LU TP 21-15
JUNE 2021

# Gribov ambiguities in non-Abelian gauge theories

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# Abstract

This thesis investigates the quantization of field theories using the functional integral formalism. Gauge invariance necessitates a gauge-fixing procedure that chooses a unique field configuration from each gauge orbit. Faddeev and Popov suggested an adjustment to the action that implements gauge fixing in this framework, however it has been found that this procedure fails to uniquely fix the gauge of non-Abelian gauge theories. This problem is known as the Gribov ambiguity. This thesis reproduces and discusses the work of Gribov, and it is illustrated how the Gribov horizons divide the functional space into regions $C_n$. Then, the Coulomb gauge of SU(2) is considered as an explicit example of Gribov ambiguities. The equation of the Gribov pendulum is derived, and the properties of its solutions are discussed. The thesis concludes with a short review of the consequences of the Gribov ambiguity, and of the possible restrictions to the integration range that have been proposed as a resolution to the problem.

# Popular-science description

The quantum world has often been described as a bizarre place full of obscure phenomena, and rightly so. Particles can exist in multiple places simultaneously, or tunnel through walls to spontaneously show up on the other side—not to mention Schrödinger's famous cat, which is both alive and dead at the same time. The quantum world appears very different from our own, and one would be forgiven for asking what impact quantum physics, and its subdiscipline quantum field theory, has on our daily lives.

Quantum field theory studies the fundamental particles in nature and how they interact with each other. It is crucial for our understanding of nuclear physics, where it helps describe the forces that hold atomic nuclei together or cause radioactive decay. Further, experiments in quantum field theory demanded the development of large-scale superconducting magnets, which later allowed for the invention of the MRI scanners found at hospitals all around the world. It is in this way that the pursuit of quantum field theory leaves behind ideas and technologies that can be used by other disciplines of science or medicine, like a kind family who plow the snow off their neighbour's driveway. These are just a few examples of how discoveries in quantum field theory can have a positive impact on our everyday life.

The modern description of quantum field theory dates back to the 1960's, when it was discovered that protons and neutrons are not indivisible as was previously thought. Rather, they have constituent particles called quarks, which bind together in groups of three. The strong force is what binds these quarks together, just like glue that holds objects together. The particles which are responsible for this type of interaction are aptly named gluons, and their behaviour is described by quantum field theory. Without the presence of the strong force, quarks would not come together to form the protons and neutrons that make up everything we see around us. These miniature glue particles are necessary to our entire existence.

Making calculations in quantum field theory is very cumbersome, so particle collisions are often studied using computer simulations. Like a busy traffic intersection at rush hour, the insides of protons and neutrons are messy and chaotic. Quarks whiz around at velocities close to the speed of light, while virtual particles pop in and out of existence, all held together by an overflowing amount of gluons. Like cars, gluons have to be aware of their surroundings on the way to their destination. And as traffic congests, the strength of the interaction between gluons and quarks increases drastically. The aim of this thesis is to address this subatomic traffic jam that arises in quantum field theory by reviewing the Gribov Ambiguity. In particular, it investigates an over-counting problem that arises because quarks and gluons can rotate in strange ways that other particles cannot, which makes these kinds of particle collisions more complex than electromagnetic interactions. The calculations in this thesis can, for instance, be used to modify the description of how slow-moving gluons are transmitted through space and time. Hopefully, this could help facilitate future computer simulations of particle collisions.

# Contents

# 1 Introduction

Gauge invariance is a property of physical theories that contain redundant degrees of freedom that have no direct impact on observable quantities. Traditional examples include shifts of the zero level of the electrostatic potential, which may be set arbitrarily without affecting the electric field, or complex rotations of quantum mechanical wave functions $\psi \to e^{i\theta}\psi$, which leave the squared amplitude unchanged. These gauge transformations have no resulting consequence on the equations of motion of the theory. Gauge invariance is a key building block in the construction of quantum field theory (QFT). It is the property that allows for the definition of force-carrying gauge fields, and describes how these fields interact with the fundamental particles of the Standard Model.

Gauge theories are classified as Abelian or non-Abelian, depending on whether the gauge transformations are commutative or not. Physically, this is manifested in a self-interaction in the force-carrying field. Quantum electrodynamics (QED) is an Abelian theory, so photons carry no electric charge and do not mutually scatter. Conversely, quantum chromodynamics (QCD) is a non-Abelian gauge theory, and consequently gluons interact with each other in the gauge field. This property adds a significant degree of complexity to non-Abelian gauge theories.

Another considerable difference between Abelian and non-Abelian gauge theories is revealed during quantization of field theories. This is canonically implemented using the formalism of creation and annihilation operators, however it can also be performed by means of the functional integral formulation of QFT. While this approach is in many ways rather convenient, it is unfortunately spoiled by gauge invariance. The gauge transformations of the field theory introduce extraneous degrees of freedom into the functional integrals that cause them to diverge. In 1967, Faddeev and Popov [1] presented a gauge-fixing procedure that was thought to cure the problem. However, ten years later Gribov [2] showed that in a non-Abelian theory such as QCD, the Faddeev-Popov procedure fails to completely fix the gauge. In essence this is an over-counting problem, and one must make further restrictions to the functional integral in order to satisfactorily quantize the theory. The only region which is free from Gribov ambiguities is a region called the fundamental modular region $\Lambda$, however it is notoriously hard to specify and it remains unknown how to implement this restriction into the functional integral [3]. The problem of the Gribov ambiguities still remains unresolved.

This thesis will aim to reproduce the work by Gribov in a Minkowski spacetime. First, section 2.2 will present the formalism of Abelian and non-Abelian gauge theories. Section 2.3 will then give an introduction to the path integral formulation of quantum mechanics. In section 3.1, this formalism will be extended to functional integrals of field configurations, with the aim to construct the generating functional of QED. This section will describe the Faddeev-Popov gauge-fixing procedure, which shall remain the primary framework of the thesis. Section 3.2 will further demonstrate the importance of the gauge-fixing procedures in this framework, and it will be shown that gauge fixing is essential for the existence of a propagator. Section 4.1 will explain the Gribov problem in non-Abelian gauge theories. Then, section 4.2 will give an explicit example of Gribov ambiguities in the Coulomb gauge of SU(2), which will further serve as a proof that Gribov ambiguities are a general feature of non-Abelian gauge theories. Section 5 will briefly discuss the relevance of these Gribov copies for lattice simulations of QCD, and give a short overview of the restrictions to the integration range that have been proposed to improve the gauge fixing.

# 2 Theoretical background

## 2.1 Conventions

This thesis adheres to the use of natural units, where $c = \hbar = 1$. Further, it uses the $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ metric, and covariant/contravariant tensor notation.

$$x^\mu = (x^0, x^1, x^2, x^3)^T \quad ; \quad x_\mu = (x^0, -x^1, -x^2, -x^3)$$

Finally, summation is implied on repeating indices according to Einstein's convention,

$$a_i b_i \equiv \sum_i a_i b_i.$$

The Fourier transform in $n$ dimensions is defined as

$$\tilde{f}(k) = \int e^{ikx} f(x)\ d^n x,$$
$$f(x) = \int \frac{1}{(2\pi)^n} e^{-ikx} \tilde{f}(k)\ d^n k.$$

## 2.2 Gauge theories

### 2.2.1 Quantum electrodynamics

The Lagrangian (density) for electromagnetism is the square of the field strength tensor (note that the current four-vector $J^\mu$ is set to zero),

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field strength tensor. It is apparent that this Lagrangian (and consequently the equations of motion) are invariant under the gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x), \tag{2.2}$$

where $\alpha(x)$ is a sufficiently differentiable function that decays faster than $1/r$ as $r \to \infty$. Fermion fields are described by the Lagrangian [4]

$$\mathcal{L}_f = \bar{\psi}_f(i\gamma^\mu D_\mu - m)\psi_f, \tag{2.3}$$

where $\gamma^\mu$ are the Dirac matrices, and $D_\mu = \partial_\mu - ig_e A_\mu$ is the covariant derivative that arises due to interaction with the gauge field. The strength of this interaction is determined by the coupling constant $g_e$. The fermion field $\psi_f$ transforms under local $U(1)$ gauge transformations, which are recovered from $\alpha(x)$ by the exponential map

$$\psi_f \to U\psi_f = e^{ig_e\alpha(x)}\psi_f. \tag{2.4}$$

The field $A_\mu$ simultaneously transforms according to equation (2.2), leaving the fermion Lagrangian invariant. The transformations $U$ form a compact Lie group, which is said to be Abelian since any two group elements commute.

