LU-TP 21-18
June 2021

Employing the little group symmetry within the spinor-helicity formalism to constrain scattering amplitudes

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Abstract

This thesis aims to describe the little group scaling and how it simplifies the calculation of scattering amplitudes. The little group has the defining property that it leaves a particular four-momentum invariant, which is used to describe how the internal structure of a particle is transformed under the little group. An important part in the study of little group scaling is the spinor-helicity formalism, which is based on spinors of helicity $h = \pm 1/2$. This formalism comes with some interesting identities which already simplify the calculation of amplitudes. Finally, we show that applying little group scaling to massless particles with complex momenta in the spinor-helicity formalism fully constrains their (mathematical) three-particle amplitudes.

Popular-science description

The framework that we use to study the fundamental building blocks of the universe is called quantum field theory. An important idea here is that there exists a 'field' of each of the fundamental particles throughout 'spacetime'. The study of the fundamental particles that make up the universe is a complicated subject. In physics, we often exploit symmetries to simplify such complicated subjects. In this thesis, the centerpiece is one such symmetry encapsulated by the so called 'little group', discovered by Eugene Wigner in 1939. The little group introduces an element of symmetry of the particles with respect to spacetime itself.

When we talk about spacetime, we must consider Einstein's special theory of relativity. It essentially gives an account of how spacetime changes from the perspective of different observers moving at different velocities. If we consider particles purely in this framework, we do not consider their internal structure. However, particles also possess an internal structure. The little group gives an insight into this internal structure and how it transforms when observed from the perspective of different observers.

Quantum field theory is probabilistic, meaning that we find probabilities for events occurring, rather than a definite answer. The calculation of these probabilities is often not an easy task. This is where we use the little group symmetry to simplify the calculations. Thus, we use the insight into the internal structure of the particles to simplify the calculation of probabilities describing the occurrence of events involving these particles.

Acknowledgements

I would like to thank both my supervisors for their constant support and feedback throughout the process of producing this thesis. I learned not only about the various topics discussed here, but also about the process of writing a scientific text.

Contents

1 Introduction

In quantum field theory, a scattering amplitude is used to calculate the cross section of a process, which gives a measure of the probability of occurrence for the process. Due to this, calculating scattering amplitudes is a very important aspect of theoretical particle physics. However, these calculations can be rather involved. To simplify these amplitude calculations, we turn to the so-called little group scaling [1]. This simplification roots from a fundamental symmetry exhibited by a set of Lorentz transformations, that go on to form the little group. The conventional method of amplitude calculation which involves Feynman diagrams, does not exploit this symmetry.

The Lorentz group describes how momentum transforms when viewed by a different observer. We simply take a momentum four-vector and perform a Lorentz transformation. However, if we want to know how a particle is transformed under a Lorentz transformation it does not suffice to give the transformation of its momentum. Apart from treating the particle as a point-like object with only momentum, we need to consider its spin. The little group, which is a subgroup of the Lorentz group, describes how the spin of a particle transforms [2, 3].

Where the little group is at the crux of the simplification, the spinor-helicity formalism [1] forms the framework wherein we will use the little group transformation. We can deconstruct the term spinor-helicity, where the spinors come from the solution to the Dirac equation, and we use the helicity to label these spinors. Using these spinors, we can go on to derive various identities, which themselves simplify the calculation of scattering amplitudes. Using the little group scaling it is possible to constrain scattering amplitudes for three and four particles of any mass [4]. This thesis focuses on amplitudes corresponding to three particles, which are all massless. Physically, massless three particle amplitudes vanish, however, it is possible to consider them with complex momenta. These massless three-particle amplitudes form the building blocks for four-particle cases [4].

In this thesis, we start with the relevant background needed from group theory in section 2, where we give a brief overview of Lie groups and further explore the Lorentz group. Section 3 introduces the little group, and we go on to study the little groups for massive and massless particles. Section 4 gives the necessary overview of the spinor-helicity formalism, starting with the possibly familiar Dirac equation. Next, section 5 demonstrates the usefulness of the spinor-helicity formalism itself in simplifying amplitude calculations. Finally, section 6 marries the concepts from the previous sections to introduce the little group scaling and further demonstrates its usefulness in calculating three-particle amplitudes.

2 Group theory background

Symmetry is an important tool in the study of particle physics, and can simplify complicated problems. Closely related to symmetries is the study of group theory, which is ubiquitous in theoretical particle physics. Particularly, the study of Lie groups is relevant, as it provides a way to study continuous symmetries in nature.

2.1 Lie groups and Lie algebras

A Lie group can be expressed using a set of continuous parameters. Considering a Lie group element G that can be expressed using a single parameter λ, we have

$$\mathrm{G} = \exp(\lambda X), \tag{2.1}$$

where X is called the generator of the Lie group. The generators of a Lie group span a tangent space to the Lie group, called the Lie algebra. To exemplify this, we consider the Lie group SO(3), which describes rotations in three dimensions. The Lie algebra for the group is denoted by $\mathfrak{so}(3)$. For simplicity, we only consider rotations around the x-axis. Here, the parameter is the angle of rotation around the x-axis, say θ; and so an SO(3) element say $R_x(\theta)$ can be represented by

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{2.2}$$

Since, a generator of the Lie group lies in the tangent space, we find it using the derivative with respect to the parameter. We shall do so for the generator corresponding to rotations around the x-axis, given by A_1. Further, since a Lie group is locally homogeneous [5], we find the derivative near the identity element of the group, which corresponds to $\theta = 0$, that is

$$A_1 = \frac{d}{d\theta} R_x(\theta)\Big|_{\theta=0} \tag{2.3}$$

We similarly derive the other generators of the Lie algebra $\mathfrak{so}(3)$, giving us

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

where A_2 and A_3 corresponds to rotations around the y-axis and z-axis respectively. Note that A_1, A_2 and A_3 are the angular momentum operators from quantum mechanics. Further these generators are related by the commutation relation

$$[A_i, A_j] = \varepsilon^{ijk} A_k, \tag{2.5}$$

where the totally antisymmetric Levi-Civita tensor ε^{ijk} acts as the structure constant of the Lie algebra.

2.2 The Lorentz group

Special relativity describes the notion of inertial reference frames, where an important property is that the speed of light is constant in all inertial reference frames. The transformations between all such reference frames are described by the Lorentz transformations.

These are Lorentz boosts and rotations. The set of all Lorentz transformations with parity and time-reversal transformations form the Lorentz group. It is a Lie group and mathematically it is described as the pseudo-orthogonal group $O(1,3)$, such that

$$O(1,3) = \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) : \Lambda\eta\Lambda^T = \eta\}. \tag{2.6}$$

Here, GL(4,$\mathbb{R}$) is the set of all real 4×4 matrices with a non-zero determinant, and the spacetime metric η is defined as

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.7}$$

The Lorentz group can be divided into four parts based on $\det(\Lambda)$ and $\mathrm{sign}({\Lambda^0}_0)$. Here, we will focus on one of these parts, namely the proper orthochronous Lorentz group, also known as the restricted Lorentz group, $\mathrm{SO(1,3)}^{\uparrow}_{+}$. The other three parts can be found using two discrete transformations called parity Λ_P and time-reversal Λ_T, where we apply Λ_P and Λ_T separately to $\mathrm{SO(1,3)}^{\uparrow}_{+}$ to find two parts, and $\Lambda_P\Lambda_T$ together to find the third. Here, the two discrete transformations are defined as

$$\mathrm{diag}(\Lambda_P) = (1,-1,-1,-1), \qquad \mathrm{diag}(\Lambda_T) = (-1,1,1,1). \tag{2.8}$$

Henceforth, when we refer to the Lorentz group, we mean the restricted Lorentz group for the sake of convenience. Bearing this in mind, we move on to describe the Lie algebra associated with the Lorentz group. The Lorentz group has three degrees of freedom from spatial rotations and three degrees of freedom from boosts; where each degree of freedom corresponds to a generator. We start with rotational generators defined in eq. (2.4). We expand the matrices with zeroes to account for one temporal dimension, and further multiply the matrices with i to make them hermitian[1], which gives us

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Since, we are multiplying the matrices with i, we need to multiply the exponent with $-i$ in the exponential map eq. (2.1) to get the same group, that is

$$G = \exp(-i\vec{\theta}\cdot\vec{J}). \tag{2.10}$$

Next, we move on to the generators for boosts K_i for $i \in \{1,2,3\}$. These are given by

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \tag{2.11}$$

[1] Physicists prefer to work with hermitian operators, as observables correspond to hermitian operators.

where the parameter for the boosts is the rapidity $\vec{\eta}$. Note that the K_i matrices are not hermitian. We will not derive the generators for the boosts, but we can motivate them by observing their form. If we consider K_1, it only has non-zero entries in the components that describe the relation between time and x-direction, which is essentially what a boost in x-direction describes. We can similarly motivate the other two boost generators.

Moreover, we will label all the generators of the Lorentz group by the two directions they mix, using the antisymmetric tensor $(M_{ij})^{\mu}{}_{\nu}$ given by (see for example [2, 3])

$$M_{i0} = K_i, \qquad \varepsilon_{ijk} M_{jk} = 2J_i. \tag{2.12}$$

Finally, the commutation relations for the Lorentz algebra are

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \qquad [K_i, K_j] = -i\varepsilon_{ijk} J_k, \qquad [J_i, K_j] = i\varepsilon_{ijk} K_k. \tag{2.13}$$

2.3 Representations of the Lorentz group

A group element is an abstract concept and its representation can be thought of as its mathematical manifestation, for example in the form of a matrix. In the previous section 2.2, we described a particular matrix representation of the Lorentz group acting on four-vectors. However, there are other possible representations acting on other objects. For example, it is possible to construct representations on the Hilbert space of particle states. We shall use such representations in the subsequent parts of the text. Now, we briefly look at some other representations of the Lorentz group.

