Brownian Motion and the Dirichlet Problem

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Abstract

In this Bachelor's thesis, a solution to the Dirichlet problem using Brownian motion is given. Brownian motion is constructed using Kolmogorov's existence and continuity theorems. Blumenthal's zero-one law and the strong Markov property in various formulations are proven. Using these results, a solution to the Dirichlet problem is given using Brownian motion. The cone condition which gives conditions on the domain guaranteeing existence of solution is proven.

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Popular Scientific Explanation

This work deals with two mathematical concepts from seemingly disparate worlds: the Dirichlet problem and Brownian motion. The Dirichlet problem deals with very smooth functions, whereas Brownian motion is prototypically the random movement of a particle suspended in a liquid. The intuition for the Dirichlet problem comes from physics. Imagine some object with a given temperature distribution on its surface. The problem is to find a function which would tell us the temperature at any point inside the object. This work culminates in formulating this function in terms of average properties of randomly moving particles.

Dedicated to my parents, who have given me
the privilege of good education and life

Contents

Notation

$\sigma(A)$	σ-algebra generated by A
$\ell(A)$	λ-system generated by A
$\wp(A)$	Power set of A
$\subseteq, \supseteq$	Subset, Superset
$\subset, \supset$	Proper subset, Proper superset
$\in$	Set membership
$\cup, \cap$	Union, Intersection
$\setminus$	Set difference
A^{c}	Complement of A
$\varnothing$	Empty set
χ_A	Characteristic function of set A
$A \times B$	Cartesian product of A and B
$\bar{A}$	Closure of A
$\inf, \sup$	Infimum, Supremum
$\mathfrak{B}(A)$	Borel σ-algebra on A
$(\Omega, \mathcal{F}, P)$	Probability space with sample space Ω, event space $\mathcal{F}$, and probability measure P
$\{\mathcal{F}_t\}$	Filtration
$\{\mathcal{F}_t^+\}$	Right-continuous filtration
$t \mapsto B_t(\omega), B.(\omega), \mathsf{p}$	Brownian motion path
$\mathcal{P}$	Space of continuous paths starting at 0
$\tilde{\mathcal{P}}$	Space of continuous paths starting at any point in $\mathbb{R}^d$
$\overset{d}{=}$	Equal in distribution
$\mathrm{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$\mathbb{R}$	Real line
$\mathbb{R}^d$	Real coordinate space of dimension d
$\mathbb{Q}_2$	Dyadic rationals

$\circ$	Function composition
E	Expected value
$a \wedge b$	Minimum of a, b
$x_n \uparrow x$	Limit from below
$x_n \downarrow x$	Limit from above
C^k	Order k differentiability class
Δ	Laplacian

Chapter 1

Introduction

1.1 Historical Background

The Dirichlet problem is as such: given some function on a boundary of the domain, find a harmonic function in the interior which extends continuously to the boundary function. This simply formulated problem is of great importance for both mathematics and related fields such as mathematical physics. Dating back two centuries, the history of its solutions is littered with famous names – the first of which is George Green, who first studied what was to become the Dirichlet Problem for domains with general boundary conditions in his 1828 *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. There, the solution of the Dirichlet problem was essentially reduced to construction of what became known as Green's functions. The next to contribute was Gauss, who approached the problem using the "Dirichlet principle", a method rooted in physical understanding of, for example, electrostatics: a charge on the boundary should by laws of electrostatics determine the electrical potential. Unfortunately, his reasoning was not without mistakes, and in 1909 Zaremba, and in 1913 Lebesgue gave examples of bounded domains with continuous boundary functions for which there was no Dirichlet problem solution. A proof of existence of a solution was given in 1900 by Hilbert.

Brownian motion gets its name from a Scottish botanist George Brown, who in 1827 observed the chaotic movement of a pollen particle when viewed under a microscope. Perhaps one of the biggest contributors to Brownian motion in a mathematical context was Norbert Wiener. His influence can be seen in the alternative name for Brownian motion – a Wiener process.

Kakutani and Doob were first to observe the connection between Brownian motion and the Dirichlet problem. Courant, Friedrichs, and Lewy noticed that a simple random walk can be used to interpret the discrete version of the Dirichlet problem, with it converging to the original problem under suitable conditions.

For some basic intuition about the topic, one may refer for example to [Kö95].

Chapter 2

Measure Theory Background

In this chapter we develop some basic tools that will be necessary to prove later theorems. The main results are Dynkin's π-λ theorem and the monotone class theorem. Filtrations of σ-algebras are also discussed, as they will be important for later sections, in particular the discussion on stopping times and Blumenthal's zero-one law.

2.1 Dynkin's π-λ Theorem

Definition 2.1.1. Let Ω be the sample space and let $\wp(\Omega)$ be its power set. A **σ-algebra** is a collection $\mathcal{F} \subseteq \wp(\Omega)$ such that

(i) $\Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^{\mathsf{c}} \in \mathcal{F}$

(iii) If $A_1, A_2, \ldots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Note that conditions (ii) and (iii) together with De Morgan's laws imply that σ-algebras are also closed under intersections. We need some terminology for a collection of subsets that satisfy less strict requirements than those of a σ-algebra.

Definition 2.1.2. A collection of events $\mathcal{P}$ is called a **π-system** if it is closed under intersection, that is if $A, B \in \mathcal{P}$ then $A \cap B \in \mathcal{P}$.

Definition 2.1.3. A collection of events $\mathcal{L}$ is called a **λ-system** if

(i) $\Omega \in \mathcal{L}$,

(ii) If $A, B \in \mathcal{L}$ and $A \subset B$, then $B \setminus A \in \mathcal{L}$,

(iii) If $\{A_n\}_{n=1}^{\infty} \in \mathcal{L}$ such that $A_i \cap A_j = \varnothing$ for $i \neq j$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$.

It can be easily verified that

Proposition 2.1.4. *A π-system that is also a λ-system is a σ-algebra.*

Proof. Let $\mathcal{A}$ be a λ-system that is also closed under intersection. The first two points of the definition of a σ-algebra are trivially satisfied, so what remains to check is point (iii). Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{A}$. We will construct a pairwise disjoint sequence $\{B_n\}_{n=1}^{\infty}$ as

$$B_n = A_n \setminus \bigcup_{m<n} A_n = A_n \cap \left(\bigcap_{m<n} A_m^{\mathsf{c}} \right)$$

$B_n \in \mathcal{A}$ since $\mathcal{A}$ is closed under intersections and complements. Since B_n are pairwise disjoint, $\cup B_n \in \mathcal{A}$. Given that $\cup B_n = \cup A_n$, the result follows. ■

The smallest σ-algebra containing some set A, called the **σ-algebra generated by** A, is denoted $\sigma(A)$. For proof of existence of $\sigma(A)$ see Corollary 1.1.3 of [Coh13]. Similarly, there exists a smallest λ-system containing A, or generated by A, denoted $\ell(A)$.

Theorem 2.1.5 (Dynkin $\boldsymbol{\pi - \lambda}$ Theorem). *Suppose $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system such that $\mathcal{P} \subseteq \mathcal{L}$. Then, $\sigma(\mathcal{P}) \subseteq \mathcal{L}$. In the case that $\mathcal{L} = \ell(\mathcal{P})$, the λ-system generated by $\mathcal{P}$, we have that $\ell(\mathcal{P}) = \sigma(\mathcal{P})$.*

Proof. Observe that $\ell(\mathcal{P}) \subseteq \sigma(\mathcal{P})$, since σ-algebras are λ-systems. To prove the reverse inclusion, we show that $\ell(\mathcal{P})$ is a σ-algebra. Since $\sigma(\mathcal{P})$ is the smallest σ-algebra, it will then be contained in $\ell(\mathcal{P})$. To do this, by Proposition 2.1.4, we need to show that $\ell(\mathcal{P})$ is closed under intersection. With this goal in mind, set

$$\mathcal{G}_1 = \{A \in \ell(\mathcal{P}) : A \cap B \in \ell(\mathcal{P}), B \in \mathcal{P}\}.$$

We show that $\mathcal{G}_1$ is a λ-system. We check every point of the definition:

(i) $\Omega \in \mathcal{G}_1$ because $\ell(\mathcal{P})$ is a λ-system.

(ii) Let $A_1, A_2 \in \mathcal{G}_1$ with $A_1 \subseteq A_2$. Then write $(A_2 \setminus A_1) \cap B = (A_2 \cap B) \setminus (A_1 \cap B) \in \ell(\mathcal{P})$, since $A_1 \cap B, A_2 \cap B \in \ell(\mathcal{P})$.

(iii) Let $\{A_n\}_{n=1}^{\infty} \in \mathcal{G}_1$ such that $A_i \cap A_j = \varnothing$ whenever $i \neq j$. Then $B \cap A_n \in \ell(\mathcal{P})$ and $(B \cap A_i) \cap (B \cap A_j) = \varnothing$ whenever $i \neq j$. Thus $B \cap (\cup_n A_n) = \cup_n (B \cap A_n) \in \ell(\mathcal{P})$.

Thus, $\mathcal{G}_1$ is a λ-system. We have that $\mathcal{P} \subseteq \mathcal{G}_1$ because $\mathcal{P} \subseteq \ell(\mathcal{P})$. We then get that $\ell(\mathcal{P}) \subseteq \mathcal{G}_1$, since $\ell(\mathcal{P})$ is the smallest λ-system containing $\mathcal{P}$. But by construction, $\mathcal{G}_1$ is at most all of $\ell(\mathcal{P})$, so we have that $\ell(\mathcal{P}) = \mathcal{G}_1$. We now lift this argument to show $\ell(\mathcal{P})$ is closed under intersection. Set

$$\mathcal{G}_2 = \{A \in \ell(\mathcal{P}) : A \cap B \in \ell(\mathcal{P}), B \in \ell(\mathcal{P})\}.$$

As above, one can show that $\mathcal{G}_2$ is a λ-system. Obviously, $\mathcal{P} \subseteq \mathcal{G}_2$ and $\mathcal{G}_2 \subseteq \ell(\mathcal{P})$. This implies that $\mathcal{G}_2 = \ell(\mathcal{P})$, since $\mathcal{G}_1 \subseteq \mathcal{G}_2$, and by the above, $\ell(\mathcal{P}) = \mathcal{G}_1$. Thus $\ell(\mathcal{P})$ is a σ-algebra, and $\ell(\mathcal{P}) = \sigma(\mathcal{P})$. Obviously, if $\mathcal{L}$ is a bigger λ-system containing $\mathcal{P}$, $\sigma(\mathcal{P}) \subseteq \mathcal{L}$. ■

We will need the following definition for much of the following work

Definition 2.1.6. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be measurable spaces. A function $f : X \to Y$ is called **measurable** if for every E in $\mathcal{F}_Y$, the preimage of E is in $\mathcal{F}_X$, that is

$$\{x \in X : f(x) \in E\} \in \mathcal{F}_X.$$

If the target space is $\mathbb{R}$, we will often implicitly assume that the σ-algebra is the Borel σ-algebra.

