The Black-Litterman Model: An Investigation of Confidence

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Abstract

This paper examines Idzorek's extension of the Black-Litterman model with respect to confidence levels and makes a general comparison with the Canonical Reference Model. To test Idzorek's method a global equities portfolio is constructed using assets representing nine different countries consisting in total of 80.9% of the global equities market. Two views are specified, one absolute view and one relative view. A sensitivity analysis of the weights is then performed by altering the investors' confidence in each view based on an interval from 0% to 100%. The portfolio weights are also derived using the Canonical Reference Model without altering the investors' confidence. Monthly return data is used spanning a 5-year period from May 2016 to April 2021, totaling 60 observations. Bayes-Stein shrinkage estimation is applied to derive the historical covariance matrix of excess returns. This paper shows that there can be both a linear and a non-linear relationship between the weight of the views and the level of confidence. And that this relationship can differ within the same portfolio. Because the relationship can be non-linear it is concluded that the marginal effect of the level of confidence can vary within the view itself. This means that the sensitivity of the views with respect to confidence will fluctuate between different levels of confidence. The results also show that the weights derived using the Canonical Reference Model are comparable to roughly 50% confidence when using Idzorek's approach. And finally, that the sensitivity of the weights involved in the views is also dependent on their market capitalization weight(s) as well the specified excess return.

Keywords: Black-Litterman model, Idzorek's extension, Confidence, Sensitivity

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1. Introduction

Markowitz (1952) introduced the concept of modern portfolio theory ("MPT") which uses a mean-variance analysis framework to derive optimal portfolio weights. This provides an intuitive approach to portfolio allocation by optimizing the expected excess return with respect to risk (variance). The practical application and usefulness of MPT is however more dubious. The main concern facing practitioners when applying MPT is providing correct estimates of the means and the covariance matrix of the excess returns. Because these variables are inherently unknown, they are subject to estimation errors. These estimation errors can result in extreme portfolio weights with relatively large long and short positions for some assets. This is because small changes in the expected excess returns can potentially greatly shift the weights in the mean-variance framework making it very input sensitive (Chopra & Ziemba, 1993) (Black & Litterman, 1990). Another concern facing investors aside from the potential risk of holding a large position are various legal or social restrictions that might be put in place. Limiting the ability of an investor to carry out the weights derived using the MPT framework in practice. This leads to investors having to use various constraints while solving the optimization problem. In the example of short selling this would lead to a nonnegativity constraint on the portfolio weights (Best & Grauer, 1991). The problem with introducing constraints on the assets when implementing the model is that it in theory moves it further away from what is supposedly a balanced portfolio. Which puts into question what the constrained weights represents and the justification for using the MPT framework in the first place (Black & Litterman, 1990). There is also cause for concern on an organizational level with a purely quantitative approach to investing. Managers using a qualitative approach might find it counter-intuitive or lack the statistical and theoretical knowledge to apply it. There is also room for potential conflict between managers favoring a qualitative approach and those favoring a quantitative approach. The management may also be hesitant as it might be difficult to synthesize the different strategies. These are some of the potential and inherent problems that contribute to the limited practical value of MPT (Michaud, 1989).

The Black-Litterman model ("BLM") was developed by Fischer Black and Robert Litterman while working at Goldman Sachs in attempt to solve some the problems facing managers trying to implement the standard mean-variance approach. Instead of requiring investors to specify expected returns for all assets directly the BLM uses the market equilibrium as a starting point (Walter, 2014).

Black and Litterman used the insights of the capital asset pricing model (CAPM) and the ICAPM to derive equilibrium returns through a reverse optimization problem, although originally derived by Sharpe (1974). This method greatly reduces the potentially large impact of estimation errors on the portfolio weights. And generally, results in more stable weights. The BLM also allows investors to deviate from the starting point of holding the equilibrium portfolio by specifying both relative and absolute views about the included assets. For example, consider a portfolio including four assets in the form of index funds from Sweden, Japan, Germany, and the US. In this case an absolute view might be that German stocks will have an expected excess return of 5%. While a relative view might be that Japanese stocks will outperform Swedish stocks by 3%. The BLM basically allows investors to express what they perceive to be deviations from the asset's equilibrium. In the standard MPT approach simply adjusting the expected returns slightly might cause a major shift in the portfolio weights. But because the BLM is based on the equilibrium returns, the weights of the assets involved in views as formulated through the BLM are not as sensitive (Black & Litterman, 1990). In the absence of views, it is often assumed that the investor will simply only hold the equilibrium portfolio. As pointed out by Walter (2014) this is not necessarily true as uncertainty in the estimation of the equilibrium portfolio can shift the weight of the risk-free asset in the original formulation of the model. One of the main points of the BLM is that it allows for views on assets to be incorporated within the mean-variance framework. This enables investors with a qualitative approach to be incorporated within the framework of a fundamentally quantitative model. The BLM thereby (partially) solves the problem of input sensitivity, the need for implementing constraints, allows the investor to focus on a certain type of assets and bridges the quantitative approach of MPT and a more traditional quantitative approach (Scowcroft & Satchell, 2000).

The BLM is however not without fault and various extensions and interpretations of the model have been suggested. Walters (2014) classifies these as three different "Reference Models" calling them "Canonical Reference Model", "Alternate Reference Models" and "Beyond Black-Litterman Reference Model". Each of which deals with different theoretical and practical problems of the BLM. The first refers to models that share a theoretical basis with the model as originally outlined by Black and Litterman (1990). The second refers to models that partially deviate from the original model but still retain certain similarities. The BLM attempts to cover the relatively large topic of portfolio allocation and so many different versions of the original model have been proposed.

