor Representations

In equation (2.15), the generators of the Lorentz group do not mutually commute; the commutator of two rotation generators corresponds another rotation generator, but so does the commutator of two boosts. By rearranging the generators in the complex linear combination

$$N_i^{\pm} \equiv \frac{1}{2}(L_i \pm iK_i), \tag{2.26}$$

we instead gain the commutation relations,

$$[N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+, \;\; [N_i^-, N_j^-] = i\epsilon_{ijk} N_k^-, \;\; [N_i^{\pm}, N_j^{\mp}] = 0, \tag{2.27}$$

so that the two sets of generators are mutually commuting.

Now, note that these commutation relations of the two sets of generators are identical to those of the $SU(2)$ algebra. The algebra of $SO(1,3)$ can hence be expressed as a direct sum of two sets of $SU(2)$ algebras. Note however that it consists of two sets of complexified $SU(2)$ algebras, because of the imaginary component in $N_i^{\pm}$, which means complex linear combinations of the generators may be included. The implications of this is however not of importance to this thesis.

Through the redefinition of the Lorentz group generators into $N_i^{\pm}$, one hence gains

$$\mathfrak{so}_{\mathbb{C}}(1,3) \cong \mathfrak{su}(2)^+_{\mathbb{C}} \oplus \mathfrak{su}(2)^-_{\mathbb{C}}. \tag{2.28}$$

Representations of the Lorentz group $SO(1,3)_{\mathbb{C}}$ can therefore be labeled by a pair of $SU(2)$ Casimir operator eigenvalues (j_1, j_2) [4].

2.3.4 The Chiral Representations of the Lorentz Group

Of central importance to this thesis are the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group, known as the left- and right-chiral representations, respectively. The transformations of these representations are two-dimensional, and carry physical significance because they both act on Weyl- and Dirac spinors. Note, however, that the choice of which representation corresponds to which chirality is a matter of convention.

For both chiral representations, one $SU(2)$ algebra is zero-dimensional, while the other corresponds to a two-dimensional representation like those considered in section 2.5.2 [4]. The left chiral representation is hence generated by

$$N_i^- = \frac{\sigma_i}{2}, \quad N_i^+ = 0, \tag{2.29}$$

and the right chiral is generated by

$$N_i^- = 0, \quad N_i^+ = \frac{\sigma_i}{2}. \tag{2.30}$$

From definition (2.16) and (2.26), any Lorentz group elements of the left- and right chiral representations can hence respectively be expressed as

$$\Lambda^{(\frac{1}{2},0)} = \exp\left(\frac{-1}{2}(i\theta^i - \eta^i)\sigma^i\right), \tag{2.31}$$

$$\Lambda^{(0,\frac{1}{2})} = \exp\left(\frac{-1}{2}(i\theta^i + \eta^i)\sigma^i\right). \tag{2.32}$$

A spinor which is the solution to equation (2.21) will transform according to equation (2.31), and is hence called a left-chiral spinor [3]. Likewise, a right chiral spinor transforms according to (2.32) and is the solution to (2.20), and the Dirac spinor like that of equation (2.19) is then a direct sum of one left- and one right chiral spinor.

A way to differentiate between the two chiralities of spinors is by using Van Der Waerden notation, where left- and right chiral spinors are represented by [4]

$$\psi_L \equiv \chi^{\dot{a}}, \;\; \psi_R \equiv \chi_a. \tag{2.33}$$

Complex conjugation of a spinor adds or removes the dot above the index [4] through

$$(\chi^{\dot{a}})^* \equiv \chi^a, \;\; (\chi_a)^* \equiv \chi_{\dot{a}}, \tag{2.34}$$

and in order to move an index up or down, the spinor is multiplied by a Levi-Civita tensor, such that

$$\epsilon_{\dot{b}\dot{a}}\chi^{\dot{a}} = \chi_{\dot{b}}, \quad \text{where} \quad \epsilon_{\dot{b}\dot{a}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{ba}, \tag{2.35}$$

$$\text{and} \quad \epsilon^{ba}\chi_a = \chi^b, \quad \text{where} \quad \epsilon^{ba} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \epsilon^{\dot{b}\dot{a}}.$$