2.3.1 Scalar and spinor representations

Before we start describing the particular representations, we describe two new sets of operators, namely N_i^- and N_i^+, where $i \in \{1, 2, 3\}$. These are defined as

$$N_i^- = \frac{J_i - iK_i}{2}, \qquad N_i^+ = \frac{J_i + iK_i}{2}. \tag{2.14}$$

An important distinction here is that J_i and K_i do not split the Lorentz algebra as seen in eq. (2.13), whereas N_i^- and N_i^+ split the algebra, which can be seen in their commutation relations,

$$[N_i^-, N_j^-] = i\varepsilon_{ijk} N_k^-, \qquad [N_i^+, N_j^+] = i\varepsilon_{ijk} N_k^+, \qquad [N_i^-, N_j^+] = 0. \tag{2.15}$$

Allowing complex coefficients, this splits the Lie algebra $\mathfrak{so}(1,3)^{\uparrow}_{+}$ into two $\mathfrak{su}(2)$ Lie algebras, which are easier to work with. A detailed discussion on the group SU(2) is out of the scope of this text, we only need to understand that it is generated by the Pauli matrices.

Further, we can also take the generators to be the Pauli matrices with a factor 1/2, which gives the commutation relations consistent with the generators in eq. (2.13), that is

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma_k}{2}. \tag{2.16}$$

We start with the (0,0) representation, which acts on scalars. Scalars have no internal structure that transforms under Lorentz transformations. Thus, we use the trivial representation such that $N_i^- = N_i^+ = 0$, and using eq. (2.14) this implies that $J_i = K_i = 0$. Now, using the exponential map as in eq. (2.1) we find a rotational element Λ_J and a boost element Λ_K, given by

$$\Lambda_J = \exp(0) = 1, \qquad \Lambda_K = \exp(0) = 1. \tag{2.17}$$

Next, we describe the $(1/2, 0)$ and $(0, 1/2)$ representations, which act on the so called two-component left-chiral spinors and the two-component right-chiral spinors respectively. In the chiral basis, the Dirac equation has a four-component spinor as a solution, where two components correspond to a left-chiral field and the other two to a right-chiral field. We will return to this discussion in section 4.1.1. Now, we use the two-dimensional representation of the algebra $\mathfrak{su}(2)$ for one set of operators and the trivial representation for the other, giving us

$$\left(\frac{1}{2}, 0\right):\ N_i^- = \frac{\sigma_i}{2},\ N_i^+ = 0; \qquad \left(0, \frac{1}{2}\right):\ N_i^- = 0,\ N_i^+ = \frac{\sigma_i}{2}, \tag{2.18}$$

where σ_i is the corresponding Pauli matrix for the value i. Using eq. (2.14), we can work out J_i and K_i, which results in the following rotational (Λ_J) and boost (Λ_K) elements

$$\left(\frac{1}{2}, 0\right): \quad \Lambda_J = \exp\left(-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right), \quad \Lambda_K = \exp\left(\vec{\eta}\cdot\frac{\vec{\sigma}}{2}\right), \tag{2.19}$$

$$\left(0, \frac{1}{2}\right): \quad \Lambda_J = \exp\left(-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right), \quad \Lambda_K = \exp\left(-\vec{\eta}\cdot\frac{\vec{\sigma}}{2}\right), \tag{2.20}$$

with the parameters $\vec{\theta}$ giving angles of rotation and $\vec{\eta}$ giving the rapidity. Here, the rotations Λ_J are the same for left and right chiral spinors, but we have a negative exponent when considering the boost element Λ_K for the right chiral spinors, which is different from left chiral spinors.

The $(1/2, 0)\oplus(0, 1/2)$ representation, where we get a 4×4 block diagonal matrix which acts on a two component left-chiral spinor stacked upon a two component right-chiral spinor, gives the Lorentz transformation of a Dirac spinor.

2.3.2 The unitary representation

We have a unitary representation U meaning $U^\dagger U = I$, where I is the identity transformation. A unitary representation is generated by a hermitian generator. To see this, we consider a unitary representation of an element infinitesimally close to identity, $U = I + i\theta J$, such that

$$U^\dagger U = I - i\theta J^\dagger + i\theta J + \mathcal{O}(\theta^2) = I \implies J^\dagger = J. \tag{2.21}$$

Hermitian operators are relevant, as physical observables correspond to hermitian operators. But, we do not have finite-dimensional hermitian generators for boosts, and so we do not have a finite-dimensional unitary representation of the Lorentz group. However, we do have an infinite-dimensional unitary representation.

2.4 The Poincaré group

We can further generalise the Lorentz group to the Poincaré group, which includes all the elements of the Lorentz group and translations. For a Lorentz transformation $\Lambda^\mu{}_\nu$ and translation a^μ, a general Poincaré group transformation of x^μ is given by

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{2.22}$$

The Poincaré group has all the generators of the Lorentz group and the translations in three spatial directions are generated by the momentum operators P_1, P_2, P_3, which correspond to the momenta in the three directions x, y, z respectively; and translations in the temporal direction are generated by $P_0 = H$, the Hamiltonian. The new commutation relations for the Poincaré algebra are

$$\begin{aligned} [J_i, P_i] = i\varepsilon^{ijk} P_k, \qquad [K_i, P_j] = -i\delta_{ij} H, \qquad [P_i, P_j] = 0, \\ [J_i, H] = 0, \qquad [K_i, H] = -iP_i, \qquad [P_i, H] = 0. \end{aligned} \tag{2.23}$$

Now, we turn to the *Casimir operators* of the Poincaré group. A Casimir operator of a group is an operator that commutes with all the generators of the group and thus with all the elements of the Lie algebra. The eigenvalues of the Casimir operators can be used to label the representations of the Poincaré group [5]. Finally, the Casimir operators of the Poincaré group are $P^\mu P_\mu$ and the Pauli-Lubanski vector squared $W^\mu W_\mu$, where W^μ is [3]

$$W^\mu \equiv -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma, \tag{2.24}$$

and we defined $M_{\nu\rho}$ in eq. (2.12). The Pauli-Lubanski vector is useful in describing spin, and we will return to it in sections 3.3 and 3.4.

3 The little group

3.1 Invariance of the standard momentum

The square of momentum $p^2 = p^\mu p_\mu \geq 0$ is invariant under Lorentz transformations. Looking further into momentum p^μ, we will only consider cases with positive energy that is $p^0 \geq 0$. This leads to three possible cases, each of which can be represented by a different form of momentum, which we will call the standard momentum k^μ:

1. We start with, $p^0 = 0$, that is the energy is zero. This corresponds to vacuum where the only possible momentum vector is $k^\mu = (0, 0, 0, 0)$.

2. Next, $p^0 > 0$ and $p^\mu p_\mu = m^2$, which corresponds to massive particles, with mass given by m. Here, we use the momentum in the rest frame to represent the case, that is we define the standard momentum $k^\mu = (m, 0, 0, 0)$. This means that the momentum vectors p^μ for particles of same mass are mapped to the same k^μ, regardless of their three-momentum.

3. Finally, $p^0 > 0$ and $p^\mu p_\mu = 0$, which corresponds to massless particles. Massless particles do not have a rest frame, so we define the standard momentum $k^\mu = (\tau, 0, 0, \tau)$.

This is where we introduce the little group element $W^\mu{}_\nu$, which is defined to be the group of Lorentz transformations that leave the standard momentum k^μ invariant [2, 6], that is

$$W^\mu{}_\nu k^\nu = k^\mu. \tag{3.25}$$

Note that, only the standard momentum k^μ is invariant under a little group transformation, other momentum vectors are transformed. The three cases considered above have three different little groups associated with them. We shall briefly describe these:

1. To start with, we consider vacuum, which is described by $k^\mu = (0, 0, 0, 0)$. Such a four-vector is left invariant by any Lorentz transformation. Thus, the little group here is the entire Lorentz group.

2. Next, we consider massive particles, where the standard momentum of choice is $k^\mu = (m, 0, 0, 0)$. Since, all the spatial components are zero, it is easy to see that all spatial rotations would leave such a four-vector invariant. Thus, the little group for massive particles is the group SO(3), describing rotations in three dimensions as discussed in section 2.1. We will further explore this case in the dedicated section 3.3.