Another issue is the fact that there are no clear guidelines and methods as how to specify some of the variables and inputs in the model and precisely how to apply it (Walter, 2014). One of these issues is how investors should specify $\mathbf{\Omega}$, the covariance matrix of the views. $\mathbf{\Omega}$ is a diagonal matrix where each value on the diagonal will represent the uncertainty of each view. For example, if the investor has two views, $\mathbf{\Omega}$ will have two diagonal values. The first diagonal value of $\mathbf{\Omega}$ will represent the uncertainty of the first view and the second the uncertainty of the second view. If investors are 100% confident in their views it becomes a straightforward process. If you are 100% confident there is no uncertainty and variance in your views and so simply set all values in $\mathbf{\Omega}$ to 0. On the other hand, if the investor has 0% confidence in the views $\mathbf{\Omega} \rightarrow \infty$, meaning that each diagonal value goes towards infinity. This is because if the investor is completely uncertain the variance of their views will become infinitely large. Leading to the weight of the views in the final portfolio being 0. This is quite intuitive as there should not be any variance in a view if you are certain of an outcome. And if you do not have any confidence in a view at all you simply should not put any weight on it. Both cases, 0% and 100%, are straight forward both conceptually and practically. The problem arises when the level of confidence lies in between one of these two extremes (Idzorek, 2005).

In their original paper Black and Litterman (1990) focus on the overall weight of the views in the final portfolio and do not provide any clear guidance as to how $\mathbf{\Omega}$ should be specified. The most popular way is to use a variance that is proportional to the prior (standard method), the prior being the historical covariances. This method involves using the historical covariance matrix of the excess returns based on how the views are specified and adjusting it using a parameter denoted τ. In the Canonical Reference Model ("CRM") τ represents the investors uncertainty in the prior (Walter, 2014). This method is used by He and Litterman (1999) and Meucci (2006), although Meucci does not diagonalize $\mathbf{\Omega}$ and has a slightly different way of parameterizing the confidence in the prior. Another method is to use a factor model, for example, the use of a GARCH based factor model by Beach and Orlov (2006). Investors can also use a confidence interval of the estimated mean return. The advantage of using a variance that is proportional to the prior is that it does not require any additional computation beyond the covariance matrix of the historical returns, τ and specifying views. All of which are already required to implement both Canonical and Alternative Reference Models. The intuition behind it is that you believe that returns will have the same variance as historically and adjust it based on your confidence in the estimated variance with τ. A relatively high historical variance would suggest uncertainty in the view and a lower weight in the portfolio and vice versa.

τ will have a similar effect. The disadvantage of this approach is that you lose the intuition behind $\mathbf{\Omega}$ as you do not directly consider your confidence in the views (Walter, 2014). As this discussion indicates there is no cookie-cutter method.

Idzorek (2005) takes a different approach in specifying $\mathbf{\Omega}$ than those described above. Idzorek uses the market capitalization weights (equal to the equilibrium weights), the weights derived using the standard method, and the weights derived while the investor has 100% confidence to show that each view has an implied confidence. The framework shows that even when the investor does not explicitly specify a confidence level and simply uses the standard approach to compute $\mathbf{\Omega}$, each view still has a level of confidence. Meaning that when the investor has computed $\mathbf{\Omega}$ they can then compute an implied level of confidence for each view. Idzorek goes on to show that using this fact an investor can start with specifying a level of confidence, C_k (k denoting the k-th view), in a view ranging from 0% to 100%. This is done through the weight-dimension by changing the tilts of the portfolio weights. Using this method, the new diagonal values of $\mathbf{\Omega}$ can be computed (Idzorek, 2005). This gives investors an intuitive and relatively easy way to express $\mathbf{\Omega}$ while not making any significant alterations to the model. His method is considered an Alternative Reference Model and extension of the BLM by Walter (2014).

Although Idzorek (2005) suggests that his method can be useful for qualitative investors he does not provide any guideline as how to specify C_k and simply leaves it up to the practitioner. The impact of changing the values of C_k is not clear as well as the sensitivity of the final weights. If an investor chooses to use Idzorek's method to specify $\mathbf{\Omega}$ by changing the values of C_k, what is an admissible value? $C_k \in [0,1]$ (0% to 100% confidence) and because it indirectly determines a value spanning $\alpha \in [0, \infty]$, small changes could have a large impact on the portfolio weights[1]. Values close to 1 should result in relatively extreme portfolio weights if the approach is taken at face value. The question then becomes how close C_k must be to the extreme end of the interval for there to be a big impact. Further, a value close to 1 should lead to a relatively extreme tilt towards the views whereas a value close to 0 should move the weights closer to that of the equilibrium portfolio. The weights of the equilibrium portfolio could by nature be considered less "extreme" as they are equivalent to the market capitalization weights. This means that starting with a confidence of 0% and moving towards 100% the weights should "get out of hand". The features of this relationship might also be of interest, is it for example linear or non-linear?

[1] In Idzorek's method C_k does not directly enter $\mathbf{\Omega}$, instead it determines α which regulates $\mathbf{\Omega}$.

If the relationship is non-linear, adjusting C_k will have different effects for different levels of confidence and so the sensitivity of the weights will vary. Is there even a point in using Idzorek's extension in the first place? These questions should be relevant to an investor wanting to implement Idzorek's extension of the model as well as how it compares to the CRM. There are papers containing empirical examples of both models, but they often do not use the same inputs. Idzorek (2005) only evaluates the model at 100% confidence and reports the weights when using the Alternative Reference Model but not the CRM. Walter (2014) for example only looks at the two extreme cases of 0% and 100% confidence. Neither further examines what happens when other values are set for C_k, the relationship between C_k and the weights and the sensitivity of the weights with respect to C_k. The purpose of this thesis will be to investigate some these questions by doing a sensitivity analysis of the final weights with respect to C_k and making a general comparison with the CRM. This should give a hypothetical investor considering using Idzorek's method or the BLM itself a better understanding of the impact of C_k and its implication when facing a real asset allocation problem. Because some of the issues previously discussed remain largely unanswered this should also provide further insight into the workings of Idzorek's method in practice. As no such research has previously been carried out. To do this I will use a portfolio containing nine ETFs each of which will represent the equities market of a certain country by, for example, replicating a certain index. The countries in question will be Japan, the US, the UK, France, Germany, Canada, Korea, Taiwan (ROC) and Australia. Similar portfolios to this typical BLM portfolio can be seen in He and Litterman (1999) for example who look at the equities market in seven different countries. Views in the context of the BLM are inherently subjective and so the views that I use will naturally be subjective as well. The purpose of the thesis is not the views in and of themselves but rather the confidence in them as measured through C_k or as scaled through τ. This does of course not mean that any view will be permissible, as a view with a relatively high expected return might have an unwanted effect on the weights, for the purpose of this thesis. For example, leading to a higher impact of C_k than might otherwise be expected. Further, I assume that the investor in this case has an investment horizon of 1 month, meaning that the views are stated over the following month. The hypothetical investor is assumed to be American and so the risk-free asset will be represented by a 1-month US Treasury Bill. If the portfolio weights are leveraged a short position in the risk-free asset will be held and vice versa if the total weight is less than 100%. To approximate the historical covariances I have used Bayes-Stein estimation as per Jorion (1986) and Jorion (1991). The return data will be based on monthly returns over a 5-year period totaling 60 observations.