Theorem 2.1.7 (**Monotone Class Theorem**). *Let $\mathcal{A}$ be a π-system that contains Ω and let $\mathcal{H}$ be a collection of real-valued functions on Ω that satisfies:*

(i) If $A \in \mathcal{A}$, then $\chi_A \in \mathcal{H}$.

(ii) If $f, g \in \mathcal{H}$, then $f + g$, and $cf \in \mathcal{H}$ for all real c.

(iii) If $f_n \in \mathcal{H}$ are non-negative and increase to a bounded function f, then $f \in \mathcal{H}$.

Then $\mathcal{H}$ contains all bounded functions measurable with respect to $\sigma(\mathcal{A})$.

Proof. Consider the collection $\mathcal{G} = \{A : \chi_A \in \mathcal{H}\}$. With the goal of applying the π-λ theorem, we show that $\mathcal{G}$ is a λ-system. By assumption, $\Omega \in \mathcal{A}$, and so by (i) $\chi_\Omega \in \mathcal{H}$ and hence $\Omega \in \mathcal{G}$. Next, assume that $A, B \in \mathcal{G}$ with $A \subset B$. We show that $\chi_{B \setminus A} \in \mathcal{H}$ given that $\chi_A, \chi_B \in \mathcal{H}$. Write

$$\chi_{B\setminus A} = \chi_B - \chi_{A\cap B} = \chi_B - \chi_A \in \mathcal{H}$$

since $A \cap B = A$ and by (ii) $\mathcal{H}$ is closed under addition and scaling by real numbers. Finally, consider the sequence $A_n \in \mathcal{G}$ such that $A_i \cap A_j = \varnothing$ whenever $i \neq j$. This is equivalent to $\chi_{A_n} \in \mathcal{H}$ with $\chi_{A_i \cap A_j} = 0$ when $i \neq j$. Since the A_n are pairwise disjoint, $\chi_{\cup_n A_n} = \sum_n \chi_{A_n} \in \mathcal{H}$. Thus, $\mathcal{G}$ is a λ-system and by the π-λ theorem, $\mathcal{G} \supset \sigma(\mathcal{A})$. By (ii), $\mathcal{H}$ contains all simple functions, and by (iii) it contains bounded functions. Thus $\mathcal{H}$ contains all bounded functions measurable with respect to $\sigma(\mathcal{A})$. ■

2.2 Stochastic Processes and Filtrations

This section gives a very brief overview of the basic definitions, but is by no means a complete account of the topic. For more details, the reader should refer to, for example, [Wen81].

Definition 2.2.1. Given a probability space $(\Omega, \mathcal{F}, P)$ and a measurable space (S, Σ), a **stochastic process** is a collection of S-valued random variables $\{X_t(\omega) : t \in T\}$ for some index set T.

The index set T is generally thought of as time, and we will deal with $T = [0, 1]$ and $T = [0, \infty)$.

Definition 2.2.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(T, \leq)$ some totally ordered index set, thought of as time. Consider σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ for every $t \in T$. Then $\{\mathcal{F}_t\}_{t\in T}$ is a **filtration** if $\mathcal{F}_k \subseteq \mathcal{F}_\ell$ for all $k \leq \ell$. The **filtered probability space** is then denoted $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, P)$, or $(\Omega, \{\mathcal{F}_t\}_{t\in T}, P)$ for brevity.

Definition 2.2.3. A process X_t is called **adapted** to the filtration $\{\mathcal{F}_t\}$ if X_t is measurable with respect to $\mathcal{F}_t$ for all t in T.

Definition 2.2.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T = [0, \infty)$. Let $X : T \times \Omega \to \mathbb{R}$ be a stochastic process. Then the **natural filtration** of $\mathcal{F}$ with respect to X is defined to be the filtration $\{\mathcal{F}_t^X\}$ with

$$\mathcal{F}_t^X = \sigma\big(\{X_s^{-1}(B) : s \in \mathbb{Q}_2, s \leq t, B \in \mathfrak{B}(\mathbb{R})\}\big)$$

where $\mathbb{Q}_2$ are the dyadic rationals $\{m2^{-n} : m, n \in \mathbb{Z}^+\}$. In other words, it is the collection of σ-algebras generated by the preimages of $\mathcal{F}$-measurable subsets of $\mathbb{R}$ for dyadic times up to t. Any stochastic process is adapted with respect to its natural filtration.

Remark. All filtrations will be natural filtrations, unless otherwise stated, so the superscript of the process in question is dropped from notation for brevity. Also note that we will generally take the σ-algebras to be complete, i.e. containing all events of measure zero and one, and events contained in them.

Chapter 3

Construction and Properties of Brownian Motion

In this chapter we show some basic properties of Brownian motion like translation invariance and Brownian scaling. We show that a process satisfying the conditions of Brownian motion exists, by first using Kolmogorov's existence theorem to construct pre-Brownian motion based on properties that can be captured in finite dimensional distributions, and then using Kolmogorov's continuity theorem to construct a continuous version of pre-Brownian motion, thus giving us the process of interest. We also show that on compact time intervals Brownian motion paths are Hölder continuous with Hölder exponents less than $1/2$.

In the second section we bring up some important properties of the sample space, and prove a result about the equivalence of two σ-algebras of the sample space.

Proofs in this section generally follow [Coh13], [Ste21]. Discussion of Kolmogorov's existence theorem and consistency of measures follows [Bil86].

3.1 Construction and Properties

Definition 3.1.1. A **Brownian motion** is a family of random variables $\{B_t\}_{t\geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ satisfying

(i) $B_0(\omega) = 0$, for all $\omega \in \Omega$,

(ii) If $0 = t_0 < t_1 < t_2 < \cdots < t_k$ then $B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}$ are independent,

(iii) for all times s, t the random variable $B_{s+t} - B_s \in \mathrm{N}(0, t)$,

(iv) for all $\omega \in \Omega$, the path $t \mapsto B_t(\omega)$ is continuous in t.

We define Brownian motion to start at 0, but this is not a necessary condition and is done for convenience. It turns out that Brownian motion starting at any point can be translated to start at 0 without losing any properties.

Proposition 3.1.2 (**Translation Invariance**). *Assume B_0 is any real number, not necessarily 0. The event $\{B_t - B_0, t \geq 0\}$ is independent of B_0 and has the same finite dimensional distributions as a Brownian motion starting at 0.*

Proof. See page 306 of [Dur19]. ■

Proposition 3.1.3 (**Brownian Scaling**). *Suppose $\{B_t\}_{t\geq 0}$ is a Brownian motion. Then if $c > 0$, $\{c^{-1/2}B_{ct}\}_{t\geq 0}$ is a Brownian motion, i.e. $\{B_t\}_{t\geq 0} \stackrel{d}{=} \{c^{-1/2}B_{ct}\}_{t\geq 0}$.*

Proof. The first condition is obviously satisfied. To prove the second, suppose we have an increasing sequence of times $t_0 < t_1 < \cdots < t_n$. Then the scaled sequence $\{ct_i\}_{i=1}^n$ is also increasing and thus, on these new times, $B_{ct_i} - B_{ct_{i-1}}$ are independent. To see that the difference of the scaled random variables has the right distribution write

$$\frac{B_{c(t+s)}}{\sqrt{c}} - \frac{B_{cs}}{\sqrt{c}} \stackrel{d}{=} \frac{1}{\sqrt{c}}\,\mathrm{N}(0, ct) \stackrel{d}{=} \mathrm{N}(0, t)$$

The scaled paths are continuous since a composition of continuous functions is continuous. ■

To begin with the construction of Brownian motion, we introduce some new concepts.

Definition 3.1.4. Suppose $X_t(\omega)$ is a stochastic process taking values in $\mathbb{R}$ and there is a sequence of times $0 = t_0 < t_1 < t_2 < \cdots < t_k$. We denote paths $t \mapsto X_t(\omega)$ as $t \mapsto \mathsf{p}(t)$. Sets of the form

$$\{\mathsf{p} : (\mathsf{p}(t_1), \mathsf{p}(t_2), \ldots, \mathsf{p}(t_k)) \in A\}$$

with $A \in \mathfrak{B}(\mathbb{R}^k)$, are called **cylindrical sets**.

On these sets we define the following

Definition 3.1.5. Finite dimensional distributions are probability measures given by

$$\mu_{t_1,\ldots,t_k}(A) = P(\{\mathsf{p} : (\mathsf{p}(t_1), \mathsf{p}(t_2), \ldots, \mathsf{p}(t_k)) \in A\})$$

for Borel A.

For discrete processes, properties are completely determined by finite dimensional distributions. Unfortunately, no such statement can be made about continuous processes. In construction of Brownian motion we will use Kolmogorov's existence theorem, which will give us a process satisfying conditions (i)-(iii). We will then construct a modification of this process that is also continuous.

The definition of finite dimensional distributions implies two consistency properties. The first is invariance under permutations, that is if π is a permutation of $\{1, 2, \ldots, k\}$ then

$$\mu_{t_1,\ldots,t_k}(A_1 \times \cdots \times A_k) = \mu_{t_{\pi 1},\ldots,t_{\pi k}}(A_{\pi 1} \times \cdots \times A_{\pi k}). \tag{3.1}$$

This follows because $(\mathsf{p}(t_1), \ldots, \mathsf{p}(t_k)) \in (A_1 \times \cdots \times A_k)$ is the same event as $(\mathsf{p}(t_{\pi 1}), \ldots, \mathsf{p}(t_{\pi k})) \in (A_{\pi 1} \times \cdots \times A_{\pi k})$. The second is that

$$\mu_{t_1,\ldots,t_{k-1}}(A_1 \times \cdots \times A_{k-1}) = \mu_{t_1,\ldots,t_{k-1},t_k}(A_1 \times \cdots \times A_{k-1} \times \mathbb{R}), \tag{3.2}$$

which follows because $(\mathsf{p}(t_1), \ldots, \mathsf{p}(t_{k-1})) \in (A_1 \times \cdots \times A_{k-1})$ if and only if $(\mathsf{p}(t_1), \ldots, \mathsf{p}(t_{k-1}), \mathsf{p}(t_k)) \in (A_1 \times \cdots \times A_{k-1} \times \mathbb{R})$. Finite dimensional distributions necessarily satisfy (3.1) and (3.2). The following theorem states the converse – if a collection of measures satisfies these consistency conditions, there must be a process with these measures as finite dimensional distributions.