### 2.2.2 Non-Abelian gauge theories

For Yang-Mills gauge theories the symmetry group is the non-Abelian SU($N$). The weak force has an SU(2) symmetry, while the strong force has an SU(3) symmetry. The pure Yang-Mills Lagrangian is once again defined by the field strength tensor

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a}, \tag{2.5}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu$ is the non-Abelian field strength tensor.[a] It can also be written in terms of the commutator $F_{\mu\nu} = \frac{i}{g_s}[D_\mu, D_\nu]$, where $D_\mu = \partial_\mu - ig_s A_\mu$ is the non-Abelian covariant derivative. The gauge field $A_\mu$ is now a vector in the Lie algebra of the group,[b] which is spanned by the traceless and Hermitian generators $T^a$ such that $A_\mu = A^a_\mu T^a$. The structure constants of the Lie group, denoted $f^{abc}$, are defined by the commutation relation between the generators according to

$$[T^a, T^b] = if^{abc}T^c. \tag{2.6}$$

The gauge transformations (Lie group elements) $S$ are recovered from the Lie algebra by the exponential map. Just like in the Abelian case, they act as gauge transformations on fermions according to

$$\psi_f \to S\psi_f = e^{ig_s\alpha^a(x)T^a}\psi_f. \tag{2.7}$$

Gauge invariance of the Lagrangian requires that the $A_\mu$-field must simultaneously transform according to

$$A_\mu \to A'_\mu = SA_\mu S^\dagger + \frac{i}{g_s}S(\partial_\mu S^\dagger). \tag{2.8}$$

[a] The subscript of $g_s$ is present simply as a reminder that this coupling constant is different from $g_e$. It could in principle be the coupling strength of any non-Abelian gauge theory, and it must not necessarily be associated with the strong force.

[b] When acting in the Lie algebra, the covariant derivative acts according to $D^{ab}_\mu = \partial^a_\mu\delta^{ab} - g_s f^{abc}A^c_\mu$.

It can be shown that the field strength tensor $F_{\mu\nu} = F^a_{\mu\nu}T^a$ consequently transforms as follows

$$F_{\mu\nu} \to F'_{\mu\nu} = SF_{\mu\nu}S^\dagger. \tag{2.9}$$

## 2.3 Path integral formulation of quantum mechanics

This section introduces the path integral formalism to the unfamiliar reader, and it concludes with the derivation of the propagator of a free particle in one dimension. We shall see that this formalism recovers the same result as can be obtained with the canonical quantization of quantum mechanics.

The path integral formalism can be thought of as a generalization of the double slit experiment, where a screen with two thin slits restricts the path of a particle travelling from a point $x_a$ in front of the screen to some point $x_b$ behind it. An interference pattern is created behind the screen, due to the superposition of the complex phases of the two paths. By introducing more slits, one can gradually lift the restriction on the path of the particle. One can consider the limiting case where the number of slits tends to infinity, and the particle can travel freely along any path $x_a \to x_b$. This limit resembles the case where the screen is removed entirely, so the propagation of a free particle can be found by integrating over all possible paths. We note that no path is more important than any other, so the total propagator $G$ is a sum of the phases of the individual paths,[a]

$$G(x_a, x_b; T) = \sum_{\text{all paths}} e^{i\cdot(\text{phase})} = \int \mathcal{D}x(t)\ e^{i\cdot(\text{phase})}. \tag{2.10}$$

The expression $\int \mathcal{D}x(t)$ is an integral over all possible paths from $x_a$ to $x_b$. The integrand $e^{i\cdot(\text{phase})}$ is called a functional, as it depends on the path $x(t)$.

The phase is associated with the action from classical mechanics, $S = \int L\ dt$.[b] This can be seen by considering the classical path, where the classical action is stationary under a small variation of this path, by construction. Similarly, the phase should also be stationary under the small variation from this path, so one can equate these two quantities [5].

Path integrals of this kind are evaluated by discretizing the time $T$ into a sequence of time intervals of duration $\varepsilon$. At each time-slice, the particle has some position $x_k$ (see figure 1). The path integral thus becomes a product of integrals of position,

$$\int \mathcal{D}x(t) = \frac{1}{C(\varepsilon)} \int \frac{dx_1}{C(\varepsilon)} \int \frac{dx_2}{C(\varepsilon)} \int \frac{dx_3}{C(\varepsilon)} \cdots \int \frac{dx_{N-1}}{C(\varepsilon)} = \frac{1}{C(\varepsilon)} \prod_{k=1}^{N-1} \int \frac{dx_k}{C(\varepsilon)}, \tag{2.11}$$

where the constant is $C(\varepsilon) = \sqrt{\frac{2\pi\varepsilon}{-im}}$ [5]. The continuum limit is recovered when $N \to \infty$, or equivalently, $\varepsilon \to 0$.

[a] We further note that the propagator can be written in bra-ket notation as $G(x_a, x_b; T) = \langle x_b | e^{-i\hat{H}T} | x_a \rangle$.

[b] There is conventionally a factor of $1/\hbar$ included here. However, in our set of units $\hbar = 1$ is omitted.

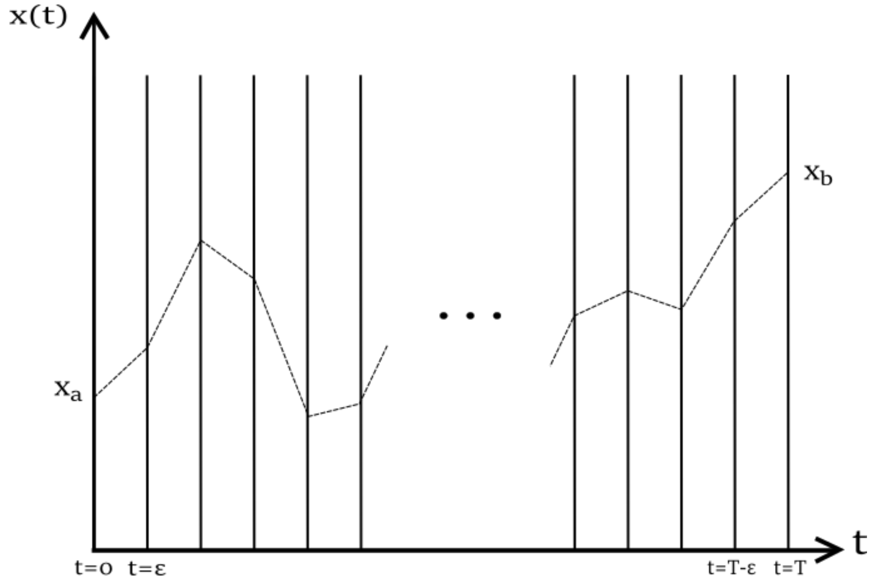


Figure 1: The functional integral is discretized into time slices. At each slice, the particle has some position $x_k$.

Note that this procedure integrates over all imaginable paths, even those that are unphysical. However, when the action is sufficiently large ($S >> 1$) the functional $e^{iS}$ is rapidly oscillating. For any such path there is a neighbouring path with the opposite phase, so these paths give on average zero contribution to the path integral [5]. The only paths which are not compensated for are those that lie sufficiently close to the classical path. That is, only the paths where $x_{k+1} \approx x_k$ will have a non-vanishing contribution to the integral.

As an example of how to evaluate a path integral, we consider a free particle travelling along one dimension from point $x_a$ to point $x_b$ in time $T$. The action is similarly discretized in slices of time $\varepsilon$,

$$S = \int \frac{m\dot{x}^2}{2}\, dt \quad \longrightarrow \quad \sum_{k=0}^{N-1} \frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\varepsilon}. \tag{2.12}$$

The path integral can then be calculated by performing the integrals over $x_k$ independently,

$$G(x_a, x_b; T) = \int \mathcal{D}x(t)\, e^{iS[x(t)]} = \frac{1}{C(\varepsilon)^N} \prod_{k=1}^{N-1} \int dx_k\, e^{\sum_{k=0}^{N-1} \frac{im}{2\varepsilon}(x_{k+1}-x_k)^2}. \tag{2.13}$$

We perform the integrals inductively, first considering the integral over $x_1$,

$$\int dx_1\, e^{\frac{im}{2\varepsilon}(x_1-x_a)^2 + \frac{im}{2\varepsilon}(x_2-x_1)^2} = \sqrt{\frac{\pi\varepsilon}{im}}\, e^{\frac{im}{4\varepsilon}(x_2-x_a)^2}, \tag{2.14}$$

where we have used the identity (C.2) and completed the square in the exponent. The same steps are taken when performing the integrals over the other coordinates $x_k$. The process is rather arduous, but for a general coordinate $x_n$ we find that

$$\int dx_n \, e^{\frac{im}{2n\varepsilon}(x_n-x_a)^2+\frac{im}{2\varepsilon}(x_{n+1}-x_n)^2} = \sqrt{\frac{2\pi\varepsilon}{im}}\sqrt{\frac{n}{n+1}}\, e^{\frac{im}{2(n+1)\varepsilon}(x_{n+1}-x_a)^2}. \tag{2.15}$$

Performing all of the integrals in succession thus yields

$$G(x_a, x_b; T) = \frac{1}{C(\varepsilon)^N}\frac{1}{\sqrt{N}}\sqrt{\frac{2\pi\varepsilon}{-im}}^{N-1} e^{\frac{im}{2N\varepsilon}(x_b-x_a)^2}. \tag{2.16}$$

Evidently, the initial and final positions $x_0$ and $x_N$ are not integrated over, as they are the fixed endpoints $x_a$ and $x_b$. We can simplify the answer by recalling that $C(\varepsilon) = \sqrt{\frac{2\pi\varepsilon}{-im}}$. Finally, the continuum limit is recovered by letting $N \to \infty$, keeping in mind that $N\varepsilon = T$,

$$G(x_a, x_b; T) = \sqrt{\frac{-im}{2\pi T}}\, e^{\frac{im}{2T}(x_b-x_a)^2}. \tag{2.17}$$

This is the standard result for the propagator of a free particle. The canonical derivation of the same result can be found in Sakurai section 2.6 [6].