Based on the previous discussion, the aim of this thesis is to investigate the following questions related to the specification of the level of confidence, C_k:

- How sensitive are the estimated portfolio weights in the Black-Litterman model to changes in C_k?
- Is the implicit mapping between C_k and the estimated portfolio weights a linear or non-linear relationship?
- How do the estimated portfolio weights of Idzorek's approach compare to those of the Canonical Reference Model?

An obvious delimitation is that of the span of the C_k that is being tested as there is of course an infinite number of solutions in its theoretical interval. I have used 10,000 values in the span of C_k and the interval between each value is $\frac{1}{10,000}$. Another delimitation is testing different possible combinations of confidence levels. In the thesis I have changed the confidence in all views simultaneously. You could also, for example, keep one view at a constant confidence level while changing that of the other views and vice versa.

2. The Building Blocks of the Black-Litterman Model

2.1 Reverse Mean-variance Optimization and Deriving the Implied Excess Returns

The basic idea of mean-variance optimization is maximizing the utility of the investor by changing the weights of the portfolio. Expected excess returns are interpreted as increasing the investors utility. The variance is treated as a risk, thereby decreasing the investors utility as the investor is assumed to be risk averse. A portfolio with a higher expected return, everything else being equal, will have a higher utility and vice versa. While a higher variance will correspond to a lower utility. This can be formulated as:

$$U = \boldsymbol{w}^T\boldsymbol{\gamma} - \left(\frac{\delta}{2}\right)\boldsymbol{w}^T\boldsymbol{\Psi}\boldsymbol{w} \tag{1}$$

U	is the investors utility, in this case described using a quadratic utility function.
$\boldsymbol{\gamma}$	is an arbitrary $Nx1$ vector containing the mean excess return of each asset.
$\boldsymbol{\Psi}$	is an arbitrary NxN covariance matrix of the excess returns of each asset.
δ	is a parameter representing the risk aversion of the investor.
$\boldsymbol{w}$	is a $Nx1$ vector containing the weights of each asset.

This utility function will have a global maximum which can be obtained by taking the derivative of U and setting it equal to 0. This will lead to the following solution:

$$\boldsymbol{\gamma} = \delta\boldsymbol{\Psi}\boldsymbol{w} \tag{2}$$

$$\boldsymbol{w} = (\delta\boldsymbol{\Psi})^{-1}\boldsymbol{\gamma} \tag{3}$$

As briefly mentioned in the introduction, the BLM uses the equilibrium excess returns, rather than the historical excess returns of an asset. These are derived through a reverse optimization problem. First, let us describe returns in the context of BLM. The returns are assumed to be normally distributed:

$$\boldsymbol{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{4}$$

This distribution is unobserved, and the mean of the distribution is treated as a random variable in and of itself:

$$\boldsymbol{\mu} \sim N(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma}) \tag{5}$$

The variance of this distribution is assumed to be $\tau\boldsymbol{\Sigma}$. That is, it is assumed to be proportional to the variance of $\boldsymbol{r}$. This is regulated through the parameter τ as briefly mentioned in the earlier discussion. $\boldsymbol{\Sigma}$ is in practice estimated as a covariance matrix of the historical excess returns of the assets where τ represents the uncertainty in the estimation. $\boldsymbol{\Pi}$ represents the equilibrium excess returns and can be derived as follows (Walter, 2014). In market equilibrium according to the CAPM all investors will hold the market portfolio representing all available assets[2]. With the only deviation being the amount invested in the risk-free asset, depending on the individual risk aversion of the investor. Meaning that the market equilibrium weights will be equivalent to the weights of the portfolio held by all investors. And so, the market equilibrium weights will be equivalent to the market capitalization weights overall. Hence the market capitalization weights can be used to derive the implied equilibrium excess returns. This is called reverse mean-variance optimization. If you use equation (2) and set the weights equal to the market capitalization weights while using the covariance matrix of the excess returns you get:

$$\boldsymbol{\Pi} = \delta\boldsymbol{\Sigma}\boldsymbol{w_{mkt}} \tag{6}$$

$\boldsymbol{\Pi}$	is a $Nx1$ vector containing the implied equilibrium excess returns.
$\boldsymbol{w_{mkt}}$	is a $Nx1$ vector containing the market capitalization weights.

This is the first step in deriving the BLM model (Black & Litterman, 1992). $\boldsymbol{\Pi}$ is an unbiased estimate of $\boldsymbol{\mu}$ as described by He and Litterman (1999):

$$\boldsymbol{\mu} = \boldsymbol{\Pi} + \boldsymbol{\varepsilon^{\mu}} \tag{7}$$

$$\boldsymbol{\varepsilon^{\mu}} \sim N(\boldsymbol{0}, \tau\boldsymbol{\Sigma}) \tag{8}$$

Assuming that δ is a correct estimation of the prevalent risk aversion on the market and $\boldsymbol{\Sigma}$ being a correct estimate of the covariance of the excess returns.

[2] See Sharpe (1964) for the capital asset pricing model.

Equation (6) can be implemented in practice. There is no set way of determining δ however. Black and Litterman (1992) simply refer to an earlier paper by Black (1989) which puts risk in the context of international equity portfolios. The parameter can however be described through a simplification using the following equation:

$$\delta = \frac{Sharpe\ Ratio}{\sigma_m} \tag{9}$$

The risk parameter is simply the Sharpe Ratio of the market divided by the market's standard deviation, σ_m. τ is usually estimated by using the fact that the variance is estimated around the mean of the returns. This allows τ to be calibrated as a maximum likelihood estimator:

$$\frac{1}{T} \tag{10}$$

When the sample size, T, increases the uncertainty of the prior decreases and vice versa. This is the most common way of estimating τ although there are other methods (Walter, 2014).