3. Finally, we consider massless particles, with $k^\mu = (\tau, 0, 0, \tau)$ as the standard momentum of choice. From here, one can see that rotations in the xy-plane would leave k^μ invariant. However, the massless case is not that simple and needs to be considered in further detail. We will explore this case in the dedicated section 3.4

3.2 The little group acting on a particle state

We can find an expression for a little group transformation acting on a particle state. To do so, we start by defining a Lorentz transformation $L^\mu{}_\nu(p)$ that maps the standard momentum k^ν to the momentum of choice p^μ

$$p^\mu = L^\mu{}_\nu(p) k^\nu. \tag{3.26}$$

Note that $L(p)$ really depends on both p^μ and k^ν. Moreover, since mass is a Lorentz invariant, such a transformation does not change the mass associated with the momentum. Now, instead of describing the particles just using their momentum, we will also consider their 'internal structure'; that is we will consider quantities such as spin. Thus, we define our particle states to be eigenstates of the momentum operator P^μ with eigenvalue p^μ such that

$$P^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle, \tag{3.27}$$

where p gives the associated momentum and σ denotes other 'internal' degrees of freedom, in particular spin. We consider the momentum transformation as in eq. (3.26) for particle states, we shall define this transformation on the particle state as [2]

$$|p,\sigma\rangle = N(p)U(L(p))|k,\sigma\rangle, \tag{3.28}$$

where the label σ is unchanged by the definition of the state, and $U(L(p))$ is some infinite-dimensional unitary representation of $L(p)$. Further, if we consider for example, a particle with spin in the x-direction, a Lorentz transformation such as a rotation around the z-axis would transform the spin. However, in eq. (3.28) even if the physical spin is changed by the Lorentz transformation, we label the state with the same σ. Moreover, this definition is only for transforming states with the standard momentum k. Finally in eq. (3.28), $N(p)$ is some normalization factor given by [2]

$$N(p) = \sqrt{\frac{k^0}{p^0}}. \tag{3.29}$$

Now, we see how eq. (3.28), where we have a state with general momentum p, transforms under a Lorentz transformation. We get

$$U(\Lambda)|p,\sigma\rangle = N(p)U(\Lambda)U(L(p))|k,\sigma\rangle = N(p)U(\Lambda L(p))|k,\sigma\rangle, \tag{3.30}$$

where we get the second equality because U is a representation of the group element and needs to exhibit closure. Next, multiplying by $U(I)$, we get

$$\begin{aligned} U(\Lambda)|p,\sigma\rangle &= N(p)U\Big(L(\Lambda p)L^{-1}(\Lambda p)\Big)U\Big(\Lambda L(p)\Big)|k,\sigma\rangle \\ &= N(p)U\Big(L(\Lambda p)\Big)U\Big(L^{-1}(\Lambda p)\Lambda L(p)\Big)|k,\sigma\rangle. \end{aligned} \tag{3.31}$$

Here, $U(L^{-1}(\Lambda p)\Lambda L(p))$ leaves the momentum part of the state vector invariant because

$$\begin{aligned} L(p) &: k \longrightarrow p, \\ \Lambda &: p \longrightarrow \Lambda p, \\ L^{-1}(\Lambda p) &: \Lambda p \longrightarrow k. \end{aligned} \tag{3.32}$$

Since, the little group leaves k invariant by definition, this is a little group element W, which we express as

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p)\Lambda L(p). \tag{3.33}$$

Due to the closure property of a group, this implies that the little group is a subgroup of the Lorentz group. Now, since k is unchanged by the little group, we express the little group acting on a particle state as

$$U(W(\Lambda, p))|k, \sigma\rangle = D_{\sigma\sigma'}(W(\Lambda, p))|k, \sigma'\rangle, \tag{3.34}$$

where $D_{\sigma\sigma'}$ is some unitary representation of the little group, depending on the particle states given by $|k, \sigma'\rangle$.

Moreover, using eq. (3.34) in eq. (3.31) gives

$$U(\Lambda)|p, \sigma\rangle = N(p) D_{\sigma\sigma'}\big(W(\Lambda, p)\big) U\big(L(\Lambda p)\big)|k, \sigma'\rangle, \tag{3.35}$$

where we can commute $D(W)$ and $U(L(\Lambda p))$ because $D(W)$ only acts on σ, leaving k invariant and $U(L(\Lambda p))$ leaves σ invariant, acting only on k. Finally, using eq. (3.28) here, we get

$$U(\Lambda)|p, \sigma\rangle = \frac{N(p)}{N(\Lambda p)} D_{\sigma\sigma'}\big(W(\Lambda, p)\big)|\Lambda p, \sigma'\rangle. \tag{3.36}$$

Thus, we see that we can directly find a representation of the Lorentz group from the representation of the little group. This is called the method of induced representations. Similarly, we can also find an induced representation of the Poincaré group from the representation of the little group. Due to this, we use the eigenvalues of the Casimir operators of the Poincaré group (recall section 2.4) to also label the representations of the little group.

3.3 Massive particles

For a massive particle, we choose the standard momentum to be in the rest frame, giving us $k^\mu = (m, 0, 0, 0)$, which is left invariant by spatial rotations. This corresponds to the little group SO(3), as discussed in section 3.1. Now, we shall use the eigenvalues of the Casimir operators of the Poincaré group to label the representations of the little group SO(3). We start with the $P^\mu P_\mu$ operator, giving us

$$P^\mu P_\mu |k, \sigma\rangle = p^\mu p_\mu |k, \sigma\rangle = m^2 |k, \sigma\rangle, \tag{3.37}$$

Next, we move on to the Pauli-Lubanksi four-vector W^μ. Since, $k^\mu = (m, 0, 0, 0)$ only P_0 will give a non-zero value and so we have

$$W^\mu |k, \sigma\rangle = -\frac{1}{2}\varepsilon^{\mu\nu\rho 0} M_{\nu\rho} P_0 |k, \sigma\rangle, \tag{3.38}$$

where the Levi-Civita tensor implies that $W^0 = 0$. Now, we consider the eigenvalue of P_0, which is simply m the particle mass, as seen by the form of k^μ. Further, we consider a permutation of the indices of the Levi-Civita tensor, $\varepsilon^{\mu\nu\rho 0} = -\varepsilon^{0\mu\nu\rho} = -\varepsilon^{ijl}$, giving us

$$W^i |k, \sigma\rangle = \frac{m}{2}\varepsilon^{ijl} M_{jl} |k, \sigma\rangle = m J_i |k, \sigma\rangle, \tag{3.39}$$

where we use eq. (2.12) in the second equality. Further, we express $W_\mu = (W_0, W_i)$ as

$$W_0 = 0, \qquad W_i|k,\sigma\rangle = -mJ_i|k,\sigma\rangle. \tag{3.40}$$

Since, W_i represents all the spatial components, it becomes negative when we lower the index according to the metric $\eta_{\mu\nu}$. Finally, we combine the results to get

$$W^\mu W_\mu|k,\sigma\rangle = -m^2 J^2|k,\sigma\rangle = -m^2 j(j+1)|k,\sigma\rangle, \tag{3.41}$$

where J^2 is the total angular momentum operator with the eigenvalue $j(j+1)$, and here j represents the spin of the particle.

Hence, we shall use the mass m and the spin j to label the representations acting on the Hilbert space of particle states. Further, if a representation is labelled by j, then the states corresponding to the representation are labelled by the z-component of spin $j_z \in \{-j, -j+1, .., 0, ..., j-1, j\}$. Thus, the dimension of this representation labelled by j is $2j+1$, and so the particle has $2j+1$ internal degrees of freedom.

3.4 Massless particles

Unlike massive particles, we cannot choose the standard momentum to be in the rest frame for massless particles. So, we take the standard momentum to be $k^\mu = (\tau, 0, 0, \tau)$. From here, we see that rotations in the xy-plane would leave k^μ invariant. Apart, from this there are less obvious Lorentz transformations that would leave k^μ invariant. We see this using a general infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$. The form of $\omega^\mu{}_\nu$ can be deduced using the form of the generators of the group in eqs.(2.9) and (2.11). Finally, for k^μ to be unchanged we must have $\omega^\mu{}_\nu k^\nu = 0$, that is

$$\begin{pmatrix} 0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3 \\ \omega^0{}_1 & 0 & -\omega^1{}_2 & -\omega^1{}_3 \\ \omega^0{}_2 & \omega^1{}_2 & 0 & -\omega^2{}_3 \\ \omega^0{}_3 & \omega^1{}_3 & \omega^2{}_3 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ 0 \\ 0 \\ \tau \end{pmatrix} = 0, \tag{3.42}$$

$$\implies \omega^0{}_3 = 0, \quad \omega^0{}_1 = \omega^1{}_3, \quad \omega^0{}_2 = \omega^2{}_3. \tag{3.43}$$

Thus, we express the matrix using only three parameters, which we rename as $\omega^0{}_1 = \alpha$, $\omega^0{}_2 = \beta$ and $\omega^1{}_2 = \theta$. Now, we get the general Lorentz transformation that leaves k^μ invariant

$$\Lambda_{\text{inv}} = \exp(-i(\alpha A + \beta B + \theta C)), \tag{3.44}$$

where

$$A^\mu{}_\nu = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad B^\mu{}_\nu = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad C^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.45}$$

Considering eq. (2.9), we see that $C = J_3$ and considering eq. (2.9) and (2.11) we find that A and B are combinations of a boost and a rotation,

$$A = K_1 - J_2 \qquad\qquad B = J_1 + K_2. \tag{3.46}$$

Interestingly, we also find these combinations of the Lorentz group generators, when we consider the eigenvalue of the Pauli-Lubanksi four-vector. To see this, we start with W^0 for which we only get a non-zero value for P_3 as we have $k_\mu = (\tau, 0, 0, -\tau)$, giving us

$$W^0 = -\frac{1}{2}\varepsilon^{0\nu\rho 3} M_{\nu\rho} P_3 = -\frac{1}{2}(M_{12} - M_{21})P_3 = -J_3 P_3, \tag{3.47}$$

where we use eq. (2.12) in the final equality. Similarly, we get $W^3 = J_3 P_0$, which gives the same result as W^0 for the state $|k, \sigma\rangle$, that is

$$W^0|k,\sigma\rangle = -J_3 P_3|k,\sigma\rangle = \tau J_3|k,\sigma\rangle, \qquad W^3|k,\sigma\rangle = J_3 P_0|k,\sigma\rangle = \tau J_3|k,\sigma\rangle. \tag{3.48}$$

Now, we move on to W^1, which can be given as

$$\begin{aligned} W^1 &= -\frac{1}{2}(\varepsilon^{1\nu\rho 0} M_{\nu\rho} P_0 + \varepsilon^{1\nu\rho 3} M_{\nu\rho} P_3) = -\frac{1}{2}\big[(M_{32} - M_{23})P_0 + (M_{20} - M_{02})P_3\big] \\ &= (J_1 P_0 - K_2 P_3), \end{aligned} \tag{3.49}$$

where we use eq. (2.12) in the final equality. Similarly, for W^2 we have

$$W^2 = -\frac{1}{2}\big[(M_{13} - M_{31})P_0 + (M_{01} - M_{10})P_3\big] = (J_2 P_0 + K_1 P_3). \tag{3.50}$$

Thus, when W^1 and W^2 act on the state $|k, \sigma\rangle$, we get

$$W^1|k,\sigma\rangle = \tau(J_1 + K_2)|k,\sigma\rangle, \qquad W^2|k,\sigma\rangle = \tau(J_2 - K_1)|k,\sigma\rangle. \tag{3.51}$$

Finally, combining all the components we get an expression for W^μ, which leads to the result

$$-W^\mu W_\mu|k,\sigma\rangle = \tau^2[(J_1 + K_2)^2 + (J_2 - K_1)^2]|k,\sigma\rangle. \tag{3.52}$$

Comparing this with eq. (3.46), we state that $-W^\mu W_\mu = \tau^2(B^2 + A^2)$.