The result of this calculation agrees with the claim that the path integral formulation of quantum mechanics is equivalent to the canonical formulation. This is further justified in appendix A, where it is shown that the path integral satisfies the Schrödinger equation, even for a general potential. However, as demonstrated by this example, the path integral formulation is typically more laborious than canonical methods when solving uncomplicated problems like a free particle. On the contrary, one of the main benefits of this formalism is in QFT, where it yields an easy route to quantization, as shown in the following section.

# 3 Functional integral methods in quantum electrodynamics

The following two subsections will describe the quantization of electrodynamics using the functional integral formalism introduced in the previous section. In order to construct QED, we must describe the time evolution of the gauge field. The important quantity then is the propagator, which is conveniently calculated using functional integrals. We will derive the photon propagator in Feynman gauge in section 3.2, but we shall see that gauge invariance spoils the equation of motion. Therefore, the gauge-fixing procedure introduced by Faddeev and Popov must first be demonstrated.

## 3.1 Faddeev-Popov procedure in QED

In QFT, the functional integral formalism does not integrate over paths $x(t)$; rather, it integrates over all possible field configurations $A_\mu(x)$. The four components are considered as separate scalar fields, so the integration measure reads $\mathcal{D}A = \mathcal{D}A^0\mathcal{D}A^1\mathcal{D}A^2\mathcal{D}A^3$. In the previous section, it was shown how the path integral formalism adds the complex phases of all paths. By analogy, one could naively attempt evaluate a functional integral over all field configurations by defining the generating functional $W$ as follows

$$W = \int \mathcal{D}A e^{iS[A]}. \tag{3.1}$$

However, this approach does not work because the integral is badly divergent. To show this, consider the action $S = \int \mathcal{L}\ d^4x$, which can be expressed as

$$\begin{aligned} S &= -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\ d^4x \\ &= -\frac{1}{4}\int (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\ d^4x \\ &= \frac{1}{4}\int \left(A_\nu\partial_\mu\partial^\mu A^\nu + A_\mu\partial_\nu\partial^\nu A^\mu - A_\mu\partial_\nu\partial^\mu A^\nu - A_\nu\partial_\mu\partial^\nu A^\mu\right)\ d^4x \\ &= \frac{1}{2}\int A_\nu(x)\left(\partial^2 g^{\mu\nu} - \partial^\nu\partial^\mu\right)A_\mu(x)\ d^4x, \end{aligned} \tag{3.2}$$

where the third step uses integration by parts. Clearly, the action vanishes for any field that satisfies $A_\mu = \partial_\mu\alpha(x)$.[a] Comparing to equation (2.2), we see that these fields are related to each other by a gauge transformation, and are all physically equivalent to the field $A_\mu = 0$. These fields form an equivalence class called a gauge orbit, along which the Lagrangian is constant. The functional integral in equation (3.1) considers all fields in the gauge orbit that intersects $A_\mu = 0$, and as a result there are an infinite number of field configurations for which the functional in equation (3.1) is unity. Consequently, the functional integral diverges. The significance of this is further discussed in section 3.2.

A remedy to this problem was first proposed by Faddeev and Popov in 1967 [1]. Their procedure restricted the functional integral to only consider one representative from each gauge orbit by a gauge-fixing procedure. A conventional choice is the Lorenz gauge, where the four-divergence of $A^\mu$ is zero at every point in spacetime. It is customary to introduce a function $G(A) = \partial^\mu A_\mu$. The gauge-fixing condition $\delta(G(A')) = \delta(\partial^\mu A_\mu + \partial^2\alpha)$ constrains the functional integral to only consider fields in the Lorenz gauge, which removes the redundant degrees of freedom. One can do so by making use of the identity[b]

$$1 \equiv \int \mathcal{D}\alpha(x)\ \delta(G(A'))\det\left(\frac{\delta G(A')}{\delta\alpha}\right). \tag{3.3}$$

The determinant is called the Faddeev-Popov determinant, which in the Lorenz gauge reduces to $\det(\partial^2)$. The operator $\partial^2$ is similarly called the Faddeev-Popov operator, and

[a] This fact is already evident on the first line of equation (3.2), since $F_{\mu\nu}$ is gauge invariant in an Abelian theory. The subsequent calculations in equation (3.2) are included for later reference.

[b] This identity is discussed in Appendix B.

it will be encountered again in section 4.1. For the present discussion, it suffices to remark that this determinant is independent of $\alpha$, so it can be treated as a constant in the functional integral.

The gauge-fixing condition is employed by inserting the identity (3.3) into the functional integral from equation (3.1),

$$W = \det\left(\partial^2\right) \int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \delta(G(A')). \tag{3.4}$$

This integral can be simplified by gauge transforming all of the fields $A_\mu \to A'_\mu$.[a] The transformed field $A'$ subsequently becomes a dummy variable which is integrated over, so it may be renamed back to $A$. We arrive at

$$W = \det\left(\partial^2\right) \int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \delta(G(A)). \tag{3.5}$$

To progress further, we generalize the gauge-fixing condition to $\delta(\partial^\mu A_\mu - \omega(x))$, where $\omega(x)$ is an arbitrary scalar function. Since observable quantities are gauge invariant, we can integrate over all functions $\omega(x)$ with a Gaussian weight factor as the functional [4],

$$\begin{aligned} W &= \det\left(\partial^2\right) \int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \, N(\xi) \int \mathcal{D}\omega \, \exp\left(-i \int d^4x \, \frac{\omega^2}{2\xi}\right) \delta\big(G(A) - \omega(x)\big) \\ &= N(\xi) \, \det(\partial^2) \int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \; \exp\left(-i \int \frac{(\partial^\mu A_\mu)^2}{2\xi} \, d^4x\right). \end{aligned} \tag{3.6}$$

The parameter $\xi$ determines the gauge, and the factor $N(\xi)$ is present in order to normalize the Gaussian distribution. Conventional gauge choices include Landau gauge $\xi \to 0^+$ (which is equivalent to Lorenz gauge) or Feynman gauge $\xi = 1$.

This concludes the derivation of the Faddeev-Popov procedure of quantization of the electromagnetic field. Effectively, the Faddeev-Popov procedure has added a gauge fixing term to the Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \longrightarrow \quad \mathcal{L}_{\text{constrained}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{(\partial^\mu A_\mu)^2}{2\xi}. \tag{3.7}$$

It must also be mentioned that the normalization of the integral in equation (3.6) remains unexplored. The properties of the prefactors $N(\xi)$ and $\det(\partial^2)$ have not been discussed, and the divergent integral over $\alpha(x)$ lingers as an infinite multiplicative constant. However, when calculating expectation values of gauge invariant operators the prefactors are unimportant—they appear both in the numerator and the denominator when calculating vacuum expectation values of the kind $\langle \hat{A} \rangle = \frac{\langle \Omega | \hat{A} | \Omega \rangle}{\langle \Omega | \Omega \rangle}$. This is further discussed in Peskin and Schröder [4].

[a] The gauge transformations form a group, so this transformation must have an inverse. It is clear, then, that this reproduces the integration measure, so $\mathcal{D}A_\mu = \mathcal{D}A'_\mu$ is a simple shift of variables.

Finally, it is pertinent to mention that the Lorenz gauge condition $\partial^\mu A_\mu = 0$ does not completely fix the gauge. It appears there is still room for gauge transformations $A_\mu \to A'_\mu$, so long as the gauge parameter $\alpha(x)$ satisfies the Laplace equation $\partial^2\alpha = 0$. However, as pointed out by Gribov, this gauge freedom is eliminated by the requirement that $\alpha(x)$ must vanish appropriately at infinity [2].

## 3.2 The photon propagator

Section 2.3 ended with a derivation of the propagator of a free particle in quantum mechanics. It would therefore be appropriate to derive the photon propagator in the Feynman gauge, as a conclusion to the discussion on QED. This section will also serve to demonstrate how imperative the Faddeev-Popov procedure is to quantization, for without it the theory would not have a well-defined propagator. This final discussion in particular will lay the foundation for the treatment of non-Abelian gauge fixing in section 4.