2.2 The Master Formula of the Black-Litterman Model

Described as "the master formula of the BLM" or "the master equations" by Walter (2014). These describe the expected return vector and the covariance matrix of the CRM. They show how the implied equilibrium excess returns (as derived earlier) are combined with those of the view portfolio. As well as the covariances of the view portfolio with the historical covariances. The equation of the posterior mean return vector is:

$$\widehat{\boldsymbol{\Pi}} = \boldsymbol{\Pi} + \tau\boldsymbol{\Sigma}\boldsymbol{P}^T[\boldsymbol{P}\tau\boldsymbol{\Sigma}\boldsymbol{P}^T + \boldsymbol{\Omega}]^{-1}[\boldsymbol{Q} - \boldsymbol{P}\boldsymbol{\Pi}] \tag{11}$$

τ is a parameter regulating Σ.

$\boldsymbol{P}$ is a KxN matrix representing the assets included in the views, K views about N assets.

$\boldsymbol{\Sigma}$ is a NxN covariance matrix of the excess returns of the assets.

$\boldsymbol{\Omega}$ is a KxK covariance matrix containing the uncertainty of each view.

$\boldsymbol{Q}$ is a $Kx1$ vector containing the excess returns of the views.

The posterior covariance matrix of the views is:

$$\boldsymbol{M} = \tau\boldsymbol{\Sigma} - \tau\boldsymbol{\Sigma}\boldsymbol{P}^T[\boldsymbol{P}\tau\boldsymbol{\Sigma}\boldsymbol{P}^{\boldsymbol{T}} + \boldsymbol{\Omega}]^{-1}\boldsymbol{P}\tau\boldsymbol{\Sigma} \qquad (12)^3$$

The total posterior covariance matrix is simply $\boldsymbol{M}$ combined with the covariance matrix of the excess returns:

$$\boldsymbol{\Sigma_p} = \boldsymbol{\Sigma} + \boldsymbol{M} \qquad (13)$$

In the complete absence of views equation (11) will just simplify to $\widehat{\boldsymbol{\Pi}} = \boldsymbol{\Pi}$. The intuition behind this is that if the investor holds no views about the assets, they will not make any deviations from holding a portfolio proportional to the equilibrium portfolio. In this case equation (13) will simplify to $\boldsymbol{\Sigma_p} = \boldsymbol{\Sigma}(1 + \tau)$ as the right-hand side of $\boldsymbol{M}$ will be equal to 0. As soon as views are introduced however, things get a bit more complicated (Walter, 2014).

The expected excess returns of the views are formulated through $\boldsymbol{Q}$. This $Kx1$ vector is altered depending on what views the investor wishes to specify. In the example in the introduction the investor expresses two different views, meaning that $\boldsymbol{Q}$ would be a $2x1$ vector. For a general case $\boldsymbol{Q}$ can be described through the following equation:

$$\boldsymbol{Q} + \boldsymbol{\varepsilon^Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix} \qquad (14)$$

Each q_k represents the expected value for each of the individual specified views. Continuing with the example from the introduction, q_1 would be equal to 5% since German stocks are believed to have an expected excess return of 5%. q_2 will in this case be 3% as Japanese stocks are believed to outperform Swedish stocks by a 3% expected excess return. Hence, it does not matter if the view is absolute or relative, q_k will be formulated in the same way. The error term of the views, $\boldsymbol{\varepsilon^Q}$, represents the investors estimation error and is assumed to have a mean of 0. As the investor should on average be correct in their view.

[3] This is the posterior covariance matrix represented using the Woodbury Matrix Identity, meaning that it is invertible even when $\boldsymbol{\Omega} = 0$ (Walter, 2014).

Making $\boldsymbol{Q}$ biased, for example thinking that the investor on average overestimates the views and making the mean positive, would add an unnecessary layer of complexity. The variance of $\boldsymbol{\varepsilon}^{\boldsymbol{Q}}$ represents the uncertainty in the views, $\boldsymbol{\Omega}$, which can be written as:

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_k \end{bmatrix} \tag{15}$$

$\boldsymbol{\Omega}$ is as previously described the diagonal covariance matrix of the views. It is assumed to be diagonal as a simplification leading to it containing only the variances of each view, ω_k. Managers are usually presumed to specify their views independently with no correlation between them. This is done for theoretical and practical reasons as adding for the possibility of non-diagonal values in $\boldsymbol{\Omega}$ would make what is already one of the most complex variables of the model even harder to compute. The two most simple cases of $\boldsymbol{\Omega}$ are when the investor is either 100% confident in the views or when the investor is completely uncertain in their views, 0% confident. If the investor is 100% certain in a view, $\omega_k = 0$. If the investor is completely uncertain $\omega_k \to \infty$. If the investor has 100% or 0% confidence in all views, then $\boldsymbol{\Omega} = 0$ and $\boldsymbol{\Omega} \to \infty$ respectively, because all diagonal values will be either zero or approach infinity. The intuition behind this is that if the investor is completely confident in their view, the view will not have any variance. Whereas if the investor is completely uncertain in their view the variance will be infinitely large as there is no confidence whatsoever. The inverse of the matrix, $\boldsymbol{\Omega}^{-1}$, is the confidence in the investors views as the inverse of uncertainty is certainty (or confidence). When the manager is completely uncertain in the views $\boldsymbol{\Omega} \to \infty$ and $\boldsymbol{\Omega}^{-1} = 0$ and the investor will have 0% confidence and so on (He & Litterman, 1999). For simplicity it is usually assumed that $\omega_k > 0$ to make sure that $\boldsymbol{\Omega}$ is invertible. However, because equation (12) is based on the Woodbury Matrix Identity it is possible to set diagonal values to zero. Unlike the standard version of the equation[4]. If the investor is completely certain in their views and specifies views on all assets, equation (11) will simplify to:

$$\widehat{\boldsymbol{\Pi}} = \boldsymbol{P}^{-1}\boldsymbol{Q} \tag{16}$$

Unless the number of views equals the amount of assets $\boldsymbol{P}$ will not be square and, of course, not invertible.