The generators A and B together with J_3 have the commutation relations

$$[J_3, A] = iB, \qquad [J_3, B] = -iA, \qquad [A, B] = 0. \tag{3.53}$$

These commutation relations are the same as the Lie algebra of group ISO(2), which gives rotations and translations in two dimensions. However, note that A and B do not commute with the Hamiltonian as seen from eq(2.23) and eq. (3.46). Due to this, their eigenvalues are not good quantum numbers. Since A and B commute, they are simultaneously diagonalizable and so we express their common eigenstates with eigenvalues a and b as

$$A|p,a,b\rangle = a|p,a,b\rangle, \qquad\qquad B|p,a,b\rangle = b|p,a,b\rangle. \tag{3.54}$$

Now, we apply a rotation by angle θ to the state, giving us a new state, which we define as

$$|p, a, b, \theta\rangle \equiv e^{-i\theta J_3}|p, a, b\rangle. \tag{3.55}$$

Further, we try to find the eigenvalue corresponding to A for this state. We start with

$$A|p, a, b, \theta\rangle = Ae^{-i\theta J_3}|p, a, b\rangle = e^{-i\theta J_3}\left(e^{i\theta J_3}Ae^{-i\theta J_3}\right)|p, a, b\rangle, \tag{3.56}$$

Now, we treat the expression in the parenthesis as a similarity transformation. To see this explicitly we observe the first few terms of the Taylor expansion of the exponential

$$\begin{aligned} e^{i\theta J_3}Ae^{-i\theta J_3} &= \Big(1 + i\theta J_3 - \frac{1}{2}\theta^2(J_3)^2 + \mathcal{O}(\theta^3)\Big)A\Big(1 - i\theta J_3 - \frac{1}{2}\theta^2(J_3)^2 + \mathcal{O}(\theta^3)\Big) \\ &= A + i\theta J_3 A - i\theta A J_3 - \frac{1}{2}\theta^2(J_3)^2 A - \frac{1}{2}\theta^2 A(J_3)^2 + \theta^2 J_3 A J_3 + \mathcal{O}(\theta^3) \\ &= A - \theta B - \frac{1}{2}\theta^2 A + \mathcal{O}(\theta^3), \end{aligned} \tag{3.57}$$

where we use the commutation relations $[J_3, A]$ and $[J_3, B]$ given in eq. (3.53) to get the third line. We continue this process for higher order terms, which gives us

$$e^{i\theta J_3}Ae^{-i\theta J_3} = A\cos\theta - B\sin\theta. \tag{3.58}$$

We perform a similar process with the operator B, and finally get that

$$A|p, a, b, \theta\rangle = (a\cos\theta - b\sin\theta)|p, a, b, \theta\rangle \tag{3.59}$$

$$B|p, a, b, \theta\rangle = (a\sin\theta + b\cos\theta)|p, a, b, \theta\rangle. \tag{3.60}$$

Since θ is arbitrary, this suggests that there exists a continuum of eigenstates . However, massless particles do not exhibit such a continuous degree of freedom [2], and thus for any physical states we must have $a = b = 0$. This means that we can only use the eigenvalues of the remaining generator J_3 to label the representations. Thus, the little group is simply given by the group SO(2), which describes rotations in a two dimensional plane. We give an SO(2) rotation by a 2×2 matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.61}$$

However, there is another way in which we can describe rotations in two dimensions, which is

$$D^h(\theta) = e^{-ih\theta}. \tag{3.62}$$

This is an element of the group U(1), which is isomorphic to the group SO(2). Isomorphism implies a one-to-one correspondence between the elements of the groups. Note that in eq. (3.62), we label the representations with h, which is the eigenvalue of the operator J_3. Further, these representations act on states labelled by the same eigenvalue, that is

$$J_3|k, h\rangle = h|k, h\rangle. \tag{3.63}$$

Since we have $k^\mu = (\tau, 0, 0, \tau)$, the eigenvalue of J_3, that is h, gives us the spin angular momentum in the direction of motion of the particle, which is the helicity. Thus, massless particles have two internal degrees of freedom, which are labelled by the helicities $\pm h$.

4 The spinor-helicity formalism

4.1 The Dirac equation

The Dirac equation, which is a linear differential equation describing relativistic particles ψ with mass m and spin-1/2, is given as

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{4.1}$$

where we use the Feynman slash notation, $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ (see eq. (A.1) for the form of γ^μ).

4.1.1 Solution and chirality

The solution to the Dirac equation is a four-component spinor $\psi(x)$, which we write as

$$\psi(x) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{4.2}$$

where ψ_L and ψ_R are two component spinors. Here, we use the chiral basis for the Dirac equation, which means that the chirality operator γ^5 is diagonalized and is given by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.3}$$

Using eqs.(4.2) and (4.3), we see that the ψ_L has an eigenvalue -1 and we call it left-chiral. Similarly, ψ_R has an eigenvalue 1 and is right-chiral. Note that chirality and helicity are different properties, for elaboration see appendix B.

Further, considering a Lorentz transformation in $(1/2, 0) \oplus (0, 1/2)$ representation described in subsection 2.3.1, we have

$$\Lambda = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \tag{4.4}$$

where the left-chiral spinors transform under Λ_L and the right-chiral spinors under Λ_R.

4.1.2 Solution for fermions and antifermions

The Dirac equation gives a free particle solution, which is of the form [1]

$$\psi(x) \sim u(p)e^{-ip\cdot x} + v(p)e^{ip\cdot x}, \tag{4.5}$$

where the first term corresponds to the fermion solutions and the second term to the antifermion solutions. In the chiral basis, $u(p)$ and $v(p)$ are spinors with left and right-chiral components such that

$$u(p) = \begin{pmatrix} u_L \\ u_R \end{pmatrix}, \qquad \text{and} \qquad v(p) = \begin{pmatrix} v_L \\ v_R \end{pmatrix}, \tag{4.6}$$

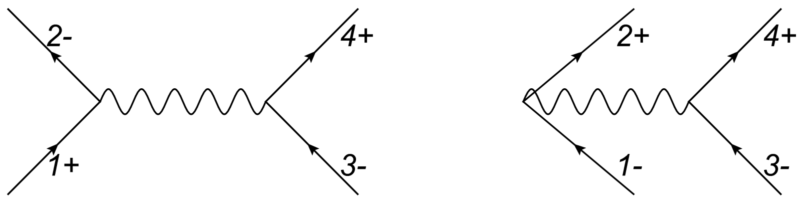

Figure 1: *Feynman diagrams where time flows from left to right. Due to crossing symmetry, the incoming particles can be expressed as outgoing particles of opposite helicity.*

where u_i and v_i for $i \in \{L, R\}$ are two component spinors. Further, we introduce the Dirac conjugate, defined as $\bar{u} \equiv u^\dagger \gamma^0$, giving us [2]

$$\begin{aligned} \bar{u}(p) &= ((u_R)^\dagger \ , \ (u_L)^\dagger) = (\bar{u}_L \ , \ \bar{u}_R) \\ \bar{v}(p) &= ((v_R)^\dagger \ , \ (v_L)^\dagger) = (\bar{v}_L \ , \ \bar{v}_R). \end{aligned} \tag{4.7}$$

Now, we distinguish between incoming and outgoing fermions. An incoming fermion is given by $u(p)$ and an incoming antifermion by $\bar{v}(p)$, whereas an outgoing fermion is given by $\bar{u}(p)$, and an outgoing antifermion is given by $v(p)$. Further, we can write the Dirac equation in momentum space, giving us

$$\begin{aligned} (\not{p} + m)u(p) &= 0, \qquad & \bar{v}(p)(\not{p} - m) &= 0, \\ \bar{u}(p)(\not{p} - m) &= 0, \qquad & (\not{p} + m)v(p) &= 0. \end{aligned} \tag{4.8}$$

4.2 The Weyl equation

For massless spinors, the Dirac equation simplifies to the Weyl equation, given by

$$\not{p}v_\pm(p) = 0, \qquad \not{p}u_\pm(p) = 0, \qquad \bar{v}_\pm(p)\not{p} = 0, \qquad \bar{u}_\pm(p)\not{p} = 0. \tag{4.9}$$

Massless particles travel at the speed of light, and since it is not possible to boost to a frame that travels faster than light, the helicity of a massless particle is the same in all physically possible reference frames. Thus, in the massless case we use helicity to label the components instead of chirality. To avoid any possible confusion with chirality, we shall refrain from using left and right to denote helicity. Instead we use the labels v_+ and v_- to denote the two opposite helicity components with $h = +1/2$ and $h = -1/2$ respectively. It is worth noting that for outgoing particles/anti-particles left-chiral corresponds to $h = +1/2$ and right-chiral corresponds to $h = -1/2$.