Thereby, one may fully change the chirality of a spinor through

$$\epsilon_{ba}(\chi^{\dot{a}})^* = \chi_b, \;\; \epsilon^{\dot{b}\dot{a}}(\chi_a)^* = \chi^{\dot{b}}. \tag{2.36}$$

Further on, equation (2.36) is also considered the charge conjugation transformation of a spinor, and applying this operation on a spinor changes which chiral representation the spinor transforms according to [4]. There are however other definitions of the same operation, a phase within definition (2.36) would, for example, not change the Lorentz transformation properties [5].

Finally, because the Levi-Civita tensors of equation (2.35) moves indices of spinors, just like the Minkowski metric for co- and contravariant fourvectors, the epsilon tensor is sometimes called the spinor metric [4]. It turns out that as with the fourvector metric, the Levi-Civita tensor also allows the construction of Lorentz invariants through combinations on the form

$$\chi^{\dot{a}}\chi_{\dot{a}} = \chi^{\dot{a}}\epsilon_{\dot{a}\dot{b}}\chi^{\dot{b}}, \quad \text{and} \quad \chi^a\chi_a = \chi_a\epsilon^{ab}\chi_b, \tag{2.37}$$

where these quantities are invariant under the chiral representations of the Lorentz group [3].

2.3.5 The $\left(\frac{1}{2}, \frac{1}{2}\right)$ Representation

This representation is the main object of the following thesis. It may be obtained from the direct product between the two chiral representations, which in terms of group elements reads as

$$\Lambda^{(\frac{1}{2},\frac{1}{2})} = \Lambda^{(\frac{1}{2},0)} \otimes \Lambda^{(0,\frac{1}{2})}. \tag{2.38}$$

This four-dimensional representation is equivalent to the standard representations of the fourvector transformations, meaning that the elements of the fourvector representation can be obtained from a similarity transformation of this representation, and vice-versa. Equation (2.38), however, offers a method to express this representation in a different basis, using the elements of the chiral representations instead. A central objective of this thesis is to derive a matrix allowing transformations between the elements of this direct product representation and the fourvector Lorentz transformations.

In many sources, the calculation of the elements of the direct product representation is not explicitly performed. Instead, one typically considers an object of two upper indices acting upon a basis of hermitian (2×2) matrices, such as [4]

$$\begin{aligned} p^{\dot{a}b} &\equiv p_\mu \sigma^{\mu\dot{a}b} \\ &= p_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} . \end{aligned} \tag{2.39}$$

By considering a Lorentz transformation on this object such as

$$(p^{\dot{a}b})' = \Lambda^{\dot{a}}{}_{\dot{\rho}} \Lambda^{b}{}_{\gamma} p^{\dot{\rho}\gamma} , \tag{2.40}$$

where the $\Lambda^{\dot{a}}{}_{\dot{\rho}}$ corresponds to a left chiral Lorentz transformations like equation (2.31), and $\Lambda^{b}{}_{\gamma}$ corresponds to a right chiral Lorentz transformations like equation (2.32), it can be shown that one recovers the fourvector Lorentz transformations of section 2.2.1 [4]. The purpose of the following thesis, however, is to perform the explicit calculation of the $(\frac{1}{2}, \frac{1}{2})$ representation by performing the direct product of the two spinors, and exploring its implications.

3 Results

The structure of this section is summarized into three main steps. First, the explicit form of every Lorentz transformation in the direct product representation is derived. In the process, the eigenvectors of these transformations are found, and these are in the second section taken as corresponding to the spacetime axes of the fourvector representation. This then allows the derivation of a linear transformation matrix between the direct product and fourvector basis. Lastly, this matrix is used to derive the form of different discrete spacetime transformations in both fourvector and direct-product basis.