Theorem 3.1.6 (**Kolmogorov's Existence Theorem**). *If $\mu_{t_1,\ldots,t_k}$ is a collection of measures satisfying* (3.1) *and* (3.2)*, then on some probability space* $(\Omega, \mathcal{F}, P)$ *there exists a stochastic process X_t with $\mu_{t_1,\ldots,t_k}$ as its finite dimensional distributions.*

For many processes, including Brownian motion, it is natural to define finite dimensional distributions only for increasing sequences of times $0 = t_0 < t_1 < t_2 < \cdots < t_k$. In such cases the following consistency condition is easier to work with

Proposition 3.1.7. *Suppose that measures $\mu_{t_1,\ldots,t_k}$ satisfy the consistency condition*

$$\begin{aligned}\mu_{s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_k}&(H_1 \times \cdots \times H_{i-1} \times H_{i+1} \times \cdots \times H_k)\\ &= \mu_{s_1,\ldots,s_k}(H_1 \times \cdots \times H_{i-1} \times \mathbb{R} \times H_{i+1} \times \cdots \times H_k),\end{aligned} \tag{3.3}$$

where times are such that $0 = t_0 < t_1 < t_2 < \cdots < t_k$. Then $\mu_{t_1,\ldots,t_k}$ satisfies consistency conditions (3.1) *and* (3.2).

Proof. Suppose that $\mu_{t_1,\ldots,t_k}$ is such that (3.3) holds. Given times $s_1 < \ldots < s_k$, let $(X_{s_1}, \ldots, X_{s_k})$ have the distribution $\mu_{s_1,\ldots,s_k}$. Suppose that $t_1, \ldots, t_k$ is a permutation of $s_1, \ldots, s_k$. Take $\mu_{t_1,\ldots,t_k}$ to be the distribution of $(X_{t_1}, \ldots, X_{t_k})$ defined as

$$\mu_{t_1,\ldots,t_k}(A_1 \times \cdots \times A_k) = P(\{X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k\}) \tag{3.4}$$

If $t_{\pi 1}, \ldots, t_{\pi k}$ is a permutation of $t_1, \ldots, t_k$, then it must also be a permutation of $s_1, \ldots, s_k$, and so by (3.4) we have that $\mu_{t_{\pi 1}, \ldots, t_{\pi k}}$ is the distribution of $(X_{t_{\pi 1}}, \ldots, X_{t_{\pi k}})$ which gives the consistency condition (3.1). Because of condition (3.3) we have that $\mu_{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k}$ is the distribution of $(X_{s_1}, \ldots, X_{s_{i-1}}, X_{s_{i+1}}, \ldots, X_{s_k})$. If we suppose that $t_k = s_i$, then $t_1, \ldots, t_{k-1}$ must be a permutation of $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k$, which are times increasing in order. By applying (3.4) to $t_1, \ldots, t_{k-1}$ we see that $\mu_{t_1, \ldots, t_{k-1}}$ is the distribution of $(X_{t_1}, \ldots, X_{t_{k-1}})$, which gives (3.2), the second consistency condition. ■

To show that the Brownian motion process B_t exists, we proceed by constructing a process that satisfies the necessary properties. Using the Kolmogorov existence theorem we can construct a "pre-Brownian motion" $\tilde{B}_t$ – a process satisfying all the conditions, but that of continuity. We begin by constructing finite dimensional distributions of a process with conditions (i)–(iii). Assuming a process taking values in $\mathbb{R}$, construct on $\mathbb{R}^k$ the following measures

$$\mu_{t_1, \ldots, t_k}(A_1, \ldots, A_k) = \prod_{i=1}^{k} \int_{A_i} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left\{-\frac{1}{2}\frac{(x_i - x_{i-1})^2}{(t_i - t_{i-1})}\right\} dx_i, \tag{3.5}$$

where $x_0 = t_0 = 0$. Since Brownian motion has independent increments, the probability of the event $\{(B_{t_1}, B_{t_2 - t_1}, \ldots, B_{t_k - t_{k-1}}) \in (A_1 \times \ldots \times A_k)\}$ is exactly (3.5). To see that these measures are consistent, think of $\mu_{t_1, \ldots, t_k}$ as the distribution of $(S_1, \ldots, S_k)$, where $S_i = \sum_{j=1}^{i} Y_j$ and the Y_j are independent $\mathrm{N}(0, t_i - t_{i-1})$ distributed, with $t_0 = 0$. If we let $g(x_1, \ldots, x_k) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$, then $g(S_1, \ldots, S_k) = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k)$ has distribution given by $\mu_{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k}$, because the sum $Y_i + Y_{i+1}$ is $\mathrm{N}(0, t_{i+1} - t_{i-1})$ distributed. Hence, we get that $\mu_{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k} = \mu_{t_1, \ldots, t_k} \cdot g^{-1}$, which is equivalent to condition (3.3). Thus, by Kolmogorov's existence theorem we have that there is some process $\tilde{B}_t$ with the finite dimensional distributions of Brownian motion.

To get Brownian motion B_t out of pre-Brownian motion $\tilde{B}_t$, we will use Kolmogorov's continuity theorem, which will give us both condition (iv) and other very appealing properties. To state and prove the theorem, we need to give some definitions.

Definition 3.1.8. A **version** of a process $\{X_t\}$ is a process $\{Y_t\}$ such that for all t it holds that $P(X_t = Y_t) = 1$.

Remark. Note that this does not imply in particular that $P(X_t = Y_t, \forall t) = 1$

Definition 3.1.9. A real-valued function f is called **α-Hölder continuous** if there are non-negative real constants C and α such that for all x, y in the domain of f

$$|f(x) - f(y)| \leq C||x - y||^{\alpha}$$

Theorem 3.1.10. *Suppose f is an α-Hölder continuous function on a subset X that is dense in $\bar{X}$. Then there exists an α-Hölder extension F to $\bar{X}$ such that $F(x) = f(x)$ for every x in X.*

Theorem 3.1.11 (Kolmogorov Continuity Theorem). *Let $\tilde{X} : [0,1] \times \Omega \to \mathbb{R}$ be a stochastic process with some underlying probability space $(\Omega, \mathcal{F}, P)$. Suppose that there exist positive constants α, β, C such that*

$$E\{|\tilde{X}_t - \tilde{X}_s|^\alpha\} \leq C|t-s|^{1+\beta}$$

Then there exists a version X_t of the process $\tilde{X}_t$ with continuous paths. Furthermore, the paths of X_t are γ-Hölder continuous for $\gamma < {}^\beta/_\alpha$.

The main step of the proof is to show that there exists $\Omega^* \subseteq \Omega$ such that $\Omega^* \in \mathcal{F}$ and for $\omega \in \Omega^*$ the process $X_t(\omega)$ is γ-Hölder continuous on the dyadic rationals $\mathbb{Q}_2$ whenever $\gamma < {}^\beta/_\alpha$. It is useful to know that γ-Hölder continuity for any $\gamma > 0$ in particular implies uniform continuity.

Proof. Fix $\gamma < {}^\beta/_\alpha$. We will consider the events

$$A_n = \left\{ \max_{1 \leq k \leq 2^n} \left| \tilde{X}\left(\frac{k}{2^n}\right) - \tilde{X}\left(\frac{k-1}{2^n}\right) \right| \geq \frac{1}{2^{n\gamma}} \right\}$$

We are interested in finding a suitable bound on $P(A_n)$, which would be a step in showing that big differences in process values don't happen very often. To this end, we use the union bound and Markov's inequality as follows

$$\begin{aligned}
P(A_n) &\leq 2^n P\left\{ \left| \tilde{X}\left(\frac{k}{2^n}\right) - \tilde{X}\left(\frac{k-1}{2^n}\right) \right| \geq \frac{1}{2^{n\gamma}} \right\} \\
&= 2^n P\left\{ \left| \tilde{X}\left(\frac{k}{2^n}\right) - \tilde{X}\left(\frac{k-1}{2^n}\right) \right|^\alpha \geq \frac{1}{2^{n\gamma\alpha}} \right\} \\
&\leq 2^n \cdot 2^{n\gamma\alpha} E\left\{ \left| \tilde{X}\left(\frac{k}{2^n}\right) - \tilde{X}\left(\frac{k-1}{2^n}\right) \right|^\alpha \right\} \\
&\leq 2^n \cdot 2^{n\gamma\alpha} C \left| \frac{1}{2^n} \right|^{1+\beta} \\
&= C \left| \frac{1}{2^n} \right|^{\beta - \gamma\alpha}
\end{aligned}$$

As an intermediary step we want to show that whenever $\omega \in B_N = \cap_{n=N}^\infty A_n^\mathsf{c}$, we have that

$$|\tilde{X}(q) - \tilde{X}(r)| \leq \frac{3}{1-2^{-\gamma}} |q-r|^\gamma$$

whenever $q, r \in \mathbb{Q}_2 \cap [0,1]$ and $|q-r| < 2^{-N}$. Suppose that $r > q$ and consider the intervals $I_i^k = [(i-1)2^{-k}, i2^{-k}]$. Suppose m is the smallest k so that q and r are in different intervals. Then,

$q \in I_i^m$ and $r \in I_{i+1}^m$ for some i, since both q and r were in the previous $k-1$ 'level' of the interval. Then there exist some increasing sequences of integers $\{r(h)\}_{h=1}^{\ell}$ and $\{q(h)\}_{h=1}^{k}$ such that we can express

$$r = i2^{-m} + \sum_{h=1}^{\ell} 2^{-r(h)}$$
$$q = i2^{-m} - \sum_{h=1}^{k} 2^{-q(h)}.$$