[4] For the standard version of the posterior covariances that does not use the Woodbury Matrix Identity see Walter (2014) for example.

What equation (16) means is that the investor will base their entire portfolio on the views. If the investor is completely uncertain in their views, then (11) will simplify in the same way as in the absence of views:

$$\widehat{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \tag{17}$$

This can be shown through a few steps of linear algebra. If $\boldsymbol{\Omega} \to \infty$, then the right-hand side of equation (9) will equal zero as the inverse of infinity is zero etc. (Walter, 2014). The process of determining the uncertainty in each view along with Idzorek's method will be described in the next chapter.

The KxN matrix $\boldsymbol{P}$ is used to describe how the views are formulated, either as absolute or relative views, and the assets involved. It has K rows, one for each view, and N columns, one for each asset. If an asset is not included in the view the corresponding value in $\boldsymbol{P}$ will simply be 0. Going back yet again to the example in the introduction, this would correspond to a $2x4$ matrix, 2 views and 4 assets:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \tag{18}$$

The first view being an absolute view that German stocks has an expected return of 5% this row will sum to 1. As German stocks is the third asset, the third column contains 1. The second view being a relative view that Japanese stocks will outperform Swedish stocks by 3% this row will instead sum to 0. The row of an absolute view will always summate to 1 and that of a relative view to 0 with the underperforming stock being negatively weighted. Another note is that all assets can be included in the relative view and the views may also contradict each other. In the case of contradicting views, they will be weighted through $\boldsymbol{P}$. The row of each view can be written as $\boldsymbol{p_k}$ which will be used later when describing the diagonal variances of $\boldsymbol{\Omega}$. When there is a large disparity in the market capitalization weights of assets in a relative view it can cause major shifts in the weights of the assets involved. For relative views including more than two assets a weighting scheme can be implemented to compensate for the difference (Idzorek, 2005).

Finally, to determine the weights, you need to go back to the result of mean-variance optimization in equation (3).

Using the new estimated excess return vector and covariance matrix of the portfolio the weights are derived:

$$\boldsymbol{w} = \left(\delta \boldsymbol{\Sigma}_{\boldsymbol{p}}\right)^{-1} \widehat{\boldsymbol{\Pi}} \tag{19}$$

These are the weights of the final portfolio using the CRM. When no views are specified or when there is a complete uncertainty in the views $\boldsymbol{\Sigma}_{\boldsymbol{p}} = \boldsymbol{\Sigma}(1 + \tau)$ and equation (19) will return weights that are proportional to, but less than those of the market capitalization weights. When using Idzorek's method, an Alternative Reference Model, the posterior covariances are not used. Instead, only the prior is used. What this means is that equation (19) will return the exact market capitalization weights in the absence of views or with 0% confidence as $\boldsymbol{w}_{\boldsymbol{mkt}} = (\delta \boldsymbol{\Sigma})^{-1} \boldsymbol{\Pi}$ (Walter, 2014).

Papers involving the BLM generally do not use any kind of constraints when deriving the final portfolio weights, including budget constraints (Walter, 2014). Idzorek (2005) does not use any constraints in his paper. He and Litterman (1999) also consider an unconstrained portfolio although they mention the fact that constraints can be applied in the final mean-variance optimization. If the total weight of the portfolio exceeds 100% the BLM suggests that the investor should leverage their portfolio. Whereas if the total weight is less than 100% the model suggests that the investor should not be fully invested into the market. In the context of the BLM using an unconstrained portfolio is consistent with the theoretical basis of the model. For example, if the investor states a relatively high degree of confidence the intuition of the model could be that the investor should leverage his or her position and vice versa (Walter, 2014).

2.3 Specifying $\boldsymbol{\Omega}$ and Idzorek's Method

The most common method of specifying $\boldsymbol{\Omega}$ is using the one outlined by He and Litterman (1999) and is required to compute $\boldsymbol{\Omega}$ as per Idzorek (2005). The standard method involves using diagonal values that are proportional to the historical covariance matrix. The covariance matrix of the views then becomes:

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{p_1 \Sigma p_1^T} * \tau & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{p_k \Sigma p_k^T} * \tau \end{bmatrix} \tag{20}$$

$\boldsymbol{p_k}$ will weight the covariance(s) depending on how row k is specified in $\boldsymbol{P}$. In the example of the view that German stocks will have an expected return of 5%, the variance (uncertainty) in the view will equal:

$$\boldsymbol{p_1 \Sigma p_1^T} * \tau = [0 \quad 0 \quad 1 \quad 0] \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{14} \\ \vdots & \ddots & \vdots \\ \sigma_{41} & \cdots & \sigma_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} * \tau = \sigma_{33} * \tau \qquad (21)$$

σ_{ij} is the historical covariance of the excess returns of assets i and j.

In this relatively simple example of an absolute view $\boldsymbol{p_k}$ will always extract the historical variance of the asset in question and turn it into a constant. In the case of relative views, the historical covariances of the assets will be weighted. In the example of Japanese stocks outperforming Swedish stocks (He & Litterman, 1999):

$$\boldsymbol{p_2 \Sigma p_2^T} * \tau = (\sigma_{11} + \sigma_{22} - \sigma_{21} - \sigma_{12}) * \tau \qquad (22)$$

Relative views with multiple assets, for example "German and US stocks will outperform Japanese and Swedish stocks by 2%", can also be specified. In this case the corresponding row values of $\boldsymbol{P}$ will be weighted by 1 divided by the number of assets out/underperforming the others. In the example all values would be ± 0.5 (Satchell & Scowcroft, 2000). Other weighting schemes can also be used in this case, for example compensating for the market capitalization weights (Idzorek, 2005).