This is a good point to describe *crossing symmetry* for massless particles. When we interchange incoming $\leftrightarrow$ outgoing and fermion $\leftrightarrow$ anti-fermion for the particles, the wavefunctions are equal with the sign of helicity flipped. That is

$$u_\pm = v_\mp \qquad \text{and} \qquad \bar{u}_\pm = \bar{v}_\mp. \tag{4.10}$$

[2]A different definition of $\bar{u}_{L/R}$, $\bar{v}_{L/R}$, that is $\bar{u}(p) = (\bar{u}_R \ , \ \bar{u}_L)$, may be found elsewhere.

For example, this can be seen in figure 1, where particle 2 can be treated as an incoming antifermion $\bar{v}_\pm$, but instead we can also treat it as an outgoing fermion $\bar{u}_\mp$ with opposite helicity. Similarly, considering particle 1, the incoming fermion $u_\pm$, can be treated as an outgoing antifermion $v_\mp$ with opposite helicity. Due to crossing symmetry, we can work entirely with outgoing particles.

4.3 Spinor helicity formalism and conventions

Henceforth, we will only use the outgoing spinors $\bar{u}(p)$ and $v(p)$ described in the previous section 4.2 to build up the spinor-helicity formalism. We start by introducing a new notation to label the spinors based on their helicity.

4.3.1 Introducing bra-kets

Here, we introduce the angle and square bra-kets with Weyl-Van der Waerden notation, which play a central role in the spinor-helicity formalism. Using the new notation for outgoing particles, we write $\bar{u}_\pm(p)$ and $v_\pm(p)$ as

$$\bar{u}_+(p) = \big([p|_{\dot{a}}\ ,\ 0\big) \qquad \bar{u}_-(p) = \big(0\ ,\ \langle p|^a\big), \tag{4.11}$$

$$v_+(p) = \begin{pmatrix} |p]^{\dot{a}} \\ 0 \end{pmatrix} \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle_a \end{pmatrix}, \tag{4.12}$$

where the square bra-kets correspond to $h = +1/2$ and the angle bra-kets to $h = -1/2$. Note that the angled kets used in section 3 of the text are not the same as the ones described here. Now, it is time to introduce some relevant definitions and conventions used in the spinor helicity formalism. We start with

$$(\sigma^\mu)^{\dot{a}b} = (1, \vec{\sigma})^{\dot{a}b} \qquad (\bar{\sigma}^\mu)_{a\dot{b}} = (1, -\vec{\sigma})_{a\dot{b}} \tag{4.13}$$

where $\vec{\sigma}$ denotes the Pauli matrices. Further, we define the $\gamma-$matrices in the chiral basis as

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)^{\dot{a}b} \\ (\bar{\sigma}^\mu)_{a\dot{b}} & 0 \end{pmatrix}. \tag{4.14}$$

Here, we have an abuse of notation as γ^μ does not have any spinor indices. Now, we go back to the Weyl equation (4.9) and consider $\not{p}$, which is

$$\not{p} = \gamma^\mu p_\mu = \begin{pmatrix} 0 & p_\mu(\sigma^\mu)^{\dot{a}b} \\ p_\mu(\bar{\sigma}^\mu)_{a\dot{b}} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & p^{\dot{a}b} \\ \bar{p}_{a\dot{b}} & 0 \end{pmatrix}, \tag{4.15}$$

such that for example,

$$\bar{p}_{a\dot{b}} \equiv p_\mu(\bar{\sigma}^\mu)_{a\dot{b}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}. \tag{4.16}$$

This gives

$$\det(\bar{p}) = p^\mu p_\mu = m^2 = 0. \tag{4.17}$$

Considering the Weyl equation (4.9) with the new definition of $\not{p}$ in eq. (4.15), gives

$$[p|_{\dot{a}} p^{\dot{a}b} = 0, \qquad p^{\dot{a}b}|p\rangle_b = 0, \qquad \langle p|^a \bar{p}_{a\dot{b}} = 0, \qquad \bar{p}_{a\dot{b}}|p]^{\dot{b}} = 0. \tag{4.18}$$

Further, we can find relations between the various square and angle bra-kets. For this, we use the definition of the Dirac conjugate and eq. (4.11) giving the following relation for real momenta p^μ

$$u(p) = \begin{pmatrix} |p]^{\dot{a}} \\ |p\rangle_a \end{pmatrix} \implies \bar{u}(p) = u^\dagger\gamma^0 = \big((|p\rangle_a)^\dagger\ ,\ (|p]^{\dot{a}})^\dagger\big) = \big([p|_{\dot{a}}\ ,\ \langle p|^a\big),$$

$$\implies (|p\rangle_a)^\dagger = [p|_{\dot{a}} \qquad\qquad (|p]^{\dot{a}})^\dagger = \langle p|^a. \tag{4.19}$$

Note that, there is an abuse of notation as $u(p)$ does not have any spinor indices. Moreover, if we consider each component of the two-component spinor, such that $\dot{a} = a$, we get

$$(|p\rangle_a)^* = [p|_{\dot{a}} \qquad\qquad (|p]^{\dot{a}})^* = \langle p|^a\ . \tag{4.20}$$

Finally, we raise or lower indices from the left using the Levi-Civita tensor

$$\langle p|^a = \varepsilon^{ab}|p\rangle_b, \qquad\qquad |p]^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}}[p|_{\dot{b}}\ . \tag{4.21}$$

where $\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab} = -\varepsilon_{\dot{a}\dot{b}}$, and the explicit form of Levi-Civita tensor is given by

$$\varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.22}$$

4.3.2 Some useful identities

The Pauli matrices have the following properties, which can be used in spinor manipulations [1, 7]

$$(\bar{\sigma}^\mu)_{a\dot{a}} = \varepsilon_{\dot{a}\dot{b}}\varepsilon_{ab}(\sigma^\mu)^{\dot{b}b}\ , \tag{4.23}$$

$$(\sigma^\mu)^{\dot{a}a}(\sigma_\mu)^{\dot{b}b} = 2\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\ , \tag{4.24}$$

$$(\bar{\sigma}^\mu)_{a\dot{a}}(\bar{\sigma}_\mu)_{b\dot{b}} = 2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}\ , \tag{4.25}$$

$$(\bar{\sigma}^\mu)_{a\dot{a}}(\sigma_\mu)^{\dot{b}b} = 2\delta_a^{\ b}\delta^{\dot{b}}_{\ \dot{a}}\ , \tag{4.26}$$

$$\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu) = 2\eta^{\mu\nu}\ . \tag{4.27}$$

Next, we consider the spin-sum completeness relation for massless particles [8], which states $u_-\bar{u}_- + u_+\bar{u}_+ = \not{p}$. If we consider crossing symmetry in eq. (4.10), along with the definitions in eqs.(4.11) and (4.12), we get the *spinor completeness relation* $\not{p} = |p\rangle[p| + |p]\langle p|$. There is an abuse of notation here because the dimensions on the left and right do not match. This relation should really be interpreted as two relations, which are

$$\bar{p}_{a\dot{b}} = |p\rangle_a [p|_{\dot{b}}, \qquad\qquad p^{\dot{a}b} = |p]^{\dot{a}} \langle p|^b. \tag{4.28}$$

Above, we considered the outer product of a square and an angle bra/ket. We can also consider the inner product of bra-kets. For momenta p^μ and q^μ,

$$\langle pq\rangle \equiv \langle p|^a |q\rangle_a, \qquad\qquad [pq] \equiv [p|_{\dot{a}} |q]^{\dot{a}}, \tag{4.29}$$

where both $\langle pq\rangle$ and $[pq]$ are invariant under Lorentz transformations. We do not have an angle bra and a square ket or vice-versa, as above. This can be seen from the structure of the product $\bar{u}_- v_+ = 0$ and $\bar{u}_+ v_- = 0$ from eq. (4.11) and (4.12). Moreover, if we were to somehow construct such a quantity, it will not be Lorentz invariant. Another interesting result is for the inner product when $p^\mu = q^\mu$, this gives us

$$[pp] = \varepsilon_{\dot{a}\dot{b}} |p]^{\dot{b}} |p]^{\dot{a}} = 0. \tag{4.30}$$

We can perform a similar calculation of angle brackets and generalise the result to

$$\langle pp\rangle = [pp] = 0. \tag{4.31}$$

The bra $\leftrightarrow$ ket transformation, corresponds to the raising and lowering of indices, which is achieved using the Levi-Civita tensor (see eq. (4.21)). Since the Levi-Civita tensor is anti-symmetric, we get that

$$\langle pq\rangle = \langle p|^a |q\rangle_a = \varepsilon_{ac} \langle q|^c \varepsilon^{ab} |p\rangle_b = \langle q|^c (-\varepsilon_{ca}) \varepsilon^{ab} |p\rangle_b = \langle q|^c (-\delta^b_c) |p\rangle_b = -\langle qp\rangle \tag{4.32}$$

We can generalise this result to square bra-kets, giving us

$$\langle pq\rangle = -\langle qp\rangle, \qquad\qquad [pq] = -[qp]. \tag{4.33}$$

At this point, it is useful to define a new term which can be seen in a fermion-vector vertex, and is also useful in describing polarization vectors. We define this as

$$\bar{u}_-(p)\gamma^\mu v_+(k) = \begin{pmatrix} 0 \ , \ \langle p|^a \end{pmatrix} \begin{pmatrix} 0 & (\sigma^\mu)^{\dot{a}b} \\ (\bar{\sigma}^\mu)_{a\dot{b}} & 0 \end{pmatrix} \begin{pmatrix} |k]^{\dot{b}} \\ 0 \end{pmatrix} \equiv \langle p|\gamma^\mu|k]. \tag{4.34}$$