Using the upper bound from event A_n^{c} we can write the following

$$\left|\tilde{X}(q) - \tilde{X}\left(\frac{i-1}{2^m}\right)\right| \leq 2^{-m\gamma} + \sum_{h=1}^{k} (2^{-\gamma})^{q(h)} \leq \sum_{h=m}^{\infty} (2^{-\gamma})^h$$
$$= \frac{2^{-m\gamma}}{1-2^{-\gamma}}$$

$$\left|\tilde{X}(r) - \tilde{X}\left(\frac{i}{2^m}\right)\right| \leq \frac{2^{-m\gamma}}{1-2^{-\gamma}}$$

To prove the intermediary step, note that $2^{-m} \leq |q-r|$ and write

$$\begin{aligned}
|\tilde{X}(q) - \tilde{X}(r)| &\leq \left|\tilde{X}\left(\frac{i}{2^m}\right) - \tilde{X}\left(\frac{i-1}{2^m}\right)\right| + \left|\tilde{X}(q) - \tilde{X}\left(\frac{i-1}{2^m}\right)\right| + \left|\tilde{X}(r) - \tilde{X}\left(\frac{i}{2^m}\right)\right| \\
&\leq 2^{-m\gamma} + \frac{2^{-m\gamma}}{1-2^{-\gamma}} + \frac{2^{-m\gamma}}{1-2^{-\gamma}} \\
&\leq (1 + \frac{2}{1-2^{-\gamma}})|q-r|^{\gamma} \\
&\leq \frac{3}{1-2^{-\gamma}}|q-r|^{\gamma}.
\end{aligned}$$

We now want to show that $P(B_N^{\mathsf{c}} \text{ i.o.}) = 0$. With the goal of applying the Borel-Cantelli Lemma, note that

$$P(B_N^{\mathsf{c}}) = P\big((\cap_{n=N}^{\infty} A_n^{\mathsf{c}})^{\mathsf{c}}\big) \leq P(\cup_{n=N}^{\infty} A_n) \leq \sum_{n=N}^{\infty} P(A_n) \leq C \sum_{n=N}^{\infty} \left(\frac{1}{2^{\beta-\gamma\alpha}}\right)^n$$
$$= \frac{C}{(2^{\beta-\gamma\alpha}-1)2^{(\beta-\gamma\alpha)(N-1)}},$$

and that we have

$$\sum_{N=1}^{\infty} P(B_N^{\mathsf{c}}) = \frac{C}{2^{\beta-\gamma\alpha}-1} \sum_{N=0}^{\infty} \left(\frac{1}{2^{\beta-\gamma\alpha}}\right)^N < \infty.$$

Thus by the Borel-Cantelli lemma we have that there exists $\Omega^* = \Omega \setminus \{B_N^{\mathsf{c}} \text{ i.o.}\} \in \mathcal{F}$ such that $P(\Omega^*) = 1$ and for all $\omega \in \Omega^*$ we have that

$$|\tilde{X}(q) - \tilde{X}(r)| \leq \frac{3}{1-2^{-\gamma}}|q-r|^{\gamma} \quad \text{for } q, r \in \mathbb{Q}_2 \cap [0,1], |q-r| < \delta(\omega),$$

where $\delta(\omega)$ must depend on ω since we can't get a single δ for all paths. To remove the restriction that $|q-r|<\delta$, partition along dyadic rationals $s_0 = q < s_1 < \cdots < s_n = r$ where $|s_i - s_{i-1}| < \delta$. Then we have that

$$\begin{aligned}
|\tilde{X}(q) - \tilde{X}(r)| &= |\tilde{X}(s_0) - \tilde{X}(s_1) + \tilde{X}(s_1) - \tilde{X}(s_2) + \ldots + \tilde{X}(s_{i-1}) - \tilde{X}(s_i)| \\
&\leq \sum_{i=1}^{n} |\tilde{X}(s_{i-1}) - \tilde{X}(s_i)| \\
&\leq \sum_{i=1}^{n} \frac{3}{1-2^{-\gamma}} |s_i - s_{i-1}|^\gamma
\end{aligned}$$

Note that

$$|q-r|^\gamma = \left| \sum_{i=1}^{n} s_{i-1} - s_i \right|^\gamma = \left(\sum_{i=1}^{n} |s_{i-1} - s_i| \right)^\gamma \geq \sum_{i=1}^{n} |s_{i-1} - s_i|^\gamma.$$

Thus we have that with probability 1

$$|\tilde{X}(q) - \tilde{X}(r)| \leq C(\omega)|q-r|^\gamma$$

for $q, r \in \mathbb{Q}_2 \cap [0,1]$.

We can now define the continuous version of the process $\tilde{X}_t$ as

$$X_t(\omega) = \begin{cases} \lim_{\substack{s \to t \\ s \in \mathbb{Q}_2}} \tilde{X}_s(\omega), & \omega \in \Omega^* \\ 0, & w \notin \Omega^*. \end{cases}$$

This extension exists by Theorem (3.1.10), since X_t is γ-Hölder continuous on the dyadic rationals which are dense in $\mathbb{R}$. To show that X_t is indeed a version of $\tilde{X}_t$, we verify that $P(X_t = \tilde{X}_t) = 1$. Obviously for the case that $t \in \mathbb{Q}_2 \cap [0,1]$, the processes are equal, so the condition is trivially satisfied. For the cases that t is not a dyadic rational, we begin by using the well-known result that almost sure convergence implies convergence in probability, so we have that $\lim_{\substack{s \to t \\ s \in \mathbb{Q}_2}} \tilde{X}_s \to X_t$ in probability. To obtain the result, we show that $\lim_{\substack{s \to t \\ s \in \mathbb{Q}_2}} \tilde{X}_s \to \tilde{X}_t$ in probability using Markov's inequality and given expectation bounds

$$P\big(|\tilde{X}_s - \tilde{X}_t| \geq \varepsilon\big) = P\big(|\tilde{X}_s - \tilde{X}_t|^\alpha \geq \varepsilon^\alpha\big) \leq \varepsilon^{-\alpha} E|\tilde{X}_s - \tilde{X}_t|^\alpha \leq \varepsilon^{-\alpha} C|s-t|^{1+\beta}$$

Thus X_t is a continuous version of $\tilde{X}_t$. ■

Corollary 3.1.12. *Brownian motion B_t exists and has α-Hölder continuous paths on compact time intervals for $\alpha < 1/2$.*

Proof. Recall that we have constructed a process $\tilde{B}_t$ which satisfies all conditions of Brownian motion but that of continuity. We show that there exist constants α, β, C as in the theorem above, for which the expectation bound is satisfied. Pick $\alpha = 2m$ and $\beta = m - 1$. We have that

$$E(|\tilde{B}_t - \tilde{B}_s|^{2m}) = E\left((t-s)^m \left|\frac{\tilde{B}_t - \tilde{B}_s}{\sqrt{t-s}}\right|^{2m}\right) = (t-s)^m E\left|\frac{\tilde{B}_t - \tilde{B}_s}{\sqrt{t-s}}\right|^{2m}.$$

Note that $(\tilde{B}_t - \tilde{B}_s)/\sqrt{t-s}$ is a standard normal variable and thus all the central moments exist. $\lim_{m\to\infty} {}^{\beta}/_{\alpha} = 1/2$, so we get that Brownian motion paths are Hölder continuous for times in $[0,1]$ for exponents up to $^1/_2$. By Brownian scaling we can get the result for any compact time interval. ■

3.2 Properties of the Sample Space

By Brownian scaling we can now extend Brownian motion from running on times $[0,1]$ to $[0,\infty)$. Note that our probability measures are on the measure space $(\mathcal{P}, \mathcal{C})$, where $\mathcal{P} = \{\text{continuous } \mathsf{p} : [0,\infty) \to \mathbb{R},\ \mathsf{p}(0) = 0\}$ and $\mathcal{C}$ is the σ-algebra generated by the cylindrical sets.

We can also construct a Brownian motion in $\mathbb{R}^d$ as a vector of independent one-dimensional Brownian motions. We then have that $\mathcal{P} = \{\text{continuous } \mathsf{p} : [0,\infty) \to \mathbb{R}^d,\ \mathsf{p}(0) = 0\}$ and $\mathcal{C}$ is once again generated by the cylindrical sets. From now on we will under B_t understand the Brownian motion starting at zero and taking values in $\mathbb{R}^d$.

Set

$$d_n(\mathsf{p}, \mathsf{p}') = \sup_{0\le t\le n} |\mathsf{p}(t) - \mathsf{p}'(t)|$$

We endow the space $\mathcal{P}$ with the metric d

$$d(\mathsf{p}, \mathsf{p}') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(\mathsf{p}, \mathsf{p}')}{1 + d_n(\mathsf{p}, \mathsf{p}')}$$

There are many equivalent metrics that can be chosen. If we were dealing with compact time intervals, i.e. paths in the space $C([0,1])$, the usual supremum norm would be appropriate. In fact, one can check that convergence with respect to d is equivalent to convergence with respect to the supremum norm on compact subsets of $[0,\infty)$.

$\mathcal{P}$ is complete with respect to d, and is separable because polynomials with rational coefficients form a countable dense subset in $\mathcal{P}$. Because of this, $\mathfrak{B}(\mathcal{P})$, the Borel sets of $\mathcal{P}$ are generated by the open balls in $\mathcal{P}$. We also have the following result

Lemma 3.2.1. *The σ-algebra $\mathcal{C}$ is the same as the σ-algebra $\mathfrak{B}(\mathcal{P})$.*

Proof. See Lemma 2.1 in Chapter 6 of [Ste11]. ■

Chapter 4

Strong Markov Property

In this section we discuss Blumenthal's zero-one law and various formulations of the strong Markov property, the latter being a key part in proving the main result of the thesis found in Chapter 5.

The proofs in this section follow [Ste11].

4.1 Stopping Times and Blumenthal's 0-1 Law

Definition 4.1.1. Given a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$, a **stopping time** is a random variable σ taking values in $[0,\infty]$, such that $\{\omega : \sigma(\omega) \leq t\} \in \mathcal{F}_t$ for all times t.

In the coming sections we will deal with **exit times** of some bounded open set $G \subset \mathbb{R}^d$, which are random variables defined as

$$\tau^x(\omega) = \inf\{t \geq 0 : B_t^x(\omega) \notin G\}$$
$$\tau_*^x(\omega) = \inf\{t > 0 : B_t^x(\omega) \notin G\}.$$

Both exit times are well-defined because almost all paths will eventually leave any ball, so the random variables are finite almost everywhere.