3.1 The Direct Product Spinor Representation

To begin, the matrix representation of a Lorentz boost purely along the x-axis in the left chiral representation is considered. By expanding the corresponding exponential map of equation (2.31), the transformation matrix of the left-chiral representation is

$$\begin{aligned} B_1^{(\frac{1}{2},0)} &= \exp\left(\frac{1}{2}\eta_1\sigma^1\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\eta_1\sigma^1\right)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{1}{2}\eta_1\sigma^1\right)^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{1}{2}\eta_1\sigma^1\right)^{2n+1} \\ &= \mathbb{I}\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{1}{2}\eta_1\right)^{2n} + \sigma^1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{1}{2}\eta_1\right)^{2n+1} \\ &= \mathbb{I}\cosh\left(\frac{\eta_1}{2}\right) + \sigma^1\sinh\left(\frac{\eta_1}{2}\right) = \begin{pmatrix} \cosh\left(\frac{\eta_1}{2}\right) & \sinh\left(\frac{\eta_1}{2}\right) \\ \sinh\left(\frac{\eta_1}{2}\right) & \cosh\left(\frac{\eta_1}{2}\right) \end{pmatrix}, \end{aligned} \tag{3.41}$$

where in the fourth step it is used that $(\sigma^i)^2 = \mathbb{I}$. In the right chiral representation, the only difference between the elements is

$$\exp\left(\frac{1}{2}\eta_1\sigma^1\right) \to \exp\left(\frac{-1}{2}\eta_1\sigma^1\right),$$

and therefore, by the same process, the matrix of a boost in x-direction in right chiral representation is

$$B_1^{(0,\frac{1}{2})} = \begin{pmatrix} \cosh\left(\frac{\eta_1}{2}\right) & -\sinh\left(\frac{\eta_1}{2}\right) \\ -\sinh\left(\frac{\eta_1}{2}\right) & \cosh\left(\frac{\eta_1}{2}\right) \end{pmatrix}.$$

Which gives the total matrix in the direct product representation as

$$B_1^{(\frac{1}{2},\frac{1}{2})} = B_1^{(\frac{1}{2},0)} \otimes B_1^{(0,\frac{1}{2})} = \begin{pmatrix} c_h^2 & -c_h s_h & c_h s_h & -s_h^2 \\ -c_h s_h & c_h^2 & -s_h^2 & c_h s_h \\ c_h s_h & -s_h^2 & c_h^2 & -c_h s_h \\ -s_h^2 & c_h s_h & -c_h s_h & c_h^2 \end{pmatrix}, \tag{3.42}$$

where $s_h \equiv \sinh\left(\frac{\eta_1}{2}\right)$, and $c_h \equiv \cosh\left(\frac{\eta_1}{2}\right)$. This matrix has the eigenvalues

$$x_1 = 1,\ \ x_2 = 1,\ \ x_3 = e^{-\eta_1},\ \ x_4 = e^{\eta_1}, \tag{3.43}$$

with associated eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ \ v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ \ v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix},\ \ v_4 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \tag{3.44}$$

where all calculations were verified by using Wolfram Mathematica.

Consider now the operator of a spatial rotation around the x-axis. Both chiral representations behave equivalently under rotations. Through the same process as in derivation (3.41), the matrices are

$$R_1^{(\frac{1}{2},0)} = R_1^{(0,\frac{1}{2})} = \begin{pmatrix} \cos\left(\frac{\theta_1}{2}\right) & -i\sin\left(\frac{\theta_1}{2}\right) \\ -i\sin\left(\frac{\theta_1}{2}\right) & \cos\left(\frac{\theta_1}{2}\right) \end{pmatrix}, \tag{3.45}$$

yielding, in total, the matrix

$$R_1^{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} c^2 & -ics & -ics & -s^2 \\ -ics & c^2 & -s^2 & -ics \\ -ics & -s^2 & c^2 & -ics \\ -s^2 & -ics & -ics & c^2 \end{pmatrix}, \tag{3.46}$$

where $s \equiv \sin\left(\frac{\theta_1}{2}\right)$, and $c \equiv \cos\left(\frac{\theta_1}{2}\right)$. This matrix has the eigenvalues,

$$x_1 = 1, \;\; x_2 = 1, \;\; x_3 = e^{-\theta_1}, \;\; x_4 = e^{\theta_1}, \tag{3.47}$$

along with the associated eigenvectors

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\; v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \;\; v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \;\; v_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \tag{3.48}$$

Note that each transformation has two eigenvectors that are left invariant, corresponding to eigenvalue 1. These will further on be referred to as invariant eigenvectors, and are later utilized to identify the fourvector basis.