The equations of motion of the $A_\mu$-field can be calculated using the Euler-Lagrange equation. We first consider the unrestricted Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, for which the equations of motion read $\partial^\mu F_{\mu\nu} = 0$. Expanding the field strength tensor yields

$$\left(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu\right) A^\mu = 0. \tag{3.8}$$

We define the linear operator $L = (\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)$, which appeared before in equation (3.2). The equations of motion can be solved if one finds a propagator $D_F^{\nu\rho}$, that is, a Green's function of this operator. It must satisfy

$$\left(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu\right) D_F^{\nu\rho}(x-y) = i\delta_\mu^\rho \delta^{(4)}(x-y), \tag{3.9}$$

or by Fourier transformation

$$\left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) \tilde{D}_F^{\nu\rho}(k) = i\delta_\mu^\rho. \tag{3.10}$$

However, the operator $L$ is not invertible, and thus it has no Green's function. This comes as a direct consequence of the fact that the equations of motion are underdetermined. If $A_\mu$ satisfies equation (3.8) then so does any gauge transformed field $A'_\mu = A_\mu + \partial_\mu\alpha$. These are the same troublesome fields that caused the action in equation (3.2) to vanish. The unphysical degrees of freedom caused by gauge invariance mean that the equations of motion do not uniquely specify the time evolution of the system. This is, in fact, the very definition of gauge invariance.

By contrast, when the gauge fixing term is added to the Lagrangian, the equations of motion instead read

$$\left(\partial^2 g_{\mu\nu} - (1-\frac{1}{\xi})\partial_\mu\partial_\nu\right) A^\mu = 0. \tag{3.11}$$

This leads to a modification of equation (3.10), so the propagator must now satisfy

$$\left(-k^2 g_{\mu\nu} + (1-\frac{1}{\xi})k_\mu k_\nu\right) \tilde{D}_F^{\nu\rho}(k) = i\delta_\mu^\rho. \tag{3.12}$$

In the Feynman gauge $\xi = 1$, the solution to this equation is

$$\tilde{D}_F^{\nu\rho}(k) = \frac{-ig^{\nu\rho}}{k^2 + i\epsilon}, \tag{3.13}$$

where the term $i\epsilon$ in the denominator is present as a choice of contour. This is the standard result for the photon propagator in Feynman gauge, and it can also be found in Weinberg section 8.5 [7].

We have thus demonstrated the importance of gauge fixing in QED. We note that the particular gauge-fixing condition is not important for the existence of a propagator; it is evident in equation (3.11) that the propagator exists for any choice of $\xi$. While the choice of the gauge parameter certainly modifies the Feynman rules, observable quantities ultimately remain unaffected. The important requirement is that the gauge condition must intersect each gauge orbit exactly once. However, as we shall see in the next section, it is not possible do to so in non-Abelian gauge theories.

# 4 Functional integral methods in non-Abelian gauge theories

## 4.1 The Gribov ambiguity

We have seen that the Faddeev-Popov procedure has been successful in quantizing the photon field. One would like to carry out the quantization of non-Abelian fields using the same procedure. The analogous gauge-fixing condition $\partial^\mu A_\mu = 0$ is known as the Landau gauge. We consider a functional integral similar to equation (3.5),

$$W = \int \mathcal{D}\alpha \int \mathcal{D}A \, \exp\left(-\frac{i}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^4x\right) \delta(\partial_\mu A^\mu) \det\left(\mathcal{M}^{ab}(A)\right). \tag{4.1}$$

where $\mathcal{M}(A) = -\partial^\mu D_\mu = -\partial^2 \cdot + ig_s[A_\mu, \partial^\mu \cdot\,]$ is the non-Abelian generalization of the Faddeev-Popov operator.[a] We note that the determinant has been left inside the integral as it depends on $A$.

However, as pointed out by Gribov [2], this procedure does not uniquely fix the gauge. In non-Abelian gauge theories there exist multiple fields in the same gauge orbit that satisfy the same gauge-fixing condition. Figure 2 below shows the possible ways in which a gauge-fixing condition can intersect some gauge orbits $L$, $L'$ and $L''$. Each gauge orbit is constructed by acting on a particular field configuration with all of the gauge transformations, such that the Lagrangian is constant along a given orbit. The Faddeev-Popov procedure should select one representative from each gauge orbit, so only orbits of the type $L$ are desired. However, in non-Abelian gauge theories there also exist orbits of the type

[a] In component form, the operator reads $\mathcal{M}^{ab} = (\partial^2\delta^a_b - \partial^\mu g_s f^{abc} A^c_\mu)$

$L'$. There are no orbits of the type $L''$. Figure 2 shows that the field $A_\mu$ has a Gribov copy $A'_\mu$, and the Faddeev-Popov procedure integrates over both configurations.

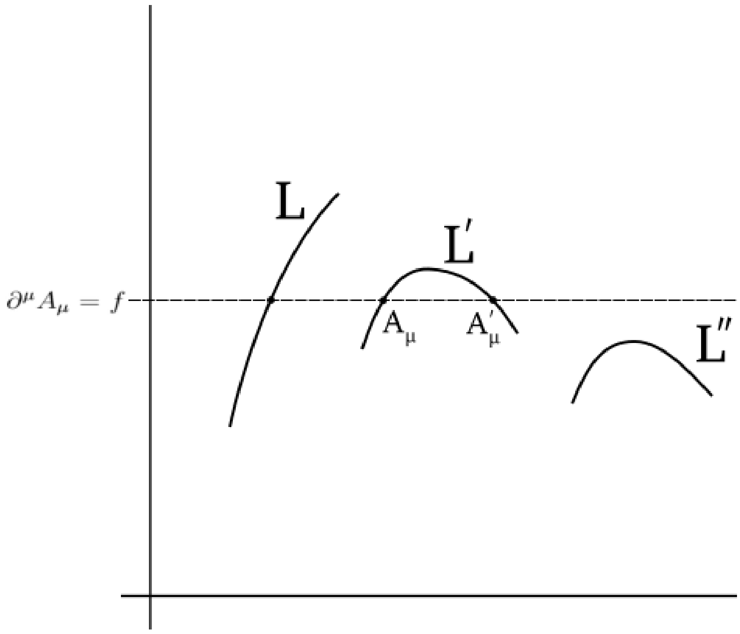


Figure 2: Schematic drawing of the functional space, where the axes show the longitudinal and transverse components of the field. The gauge-fixing condition $\partial^\mu A_\mu = f$ should intersect each gauge orbit exactly once, however the orbit $L'$ is intersected at two distinct fields $A_\mu$ and $A'_\mu$.

To show this, let $A_\mu$ satisfy the Landau gauge condition $\partial^\mu A_\mu = 0$ and consider a gauge transformation $A'_\mu = SA_\mu S^\dagger + \frac{i}{g_s}S(\partial_\mu S^\dagger)$. The divergence of the transformed field is

$$\begin{aligned}\partial^\mu A'_\mu &= \partial^\mu \left( SA_\mu S^\dagger + \frac{i}{g_s}S(\partial_\mu S^\dagger)\right) \\ &= (\partial^\mu S)A_\mu S^\dagger + SA_\mu(\partial^\mu S^\dagger) + \frac{i}{g_s}(\partial_\mu S)(\partial^\mu S^\dagger) + \frac{i}{g_s}S(\partial^2 S^\dagger).\end{aligned} \tag{4.2}$$

Therefore, if $S$ satisfies the equation

$$(\partial^\mu S)A_\mu S^\dagger + SA_\mu(\partial^\mu S^\dagger) + \frac{i}{g_s}(\partial_\mu S)(\partial^\mu S^\dagger) + \frac{i}{g_s}S(\partial^2 S^\dagger) = 0, \tag{4.3}$$

then $\partial^\mu A'_\mu = 0$, and the gauge transformed field is a Gribov copy of $A_\mu$. We investigate this condition by linearizing $S \approx 1 + ig_s\alpha$ and discarding terms proportional to $\alpha^2$. The expression reduces to[a]

$$\begin{aligned} 0 &= ig_s(\partial^\mu \alpha)A_\mu - ig_s A_\mu(\partial^\mu \alpha) + \partial^2 \alpha \\ &= \partial^2 \alpha - ig_s[A_\mu, \partial^\mu \alpha] \\ &= -\mathcal{M}(A)\alpha. \end{aligned} \tag{4.4}$$

So, for gauge transformations close to unity, it appears that the existence of Gribov copies is governed by the solutions to the equation $-\mathcal{M}(A)\alpha = 0$. This is an eigenvalue equation for the Faddeev-Popov operator

$$-\partial^2 \alpha + ig_s[A_\mu, \partial^\mu \alpha] = \epsilon(A)\alpha. \tag{4.5}$$

As pointed out in a paper by Sobreiro and Sorella [8], this eigenvalue equation can be thought to resemble the Schrödinger equation, where the gauge field $A_\mu$ plays the role of a potential. Therefore, we can use what is known about solutions to the Schrödinger equation to discuss properties of equation (4.5).