The standard method described above shifts the intuitive focus of $\boldsymbol{\Omega}$ as a measure of uncertainty to the historical covariances. If the investor believes that these historical covariances can be used to accurately describe the uncertainty of the views it then becomes an issue of specifying τ. The uncertainty that the investor has in $\boldsymbol{\Sigma}$ and how it is scaled in proportion to the real covariances of the returns. This is one of the issues with using this approach, the intuition behind what $\boldsymbol{\Omega}$ represents becomes diffuse. It will now represent a combination of how the views are specified, the estimated historical covariances and the degree of confidence in the estimated historical covariances. It is no longer a clear description of the uncertainty that the investor has in their beliefs. The two simple cases previously describes where the investor has a 0% or a 100% certainty in their views are still intuitive. But what happens if a manager is say, 60% or 50% confident in their view? Representing this numerically is not as clear.

In an example He and Litterman (1999) simply doubles the variance in $\boldsymbol{\Omega}$ when a 50% confidence is stated. Idzorek (2005) points out that there is an implied level of confidence in all the views while using this approach. If the investor uses the standard method, it will carry with it a level of implied confidence regardless of if a level of confidence has or has not been formulated. To calculate this degree of confidence the investor needs to compute the weight vector based on the standard method and the weights derived under 100% certainty view by view. The last weights need to be computed k number of times, one time for each view. In practice it is done by considering each view one by one like it was the only view and setting $\boldsymbol{\Omega} = 0$. After this has been carried out k times a new posterior mean return vector can be used to derive the weights under 100% confidence, using the mean-variance formula shown earlier. The implied confidence level can be shown with the following equation (Idzorek, 2005):

$$C_k = \frac{\widehat{\boldsymbol{w}}_{\boldsymbol{k}} - \boldsymbol{w}_{\boldsymbol{k,mkt}}}{\boldsymbol{w}_{\boldsymbol{k,100\%}} - \boldsymbol{w}_{\boldsymbol{k,mkt}}} \tag{23}$$

$$C_k \in [0,1]$$

C_k	is the confidence in view k.
$\widehat{\boldsymbol{w}}_{\boldsymbol{k}}$	is a constant representing the weight derived using the standard approach in specifying Ω.
$\boldsymbol{w}_{\boldsymbol{k,mkt}}$	is a constant representing the market capitalization weight.
$\boldsymbol{w}_{\boldsymbol{k,100\%}}$	is a constant representing the weight derived under 100% confidence in the view.

Idzorek uses this fact to derive a method of going the other way around. Making it possible for the investor to specify a level of confidence in each view and then computing ω_k for each view. The method described by Idzorek involves seven steps using a least squares method[5]. Walter (2014) shows that an easier method can be applied to derive the same results as those described by Idzorek. Since the goal of this thesis is not to provide any further insight into the method in and of itself, I will use the one outlined by Walter (2014).

[5] See Idzorek (2005) for the full method.

First, $\boldsymbol{\Omega}$ can be described using the following equation:

$$\boldsymbol{\Omega} = \tau\alpha\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^T \tag{24}$$

$$\alpha \in [0, \infty]$$

Where α is the coefficient of uncertainty and can be calculated through the following formula:

$$\alpha = \frac{1 - C_k}{C_k} \tag{25}$$

Note that Walter (2014) does not include τ in equation (24), arguing that it can be dropped. But since Idzorek includes it, I will as well to stay consistent with his method. To derive the final covariance matrix based on the specified confidence levels, $\boldsymbol{\Omega_C}$, the investor must only use equations (24) and (25). This is done by first deciding on a confidence level for view k and calculating α with equation (25). Then plugging this value into equation (24) and extracting the variance, ω_k, from $\boldsymbol{\Omega}$. After this has been done once for each view the investor can now insert them into $\boldsymbol{\Omega_C}$. The diagonal values of this matrix will all be based on a level of confidence from 0% to 100% depending on what the investor has specified. Another thing to note is that as $C_k \to 1 \Leftrightarrow \alpha \to 0$ and vice versa. Meaning that a higher degree of confidence translates into reducing the variance of the view by reducing the historical variance. C_k will have a similar role to that of τ as it will scale $\boldsymbol{\Sigma}$ through α for the variance of each view. If for example, $C_k = 0.5$ then $\alpha = 1$ and equation (24) will be indifferent from that of the standard method. The difference between Idzorek's method and the standard method will then be how the investor determines the other variables and inputs. And the method developed by Idzorek is simply another way of specifying the same thing as the standard method. What this shows is the importance of $\boldsymbol{\Sigma}$ in both approaches as it will have a big effect on how $\boldsymbol{\Omega}$ will be determined. It is also related to the fact that τ is not of the same importance if Idzorek's method is used. This is because it partially loses its original theoretical interpretation and now purely becomes a scalar (Walter, 2014). Idzorek (2005) does not provide any insights or suggestions as to how C_k should be specified. He points out the fact that more information might be relevant to the investor than the historical covariances. And the fact that it can be used by qualitative investors to more easily specify $\boldsymbol{\Omega}$. The similarities with how τ regulates the variance of the views might make an investor hesitant to its usefulness. It does however enable investors to regulate the variance of the views more freely and easily than to only have τ regulate all of them (Idzorek, 2005).

2.4 Estimating the Historical Covariance Matrix

One of the primary issues of the MPT approach to asset allocation is providing adequate estimates of the means and the covariances of the returns. As pointed out by Michaud (1989) and as early as 1955 by Stein (1955) amongst others, the sample means and covariances cannot be used to accurately predict future returns. In the literature the focus often lies on the means but the covariances can heavily affect the process as well. Relatively extreme values can heavily distort the final weights when using mean-variance optimization. Further, different estimation methods can result in vastly different weights. Portfolio weights derived using the MPT framework can even perform similarly or worse than equally weighted portfolios. This is because the estimation errors can drive the process if prevalent enough and because of the inherent nature of financial data it is prone to estimation errors (Michaud, 1989).