In the following discussion the exact same process as derivation (3.41) has been repeated for all transformations and directions. In all cases, the matrices in the direct product basis have the same eigenvalues as either equation (3.43) or (3.47) depending on the transformation, with only the index of each parameter changing. Furthermore, the derivations following this section only utilize the invariant eigenvectors of each direction, therefore only these are shown.

For the transformations in y-direction, the total matrices are

$$B_2^{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} c_h^2 & -ic_h s_h & ic_h s_h & s_h^2 \\ ic_h s_h & c_h^2 & -s_h^2 & ic_h s_h \\ -ic_h s_h & -s_h^2 & c_h^2 & -ic_h s_h \\ s_h^2 & -ic_h s_h & ic_h s_h & c_h^2 \end{pmatrix}, \;\; R_2^{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} c^2 & -cs & -cs & s^2 \\ cs & c^2 & -s^2 & -cs \\ cs & -s^2 & c^2 & -cs \\ s^2 & cs & cs & c^2 \end{pmatrix}, \tag{3.49}$$

with invariant eigenvectors,

$$v(B)_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \;\; v(B)_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \;\; \text{and} \;\; v(R)_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\; v(R)_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}. \tag{3.50}$$

Finally, the transformations in z-direction take the form,

$$B_3^{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\eta_3} & 0 & 0 \\ 0 & 0 & e^{-\eta_3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \;\; R_3^{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} e^{-i\theta_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix}. \tag{3.51}$$

Since the matrices are diagonal, the invariant eigenvectors are the basis vectors corresponding to diagonal elements 1.

This section has thereby derived the explicit form of the elements of the direct product representation of the Lorentz group.

3.2 Derivation of the Basis Transformation Matrix

Now the invariant eigenvectors are used to deduce a transformation matrix between the direct product spinor basis and the fourvector basis. Firstly, note that a Lorentz boost will leave fourvectors in the non-boosted spatial directions invariant. Likewise, spatial rotations in the fourvector basis will leave the rotational axis and the time-axis invariant. The invariant eigenvectors of each transformation in the direct product basis must hence correspond to some linear combination of these invariant fourvector axes.

To illustrate this, note that one invariant vector of the boost in x-direction is shared with the boost in y-direction. This vector must be proportional to the only axis left invariant by both simultaneously,

$$\text{thus,} \quad \hat{z} \propto \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \tag{3.52}$$

Likewise, the time axis is shared by both rotations, and must hence correspond to

$$\hat{t} \propto \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}. \tag{3.53}$$

Using the same argument, the remaining invariant eigenvectors correspond to,

$$\hat{x} \propto \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{y} \propto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.54}$$

Simply putting these together in the form of a linear basis transformation matrix yields

$$\mathbb{S} \equiv \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{3.55}$$

This matrix however fails to reproduce the correct fourvector transformations. Instead, in order to gain the correct basis transformation matrix, note that if v_i is an eigenvector of eigenvalue ϵ_i, then so is $\pm\, i\, v_i$. Using this, the vectors may be scaled as to reproduce the correct fourvector transformations like those shown in section 2.2.1 and appendix A. By rescaling the rows of $\mathbb{S}$ and performing a similarity transformation on a direct product Lorentz transformations with the resulting matrix, it is found that one matrix that reproduces all the correct fourvector transformations is,

$$\mathbb{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \end{pmatrix}, \tag{3.56}$$

which has the inverse,

$$\mathbb{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -i & 0 \end{pmatrix} = \mathbb{T}^{\dagger}, \tag{3.57}$$