We first note that the potential term could be attractive or repulsive, depending on the direction of $A_\mu$. For $A_\mu$-fields with a particular direction and a particular magnitude, equation (4.5) can have solutions with eigenvalue $\epsilon_1(A) = 0$, which can be thought of as a bound state of a potential well. These fields lie on the first Gribov horizon, denoted $l_1$ in figure 3 on the next page. For even larger magnitude gauge fields $A_\mu$ the eigenvalue becomes increasingly negative, until a second bound state appears with eigenvalue $\epsilon_2(A) = 0$. This set of fields lie on the second horizon $l_2$. These horizons divide the functional space into regions denoted $C_0, C_1, ..., C_n$ where the Faddeev-Popov operator has $0, 1, ..., n$ negative eigenvalues, respectively. The Gribov regions and horizons are sketched in figure 3.

Finally, many authors point out that Gribov ambiguities are present in all non-Abelian gauge theories [2, 3, 8]. This fact shall be formally proven in the next section, where the example of Gribov copies in SU(2) is worked out in detail. Hall [9] writes that every compact, non-Abelian Lie group has an SU(2) subgroup, so the problem must persist in all non-Abelian gauge theories. Furthermore, Singer [10] showed that the existence of Gribov ambiguities is independent of the choice of gauge-fixing condition. The example in the next section will be worked out in the Coulomb gauge for convenience, however there is thus no other choice of gauge-fixing condition that would remove the Gribov copies.

[a] Note that we also discard terms proportional to $(\partial^\mu \alpha)^2$ and $\alpha(\partial^\mu \alpha)$.

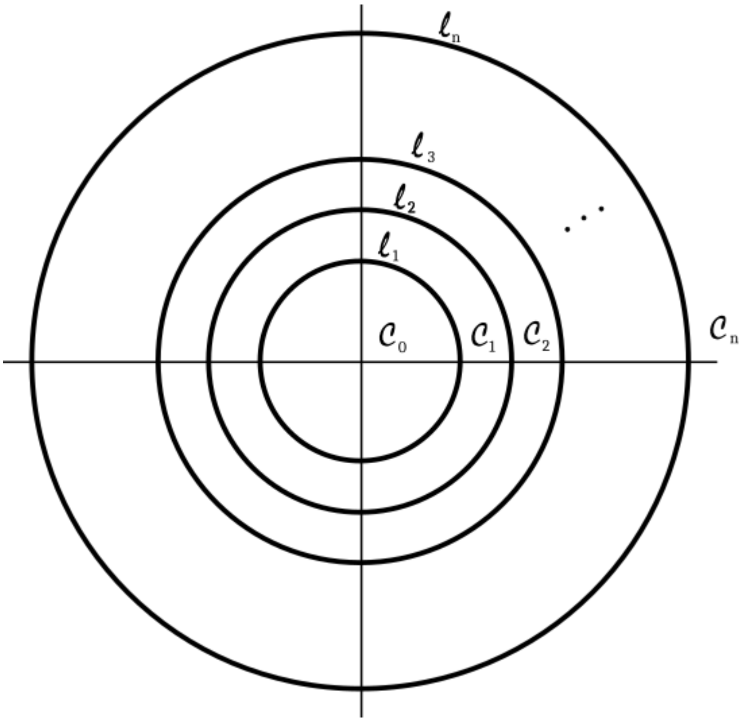


Figure 3: The Gribov regions $C_n$, divided by the Gribov horizons $l_n$. At each horizon the operator has a zero eigenvalue, and consequently the Faddeev-Popov determinant vanishes.

## 4.2 The Gribov pendulum

This section will explore the properties of the simplest examples of Gribov copies. As mentioned, we consider a three-dimensional gauge field in the Coulomb gauge ($\partial_i A_i = 0$), for the gauge group SU(2). As the time-like component of $A_\mu$ has been disregarded, the metric is for this section taken to be Euclidean, $g_{ij} = \delta_{ij}$.

The generators of SU(2) are the Pauli matrices $\sigma_i$ (divided by two),

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.6}$$

The Pauli matrices satisfy the relation

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{4.7}$$

where $\epsilon_{ijk}$ is the Levi-Civita tensor.

This section will contain a significant amount of algebra. To simplify the calculations, the gauge field is assumed to be spherically symmetric, $A_i = A_i(r)$. We define the unit vector $n_i = x_i/r$, and the corresponding Lie algebra vector $\hat{n} = in_j\sigma_j$. Identity (4.7) allows us to calculate

$$(\vec{n}\cdot\vec{\sigma})^2 = n_i\sigma_i n_j\sigma_j = 1, \qquad \hat{n}^2 = -n_i\sigma_i n_j\sigma_j = -1. \tag{4.8}$$

We consider a Gribov copy $A'_i(r)$, which must lie in the same gauge orbit and also satisfy the same gauge condition,

$$A'_i = SA_iS^\dagger + \frac{i}{g_s}S(\partial_i S^\dagger), \tag{4.9}$$

$$\partial_i A'_i = \partial_i A_i. \tag{4.10}$$

Since $A_i$ is a three-dimensional gauge field, it is described by three independent parameters. As it is spherically symmetric, we can employ Gribov's parametrization [2] as an ansatz,

$$A_i(r) = if_1(r)(\partial_i\hat{n}) + if_2(r)\hat{n}(\partial_i\hat{n}) + if_3(r)\hat{n}n_i, \tag{4.11}$$

where $\partial_i\hat{n} = \frac{\partial\hat{n}}{\partial x_i}$. Note that a factor of $i$ has been included, to ensure that the parameters $f_{1,2,3}(r)$ are real-valued. $A'_i$ is similarly parametrized by

$$A'_i(r) = if'_1(r)(\partial_i\hat{n}) + if'_2(r)\hat{n}(\partial_i\hat{n}) + if'_3(r)\hat{n}n_i \tag{4.12}$$

We recover a gauge transformation from $\alpha(r)$ by the exponentiation $S = e^{ig_s\alpha(r)(\vec{n}\cdot\vec{\sigma}/2)}$,

$$\begin{aligned} S = e^{ig_s\alpha(r)(\vec{n}\cdot\vec{\sigma}/2)} &= \cos\frac{g_s\alpha}{2} + \hat{n}\sin\frac{g_s\alpha}{2} \\ S^\dagger = e^{-ig_s\alpha(r)(\vec{n}\cdot\vec{\sigma}/2)} &= \cos\frac{g_s\alpha}{2} - \hat{n}\sin\frac{g_s\alpha}{2}. \end{aligned} \tag{4.13}$$

The gauge transformation from equation (4.9) can thus be expanded using equations (4.11) and (4.13).

$$\begin{aligned} A'_i = \quad & SA_iS^\dagger + \frac{i}{g_s}S(\partial_i S^\dagger) \\ = & \left(\cos\frac{g_s\alpha}{2} + \hat{n}\sin\frac{g_s\alpha}{2}\right)\left(if_1(r)(\partial_i\hat{n}) + if_2(r)\hat{n}(\partial_i\hat{n}) + if_3(r)\hat{n}n_i\right)\left(\cos\frac{g_s\alpha}{2} - \hat{n}\sin\frac{g_s\alpha}{2}\right) \\ & + \frac{i}{g_s}\left(\cos\frac{g_s\alpha}{2} + \hat{n}\sin\frac{g_s\alpha}{2}\right)\left(\left(-\sin\frac{g_s\alpha}{2} - \hat{n}\cos\frac{g_s\alpha}{2}\right)\frac{g_s}{2}(\partial_i\alpha) - (\partial_i\hat{n})\sin\frac{g_s\alpha}{2}\right) \end{aligned} \tag{4.14}$$

We perform the multiplication, and simplify the expression with the help of the identities (C.3), (C.4) and (C.9) in the Appendix. We obtain

$$\begin{aligned} A'_i = \quad & \left(if_1\cos(g_s\alpha) - \left(if_2 + \frac{i}{2g_s}\right)\sin(g_s\alpha)\right)(\partial_i\hat{n}) \\ & + \left(\left(if_2 + \frac{i}{2g_s}\right)\cos(g_s\alpha) + if_1\sin(g_s\alpha) - \frac{i}{2g_s}\right)\hat{n}(\partial_i\hat{n}) \\ & + \left(if_3 - \frac{i}{2}\frac{\partial\alpha}{\partial r}\right)\hat{n}n_i. \end{aligned} \tag{4.15}$$

We can therefore identify

$$
\begin{aligned}
f_1' &= f_1\cos(g_s\alpha) - \big(f_2 + \frac{1}{2g_s}\big)\sin(g_s\alpha) \\
f_2' &= \big(f_2 + \frac{1}{2g_s}\big)\cos(g_s\alpha) + f_1\sin(g_s\alpha) - \frac{1}{2g_s} \\
f_3' &= f_3 - \frac{1}{2}\frac{\partial\alpha}{\partial r}
\end{aligned}
\tag{4.16}
$$