To show that the exit times above are stopping times, we will need the lemma below. Before we state and prove it, we need to introduce a new concept.

Definition 4.1.2. A **right-continuous filtration** corresponding to $\{\mathcal{F}_t\}$ is a filtration defined as $\{\mathcal{F}_t^+\}$ with $\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s$. The name is justified by noticing that

$$\bigcap_{s>t} \mathcal{F}_s^+ = \bigcap_{s>t} \left(\bigcap_{h>s} \mathcal{F}_h \right) = \bigcap_{h>t} \mathcal{F}_h = \mathcal{F}_t^+.$$

Also note that all σ-algebras are assumed to be complete, that is containing all events of probability zero and one, and that $\mathcal{F}_t$ is assumed to be the natural σ-algebra.

Lemma 4.1.3 (Blumenthal's Zero-One Law). *$\mathcal{F}_0 = \mathcal{F}_0^+$, and for any $A \in \mathcal{F}_0^+$, $P(A)$ is 0 or 1.*

Proof. We fix an arbitrary measurable bounded continuous function f on $\mathbb{R}^{kd}$ together with a sequence of increasing times $0 \leq t_1 < t_s < \cdots < t_k$. For any $\delta > 0$ we set

$$f_\delta = f(B_{t_1+\delta} - B_\delta, B_{t_2+\delta} - B_{t_1+\delta}, \ldots, B_{t_k+\delta} - B_{t_{k-1}+\delta}).$$

If $A \in \mathcal{F}_0^+$, then $A \in \mathcal{F}_\delta$ for $\delta > 0$, giving us $\mathcal{F}_0^+ \subseteq \mathcal{F}_\delta$. By the independence of increments above from B_δ, we have

$$\int_A f_\delta \, dP = P(A) \int_\Omega f_\delta \, dP\,.$$

Since the paths are continuous we can let $\delta \to 0$,

$$\int_A f_0 \, dP = P(A) \int_\Omega f_0 \, dP\,.$$

Any bounded continuous function can be written in the form $g(x_1, \ldots, x_k) = f(x_1, x_2 - x_1, \ldots, x_k - x_{k-1})$. From this we have

$$\int_A g(B_{t_1}, \ldots, B_{t_k}) \, dP = P(A) \int_\Omega g(B_{t_1}, \ldots, B_{t_k}) \, dP\,.$$

This equality holds for g that are characteristic functions of cylindrical sets. Thus, we have $P(A \cap E) = P(A)P(E)$ whenever E is a cylindrical set. By the π-λ theorem we have that this extends to all Borel E. As a consequence we have that $P(A) = P(A)^2$, giving us that $P(A) = 0$ or $P(A) = 1$. Note that since at time 0 the value of B_0 is deterministic, $\mathcal{F}_0 = \sigma\big(\{B_0^{-1}(A) : A \in \mathfrak{B}(\mathbb{R})\}\big)$, and is thus equal to the trivial σ-algebra $\{\varnothing, \Omega\}$ up to null sets. This gives us that $\mathcal{F}_0 = \mathcal{F}_0^+$, since both are complete. ∎

Proposition 4.1.4. *Exit times τ^x and τ_*^x are stopping times.*

Proof. By translation invariance of Brownian motion, we can assume that $x = 0$. We begin by defining for any open set $\mathcal{O} \subset \mathbb{R}^d$ the "entrance time" $\tau_{\mathcal{O}} = \inf\{t \geq 0 : B_t(\omega) \in \mathcal{O}\}$. Up to a set of measure zero we have that

$$\{\tau_{\mathcal{O}}(\omega) < t\} = \bigcup_{r<t} \{B_r(\omega) \in \mathcal{O}\},$$

where the union is taken over rational r. This holds because by continuity, a path is in $\mathcal{O}$ before time t, if and only if it is in $\mathcal{O}$ at some rational times $r < t$. This gives us that $\{\tau_{\mathcal{O}}(\omega) < t\} \in \mathcal{F}_t$.

Now consider regions near the boundary of G defined as $\mathcal{O}_n = \{x : d(x, G^{\mathsf{c}}) < 1/n\}$. For $t > 0$ we have that

$$\{\tau(\omega) \leq t\} = \bigcap_{n=1}^{\infty} \{\tau_{\mathcal{O}_n}(\omega) < t.\}$$

This holds because a path exits G by time t if and only if it is in $\mathcal{O}_n$ before time t. Hence, for $t > 0$, $\{\tau(\omega) \leq t\} \in \mathcal{F}_t$. As for the event $\{\tau(\omega) = 0\}$, note that if $x \in G$, this event is the empty set $\varnothing$, and if $x \notin G$, it is Ω. Thus τ is a stopping time.

For τ_*, note that $\tau^x = \tau_*^x > 0$ for all paths if $x \in G$, and $\tau^x = \tau_*^x = 0$ if $x \notin \bar{G}$. Thus, the only differences that can occur between these exit times are when the path starts on the boundary. As above, for $t > 0$

$$\{\tau_*^x(\omega) \leq t\} \in \mathcal{F}_t.$$

For $t = 0$ we then have $\{\tau_*^x(\omega) = 0\} \in \bigcap_{t>0} \mathcal{F}_t$. The result follows by the lemma above. ■

Thus, for points $x \in \partial G$, we have that $P(\tau_*^x = 0)$ is either 0 or 1. We have an important definition which will play a crucial role in the solution of the Dirichlet problem.

Definition 4.1.5. If $P(\tau_*^x = 0) = 1$, the point x is called **regular**.

It will turn out that the Dirichlet problem has a solution if every point of the boundary is regular. The above discussion will also prove useful in showing an easy to check condition for the regularity of a point.

4.2 Strong Markov Property

Lemma 4.2.1. *Let σ be a stopping time and consider $\sigma^{(n)} = ([2^n\sigma] + 1)2^{-n}$. Then, $\sigma^{(n)}$ is a sequence of stopping times.*

Proof. Since $\sigma^{(n)}$ is some time of the form $k2^{-n}$ after σ, and we are interested in the measurability of $\sigma^{(n)}$ with respect to $\mathcal{F}_t$, we have that

$$\frac{m}{2^n} \leq \sigma < \sigma^{(n)} \leq \frac{m+1}{2^n} < t.$$

We know that σ is a stopping time, so by definition $\{\sigma \leq t\} \in \mathcal{F}_t$. From the inequality above, $\{\sigma^{(n)} \leq t\} = \{\sigma \leq (m+1)2^{-n}\} \in \mathcal{F}_{(m+1)2^{-n}} \subset \mathcal{F}_t$. ■

We will need to introduce a new quantity $\mathcal{F}_\sigma$ to denote the information known at time σ. We will define it to be

$$\mathcal{F}_\sigma = \{A : A \cap \{\sigma \leq t\} \in \mathcal{F}_t \text{ for all } t\}.$$

Definition 4.2.2. Suppose that μ_n and ν are probability measures on $\mathbb{R}^d$. It is said that $\mu_n \to \nu$ **weakly** if

$$\int_{\mathbb{R}^d} \varphi \, d\mu_n \to \int_{\mathbb{R}^d} \varphi \, d\nu \quad \text{as } n \to \infty,$$

for all continuous and bounded functions φ on $\mathbb{R}^d$.

Definition 4.2.3. Suppose f is an $\mathbb{R}^d$-valued function on some space (X, m). The **distribution measure** μ of f is defined as

$$\mu(B) = m(f^{-1}(B)), \quad B \in \mathfrak{B}(\mathbb{R}^d).$$

Lemma 4.2.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{f_n\}$ be a sequence of random variables. If $f_n \to f$ almost everywhere as $n \to \infty$, then $f_n \to f$ as $n \to \infty$ in terms of weak convergence of their distribution measures.*

Proof. This is a standard probability theory result. Convergence almost everywhere implies convergence in probability, and convergence in probability implies convergence in distribution. For a proof see for example Theorem 2 on p.256 of [Shi96]. ■

Lemma 4.2.5. *Suppose $\{\mu_n\}_{n=1}^{\infty}$ and ν are probability measures on $\mathbb{R}^d$, and ν is absolutely continuous with respect to the Lebesgue measure. If $\mu_n \to \nu$ weakly as $n \to \infty$, then $\mu_n(\mathcal{O}) \to \nu(\mathcal{O})$ for all open sets $\mathcal{O}$.*

Proof. For a proof of the result for measures on $\mathbb{R}$ see Lemma 2.15 in Chapter 5 of [Ste11]. ■

Theorem 4.2.6. *Suppose B_t is a Brownian motion and $\sigma(\omega)$ is a stopping time. Then the process B_t^* defined as*

$$B_t^*(\omega) = B_{t+\sigma(\omega)}(\omega) - B_{\sigma(\omega)}(\omega)$$

is also a Brownian motion and is independent of $\mathcal{F}_\sigma$.

Proof. Assume that $\sigma(\omega)$ takes on a countable set of values $s_1 < s_2 < \cdots < s_\ell < \cdots$, and that $0 \leq t_1 < t_2 < \cdots < t_k$ are fixed. We will consider the cylindrical sets for a Brownian motion in $\mathbb{R}^d$, so we introduce the vectors

$$\begin{aligned}
\mathbf{B} &= (B_{t_1}, B_{t_2}, \ldots, B_{t_k}) \\
\mathbf{B}^* &= (B_{t_1}^*, B_{t_2}^*, \ldots, B_{t_k}^*) \\
\mathbf{B}_\ell^* &= (B_{t_1+s_\ell} - B_{s_\ell}, B_{t_2+s_\ell} - B_{s_\ell}, \ldots, B_{t_k+s_\ell} - B_{s_\ell})
\end{aligned}$$

taking values in $\mathbb{R}^{kd}$. If E is a Borel set in $\mathbb{R}^{kd}$ then we can write

$$\{\omega : \mathbf{B}^* \in E\} = \bigcup_\ell \{\omega : \mathbf{B}^*_\ell \in E\} \cap \{\sigma(\omega) = s_\ell\}.$$

Now suppose that we have a set $A \in \mathcal{F}_\sigma$. Then, $A \cap \{\sigma(\omega) = s_\ell\} \in \mathcal{F}_{s_\ell}$. Note that $\mathcal{F}_\sigma = \mathcal{F}_s$ if $\sigma(\omega)$ is constant and equal to s. Since $A \cap \{\sigma(\omega) = s_\ell\}$ is independent of $\{\mathbf{B}^*_\ell \in E\}$ we can express the probability of $\{\omega : \mathbf{B}^* \in \mathbf{E}\} \cap A$ as

$$\begin{aligned} P\big(\{\omega : \mathbf{B}^* \in E\} \cap A\big) &= P\Big(\bigcup_l \{\omega : \mathbf{B}^*_\ell \in E\} \cap A \cap \{\sigma(\omega) = s_\ell\}\Big) \\ &= \sum_l P\big(\{\omega : \mathbf{B}^*_\ell \in E\}\big) P\big(A \cap \{\sigma(\omega) = s_\ell\}\big). \end{aligned}$$

Since $P(\mathbf{B}^*_\ell \in E) = P(\mathbf{B} \in E)$, and $\{\sigma(\omega) = s_\ell\}$ forms a countable partition of the sample space, we can write

$$\begin{aligned} P\big(\{\omega : \mathbf{B}^* \in E\} \cap A\big) &= P\big(\{\omega : \mathbf{B} \in E\}\big) \sum_\ell P\big(A \cap \{\sigma(\omega) = s_\ell\}\big) \\ &= P\big(\{\omega : \mathbf{B} \in E\}\big) P(A). \end{aligned}$$

Taking $A = \Omega$ we can see that the conditions in the definition of Brownian motion are satisfied. Taking any $A \subset \mathcal{F}_\sigma$ gives independence of $\mathbf{B}^*$ from $\mathcal{F}_\sigma$.