For example, this matrix produces a Lorentz boost in x-direction as

$$(\mathbb{T}\Lambda_1^{(\frac{1}{2},\frac{1}{2})}\,\mathbb{T}^{-1})p^{\mu} = \begin{pmatrix} p^0\cosh(\eta_1) - p^1\sinh(\eta_1) \\ p^1\cosh(\eta_1) - p^0\sinh(\eta_1) \\ p^2 \\ p^3 \end{pmatrix}, \tag{3.58}$$

and likewise, a spatial rotation around the x-axis as

$$(\mathbb{T}\Lambda_1^{(\frac{1}{2},\frac{1}{2})}\,\mathbb{T}^{-1})p^{\mu} = \begin{pmatrix} p^0 \\ p^1 \\ p^2\cos(\theta_1) - p^3\sin(\theta_1) \\ p^3\cos(\theta_1) + p^2\sin(\theta_1) \end{pmatrix}. \tag{3.59}$$

Hence, $\mathbb{T}$ manages to reproduce the fourvector transformations of section 2.2.1 and appendix A, from the direct-product spinor transformations. Note however that $\mathbb{T}$ may be multiplied by a phase on the form $e^{i\xi}$ and still reproduce the same transformations, which is a degree of ambiguity present in the following results. Generally, any constant could actually be introduced into this derivation. For example of $\frac{1}{\sqrt{2}}$ was introduced into the matrix in order to make it unitary.

The $\mathbb{T}$ matrices also manage to reproduce the correct inverse transformations going from fourvector- to direct product basis. Through the use of Wolfram Mathematica, all of the Lorentz group elements were verified to be preserved in both bases under the action of $\mathbb{T}$.

This section has thereby derived a unitary basis transformation matrix between a direct product of the two chiral representations and the fourvector representation of the Lorentz group.

3.3 Discrete Spacetime Transformations

Using the found transformation matrix, the form of some known discrete transformations of fourvectors will now be evaluated in the direct product basis, and vice versa.

For this discussion, the form of the direct product of two Weyl spinors is written as

$$(\chi^{\dot{a}} \otimes \chi_a) \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix} \otimes \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}^1 \lambda_1 \\ \tilde{\lambda}^1 \lambda_2 \\ \tilde{\lambda}^2 \lambda_1 \\ \tilde{\lambda}^2 \lambda_2 \end{pmatrix}, \tag{3.60}$$

such that the derived matrix acts on the direct product basis as

$$\mathbb{T} \begin{pmatrix} \tilde{\lambda}^1 \lambda_1 \\ \tilde{\lambda}^1 \lambda_2 \\ \tilde{\lambda}^2 \lambda_1 \\ \tilde{\lambda}^2 \lambda_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\lambda}^1 \lambda_2 - \tilde{\lambda}^2 \lambda_1 \\ \tilde{\lambda}^1 \lambda_1 - \tilde{\lambda}^2 \lambda_2 \\ i(\tilde{\lambda}^1 \lambda_1 + \tilde{\lambda}^2 \lambda_2) \\ -(\tilde{\lambda}^1 \lambda_2 + \tilde{\lambda}^2 \lambda_1) \end{pmatrix} = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}, \tag{3.61}$$

for some fourmomentum vector on the right.

First, consider the form of the discrete $\mathcal{P}$ and $\mathcal{T}$ transformations in fourvector representation, as shown in equations (2.8) and (2.9). The form of $\mathcal{P}$ in the direct product basis is then

$$\mathbb{T}^{-1} \mathcal{P}\, \mathbb{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.62}$$

such that a parity transformed direct product of two Weyl spinors has the components

$$(\mathbb{T}^{-1} \mathcal{P}\, \mathbb{T})(\chi^{\dot{a}} \otimes \chi_a) = - \begin{pmatrix} \tilde{\lambda}^1 \lambda_1 \\ \tilde{\lambda}^2 \lambda_1 \\ \tilde{\lambda}^1 \lambda_2 \\ \tilde{\lambda}^2 \lambda_2 \end{pmatrix}. \tag{3.63}$$