Another issue is related to the distribution of the returns. In standard financial theory it is often assumed that returns are normally distributed as well as stationary. Empirical evidence suggests otherwise. The distribution of financial data may vary over time as stationarity does not necessarily hold. Meaning that the assets risk could vary over time. This can become an issue while using data spanning a relatively long timespan. As the assets risk in the earlier period might be different than that of the later period. If an investor suspects that this might be the case one potential remedy is giving different weights to each observed excess return as well as using higher frequency (usually daily) data. Shifting the impact towards more recent observations. This should in theory more accurately describe the present distribution. One disadvantage of this approach is serial correlation which violates the assumption of identically independently distributed returns. An example of this is when an investor uses global assets, for example, funds in countries with different opening hours for their financial markets. Although new information will become available at the same time, it might be incorporated into the price at different trading days. Using lags is a way to address this issue. Further, the distribution of returns often have fait tails meaning that outcomes at the extreme ends of the distribution are more likely than a normal distribution would suggest. There is no ad hoc way to solve these problems and economic intuition must often be applied to attain a practical solution. For example, serial correlation can be a larger issue while using daily data. While using monthly data you might partially avoid this problem you instead run into the problem of the risk varying over time as well as fewer observations (Litterman, 2003). The method described above is one of two common approaches when estimating the historical covariance matrix of the BLM.

The second one is using what is known as shrinkage estimation (Walter, 2014). The basic idea of shrinkage estimation is that the sample mean is an undesirable approximation of the assets true mean. And so, a technique of "shrinking" the sample mean towards what is perceived as a more acceptable value is applied. The extent of the shrinkage is determined by a shrinkage factor, w, which will be demonstrated in an equation below[6]. First, the expected return vector of this method, Bayes-Stein estimation, can be written as (Jorion, 1986):

$$E(\boldsymbol{r_{BS}}) = (1 - w)\boldsymbol{Y} + wY_0\mathbf{1} \tag{26}$$

$\mathrm{E}(\boldsymbol{r_{BS}})$	is a $Nx1$ vector containing the new Bayes-Stein expected returns.
w	is a constant representing the shrinkage factor.
$\boldsymbol{Y}$	is a $Nx1$ vector containing the sample means.
Y_0	is a constant representing the mean of the mean-variance portfolio.
$\mathbf{1}$	is a $Nx1$ vector simply containing 1's at each position.

The shrinkage factor basically functions as a weight by shifting the impact of the sample means towards Y_0, basically shrinking them. Y_0 is attained by performing the standard mean-variance optimization described earlier by using the sample means and covariances of the excess returns. The reasoning behind this is primarily based on statistics and not standard financial theory[7]. Bayes-Stein estimation does however have desirable properties while using financial data (Michaud, 1989). The equation of the shrinkage factor is the following (Jorion, 1991):

$$w = \frac{\lambda}{(T + \lambda)} \tag{27}$$

$$\lambda = \frac{(N + 2)(T - 1)}{[(\boldsymbol{Y} - Y_0\mathbf{1})^T\boldsymbol{S}^{-1}(\boldsymbol{Y} - Y_0\mathbf{1})(T - N - 2)]} \tag{28}$$

T	is the sample size.
$\boldsymbol{S}$	is a NxN matrix containing the sample covariances.

[6] Do not confuse this with the weight vector $\boldsymbol{w}$.

[7] It is however noteworthy that in the CAPM world all investors are believed to be mean-variance optimizers (Sharpe, 1964). Because the BLM is based on a CAPM world, basing the shrinkage on a mean-variance portfolio should be reasonable.

Next, the sample covariances are regulated through the following equation:

$$\boldsymbol{\Sigma}_{\boldsymbol{BS}} = \boldsymbol{S}\left(1 + \frac{1}{T + \lambda}\right) + \frac{\lambda}{T(\lambda + T + 1)} * \frac{\mathbf{1}\mathbf{1}^T}{\mathbf{1}^T\boldsymbol{S}^{-1}\mathbf{1}} \tag{29}$$

$\boldsymbol{\Sigma}_{\boldsymbol{BS}}$ is the historical covariance matrix adjusted using Bayes-Stein estimation. The left-hand side of the equation regulates the sample covariance matrix depending on the uncertainty of the sample mean. Large samples will correspond to a relatively small adjustment of the sample covariances and vice versa. And so, with a large enough sample using Bayes-Stein estimation will not have a noticeable effect on neither the sample mean or covariances. Smaller covariances correspond to a smaller adjustment and vice versa by changing the right-hand side of the equation (Jorion, 1986). Bayes-Stein estimation basically diminishes the effect of estimation errors in the standard mean-variance approach and can potentially greatly improve portfolio selection (Michaud, 1989).

3. Method

3.1 Sample of Assets Included in the BLM Portfolio

As briefly mentioned in the introduction I will use index funds to represent the equities market in nine different countries. When building a portfolio using the framework originally developed by Black and Litterman (1990) there are no set guidelines for exactly what kind of assets that are appropriate to use. In their paper they do however focus on constructing a global portfolio by using assets such as cash and bonds. He and Litterman (1999) use a global portfolio consisting of assets intended to represent the equities market in seven different countries. Idzorek (2005) takes a slightly different approach by looking at broad assets within the US market such as US large growth stocks and US bonds for example. In these papers the assets usually function as examples to illustrate the workings of the BLM such as in He and Litterman (1999) while Idzorek (2005) uses it to show how confidence levels work. In this paper I will basically follow a similar example to that of He and Litterman (1999). To approximate each market, that of Japan, the US, the UK, France, Germany, Canada, Korea, Taiwan (ROC) and Australia, I have used ETFs (exchange traded fund) from BlackRock. These ETFs are aimed at giving investors exposure to specific countries, usually by tracking a domestic index. For example, in the case of Japan the iShares JPX-Nikkei 400 ETF is employed as it tracks the JPX-Nikkei 400, which is in turn an index tracking large- and mid-cap stocks in Japan[8]. There is no specific reason as to why BlackRock ETFs are used other than the fact that all their ETFs that track indices have a market beta relatively close to 1. Meaning that they fulfill their purpose of tracking each of their respective market. As well as of course, readily available data. The return data is based on monthly NAV (net asset value) returns provided directly by BlackRock. In total there are 60 observations for each asset stretching a 5-year period from May 2016 until April 2021 (BlackRock, 2021). Other assets tracking the same indices and markets should provide reasonably comparable estimates as these ETFs are essentially just index funds. As described at the end of last chapter both daily and monthly data can be used, and both have their pros and cons. Usually daily data is used although there is nothing to suggest that monthly data cannot be used if the adequate estimation method is employed. When monthly data is used the most common sample size is 60, the same as this paper uses (Walter, 2014).

[8] A full list of all assets is included in the appendix.