We have thus found a description of how the parameters $f(r)$ transform under a gauge transformation. Next, the divergence of $A_i$ can be addressed. Using the identities (C.3), (C.5), (C.6), (C.7) and (C.8) we find that the divergence reduces to

$$
\partial_i A_i = i\frac{\partial f_3}{\partial r}\hat{n} + \frac{2i}{r}f_3\hat{n} - \frac{2i}{r^2}f_1\hat{n} \tag{4.17}
$$

and equivalently

$$
\partial_i A_i' = i\frac{\partial f_3'}{\partial r}\hat{n} + \frac{2i}{r}f_3'\hat{n} - \frac{2i}{r^2}f_1'\hat{n}. \tag{4.18}
$$

We demand that Gribov copies must have the same divergence, $\partial_i A_i = \partial_i A_i'$. By inserting the transformation law of the parameters from equation (4.16), we find the differential equation

$$
\frac{\partial^2\alpha(r)}{\partial r^2} + \frac{2}{r}\frac{\partial\alpha(r)}{\partial r} - \frac{4}{r^2}\left[\left(f_2 + \frac{1}{2g_s}\right)\sin(g_s\alpha) - f_1\big(\cos(g_s\alpha) - 1\big)\right] = 0. \tag{4.19}
$$

Finally, we can multiply both sides by $g_s$, and get rid of the unfortunate prefactors by substitution of variables $\tau = log(r)$,

$$
\frac{\partial^2\big(g_s\alpha\big)}{\partial\tau^2} + \frac{\partial\big(g_e\alpha\big)}{\partial\tau} - \left[\big(4g_sf_2 + 2\big)\sin(g_s\alpha) - 4g_sf_1\big(\cos(g_s\alpha) - 1\big)\right] = 0. \tag{4.20}
$$

This is the equation of a damped pendulum, and consequently it is known as the Gribov pendulum. The parameters $f_1, f_2$ in the gauge field can be viewed as external agents acting on the pendulum as driving forces. In addition, there is a constant force acting "downwards" on the pendulum, as shown in figure 4. As it is the equations of motion of a pendulum, we know that it must always have solutions even for arbitrary parameters $f_1, f_2$. The variable $\tau$ can be seen as a time coordinate that describes the evolution of the pendulum.

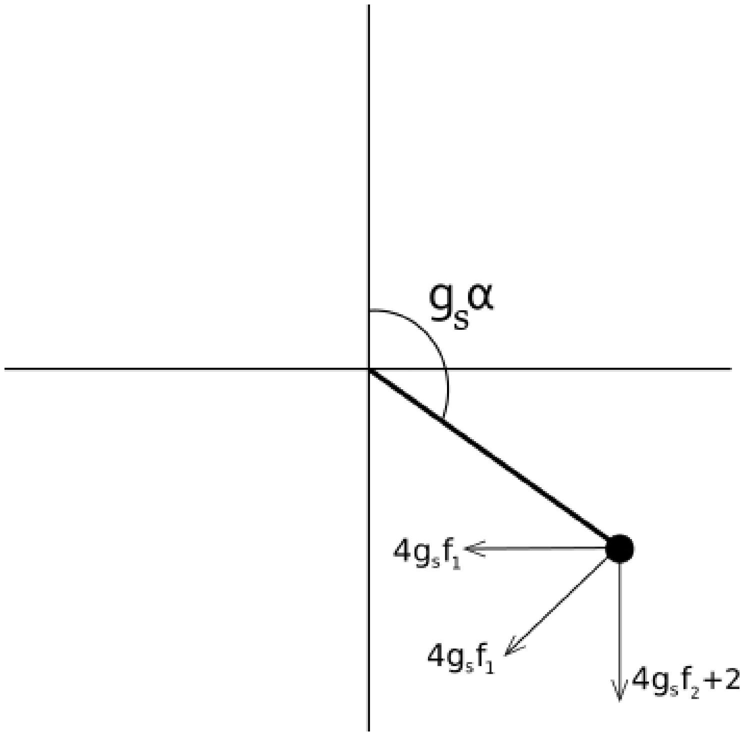


Figure 4: Equation (4.20) describes a damped pendulum, whose angle $g_s\alpha$ evolves as a function of the time coordinate $\tau$.

## 4.3 Dynamics of the Gribov pendulum

We shall now discuss the properties of the solutions $g_s\alpha(e^\tau)$, which in turn will give insight into the nature of the Gribov copies $A_i'$. While it is not possible to solve the differential equation (4.20) analytically, we shall turn our attention to a discussion on the boundary conditions of the gauge field as $\tau \to \pm\infty$.

We require that the gauge field $A_i$ must be regular on the domain, meaning that it must be differentiable and cannot contain singularities. We note that the quantity $\partial_i \hat{n} = \frac{\partial \hat{n}}{\partial x_i}$ diverges close to the origin, because the unit vector $\hat{n}$ is very sensitive to displacements in $x_i$. Looking at the parametrization of the field from equation (4.11), we thus demand that the parameters $f_1, f_2 \to 0$ as $\tau \to -\infty$, for any other values of these parameters would introduce a singularity into the gauge field at the origin. Then, we introduce two kinds of boundary conditions at $\tau \to \infty$,

$$\begin{aligned} &\text{Weak boundary condition (WBC): } A_i \text{ decays as } 1/e^\tau \text{ as } \tau \to \infty, \\ &\text{Strong boundary condition (SBC): } A_i \text{ decays faster than } 1/e^\tau \text{ as } \tau \to \infty. \end{aligned} \tag{4.21}$$

We note that $\partial_i \hat{n} \sim 1/e^\tau$ as $\tau \to \infty$, and hence the demands on the parameters $f_1, f_2$ are as follows

$$\begin{aligned} &\text{WBC demands that } f(e^{\tau}) \to C \text{ as } \tau \to \infty, \\ &\text{SBC demands that } f(e^{\tau}) \to 0 \text{ as } \tau \to \infty. \end{aligned} \tag{4.22}$$

We first investigate the vacuum field where $f_1 = f_2 = f_3 = 0$. The equation of the Gribov pendulum (4.20) simplifies to

$$\frac{\partial^2 \big(g_s\alpha\big)}{\partial\tau^2} + \frac{\partial\big(g_s\alpha\big)}{\partial\tau} - 2\sin(g_s\alpha) = 0, \tag{4.23}$$

and the parametrization of $A_i'$ reduces to

$$A_i'(e^{\tau}) = -\frac{i}{2g_s}\sin(g_s\alpha)(\partial_i\hat{n}) + \frac{i}{2g_s}\left(\cos(g_s\alpha) - 1\right)\hat{n}(\partial_i\hat{n}) - \frac{i}{2r}\frac{\partial\alpha}{\partial\tau}\hat{n}n_i \tag{4.24}$$

The regularity condition on the Gribov copy $A_i'$ at the origin implies that $\sin(g_s\alpha)$ and $\big(\cos(g_s\alpha) - 1\big)$ must vanish as $\tau \to -\infty$, to remove the contribution from the diverging quantity $\partial_i\hat{n}$. This implies that the power series expansion of $(g_s\alpha)$ in terms of $e^{\tau}$ must satisfy

$$g_s\alpha(e^{\tau}) \sim 2\pi n + \gamma e^{\tau} + \mathcal{O}\big((e^{\tau})^2\big) \quad \text{as } \tau \to -\infty, \tag{4.25}$$

for some integer $n$ and a real number $\gamma$. Hence, the pendulum must start at the unstable equilibrium position at the top of figure 4. The angular velocity at the start is

$$\frac{\partial}{\partial\tau}\big(g_s\alpha(e^{\tau})\big) \sim \gamma e^{\tau} + \mathcal{O}\big((e^{\tau})^2\big) \quad \text{as } \tau \to -\infty. \tag{4.26}$$

Neglecting higher order terms, we note that the value of $\gamma$ determines the initial conditions of the pendulum. We first consider $\gamma > 0$, in which case the initial angular velocity $\gamma e^{\tau}$ must point to the right in figure 4. As a consequence, the pendulum will leave the unstable equilibrium and perform some number of oscillations in the field. The damping term eventually brings the pendulum to a stop at the stable equilibrium at the bottom of figure 4. Therefore, at the boundary $\tau \to \infty$ the solution must behave according to

$$g_s\alpha(e^{\tau}) \sim (2m+1)\pi \quad \text{as } \tau \to \infty. \tag{4.27}$$

Next, we note that the case $\gamma < 0$ will describe the motion of a pendulum that instead falls to the left in figure 4. Although the sign of the solution $g_s\alpha$ changes, the behaviour of the pendulum will otherwise be the same. This is a manifestation of the fact that whenever $g_s\alpha$ is a solution to equation (4.23), then $-g_s\alpha$ must also solve the same equation. Thus, the case where $\gamma$ is negative only reproduces the same solutions as when $\gamma$ was taken to be positive. Finally, if $\gamma = 0$ the pendulum will remain at rest at the top of figure 4, and the solution $g_s\alpha(e^{\tau}) = 2\pi n$ corresponds to the trivial transformations $S = \pm 1$. These gauge transformation simply map $A_i$ to itself. It therefore suffices to only consider the case $\gamma > 0$.