To lift the result to general stopping times we define $\sigma^{(n)}$ as in 4.2.1. We obviously have that $\sigma^{(n)}(\omega) \downarrow \sigma(\omega)$ for all ω as $n \to \infty$. We also have that $\mathcal{F}_\sigma \subseteq \mathcal{F}_{\sigma^{(n)}}$, because if $A \in \mathcal{F}_\sigma$ then $A \cap \{\sigma^{(n)} \leq t\} = A \cap \{\sigma \leq k2^{-n}\} \in \mathcal{F}_{k2^{-n}} \subseteq \mathcal{F}_t$. Let $B^{*(n)}_t(\omega) = B_{t+\sigma^{(n)}(\omega)}(\omega) - B_{\sigma^{(n)}(\omega)}(\omega)$ and $\mathbf{B}^{*(n)} = (B^{*(n)}_{t_1}, \ldots, B^{*(n)}_{t_k})$. Assuming $A \subseteq \mathcal{F}_\sigma$ we have that

$$P\big(\{\omega : \mathbf{B}^* \in E\} \cap A\big) = P\big(\{\omega : \mathbf{B} \in E\}\big) P(A).$$

Letting $n \to \infty$ we get that the equality above holds for general σ pointwise. Applying Lemmas 4.2.4 and 4.2.5 we get that the equality holds when E is an open set, in particular open cubes. Since open cubes form a π-system and the collection where two measures agree is a λ-system, we get that the result extends to all Borel sets E by 2.1.5, the π-λ theorem. ■

With the goal of using the strong Markov property in the solution of the Dirichlet problem, we will prove versions of the property that are more directly applicable. To work with these results we introduce some new notions. Define $\tilde{\mathcal{P}}$ to be the space of all paths in $\mathbb{R}^d$, i.e. paths not necessarily starting at 0. We can express our new space as $\tilde{\mathcal{P}} = \mathcal{P} \times \mathbb{R}^d$, and hence every path $\tilde{\mathsf{p}}$ in $\tilde{\mathcal{P}}$ as a pair (p, x) where $\mathsf{p} \in \mathcal{P}$ and $x \in \mathbb{R}^d$; moreover $\mathsf{p} = \tilde{\mathsf{p}} - \tilde{\mathsf{p}}(0)$ and $x = \tilde{\mathsf{p}}(0)$. Any function f on $\tilde{\mathcal{P}}$ can be written as $f(\tilde{\mathsf{p}}) = f_1(\mathsf{p}, x)$. Note also that $\tilde{\mathcal{P}}$ inherits a metric from $\mathcal{P}$ and $\mathbb{R}^d$ and a corresponding class of Borel subsets.

We write $B_\cdot(\omega)$ for the path $t \mapsto B_t(\omega)$. Similarly, $B_{\sigma(\omega)+\cdot}$ is identified with $t \mapsto B_{\sigma(\omega)+t}$, and $B^*_\cdot(\omega)$ with $t \mapsto B_{\sigma(\omega)+t}(\omega) - B_{\sigma(\omega)}(\omega)$.

Theorem 4.2.7. *Let f be a bounded Borel function on the space $\tilde{\mathcal{P}}$ of all paths. Then*

$$\int_\Omega f\big(B_{\sigma(\omega)+\cdot}(\omega)\big)\, dP(\omega) = \iint_{\Omega\times\Omega} f\big(B_\cdot(\omega) + B_{\sigma(\omega')}(\omega')\big)\, dP(\omega)dP(\omega')$$

Proof. Rewrite f as a function f_1 on the product space $\mathcal{P}\times\mathbb{R}^d$. We first consider separable functions f_1, so that we can write $f_1(\mathsf{p}, x) = f_2(\mathsf{p})f_3(x)$. Noting that $B_{\sigma(\omega)+t}(\omega) = B^*_t(\omega) + B_{\sigma(\omega)}(\omega)$ we rewrite the statement of the theorem as

$$\int_\Omega f_1\big(B^*_\cdot(\omega), B_{\sigma(\omega)}(\omega)\big)\, dP(\omega) = \iint_{\Omega\times\Omega} f_1\big(B_\cdot(\omega), B_{\sigma(\omega')}(\omega')\big)\, dP(\omega)dP(\omega'),$$

and then applying assumed separability of f_1 as

$$\int_\Omega f_2\big(B^*_\cdot(\omega)\big) f_3\big(B_{\sigma(\omega)}(\omega)\big)\, dP(\omega) = \iint_{\Omega\times\Omega} f_2\big(B_\cdot(\omega)\big) f_3\big(B_{\sigma(\omega')}(\omega')\big)\, dP(\omega)dP(\omega')\,.$$

We can manipulate the right-hand side of the expression

$$\iint_{\Omega\times\Omega} f_2\big(B_\cdot(\omega)\big) f_3\big(B_{\sigma(\omega')}(\omega')\big)\, dP(\omega)dP(\omega') = \int_\Omega f_2\big(B_\cdot(\omega)\big)\, dP(\omega) \int_\Omega f_3\big(B_{\sigma(\omega')}(\omega')\big)\, dP(\omega') \tag{4.1}$$

$$= \int_\Omega f_2\big(B^*_\cdot(\omega)\big)\, dP(\omega) \int_\Omega f_3\big(B_{\sigma(\omega')}(\omega')\big)\, dP(\omega') \tag{4.2}$$

$$= \int_\Omega f_2\big(B^*_\cdot(\omega)\big) f_3\big(B_{\sigma(\omega)}(\omega)\big)\, dP(\omega) \tag{4.3}$$

where the second line follows from theorem 4.2.6, and the third from independence guaranteed in that theorem.

We have shown that this equality holds for all f that are characteristic functions of Borel sets of the form $E = E_2 \times E_3$, where $E_2 \in \mathfrak{B}(\mathcal{P})$ and $E_2 \in \mathfrak{B}(\mathbb{R}^d)$. By 2.1.7, the monotone class theorem, this result extends to all bounded Borel functions on $\tilde{\mathcal{P}}$. ■

Recall that we denote Brownian motion starting at y with $B^y_t(\omega) = B_t(\omega) + y$. We will also use the superscript in stopping times to denote where the Brownian motion started, i.e. given some bounded open set G, the exit time is $\tau^y = \inf\{t \geq 0 : B^y_t \notin G\}$. To state and prove the final version of the strong Markov property, we introduce the notion of a stopped Brownian motion, defined as

$$\hat{B}^y_t(\omega) = y + B_{t\wedge\tau^y(\omega)}(\omega),$$

where $a \wedge b = \min(a, b)$. With these notions in mind, we can state and prove a version of the strong Markov property that will be most readily applied in the solution of the Dirichlet problem.

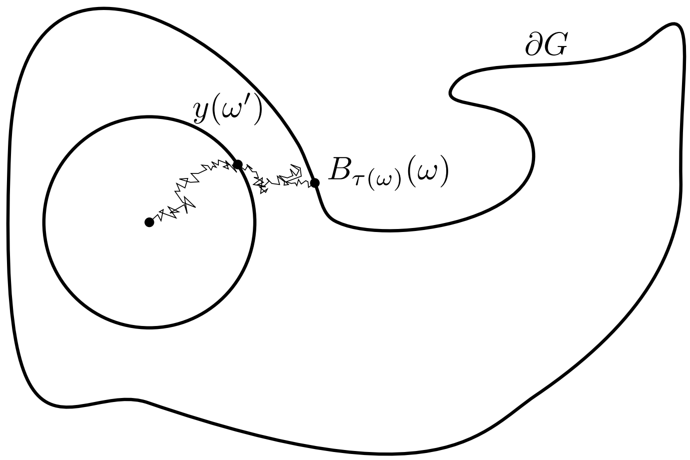


Figure 4.1: Brownian whale

Theorem 4.2.8. *Let σ and τ be such stopping times that $\sigma(\omega) \leq \tau(\omega)$ for all ω. If F is a bounded Borel function on $\mathbb{R}^d$, then for every $t \geq 0$*

$$\int_\Omega F\big(\hat{B}_{\sigma(\omega)+t}(\omega)\big)\, dP(\omega) = \iint_{\Omega\times\Omega} F\big(\hat{B}_t^{y(\omega')}(\omega)\big)\, dP(\omega)\, dP(\omega')$$

where $y(\omega') = \hat{B}_{\sigma(\omega')}(\omega')$.