Typically, the parity operation is defined to exchange the chirality of Weyl spinors and the generators of the chiral Lorentz group representations. Exchanging the chirality of the spinors in equation (3.60) would leave the first and fourth component unchanged, while the second and third would switch positions. The $\mathcal{P}$ operator does have this effect on the direct product, and could therefore correspond to an exchange of chirality. However, equation (3.63) also introduces a phase of -1 in front of the direct product, which differs from the typical definition of the parity operator. This phase could, for example, be compensated for by introducing a phase of i in front of both Weyl spinors, or more generally by any phase

$$\chi^{\dot{a}} \to e^{i\xi} \chi^{\dot{a}}, \;\; \chi_a \to e^{i\tilde{\xi}} \chi_a, \;\; \text{such that}, \;\; \xi + \tilde{\xi} = \pi. \tag{3.64}$$

Moreover, considering the time reversal operation in the direct product basis, one gains

$$\mathbb{T}^{-1}\mathcal{T}\,\mathbb{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.65}$$

which only differs from $\mathcal{P}$ by a phase of -1. A time-reversed direct product hence has the components

$$(\mathbb{T}^{-1}\mathcal{T}\,\mathbb{T})(\chi^{\dot{a}} \otimes \chi_a) = \begin{pmatrix} \tilde{\lambda}^1\lambda_1 \\ \tilde{\lambda}^2\lambda_1 \\ \tilde{\lambda}^1\lambda_2 \\ \tilde{\lambda}^2\lambda_2 \end{pmatrix}. \tag{3.66}$$

Hence, the effect of the $\mathcal{T}$ operator of equation (3.65) seems to rather reproduce the chirality inversion which would typically be expected from the $\mathcal{P}$ operator. At the level of the two Weyl spinors, the $\mathcal{T}$ operation could hence be compensated for by replacing χ^L with χ^R, and vice versa. Neither of the operators $\mathcal{P}$ and $\mathcal{T}$ thus produce the conventional results of the Weyl spinors. However, it is not conventional to explicitly express fourvectors using direct products either, and one might therefore have to reconsider the structure of the $\mathcal{T}$ and $\mathcal{P}$ operators in this fourvector basis. Furthermore, the time reversal operator of a Weyl-spinor is commonly defined as an antiunitary operator, meaning it complex conjugates any constants along with acting on functions, which is not the case in equation (3.65) [2].

Finally, the charge conjugation operator $\mathcal{C}$ is considered. This operator is not a spacetime transformation and does not have an analogue for fourvectors. Instead, the form of this in each chiral representation is defined according to equation (2.36). The direct product vector then takes the form,

$$\mathcal{C}(\chi^{\dot{a}} \otimes \chi_a) = (\mathcal{C}\chi^{\dot{a}} \otimes \mathcal{C}\chi_a) = \begin{pmatrix} -(\tilde{\lambda}^2\lambda_2)^* \\ (\tilde{\lambda}^2\lambda_1)^* \\ (\tilde{\lambda}^1\lambda_2)^* \\ -(\tilde{\lambda}^1\lambda_1)^* \end{pmatrix}. \tag{3.67}$$

The $\mathcal{C}$ operator applied to the fourvector basis is then

$$\mathcal{C}p^\mu = \mathbb{T}(\mathcal{C}\,(\chi^{\dot{a}} \otimes \chi_a)) = -\begin{pmatrix} (\tilde{\lambda}^1\lambda_2 - \tilde{\lambda}^2\lambda_1)^* \\ (\tilde{\lambda}^2\lambda_2 - \tilde{\lambda}^1\lambda_1)^* \\ (-i(\tilde{\lambda}^2\lambda_2 + \tilde{\lambda}^1\lambda_1))^* \\ (\tilde{\lambda}^1\lambda_2 + \tilde{\lambda}^2\lambda_1)^* \end{pmatrix}. \tag{3.68}$$

Comparing this result to that applied to the non-charge-conjugated spinor component fourvector shown in equation (3.61), this corresponds to the fourmomentum components

$$\mathcal{C}p^{\mu} = \begin{pmatrix} (p^0)^* \\ -(p^1)^* \\ -(p^2)^* \\ -(p^3)^* \end{pmatrix} = \begin{pmatrix} p^0 \\ -p^1 \\ -p^2 \\ -p^3 \end{pmatrix}, \tag{3.69}$$

assuming that momentum is a real quantity.