Plugging equation (4.27) into equation (4.24), we see that the Gribov copy $A'_i$ must behave like

$$A'_i(e^\tau) \sim \big(\cos\big((2m+1)\pi\big) - 1\big)\hat{n}(\partial_i \hat{n}) \sim \frac{1}{e^\tau} \quad \text{as } \tau \to \infty. \tag{4.28}$$

So, even though $A_i = 0$ satisfies the SBC, the Gribov copy $A'_i$ only satisfies the WBC. As discussed previously, in this approach to quantization we demand that the gauge field must obey the SBC. We have thus shown that the vacuum field $A_i = 0$ has no Gribov copy, given this restriction.

However, there are circumstances which can create Gribov copies that satisfy the SBC. Consider the field $A_i$ parametrized by $f_1 = f_3 = 0$ and $f_2 = f_2(e^\tau)$. In this case, the condition $\partial_i A'_i = 0$ instead reads

$$\frac{\partial^2\big(g_s\alpha\big)}{\partial\tau^2} + \frac{\partial\big(g_s\alpha\big)}{\partial\tau} - \Bigg(4g_s f_2 + 2\Bigg)\sin(g_s\alpha) = 0, \tag{4.29}$$

Just like before, we demand that $f_2 \to 0$ as $\tau \to -\infty$, so the pendulum must start at the unstable equilibrium position at the top of figure 4. It may then perform some oscillations in the field. However, if $f_2 < -\frac{1}{2g_s}$ for a significant interval of time $\tau$, then this force acts to asymptotically restore the pendulum to the unstable equilibrium position at the top of the figure. The damping term helps ensure that the displacement of the pendulum decays exponentially. As a result, the pendulum will remain at the unstable equilibrium even as $\tau \to \infty$, in which case

$$g_s\alpha(e^\tau) = 2\pi n \quad \text{as } \tau \to \infty. \tag{4.30}$$

Putting this back into the parametrization of $A'_i$, we see that it now satisfies the SBC, as required. We conclude that any field $A_i$ for which $\int f_2(e^\tau)\, d\tau$ is sufficiently large (and negative) will have a Gribov copy. This proves that Gribov ambiguities are present in the Coulomb gauge of SU(2). Together with the results from Hall [9] and Singer [10], the preceding derivation completes a formal proof that Gribov ambiguities exist in all gauges of all compact, non-Abelian gauge theories.

# 5 Discussion

We turn our attention to a discussion on the consequences of the Gribov ambiguity. We have shown in section 4 that Gribov ambiguities exist in all non-Abelian gauge theories, however it turns out that they are more consequential in some gauge theories than in others. It is evident in equation (4.5) that the magnitude of the coupling constant $g_s$ determines the strength of the interaction between $A_\mu$ and $\partial_\mu\alpha$. The coupling affects what particular magnitude $A_\mu$ must have in order to create zero-modes for the Faddeev-Popov operator, so Gribov ambiguities will be particularly significant in strongly interacting theories. For instance, Gribov ambiguities are of little relevance when simulating weak interactions, because the Gribov horizons are far away from the origin in configuration space. Conversely, they are perhaps a more important consideration when simulating strong interactions, particularly in the infrared due to the running coupling of the theory.

The Gribov problem is of particular significance in lattice gauge theory, where spacetime is discretized in order to allow for simulations. When implementing functional integrals in these theories, it may be important to consider the effects of Gribov ambiguities to prevent some physical fields from appearing multiple times in the simulations. For instance, a paper by Silva and Oliveira [11] investigates a modification to the gluon propagator that arises due to the presence of Gribov ambiguities. In particular, they used lattice simulations to study the gluon propagator in the Landau gauge. Their results show that the existence of Gribov copies modifies the lowest momenta components of the gluon propagator ($q < 2.6$ GeV), but that the effect is small ($\lesssim 10\%$).

In section 3.2, it was established that a photon propagator could only be defined once a gauge-fixing procedure had been implemented. This followed because gauge transformations could be made arbitrarily small, and consequently there were equivalent fields $A'_\mu$ in the neighbourhood of $A_\mu$ that satisfied the same equations of motion. As a result, no propagator could uniquely describe the time evolution of the field, and it was necessary first to eliminate the equivalent fields $A'_\mu$. One may wonder if the Gribov copies that appear in QCD lead to a the same problem for the gluon propagator. However, that turns out not to be the case, because every field will have a discrete number of Gribov copies. As is pointed out in Gribov's paper [2], the copies will likely be situated in an entirely different region of configuration space, far away from $A_\mu$ (particularly in a weakly interacting theory, for the reasons pointed out above). They then have little or no significance for the propagator,[a] which is a perturbative quantity. More generally, we can say that Gribov copies are unimportant when treating QFT perturbatively.

Finally, we must discuss the possible resolutions to the Gribov problem. This serves to give a bit more historical context to the study of Gribov ambiguities and the progress that has been made in recent decades, but the details of these results are beyond the scope of this project.

Gribov [2] attempted to resolve the problem of Gribov copies by restricting the integral to only consider field configurations inside the first Gribov region $C_0$ in figure 3. In this region, the Faddeev-Popov operator has only positive eigenvalues,[b] so it can be defined as the set of field configurations $A_\mu$ that satisfy

$$C_0 = \left\{ A_\mu : \partial^\mu A_\mu = 0 \text{ and } -\partial^2 \,\cdot\, + i g_s \partial^\mu [A_\mu, \,\cdot\,] > 0 \right\}. \tag{5.1}$$

It has been found that the region $C_0$ is convex and bounded in every direction [12], such that no arbitrarily large field configuration is present. In addition, it was shown by Dell'Antonio and Zwanziger [13] that every gauge orbit intersects the Gribov region, so no physical

[a] Beyond, of course, the adjustment of the propagator that was observed in the infrared, where the interaction was very strong.

[b] Returning to the analogy with the Schrödinger equation, we note that the fields in $C_0$ correspond to potential wells which are too shallow to have bound state solutions.

configuration would be left out of the functional integral. Finally, while Gribov was aware that fields $A_\mu \in C_0$ close to the boundary $l_1$ had Gribov copies, he demonstrated that these copies must lie in the region $C_1$. For these reasons, restricting to the integration range to $C_0$ appeared an attractive option to resolve the Gribov ambiguity [2]. However, it turned out that this restriction failed, because there are fields which have Gribov copies inside $C_0$. While Gribov himself was aware of this possibility, the conclusive proof of this fact was first presented by Semenov-Tyan-Shanskii and Franke [14].

At present, it instead appears that a more appropriate endeavour to improve the gauge fixing instead involves restricting the integration even further, to the fundamental modular region $\Lambda$. From each gauge orbit, this region selects the configuration $A_\mu$ which lies closest to the origin (i.e. the configuration that minimizes $\left|\left|A^a_\mu\right|\right|^2 = \left|\mathrm{Tr}\int d^4x A^a_\mu A^{\mu,a}\right|$). The region $\Lambda$ is a proper subset of $C_0$, so it inherits many of the important properties this region has. Further, it has be shown [3] that the interior of the fundamental modular region is intersected by every gauge orbit, as desired. Zwanziger also writes that every gauge orbit has a unique global minimum in the interior of this region, which makes it free from Gribov copies. However, it turns out that the boundary of $\Lambda$ contains degenerate minima, which give rise to Gribov copies on the boundary [3]. The boundary of $\Lambda$ is hard to specify exactly, and it is not yet known how to eliminate these Gribov copies [8].

# 6 Conclusion

This thesis has introduced the path integral formalism of quantum mechanics, and made a connection to the canonical formulation. This formalism was subsequently expanded to QFT, and the significance of the Faddeev-Popov gauge-fixing procedure was presented. Then, the Gribov ambiguity was demonstrated, using SU(2) Yang-Mills theory as an explicit example. Subsequently, it was in short discussed that this problem results in a modification of the gluon propagator at low energy scales. Finally, while the resolution to the Gribov ambiguity remains a conundrum, current research suggests that it is possible to define a region in the functional space which is free from Gribov copies.

A possible extension to this study would be to more thoroughly examine the properties of the fundamental modular region $\Lambda$, and how one can restrict the functional integral to this domain. A more detailed analysis could be based on the results from Zwanziger [3, 12, 13, 15], who covers this topic comprehensively. In addition, it remains to be studied in more detail what implications the Gribov ambiguity has for the gluon propagator. It would be appropriate to verify that the modification found in the paper by Silva and Oliveira [11] agrees with the theoretical calculation produced in the paper by Sobreiro and Sorella [8].