Proof. We begin by taking a look at the left-hand side of the expression

$$\begin{aligned}
\int_\Omega F\big(\hat{B}_{\sigma(\omega)+t}(\omega)\big)\, dP(\omega) &= \int_\Omega F\big(B_{(\sigma(\omega)+t)\wedge\tau(\omega)}(\omega)\big)\, dP(\omega) \\
&= \int_\Omega F\big(B_{\sigma(\omega)+t}(\omega)\big)\chi_{\tau(\omega)\geq\sigma(\omega)+t}\, dP(\omega) \\
&+ \int_\Omega F\big(B_{\tau(\omega)}(\omega)\big)\chi_{\tau(\omega)<\sigma(\omega)+t}\, dP(\omega) \\
&= I_1 + I_2.
\end{aligned}$$

We will address the two integrals separately, beginning with I_1. Consider the function $f : \tilde{\mathcal{P}} \to \mathbb{R}^d$ given by

$$f(\tilde{\mathsf{p}}) = F\big(\tilde{\mathsf{p}}(t)\big)\chi_{\tau(\tilde{\mathsf{p}})\geq t}.$$

Here we denote with $\tau(\tilde{\mathsf{p}}) = \inf\{s \geq 0 : \tilde{\mathsf{p}}(s) \notin G\}$, the exit time of a path from G. Given some ω, set $\tilde{\mathsf{p}}(\cdot) = B_{\sigma(\omega)+\cdot}(\omega)$. We can then see that

$$\tau(\tilde{\mathsf{p}}) = \inf\{s \geq 0 : B_{\sigma(\omega)+s}(\omega) \notin G\} = \tau(\omega) - \sigma(\omega),$$

because the path $B_\cdot(\omega)$ exits at time $\tau(\omega)$ and so the path $B_{\sigma(\omega)+\cdot}(\omega)$ must exit at time $\tau(\omega) - \sigma(\omega)$. We thus have that

$$f(\tilde{\mathsf{p}}) = f\big(B_{\sigma(\omega)+\cdot}(\omega)\big) = F\big(B_{\sigma(\omega)+t}(\omega)\big)\chi_{\tau(\omega)\geq\sigma(\omega)+t},$$

which can be recognized as the integrand in question. We can now apply the version of the strong Markov property above, Theorem 4.2.7. We then get

$$I_1 = \iint_{\Omega\times\Omega} f\big(B_\cdot(\omega) + B_{\sigma(\omega')}(\omega')\big)\, dP(\omega)\, dP(\omega')$$

The integrand is now equal to $F\big(B_t(\omega) + B_{\sigma(\omega')}(\omega')\big)\chi_{\tau\left(B_\cdot(\omega)+B_{\sigma(\omega')}(\omega')\right)\geq t}$. Note that the stopping time term in the characteristic function above can be thought of as the exit time of a path starting at the point $y(\omega')$, that is, the point where another independent path stopped. It is thus a reasonable expectation that $\tau^{y(\omega')}(\omega) = \tau\big(B_\cdot(\omega)+B_{\sigma(\omega')}(\omega')\big)$. This intuition is easily verified via the definitions of terms involved:

$$\begin{aligned}
\tau^{y(\omega')}(\omega) &= \inf\{t \geq 0 : B_t^{y(\omega')}(\omega) \notin G\} \\
&= \inf\{t \geq 0 : B_t^{B_{\sigma(\omega')\wedge\tau(\omega')}(\omega')}(\omega) \notin G\} \\
&= \inf\{t \geq 0 : B_{\sigma(\omega')}(\omega') + B_t(\omega) \notin G\} \\
&= \tau\big(B_\cdot(\omega) + B_{\sigma(\omega')}(\omega')\big).
\end{aligned}$$

Note also that $B_t(\omega) + B_{\sigma(\omega')}(\omega') = B_t^{y(\omega')}(\omega)$, and that

$$\hat{B}_t^{y(\omega')}(\omega) = B_{\sigma(\omega')\wedge\tau(\omega')}(\omega') + B_{t\wedge\tau^{y(\omega')}(\omega)}(\omega) = B_{\sigma(\omega')}(\omega') + B_t(\omega) = B_t^{y(\omega')}(\omega)$$

whenever $\tau^{y(\omega')}(\omega) \geq t$. With these results in mind we can finish the computation of the first integral

$$\begin{aligned}
I_1 &= \iint_{\Omega\times\Omega} F\big(B_t(\omega) + B_{\sigma(\omega')}(\omega')\big)\chi_{\tau\left(B_\cdot(\omega)+B_{\sigma(\omega')}(\omega')\right)\geq t}\, dP(\omega)\, dP(\omega') \\
&= \iint_{\Omega\times\Omega} F\big(B_t^{y(\omega')}(\omega)\big)\chi_{\tau^{y(\omega')}(\omega)\geq t}\, dP(\omega)\, dP(\omega') \\
&= \iint_{\Omega\times\Omega} F\big(\hat{B}_t^{y(\omega')}(\omega)\big)\chi_{\tau^{y(\omega')}(\omega)\geq t}\, dP(\omega)\, dP(\omega')\,.
\end{aligned}$$

To deal with the second integral I_2, we define a function $g : \tilde{\mathcal{P}} \to \mathbb{R}^d$ as

$$g(\tilde{\mathsf{p}}) = F\Big(\tilde{\mathsf{p}}\big(\tau(\tilde{\mathsf{p}})\big)\Big)\chi_{\tau(\tilde{\mathsf{p}})<t}.$$

Once again consider $\tilde{\mathsf{p}}(\cdot) = B_{\sigma(\omega)+\cdot}(\omega)$, then

$$\begin{aligned}
g\big(B_{\sigma(\omega)+\cdot}(\omega)\big) &= F\big(B_{\sigma(\omega)+\tau(B_{\sigma(\omega)+\cdot}(\omega))}(\omega)\big)\chi_{\tau(\omega)<\sigma(\omega)+t} \\
&= F\big(B_{\sigma(\omega)+\tau(\omega)-\sigma(\omega)}(\omega)\big)\chi_{\tau(\omega)<\sigma(\omega)+t} \\
&= F\big(B_{\sigma(\omega)}(\omega)\big)\chi_{\tau(\omega)<\sigma(\omega)+t}.
\end{aligned}$$

Recognizing that this is the integrand of I_2, we can apply the strong Markov property to the left-hand side to obtain

$$I_2 = \iint_{\Omega\times\Omega} g\big(B_\cdot(\omega) + B_{\sigma(\omega')}(\omega')\big)\, dP(\omega)\, dP(\omega')\,.$$

We can express the integrand above using F

$$\begin{aligned} g\big(B_\cdot(\omega) + B_{\sigma(\omega')}(\omega')\big) &= F\big(B_{\tau\left(B_\cdot(\omega)+B_{\sigma(\omega')}(\omega')\right)}(\omega) + B_{\sigma(\omega')}(\omega')\big)\chi_{\tau\left(B_\cdot(\omega)+B_{\sigma(\omega')}(\omega')\right)<t} \\ &= F\big(B_{\tau^{y(\omega')}(\omega)}(\omega) + B_{\sigma(\omega')}(\omega')\big)\chi_{\tau^{y(\omega')}(\omega)<t}. \end{aligned}$$

Note also that

$$\hat{B}_t^{y(\omega')}(\omega) = B_{\tau^{y(\omega')}(\omega)}(\omega) + B_{\sigma(\omega')}(\omega')$$

whenever $\tau^{y(\omega')}(\omega) < t$. Combining these results we have that

$$\begin{aligned} I_2 &= \iint_{\Omega\times\Omega} F\big(B_{\tau^{y(\omega')}(\omega)}(\omega) + B_{\sigma(\omega')}(\omega')\big)\chi_{\tau^{y(\omega')}(\omega)<t}\, dP(\omega)\, dP(\omega') \\ &= \iint_{\Omega\times\Omega} F\big(\hat{B}_t^{y(\omega')}(\omega)\big)\chi_{\tau^{y(\omega')}(\omega)<t}\, dP(\omega)\, dP(\omega')\,. \end{aligned}$$

We can now combine I_1 and I_2 to obtain the right-hand side of the theorem statement

$$I_2 + I_2 = \iint_{\Omega\times\Omega} F\big(\hat{B}_t^{y(\omega')}(\omega)\big)\, dP(\omega)\, dP(\omega')\,.$$

■

Chapter 5

Dirichlet Problem

Having developed some theory for Brownian motion, we can finally study the Dirichlet problem. We will show that if there exists a bounded solution, then the Brownian motion solution agrees with it. Conditions on the boundary guaranteeing the validity of the solution, and the relatively intuitive Cone condition will be studied.

The proofs in this section follow [Ste11].

5.1 Brownian Motion Solution to the Dirichlet Problem

Consider a bounded open set G in $\mathbb{R}^d$ and some bounded function f. The Dirichlet problem is to find a function u that satisfies

(a) $u \in C^2$ and $\Delta u = 0$ for $x \in G$

(b) For each $x \in \partial G$, u is continuous and $u = f$.

The main result of this section is that the harmonic measure leads to the solution of the Dirichlet problem given some restrictions on the boundary of the domain.

Definition 5.1.1. The **harmonic measure** on the boundary ∂G of some open set $G \in \mathbb{R}^d$ is given by

$$\mu^x(E) = P\big(\{\omega : B^x_{\tau^x(\omega)}(\omega) \in E\}\big),$$

where E are Borel sets of ∂G.

To prove the main result we will need to introduce some new concepts

Theorem 5.1.2 (**Arzelà-Ascoli Theorem**). *Consider a sequence of continuous functions $f_n : [a,b] \to \mathbb{R}$. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly.*

Corollary 5.1.3. *A family $\mathcal{H}$ of α-Hölder uniformly bounded functions $f : [a,b] \to \mathbb{R}$ is relatively compact in $C([a,b])$.*

Compactness of Brownian motion paths will allow us to use the Riesz representation theorem.

Theorem 5.1.4 (**Riesz Representation Theorem**). *Let X be a locally compact Hausdorff space, and let T be a positive linear functional on the vector space K of continuous real-valued functions on X with compact support. Then there is a unique regular Borel measure μ on X such that*

$$T(f) = \int f \, d\mu$$

for every f in K.

Definition 5.1.5. Let $\mathcal{M}$ be a metric space. A function $f : \mathcal{M} \to \mathbb{R}$ is called **upper semicontinuous** if

$$\limsup_{x \to y} f(x) \leq f(y)$$

for all points $y \in \mathcal{M}$.

Proposition 5.1.6. *The limit of a sequence of decreasing continuous functions is upper semicontinuous.*

We will also need the following inequality. The result is a consequence of Brownian motion being a martingale, and follows from Doob's martingale inequality.