Hence, charge conjugating the two Weyl spinors in the direct product results in changing the sign of the spatial components of the corresponding fourmomentum vector. However, a significant difference in this derivation, compared to that of the $\mathcal{P}$ and $\mathcal{T}$ operators, stems from $\mathbb{T}$ only being applied once in equation (3.68). As a result, any phase introduced into $\mathbb{T}$ would not cancel with its inverse, as in the similarity transformation, and would therefore change the result of the charge conjugation. The result of equation (3.69) is however invariant under complex conjugation of the two Weyl spinors, in the case of real momenta.

In summary, this section has hence derived the explicit form $\mathcal{P}$ and $\mathcal{T}$ in the direct product basis, and deduced that this corresponds to interchanging the chiralities of the Weyl spinors, up to a phase. The effect of the $\mathcal{C}$ operator on the fourvector basis was also derived, and was shown to corresponds to an inversion of the spatial components of the fourvector.

4 Conclusion

In conclusion, the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group was successfully derived by performing a direct product of its elements in both chiral representations. Using this, a transformation matrix rotating between the direct product basis and the fourvector basis was derived, and it was shown to recover all of the standard fourvector Lorentz transformations from the direct product representation. Worth noting, however, is that during the derivation, any phase of the form $e^{i\xi}$ could be introduced into the transformation matrix, while still recovering the correct fourvector transformations.

To explore the implications of this basis transformation matrix, the parity and time-reversal operators in the fourvector basis were transformed into the direct product basis, to find which discrete operations on the Weyl spinors the operators could correspond to. It was found that the fourvector time reversal operator could be obtained by exchanging the chiralities of the left- and right chiral spinors in the direct product. This effect is however typically how the parity operator, not the time reversal operator, is defined for a Weyl spinor. When considering the fourvector $\mathcal{P}$ operator however, the results in the direct-product basis introduced an additional phase of -1 in front of the direct product, while also exchanging the chiralities of the spinors. This hence differs from the typical parity operation on Weyl spinors by a sign. One possible compensation for this difference would be introducing a phase of i in front of both Weyl spinors in the direct product, and a possible cause could be the unconventional treatment of fourvectors as direct products, which might require a redefinition of the $\mathcal{P}$ and $\mathcal{T}$ symmetry operators.

Finally, the right- and left chiral spinors were charge conjugated before once again considering their direct product. It was then found that the transformation into the fourvector basis changed the sign of the spatial components of the corresponding real fourmomentum vector. The results of the $\mathcal{C}$ operation derivations did however depend on which phase was introduced into the basis transformation matrix, unlike the results of the $\mathcal{P}$ and $\mathcal{T}$ operators.

Appendix A: All Lorentz group transformations and the Pauli Matrices

This appendix contains the explicit forms used for the Lorentz transformations in the four-vector representation, which were not shown in section 2.4.1. along with their generators, and the explicit form of the Pauli matrices.

The operators for spatial rotations are

$$R(\theta_2, \hat{y}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 0 & 0 & 0 \\ 0 & -\sin(\theta_2) & 0 & \cos(\theta_2) \end{pmatrix}, \quad R(\theta_3, \hat{z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) & 0 \\ 0 & \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with corresponding generators

$$L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. The matrices for Lorentz boosts are

$$\Lambda(\eta_2, \hat{y}) = \begin{pmatrix} \cosh(\eta_2) & 0 & -\sinh(\eta_2) & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh(\eta_2) & 0 & \cosh(\eta_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda(\eta_3, \hat{z}) = \begin{pmatrix} \cosh(\eta_3) & 0 & 0 & -\sinh(\eta_3) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\eta_3) & 0 & 0 & \cosh(\eta_3) \end{pmatrix},$$

respectively corresponding to the generators

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In definition [2.17] the Pauli matrices were used. These are defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.70}$$

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