It is worth noting here that really, $\langle p|\gamma^\mu|k] = \langle p|(\bar{\sigma}^\mu)_{a\dot{b}}|k]$. Similarly, $[p|\gamma^\mu|k\rangle = [p|(\sigma^\mu)^{\dot{a}b}|k\rangle$. From here, it can be deduced that $\langle p|\gamma^\mu|k\rangle = [p|\gamma^\mu|k] = 0$. Moreover, we have the properties [1]

$$\langle p|\gamma^\mu|k] = [k|\gamma^\mu|p\rangle, \qquad\qquad \langle p|\gamma^\mu|k]^* = \langle k|\gamma^\mu|p]. \tag{4.35}$$

Further, we use this to define

$$\langle p|P|k] \equiv P_\mu \langle p|\gamma^\mu|k], \tag{4.36}$$

where, p^μ and k^μ are taken to be lightlike vectors. Now, if P_μ is also a lightlike vector, we get

$$\langle p|P|k] = \langle p|^a P_{a\dot{b}}|k]^{\dot{b}} = \langle p|^a|P\rangle_a[P|_{\dot{b}}|k]^{\dot{b}} = \langle pP\rangle[Pk]. \tag{4.37}$$

Using the above result eq. (4.37), we explore momentum conservation in the light of the spinor-helicity formalism. We start with the following expression for n particles where $|1\rangle = |p_1\rangle$, and similar notation is used for other bra-kets and other particles,

$$\begin{aligned}\sum_{i=1}^{n}\langle qi\rangle[ik] &= \langle q1\rangle[1k] + ... + \langle qn\rangle[nk] = \langle qp_1\rangle[p_1k] + ... + \langle qp_n\rangle[p_nk]\\ &= \langle q|p_1|k] + ... + \langle q|p_n|k] = \big((p_1)_\mu + ... + (p_n)_\mu\big)\langle q|\gamma^\mu|k]\\ &= \sum_{i=1}^{n}(p_i)_\mu\langle q|\gamma^\mu|k].\end{aligned} \tag{4.38}$$

Now, by the conservation of momentum we know that $\sum_{i=1}^{n}(p_i)_\mu = 0$, since all the particles are outgoing. This means that for any lightlike vectors q^μ and k^μ, we get

$$\sum_{i=1}^{n}\langle qi\rangle[ik] = 0. \tag{4.39}$$

This is a useful identity. For example, we can use this for four particles to get that

$$\begin{aligned}&\langle 11\rangle[13] + \langle 12\rangle[23] + \langle 13\rangle[33] + \langle 14\rangle[43] = 0\\ \Longleftrightarrow\ &\langle 12\rangle[23] = -\langle 14\rangle[43],\end{aligned}$$

where we use eq. (4.31) to find that the first and third term are zero.

Next, we turn to the *Fierz identity*. To derive this, we start with

$$\begin{aligned}\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] &= \langle 1|(\bar{\sigma}^\mu)_{a\dot{b}}|2]\langle 3|(\bar{\sigma}_\mu)_{c\dot{d}}|4] = \langle 1|^a|2]^{\dot{b}}\langle 3|^c|4]^{\dot{d}}(\bar{\sigma}^\mu)_{a\dot{b}}(\bar{\sigma}_\mu)_{c\dot{d}}\\ &= 2\langle 1|^a|2]^{\dot{b}}\langle 3|^c|4]^{\dot{d}}\varepsilon_{ac}\varepsilon_{\dot{b}\dot{d}} = 2\langle 1|^a|2]^{\dot{b}}\varepsilon_{ac}\langle 3|^c\varepsilon_{\dot{b}\dot{d}}|4]^{\dot{d}}\\ &= 2\langle 13\rangle[42] = 2\langle 31\rangle[24],\end{aligned} \tag{4.40}$$

where in the third equality, we use eq. (4.25).

Finally, before we move to the next subsection, we shall explore another relation, which is useful for calculating scattering amplitudes. We start with

$$(p+q)^2 = p^2 + q^2 + 2p\cdot q = 2p^\mu \cdot q_\mu,$$

where we recall that we have massless particles. Further, using eq. (4.27), we get

$$\begin{aligned}2p_\mu q_\nu \eta^{\mu\nu} &= p_\mu q_\nu \mathrm{tr}((\bar{\sigma}^\mu)(\sigma^\nu)) = p_\mu q_\nu (\bar{\sigma}^\mu)_{a\dot{b}}(\sigma^\nu)^{\dot{b}a}\\ &= (\bar{p}_{a\dot{b}})(q^{\dot{b}a}) = (|p\rangle_a[p|_{\dot{b}})(|q]^{\dot{b}}\langle q|^a)\\ &= \langle qp\rangle[pq] = \langle pq\rangle[qp].\end{aligned} \tag{4.41}$$

Here, in the second equality in the second line, we use eq. (4.15), in the fourth equality, we use the spinor completeness relation eq. (4.28), and in the final equality we use the antisymmetry from eq. (4.33). This finally gives the result

$$(p+q)^2 = \langle qp\rangle[pq] = \langle pq\rangle[qp]. \tag{4.42}$$

4.4 Polarization vectors

Massless particles of spin-1 are denoted by their polarization vectors. In the spinor helicity formalism the outgoing polarization vectors are given by [1]

$$(\epsilon^\mu)^*_-(p,q) = \frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]} \qquad (\epsilon^\mu)^*_+(p,q) = \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}, \tag{4.43}$$

where q is an arbitrary reference spinor, such that $q^2 = 0$ and $q \cdot p \neq 0$. Note that, (ϵ^μ) denotes incoming polarization vectors and the conjugate $(\epsilon^\mu)^*$ denotes outgoing polarization vectors [8]. The arbitrariness of q is due to gauge invariance.

Deriving the polarization vectors, is out of the scope of this text, but we can show that the above expression eq. (4.43), has the properties of a polarization vector. We need to show that $\epsilon^2_\pm = -1$, for that we start by computing the conjugate of $(\epsilon^\mu_+)^*$, that is

$$(\epsilon^\mu_+) = \frac{\langle q|\gamma^\mu|p]^*}{\sqrt{2}\langle qp\rangle^*} = \frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]} = (\epsilon^\mu_-)^*, \tag{4.44}$$

where we use eqs.(4.20) and (4.35) in the second equality. This gives us

$$(\epsilon^\mu_+)^*(\epsilon_{\mu+}) = \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}\frac{\langle p|\gamma_\mu|q]}{\sqrt{2}[pq]} = \frac{\langle pq\rangle[pq]}{\langle qp\rangle[pq]} = -1, \tag{4.45}$$

where we use the Fierz identity eq. (4.40) in the second equality. Similarly, we can also prove $(\epsilon^\mu_-)^*(\epsilon_{\mu-}) = -1$. Another property is

$$p^\mu(\epsilon_+(p,q))_\mu = p^\mu\frac{\langle q|\gamma_\mu|p]}{\sqrt{2}\langle pq\rangle} = \frac{\langle q|^a\bar{p}_{a\dot{b}}|p]^{\dot{b}}}{\sqrt{2}\langle pq\rangle} = \frac{\langle q|^a|p\rangle_a[p|_{\dot{a}}|p]^{\dot{a}}}{\sqrt{2}\langle pq\rangle} = 0, \tag{4.46}$$

where we use eq. (4.31) in the final step. And finally, we prove that ϵ_- and ϵ_+ are orthogonal,

$$(\epsilon^\mu_-)^*(\epsilon_{\mu+}) = \frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]}\frac{\langle p|\gamma_\mu|q]}{\sqrt{2}[pq]} = \frac{\langle pp\rangle[qq]}{[pq][pq]} = 0, \tag{4.47}$$

where we again use the Fierz identity eq. (4.40) in the second equality.

5 Calculating scattering amplitudes

Now that we have introduced the spinor-helicity formalism, it is worth considering some calculations of scattering amplitudes within this formalism.

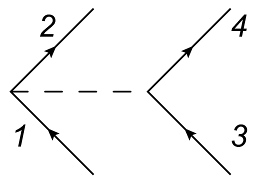

Figure 2: *Interaction between Dirac fermions and a scalar field, with all outgoing particles. Here, time flows form left ot right.*

5.1 Example from Yukawa theory

As an example of scattering amplitude calculations, we examine the interaction between a Dirac field and a scalar field, given by the Yukawa interaction. We consider a four-fermion Feynman diagram given by figure 2. Here, we will calculate the scattering amplitude $A_4(\bar{f}^{h_1}, f^{h_2}, \bar{f}^{h_3}, f^{h_4})$, where f denotes a fermion and $\bar{f}$ denotes an anti-fermion and the indices h_i denote the helicity of the particles. Here, we shall consider all particles to be outgoing, for which we employ crossing symmetry. Finally, using the Feynman rules [1] we get the following expression

$$A_4(\bar{f}^{h_1}, f^{h_2}, \bar{f}^{h_3}, f^{h_4}) = ig[\bar{u}_4 v_3] \times \frac{-i}{(p_1+p_2)^2} \times ig[\bar{u}_2 v_1]. \tag{5.1}$$

Let us start by considering different possibilities for the helicity of $\bar{u}_2$ and v_1,

$$\bar{u}_+(p)v_-(p) = \begin{pmatrix}[2|_{\dot{a}} & 0\end{pmatrix} \begin{pmatrix} 0 \\ |1\rangle_a \end{pmatrix} = 0, \tag{5.2}$$

It is clear from eq. (5.2) that if they have opposite helicities, the amplitude will be zero. Thus, we conclude that two fermions on the same side of the Feynman diagram for this case must have the same helicity.