Finally, it would be pertinent to investigate how gauge fixing of non-Abelian gauge theories is performed. As we introduced in section 3.1, the restriction to the path integral is implemented by adding a gauge fixing term to the action. However, gauge fixing of non-

Abelian gauge theories necessitates the definition of Faddeev-Popov ghosts $c$ and $\bar{c}$, a topic which has been left beyond the scope of this thesis. Studying these ghost fields would perhaps give further insight into the properties of Gribov ambiguities.

# References


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# A Path integrals and the Schrödinger equation

In section 2.3 the propagator of a free particle was derived using the path integral formalism. It can be straightforwardly shown that this propagator satisfies the Schrödinger equation.

$$i\frac{\partial}{\partial T}\left(\sqrt{\frac{-im}{2\pi T}}e^{\frac{im}{2T}(x_a-x_b)^2}\right) = -\frac{1}{2m}\frac{\partial^2}{\partial x_a^2}\left(\sqrt{\frac{-im}{2\pi T}}e^{\frac{im}{2T}(x_a-x_b)^2}\right) \tag{A.1}$$

We will now show that this is a general result. The path integral formulation satisfies the Schrödinger equation even for a general potential $V(x)$.

As demonstrated in section 2.3, functional integrals $\int \mathcal{D}x(t)$ can be evaluated as a product of position integrals $\prod_j \int dx_j$. We consider the final integral over $dx' = dx_{N-1}$.

$$G(x_a, x_b; T) = \int \frac{dx'}{C(\varepsilon)} \exp\left[\frac{im}{2\varepsilon}(x'-x_b)^2 - i\varepsilon V\left(\frac{x'+x_b}{2}\right)\right] G(x_a, x', T-\varepsilon) \tag{A.2}$$

As discussed, we only consider paths for which $x' \approx x_b$. It is therefore justified to Taylor expand the final two terms around $x_b$.

$$\begin{aligned} G(x_a, x_b; T) = \int \frac{dx'}{C(\varepsilon)} \exp\left(\frac{im}{2\varepsilon}(x'-x_b)^2\right) \left[1 - i\varepsilon V(x_b) + \mathcal{O}(\varepsilon^2)\right] \\ \times \left[1 + (x'-x_b)\frac{\partial}{\partial x_b} + \frac{1}{2}(x'-x_b)^2\frac{\partial^2}{\partial x_b^2} + \mathcal{O}\big((x'-x_b)^3\big)\right] G(x_a, x_b, T-\varepsilon) \end{aligned} \tag{A.3}$$

Using the Gaussian integral identities in equation (C.1), we obtain

$$G(x_a, x_b; T) = \left(1 - i\varepsilon V(x_b) + \frac{i\varepsilon}{2m}\frac{\partial^2}{\partial x_b^2} + ...\right) G(x_a, x_b, T-\varepsilon) \tag{A.4}$$

Rearranging and multiplying by $i/\varepsilon$ yields

$$i\frac{G(x_a, x_b; T) - G(x_a, x_b; T-\varepsilon)}{\varepsilon} = V(x_b)G(x_a, x_b; T) - \frac{1}{2m}\frac{\partial^2}{\partial x_b^2}G(x_a, x_b; T) \tag{A.5}$$

Taking the continuum limit $\varepsilon \to 0$ and restoring powers of $\hbar$, one obtains the Schrödinger equation.

$$i\hbar\frac{\partial}{\partial T}G(a_x, x_b; T) = \left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right] G(x_a, x_b; T). \tag{A.6}$$

This justifies the statement that the path integral formulation is equivalent to other formalisms of non-relativistic quantum mechanics.

# B Delta function identity

Consider a differentiable function $f$ which vanishes at exactly one point $x_0$ of the domain. As discussed in section 13.1 of Riley [16], delta functions satisfy

$$\left|\frac{df(x)}{dx}\right|_{x=x_0} \delta(f(x)) = \delta(x - x_0). \tag{B.1}$$

Integrating over $x$ on both sides yields

$$\left|\frac{df(x)}{dx}\right|_{x=x_0} \int dx\ \delta(f(x)) = 1 \tag{B.2}$$

This result can be generalized to a vector field $\mathbf{g}(\mathbf{x})$ of $n$ discrete variables. Let $\mathbf{g}(\mathbf{x_0}) = \mathbf{0}$ and consider the first order Taylor expansion around $\mathbf{x_0}$

$$g_i(\mathbf{x}) = \underbrace{g_i(\mathbf{x_0})}_{=0} + \sum_j J_{ij}(x^j - x_0^j) + ... \tag{B.3}$$

where $J_{ij}$ is the Jacobian matrix evaluated at $\mathbf{x_0}$. If the Jacobian is diagonalized and has positive determinant, then $\sum_j J_{ij}(x^j - x_0^j) = J_{ii}(x^i - x_0^i)$ and $\prod_i |J_{ii}| = \det J$. Thus

$$\delta^{(n)}(\mathbf{g}(\mathbf{x})) = \prod_i \delta(g_i(\mathbf{x})) = \prod_i \delta(J_{ii}(a^i - a_0^i)) = \prod_i \frac{\delta(a^i - a_0^i)}{|J_{ii}|} = \frac{\delta^{(n)}(\mathbf{a} - \mathbf{a_0})}{\det J} \tag{B.4}$$

Moving the determinant to the left side and integrating over all the variables $dx^j$ yields the identity

$$\left(\prod_j \int dx^j\right) \delta^{(n)}(\mathbf{g}(\mathbf{x})) \det\left(\frac{\partial g^i}{\partial x^j}\right) = 1 \tag{B.5}$$

The identity (3.3) is the generalization of this identity for a continuous vector field.

# C General identities

## Gaussian identities

The standard Gaussian integrals are [4]

$$\int e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \quad \int x e^{-ax^2} dx = 0, \quad \int x^2 e^{-ax^2} dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}. \tag{C.1}$$

The first identity in equation (C.1) also implies that

$$\begin{aligned} \int e^{-ax^2+bx+c} dx &= \int e^{-a(x-\frac{b}{2a})^2+\frac{b^2}{4a}+c} \\ &= \left(\int e^{-ax^2} dx\right) e^{\frac{b^2}{4a}+c} \\ &= \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}, \end{aligned} \tag{C.2}$$

where the second line moves the constant $\exp\left(\frac{b^2}{4a}+c\right)$ out from the integral, and changes variables $x \to x - \frac{b}{2a}$.

## Gribov pendulum identities

First, we find that differentiating a radial function $g(r)$ gives

$$\partial_i g(r) = \frac{\partial g(r)}{\partial x_i} = \frac{\partial g(r)}{\partial r}\frac{\partial r}{\partial x_i} = g'(r) n_i. \tag{C.3}$$

We also note that

$$(\partial_i \hat{n})\hat{n} + \hat{n}(\partial_i \hat{n}) = \partial_i \hat{n}^2 = 0,$$

and consequently

$$(\partial_i \hat{n})\hat{n} = -\hat{n}(\partial_i \hat{n}), \qquad \hat{n}(\partial_i \hat{n})\hat{n} = \partial_i \hat{n}. \tag{C.4}$$

We also need the following four identities for the normal vector in the Lie algebra.

$$\partial_i n_i = \partial_i\left(\frac{x_i}{r}\right) = \frac{(\partial_i x_i) r - x_i(\partial_i r)}{r^2} = \frac{3r - x_i n_i}{r^2} = \frac{3 - n_i^2}{r} = \frac{2}{r}. \tag{C.5}$$

$$(\partial_i \hat{n}) n_i = i(\partial_i n_j) n_i \sigma_j = i\frac{(\partial_i x_j) r - x_j(\partial_i r)}{r^2} n_i \sigma_j = \frac{i\delta^{ij} n_i \sigma_j - i n_j n_i^2 \sigma_j}{r} = 0. \tag{C.6}$$

$$(\partial_i \hat{n})(\partial_i \hat{n}) = i^2\left(\frac{\delta^{ij}}{r} - \frac{n_i n_j}{r}\right)\left(\frac{\delta^{ik}}{r} - \frac{n_i n_k}{r}\right)\sigma_j \sigma_k = \frac{-\sigma_i\sigma_i}{r^2} + \frac{\hat{n}^2}{r^2} - \frac{\hat{n}^2}{r^2} - \frac{\hat{n}^2}{r^2} = \frac{-2}{r^2}. \tag{C.7}$$

$$\partial^2 \hat{n} = i\partial_i \left( \frac{\delta^{ij}}{r} - \frac{n_i n_j}{r} \right) \sigma_j = i \left( -\frac{\delta^{ij} n_i}{r^2} - \frac{2n_j + 0 - n_i n_j n_i}{r^2} \right) \sigma_j = \frac{-2\hat{n}}{r^2}. \tag{C.8}$$

Finally, we recall the trigonometric identities

$$\begin{aligned} 2\cos(x)\sin(x) &= \sin(2x) \\ \cos^2(x) - \sin^2(x) &= \cos(2x) \\ 1 - 2\sin^2(x) &= \cos(2x). \end{aligned} \tag{C.9}$$