Lemma 5.1.7 (**Brownian motion maximal inequality**). *For all $T > 0$ and $\alpha > 0$ we have that*

$$\begin{aligned} P(\{\omega : \sup_{t \leq T} |B_t(\omega)| > \alpha\}) &\leq \frac{1}{\alpha} \|B_T\|_{L^1} \\ &= \frac{1}{\alpha} \sqrt{\frac{T}{2\pi}} \end{aligned}$$

Proof. The proof is outside the scope of this work, see the chapter on Brownian motion in [Ste11] for details. ■

Theorem 5.1.8. *If a function u on $\mathbb{R}^d$ is defined by*

$$u(x) = \int_{\partial G} f(y) \, d\mu^x(y), \quad x \in G,$$

where f is a function on ∂G, then *(a) u is harmonic in G,*

(b) if y is a regular point of ∂G and $x \in G$, then $u(x) \to f(y)$, as $x \to y$.

Proof. To prove (a) we will show that u satisfies the mean value property possessed by harmonic functions. Fix some $x \in G$ and let S be a sphere around x such that S and the interior ball are contained in G. With m denoting the standard measure on the sphere normalized to have $m(S) = 1$, we want to show that

$$u(x) = \int_S u(y)\, dm(y)\,.$$

Let $\sigma(\omega)$ be the hitting time of S by $B_t^x(\omega)$ and consider the stopped process $\hat{B}_t^x$. Since a path starting at x will hit S before it hits ∂G, we have that $\hat{B}_{\sigma(\omega)}(\omega) = y(\omega) \in S$. Suppose F is some continuous bounded extension of f to $\mathbb{R}^d$. Taking the statement of the two stopping times version of the strong Markov property, Theorem 4.2.8, and letting $t \to \infty$ we get

$$\begin{aligned}&\int_\Omega F\big(\hat{B}_{\sigma(\omega)+t}(\omega)\big)\, dP(\omega) = \iint_{\Omega\times\Omega} F\big(\hat{B}_t^{y(\omega')}(\omega)\big)\, dP(\omega)\, dP(\omega')\\ &= \int_\Omega F\big(B^x_{\tau^x(\omega)}(\omega)\big)\, dP(\omega) = \iint_{\Omega\times\Omega} F\big(B^{y(\omega')}_{\tau^{y(\omega')}(\omega)}(\omega)\big)\, dP(\omega)\, dP(\omega') \quad \text{as } t \to \infty.\end{aligned}$$

Note that the left hand side equals $u(x)$ since

$$\begin{aligned}u(x) &= \int_{\partial G} f(y)\, d\mu^x(y)\\ &= \int_\Omega f\big(B^x_{\tau^x(\omega)}(\omega)\big)\, dP(\omega)\\ &= \int_\Omega F\big(B^x_{\tau^x(\omega)}(\omega)\big)\, dP(\omega), \quad \text{for } x \in G.\end{aligned}$$

Similarly, the right-hand side can be expressed as

$$\iint_{\Omega\times\Omega} F\big(B^{y(\omega')}_{\tau^{y(\omega')}(\omega)}(\omega)\big)\, dP(\omega)\, dP(\omega') = \int_\Omega u\big(y(\omega')\big)\, dP(\omega')$$

To deal with this expression, note that for any continuous function G on S we can write

$$\int_\Omega G(B^x_{\sigma(\omega')}(\omega'))\, dP(\omega') = \int_S G(y)\, dm(y)\,.$$

To see why this equality must hold suppose $x = 0$ for simplicity. The left-hand side defines a continuous linear functional on the continuous functions on S, and so by Riesz representation theorem there must be some measure μ on S so that the left-hand side can be expressed as $\int_S G(y)\, d\mu(y)$. Since rotation is an orthogonal transformation, and Brownian motion is invariant under such transformations, we get that $\mu = m$. Now since $y(\omega') = B^x_{\sigma(\omega')}(\omega')$, we can see by setting $u = G$ that

$$\int_\Omega u\big(y(\omega')\big)\, dP(\omega') = \int_S u(y)\, dm(y)\,.$$

Thus we have that

$$u(x) = \int_S u(y)\,dm(y),$$

and thus part (a) is established.

For part (b) we begin by writing

$$\begin{aligned}
u(x) - f(y) &= \int_{\partial G} (f(y') - f(y))\,d\mu^x(y') \\
&= \int_{\partial G_1} (f(y') - f(y))\,d\mu^x(y') + \int_{\partial G_2} (f(y') - f(y))\,d\mu^x(y') \\
&= I_1 + I_2.
\end{aligned}$$

Here we assume that y is regular and let $\partial G_1 = \{y' \in \partial G : |y' - y| \leq s\}$, and $\partial G_2 = \partial G \setminus \partial G_1$. Since f is continuous we can make $|f(y') - f(y)| < \varepsilon$ on ∂G_1 if s is small enough. As a consequence, we can make I_1 to be less than ε as well. To deal with I_2, we first show that

$$\lim_{x \to y, x \in G} P\big(\{\tau^x > \delta\}\big) = 0,$$

for all $\delta > 0$. Note that $x \mapsto P\big(\{B_t^x \in G, \text{ for all } \varepsilon \leq t \leq \delta\}\big)$ is continuous. To see this, recall that $P(A) = E(\chi_A)$, and notice that $\lim_{x\to y} \chi_{\{B_t^x \in G, \forall \varepsilon \leq t \leq \delta\}} = \chi_{\{B_t^y \in G, \forall \varepsilon \leq t \leq \delta\}}$. Note also that $P\big(\{B_t^x \in G, \text{ for all } \varepsilon \leq t \leq \delta\}\big)$ are decreasing as $\varepsilon \downarrow 0$. The limit is upper semicontinuous by Proposition 5.1.6

$$\begin{aligned}
\lim_{\epsilon \downarrow 0} P\big(\{B_t^x \in G, \text{ for all } \varepsilon \leq t \leq \delta\}\big) &= P\big(\{B_t^x \in G, \text{ for all } 0 \leq t \leq \delta\}\big) \\
&= P\big(\{\tau^x > \delta\}\big).
\end{aligned}$$

By the definition of upper semicontinuity and by regularity of y we have that

$$\limsup_{x \to y} P\big(\{\tau^x > \delta\}\big) \leq P\big(\{\tau^y > \delta\}\big) = 0. \tag{5.1}$$

Now we consider

$$\begin{aligned}
P\big(\{\omega : |B_\tau(\omega)| > s\}\big) &= P\big(\{\omega : |B_\tau(\omega)| > s\}\big)P(\tau \leq \delta) + P\big(\{\omega : |B_\tau(\omega)| > s\}\big)P(\tau > \delta) \\
&\leq P\big(\{\omega : \sup_{t \leq \delta} |B_t(\omega)| > s\}\big) + P(\tau > \delta). \qquad (5.2)
\end{aligned}$$

The first term can be made less than $\varepsilon/2$ by the Brownian motion maximal inequality, by picking $\varepsilon = \dfrac{\pi\varepsilon^2}{2s^2}$. The second can be made less than $\varepsilon/2$, by (5.1). Thus combining these two bounds with (5.2) we obtain that for some given $s > 0$ and $\varepsilon > 0$, if x is close enough to $y \in \partial G$

$$P\big(\{\omega : |y - B^x_{\tau^x(\omega)}(\omega)| > s\}\big) < \varepsilon.$$

Now we go back to the integral I_2. Observe that points $y' \in \partial G_1$ can be expressed as $B^x_{\tau^x(\omega)}(\omega)$. We can thus write $\mu^x(\partial G_2) = P(\{\omega : |y - B^x_{\tau^x(\omega)}(\omega)| > s\})$. Since we can make this contribution less than ε, we can see that

$$I_2 \leq 2 \sup |f| \, \mu^x(\partial G_2)\varepsilon.$$

Thus combining these bounds on I_1 and I_2 we see that $u(x) - u(y)$ is dominated by a multiple of ε if x is close enough to y. Since the choice of ε was arbitrary, the result follows. ■

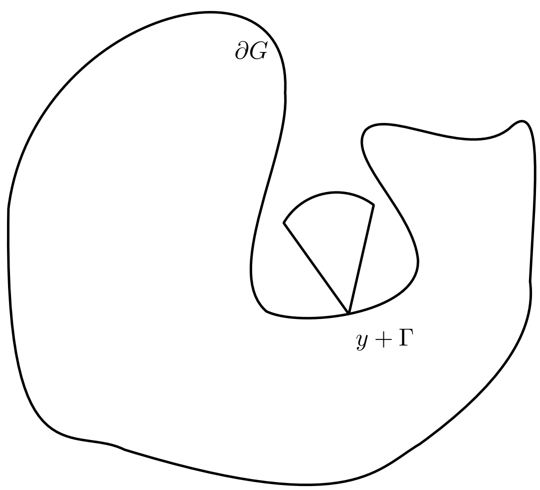
5.2 Cone Condition

Definition 5.2.1. A **truncated cone** Γ pointing in direction γ with aperture α and radius δ is the open set

$$\Gamma = \{y \in \mathbb{R}^d : |y| < \alpha\langle y, \gamma\rangle, |y| < \delta\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Theorem 5.2.2 (Cone Condition). *Suppose $y \in \partial G$ and there exists a cone Γ such that $y + \Gamma \subset G^{\mathsf{c}}$. Then y is a regular point.*



Proof. Assume $y = 0$ and set $A_n = \bigcup_{r_k < 1/n}\{\omega : B_{r_k}(\omega) \in \Gamma\}$ where r_k is an enumeration of positive rationals. Then $A = \bigcap_{n=1}^{\infty} A_n$ is the collection of paths starting at the origin that enter Γ for a sequence of times tending towards zero. Note, that $A_n \in \mathcal{F}_n$ for all n, and hence $A \in \mathcal{F}_0^+ = \mathcal{F}_0$, by Blumenthal's zero-one law. Hence, $m(A) = 0$ or $m(A) = 1$. We show it is the latter. For contradiction, assume $m(A) = 0$. By rotation invariance of Brownian motion this result would hold for any rotation of Γ. Finitely many rotations will cover a punctured ball $B(0, \delta) \setminus \{0\}$. However, every path enters this ball at arbitrarily small times, so we get a contradiction. Thus, $m(A) = 1$.

Now if $y + \Gamma$ is contained in G^{c}, then for each ω there are arbitrarily small times t for which $B_t(\omega) \in \Gamma$, and thus $B_t(\omega) \notin G$, meaning y is a regular point. ■

Corollary 5.2.3. *If the cone condition is satisfied for every point of the boundary ∂G, there exists a solution to the Dirichlet problem with G as the domain.*

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