$$\bar{u}_+(p)v_+(p) = \begin{pmatrix}[2|_{\dot{a}} & 0\end{pmatrix} \begin{pmatrix} |1]^{\dot{a}} \\ 0 \end{pmatrix} = [21] \tag{5.3}$$

and similarly,

$$\bar{u}_-(p)v_-(p) = \langle 21\rangle. \tag{5.4}$$

Now, we resume our calculation for $A_4(\bar{f}^{h_1}, f^{h_2}, \bar{f}^{h_3}, f^{h_4})$. We know that particles 1 and 2 must have the same helicity and so must particles 3 and 4. We consider a singular case where 1 and 2 have positive helicity and 3 and 4 have negative helicity. This gives us

$$iA_4(\bar{f}^+, f^+, \bar{f}^-, f^-) = ig^2\langle 43\rangle \frac{1}{2p_1\cdot p_2}[21] = ig^2\langle 43\rangle \frac{1}{\langle 12\rangle[21]}[21] = ig^2\frac{\langle 43\rangle}{\langle 12\rangle}, \tag{5.5}$$

where we use eq. (4.42) in the second equality. We can apply momentum conservation here to get an interesting result. We start with

$$\langle 12\rangle[21] = (p_1+p_2)^2 = (p_3+p_4)^2 = \langle 43\rangle[34]. \tag{5.6}$$

Now, we use this result in the expression for $iA_4(\bar{f}^+, f^+, \bar{f}^-, f^-)$, which gives

$$iA_4(\bar{f}^+, f^+, \bar{f}^-, f^-) = ig^2\langle 43\rangle \frac{1}{\langle 43\rangle[34]}[21] = ig^2\frac{[21]}{[34]}. \tag{5.7}$$

5.2 Three-particle special kinematics

Before we move to another example, we consider a three-particle process with lightlike momentum vectors for each particle, such that

$$p_1^\mu + p_2^\mu + p_3^\mu = 0. \tag{5.8}$$

In such a process, the amplitude either consists of only square brackets or only angle brackets. To prove this, we start with

$$\langle 12\rangle[12] = (p_1 + p_2)^2 = (p_3)^2 = 0, \tag{5.9}$$

where we use eq. (4.42) in the first equality. Now, this means that either $\langle 12\rangle$ or $[12]$ has to be zero. They can both be zero, but that is an uninteresting result, and so we ignore that.

1. *Case 1*: We assume that $\langle 12\rangle$ is non-zero and thus $[12] = 0$. Then we have

   $$\langle 12\rangle[23] = \langle 1|p_2|3] = -\langle 1|(p_1 + p_3)|3] = 0, \tag{5.10}$$

   where we use eq. (4.37) in the first equality and the Weyl equation eq. (4.18) in the last equality. Now, since we assume that $\langle 12\rangle$ is non-zero, this means $[23]$, must be zero. Similarly, we say

   $$\langle 21\rangle[13] = 0. \tag{5.11}$$

   Now, $\langle 12\rangle = -\langle 21\rangle$, thus if one of them is non zero, so is other. And thus, $[13]$ must be zero. Thus we have shown $[12] = [13] = [23] = 0$. So, we have shown that all possible square brackets for three particles are zero, since $[pq] = -[qp]$. Thus, in this case a non-zero amplitude would only consist of angle brackets.

2. *Case 2*: We assume that $[12]$ is non-zero and thus $\langle 12\rangle = 0$. We can use similar techniques as the previous case to show that

   $$[12]\langle 23\rangle = 0 \tag{5.12}$$
   $$[21]\langle 13\rangle = 0. \tag{5.13}$$

   Further, using the same reasoning as before, we have that $\langle 12\rangle = \langle 13\rangle = \langle 23\rangle = 0$; and thus, a non-zero amplitude would only consist of angle brackets.

Thus, we have shown that the amplitude for a three particle process either consists of only square brackets or only angle brackets.

5.3 Example from QED with spin-1 particle

Here, we shall study an example of a three-particle vertex from QED. Note that the following example is not physically possible, since a photon cannot decay to e^+e^-. However, this process is possible if we consider particles with complex momenta. We have $A_3((e^-)^{h_1}, (e^+)^{h_2}, \gamma^{h_3})$, that is a vertex with an electron, a positron and a photon with h_i denoting the helicity of the respective particles. We take the helicities to be $h_1 = -1/2, h_2 = +1/2$ and $h_3 = +1$, which gives

$$\begin{aligned} iA_3((e^-)^-, (e^+)^+, \gamma^+) &= \bar{u}_-(p_1) ie\gamma_\mu v_+(p_2)\epsilon^\mu_+(p_3, q) \\ &= ie\langle 1|\gamma_\mu|2] \frac{\langle q|\gamma^\mu|3]}{\sqrt{2}\langle q3\rangle} = ie\sqrt{2}\frac{\langle 1q\rangle[32]}{\langle q3\rangle}. \end{aligned}$$

However, we recall that q is picked arbitrarily, and thus the amplitude should be free of q. To obtain such a form, we start with

$$\frac{A_3}{e\sqrt{2}} = \frac{\langle 1q\rangle[32]}{\langle q3\rangle} = \frac{[21]\langle 1q\rangle[32]}{[21]\langle q3\rangle} = \frac{[2|p_1|q\rangle[32]}{[21]\langle q3\rangle},$$

where, we multiply by one in the second equality and use eq. (4.37) in the last step. Now, by momentum conservation we have $p_1 + p_2 + p_3 = 0$, this gives us

$$\frac{A_3}{e\sqrt{2}} = \frac{[2|(-p_2 - p_3)|q\rangle[32]}{[21]\langle q3\rangle} = \frac{[2|p_3|q\rangle[32]}{[12]\langle q3\rangle} = \frac{[23]\langle 3q\rangle[32]}{[12]\langle q3\rangle}, \tag{5.14}$$

where in the second equality, we use the Weyl equation eq. (4.18) to eliminate the term with p_2 and the negative sign flips [21] to [12]. Finally, this gives

$$iA_3((e^-)^-, (e^+)^+, \gamma^+) = ie\sqrt{2}\frac{[23]^2}{[12]}. \tag{5.15}$$

Thus, we arrive at an expression for the amplitude independent of the arbitrary momentum q. Moreover, the expression for this three particle amplitude only contains square brackets, which is consistent with the result from section 5.2.

6 The little group scaling

In this section of the thesis, we will bring together the concepts explored in the previous sections to derive the final result that fixes three-particle amplitudes. To start with, we will apply the little group transformation to the spinors from the spinor-helicity formalism.

The little group transformation for massless particles can be realised by a single complex number, given by $D^h(\theta) = e^{-ih\theta}$ as seen in eq. (3.62). When we apply this complex number to a spinor, we refer to it as scaling. To simplify notation, we define $t \equiv e^{i\theta/2}$, and give the scaling relation using the notation for particle states $|k, h\rangle$ as in eq. (3.63),

$$|k, h\rangle \to t^{-2h}|k, h\rangle, \tag{6.1}$$

where the factor 1/2 in the definition $t \equiv e^{i\theta/2}$ can be motivated by the fact that spinors are mapped back to themselves after a rotation of $\theta = 4\pi$ instead of $\theta = 2\pi$, as seen in eqs.(2.19) and (2.20). Now, we consider the basic elements of the spinor-helicity formalism, which are the angle spinors $|p\rangle$ with $h = -1/2$ and square spinors $|p]$ with $h = +1/2$. Using eq. (6.1) here we get

$$|p\rangle \to t|p\rangle, \qquad\qquad |p] \to t^{-1}|p]. \tag{6.2}$$

Further, we turn to scalars, which have $h = 0$, and thus by eq. (6.1), we get that they are scaled by 1. Next, we discuss outgoing polarization vectors, where we start with $(\epsilon^\mu)^*_-(p, q)$, with $h = -1$ which is scaled as

$$(\epsilon^\mu)^*_- \to \frac{t_p\langle p|\gamma^\mu|q]t_q^{-1}}{\sqrt{2}t_p^{-1}[pq]t_q^{-1}} = t_p^2\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]} = t_p^2(\epsilon^\mu)^*_- \ , \tag{6.3}$$

where we use eq. (6.2) to scale the angle and square spinors. Similarly for $(\epsilon^\mu)^*_+(p, q)$ with $h = +1$, we get

$$(\epsilon^\mu)^*_+ \to \frac{t_q\langle q|\gamma^\mu|p]t_p^{-1}}{\sqrt{2}t_q\langle qp\rangle t_p} = t_p^{-2}\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle} = t_p^{-2}(\epsilon^\mu)^*_+ \ . \tag{6.4}$$

Note that the scaling of the polarization vector depends on the scaling of spinors with p (not q, as q is an arbitrary reference momentum).

We use this result to find how an amplitude is scaled. If we have an $n-$particle amplitude, and say particle i with helicity h_i is transformed under the little group scaling, then the amplitude is scaled by $t_i^{-2h_i}$. This is because the amplitude is simply a construction of the basic spinors of the spinor-helicity formalism (with a coupling factor). This can be given as

$$A_n(\psi_1, ..., t_i^{-2h_i}\psi_i, ...\psi_n) = t_i^{-2h_i}A_n(\psi_1, ..., \psi_i, ..., \psi_n). \tag{6.5}$$

Note that, we have only discussed particles of spins: 0, 1/2 and 1 but the above result (eq. (6.5)), is valid for particles of any spin. [4]

6.1 Scaling examples

Now, that we have described the little group scaling and how it affects amplitudes, it is worth returning to the examples of amplitudes from the section 5.

Example 1. Starting with the four-particle amplitude from section 5.1, we have the result eq. (5.5)

$$A_4(\bar{f}^+, f^+, \bar{f}^-, f^-) = g^2\frac{\langle 43\rangle}{\langle 12\rangle}. \tag{6.6}$$

Let us consider particle 4, which has a helicity $h_4 = -1/2$. If we scale $\langle 4| \to t_4 \langle 4|$, according to eq. (6.5) the amplitude A_4 should scale as $t_4^{-2h_4} A_4 = t_4 A_4$. To verify, we scale the right side, which gives us

$$g^2 \frac{t_4 \langle 43 \rangle}{\langle 12 \rangle} = t_4 A_4. \tag{6.7}$$

Thus, we get the expected result and we can do a similar analysis for any of the particles and see that the little group scaling relation in eq. (6.5) holds.

Example 2. We consider another example using the result from section 5.3, eq. (5.15)

$$A_3((e^-)^-, (e^+)^+, \gamma^+) = e\sqrt{2} \frac{[23]^2}{[12]}. \tag{6.8}$$

Let us consider particle 3, the photon with $h_3 = +1$. If we scale the spinor such that $|3] \to t_3^{-1}|3]$, the amplitude A_3 is scaled as $t_3^{-2h_3} A_3 = t_3^{-2} A_3$. To verify, we see

$$e\sqrt{2} \frac{\left([23] t_3^{-1}\right)^2}{[12]} = e\sqrt{2} \frac{t_3^{-2} [23]^2}{[12]} = t_3^{-2} A_3. \tag{6.9}$$

Example 3. Finally, for the last example we consider particle 2, the positron with $h_2 = +1/2$. We scale it such that $|2] \to t_2^{-1}|2]$, giving $t_2^{-2h_2} A_3 = t_2^{-1} A_3$. To verify, we see

$$e\sqrt{2} \frac{\left(t_2^{-1} [23]\right)^2}{[12] t_2^{-1}} = e\sqrt{2} \frac{t_2^{-2} [23]^2}{t_2^{-1} [12]} = t_2^{-1} A_3. \tag{6.10}$$

Once again, we see that the scaling relation holds.

6.2 Three-particle amplitudes

The little group scaling has especially important implications for a three-particle amplitude, because it can be used to completely determine the kinematic part of the amplitude. From the results of section 5.2, we know that a three-particle amplitude is given either using only square brackets or only angle brackets, that is

$$A_s(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\ [12]^{x_{12}} [13]^{x_{13}} [23]^{x_{23}}, \tag{6.11}$$

$$A_a(1^{h_1}, 2^{h_2}, 3^{h_3}) = c'\ \langle 12 \rangle^{y_{12}} \langle 13 \rangle^{y_{13}} \langle 23 \rangle^{y_{23}} \tag{6.12}$$

Now the little group scaling of $|1] \to t_1^{-1}|1]$, gives $A_s(1,2,3) \to t_1^{-2h_1} A_s(1,2,3)$, that is

$$\begin{aligned} t_1^{-2h_1} A_s(1^{h_1}, 2^{h_2}, 3^{h_3}) &= c\ (t_1^{-1}[12])^{x_{12}} (t_1^{-1}[13])^{x_{13}} [23]^{x_{23}} \\ &= c\ (t_1^{-x_{12}-x_{13}}) [12]^{x_{12}} [13]^{x_{13}} [23]^{x_{23}}. \end{aligned} \tag{6.13}$$

Similarly, from $|1\rangle \to t_1|1\rangle$ we get

$$t_1^{-2h_1} A_a(1^{h_1}, 2^{h_2}, 3^{h_3}) = c'\ (t_1^{y_{12}+y_{13}}) \langle 12 \rangle^{y_{12}} \langle 13 \rangle^{y_{13}} \langle 23 \rangle^{y_{23}}. \tag{6.14}$$

For, the square bra-kets, this gives us the result $2h_1 = x_{12} + x_{13}$, which can be generalised to get the result

$$2h_1 = x_{12} + x_{13} \qquad 2h_2 = x_{12} + x_{23} \qquad 2h_3 = x_{13} + x_{23}. \tag{6.15}$$

For the angle bra-kets, we get the generalised result

$$-2h_1 = y_{12} + y_{13} \qquad -2h_2 = y_{12} + y_{23} \qquad -2h_3 = y_{13} + y_{23}. \tag{6.16}$$

This is an easily solvable system of equations, which gives us the values for all the exponents in the amplitude, and thus the complete kinematic part of the amplitude. Using eq. (6.15) and (6.16), gives us the following amplitude respectively,

$$A_s(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\ [12]^{h_1+h_2-h_3}[13]^{h_1-h_2+h_3}[23]^{-h_1+h_2+h_3}, \tag{6.17}$$

$$A_a(1^{h_1}, 2^{h_2}, 3^{h_3}) = c'\ \langle 12\rangle^{-(h_1+h_2-h_3)}\langle 13\rangle^{-(h_1-h_2+h_3)}\langle 23\rangle^{-(-h_1+h_2+h_3)}. \tag{6.18}$$

We still have to choose whether to use the expression with square brackets or angle brackets. To deduce the appropriate expression, we use dimensional analysis. For this, we directly use the result that, an n-particle amplitude in the four-dimensional spacetime has dimension, $(\text{mass})^{4-n}$ [1]. From eq. (4.42) we see that both square and angle brackets have dimension $(\text{mass})^1$, and assuming we have a dimensionless Yang-Mills coupling, we can compute the dimensions of each of the expressions. Finally, for three particles, we shall pick the expression with dimension $(\text{mass})^{4-3} = (\text{mass})^1$.

To conclude this section, we revisit the example from section 5.3, where we calculate $A_3((e^-)^-, (e^+)^+, \gamma^+)$. To be able to compare our results we will employ $h_1 = -1/2$, $h_2 = +1/2$ and $h_3 = +1$ in the little group scaling result eqs.(6.17) and (6.18). This gives us

$$A_{3,s}((e^-)^-, (e^+)^+, \gamma^+) = c\frac{[23]^2}{[12]}, \qquad A_{3,a}((e^-)^-, (e^+)^+, \gamma^+) = c'\frac{\langle 12\rangle}{\langle 23\rangle^2}. \tag{6.19}$$

Here, $A_{3,s}$ has dimension $(\text{mass})^1$, and $A_{3,a}$ has dimension $(\text{mass})^{-1}$. Thus, we get $A_3 = A_{3,s}$. Comparing with eq. (5.15), we get the same answer. Thus, we clearly see how the little group scaling fixes the amplitude based on the helicity of the particles involved.

7 Conclusion

In this thesis, we started by establishing the relevant background about the Lorentz group, the Poincaré group and their Lie algebras. Building on this, we introduced the little group, which has the defining property of leaving a standard momentum of choice invariant. This property is used to find how the little group acts on a particle state and further understand the internal degrees of freedom for massive and massless particles. A massive particle of mass m and spin j has $2j + 1$ spin degrees of freedom, and a massless particle has two helicity degrees of freedom, with $h = \pm j$, for j labelling the representation of the little group. Finally, we also found an explicit one-dimensional representation of the effective little group acting on massless particle states, which was labelled by the helicity h.

Next, we introduced the spinor-helicity formalism built on the spinors from the Weyl equation, which were labelled by their helicities $h = \pm 1/2$. Since helicity provides the label for the spin degrees of freedom in massless particles, the formalism allowed us to treat the massless spinors in their fundamental form. Further, we saw that the calculation of scattering amplitudes are simplified just by using the spinor-helicity formalism.

Finally, we combined the representation of the little group acting on massless states with the already powerful tool in the form of the spinor-helicity formalism, giving us the little group scaling. Using the little group scaling for three-particle amplitudes, we get the final result of the thesis where the little group scaling fully constrained three-particle amplitudes for massless particles with complex momenta.

This thesis only delved into massless three-particle amplitudes and how they are fully constrained by the little group scaling. However, the application of the little group scaling is neither limited to three-particle amplitudes nor to massless particles. It can be used to constrain further scattering amplitudes, with more particles which are of any mass [4].

A Appendix: More on the spinor-helicity formalism

The gamma matrices used in the text are such that $\gamma^\mu = (\gamma^0, \gamma^i)$, and we use the chiral basis given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{A.1}$$

The Pauli matrices with the identity matrix are given by σ^μ, explicitly

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad & \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{A.2}$$

Further, certain texts [7] employ τ^μ matrices instead of σ^μ matrices. The relation is given by

$$\tau^\mu = \frac{1}{\sqrt{2}}\sigma^\mu. \tag{A.3}$$

Another common convention in spinor-helicity formalism is to use

$$\lambda_p^a \equiv \langle p|^a, \quad \lambda_{p,a} \equiv |p\rangle_a, \quad \tilde{\lambda}_{p,\dot{a}} \equiv [p|_{\dot{a}}, \quad \tilde{\lambda}_p^{\dot{a}} \equiv |p]^{\dot{a}}. \tag{A.4}$$

B Appendix: Helicity and chirality

The helicity of a particle is defined to be the projection of its spin along the direction of its momentum. Positive helicity corresponds to the spin component being parallel to the direction of momentum and a negative helicity corresponds to the spin component being anti-parallel to the direction of momentum.

Chirality is defined by the chirality operator. In the chiral basis, the chirality operator γ^5 is diagonalized and is given by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.5}$$

The spinor component corresponding to the eigenvalue -1 is called left-chiral and the one corresponding to the eigenvalue 1 is called right-chiral.

Chirality is a defining property of a particle, and does not change depending on the reference frame. However, helicity of a massive particle can change when boosting to a different reference frame. This is because the direction of momentum can be changed by boosting to a frame that is moving faster than the particle. However, for massless particles, the helicity stays the same irrespective of the reference frame. Thus, for outgoing massless particles/anti-particles left-chiral corresponds to positive helicity and right-chiral corresponds to negative helicity.

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