The COVID-19 pandemic has heavily impacted stock markets throughout the world during the past year (McKinsey & Company, 2021). For this reason, I decided to look at a longer period, five years, rather than a shorter time frame. Looking at period of say one year, the initial shock of COVID-19 might provide unjustifiably big risk estimates. An even shorter period could provide inadequate return data as it would basically only capture the recovery period. This is of course based on my personal judgement and there are arguments as to why it might be of interest to choose a relatively short time frame as well. Finally, the risk-free rates for each country have been obtained through Investing. They are based on the 1-month yield of government bonds in each country. Whenever available I used 1-month bonds and if not, I used the shortest available maturity, for Germany it is based on 3-month government bonds for example (Investing, 2021).

3.2 Inputs

There are several different inputs required to compute the BLM weights. First, to compute the historical covariance matrix, I used the shrinkage estimation method outlined in the previous chapter. The sample covariances and means were computed from the ETFs NAV excess returns and the covariances where then adjusted by computing lambda as per equation (28) with a sample size, T, of 60. When using monthly data an investor will of course have a limited sample size unless you are willing to stretch the time period over decades (for a $T > 120$). This is the reason as to why I have chosen Bayes-Stein estimation as it will have a larger effect for smaller sample sizes (Michaud, 1989). The risk aversion parameter is estimated by using equation (9). To capture the global market risk aversion, I have used 10 different global funds containing assets in various countries. The Sharpe ratio of each fund has been divided by its standard deviation. The average of each of these values is used to capture an approximation of δ which in this case is approximately 4.946. The standard deviation of δ is approximately 0.35 which is relatively low and indicates that ten assets is appropriate. The Sharpe ratios and standard deviations have been provided through Morningstar (Morningstar, 2021).

The market capitalization weights have been obtained through Statista. The market capitalization represents the size of each country’s equities market compared to worlds total equity market value as of January 2021. The portfolio’s market capitalization weights are computed by putting them in relation to the combined global weight of each of the nine included countries. The sum of all included countries equals roughly 81% of the global equities market.

The US equities market is equivalent to roughly 56% of the world's total equities market and 69% of the market capitalization of the portfolio. Japan represents the second largest market with a market capitalization weight of 9%. The rest of the countries fall into the range of 5% to 2% (Statista, 2021). Using these values as well as the Bayes-Stein historical covariance matrix and δ the monthly implied equilibrium excess returns can be derived using equation (6).

Table 1. Market Capitalization Weights (Statista, 2021).

Asset:	Global Equity Market Weights:	Portfolio Market Capitalization Weights:
Japan	7.40%	9.15%
USA	55.90%	69.10%
UK	4.10%	5.07%
France	2.90%	3.58%
Germany	2.60%	3.21%
Canada	2.40%	2.97%
Korea	1.80%	2.22%
Taiwan	1.70%	2.10%
Australia	2.10%	2.60%
Sum:	80.90%	100.00%

The sample excess returns are quite different from the ones derived using the BLM. This could in some cases likely be owed to δ which has a value upwards of 5. If the prevailing risk aversion is relatively high investors will demand relatively high returns and vice versa. The covariances function in a similar manner by increasing or decreasing the returns.

Table 2. Sample Excess Returns and Implied Excess Returns (BlackRock, 2021) (Investing, 2021) (Statista, 2021).

Asset:	Sample Excess Returns:	Implied Excess Returns:
Japan	0.87%	0.66%
USA	0.35%	0.97%
UK	0.43%	0.90%
France	1.57%	0.86%
Germany	1.56%	0.93%
Canada	-0.02%	1.04%
Korea	-0.02%	0.98%
Taiwan	1.52%	0.79%
Australia	-0.28%	1.08%

Higher market capitalization weights will also contribute to increasing the implied excess returns (Bodie, Kane & Marcus, 2014). They are however relatively stable compared to the sample excess returns when observing table (2).

3.3 Views

The following two views have been specified to test the portfolio:

1. Canadian equities will outperform US equities by an excess return of 0.3% over the coming month.

2. Taiwanese equities will have an absolute performance of 1% excess return over the coming month.

The first view is a relative view, Canadian equities will perform better relative to US equities. The second view is an absolute view stating the belief that Taiwanese equities will have an excess return of 1%. Canada's implied excess return is 1.04% whereas the US implied excess return is 0.97% and so the view expresses a belief that the outperformance will be higher than expected. Taiwan's implied equilibrium excess return is 0.79% meaning that the view states that Taiwanese equities will perform 0.21% higher than what is implied. There is of course an infinite number of potential views and a larger number of views in total can be expressed. The views are arbitrary and not based on any forecast or personal judgement. They simply serve the purpose of testing the portfolio by adjusting C_k and the portfolio of the CRM. These relatively small adjustments compared to the implied excess returns are in line with, for example Idzorek (2005). The reasoning behind this is that relatively large values in the $\boldsymbol{Q}$ vector can result in large long and or short positions (Walter, 2014). Such views can of course be implemented if the investor believes it justifiable.

3.4 Sensitivity Analysis

The sensitivity analysis is performed by changing the value of C_k for each of the two views based on an interval I, of 10,000 values, ranging from $\frac{1}{10{,}000}$ to 1. C_k cannot be set to 0 as Idzorek's method does not work in this case, because of equation (25). However, since the lowest value will be $\frac{1}{I}$, as $I \to \infty$ the lowest value of $C_k \to 0$.

The interval size of 10,000 is easily adjustable, however, because of a limitation in computer power as well as ease of handling the data I decided on that number. It is also questionable whether increasing I would have any impact on the results. The final weights are computed through a loop created in MATLAB which carries out BLM portfolio optimization one time for each value in an array containing the interval. No constraints have been placed on the portfolio optimization process including a budget constraint. This is line with Idzorek (2005) and the CRM (He & Litterman, 1999). Adding constraints would alter the models and affect the results. Because the purpose is to compare and evaluate the models, I decided to not include any constraints. The risk-free asset is assumed to be a 1-month US Treasury Bill. The weights of the CRM using the standard approach to specifying $\boldsymbol{\Omega}$ through τ is also accomplished, separately from the loop. τ is defined through equation (10) and is equal to $\frac{1}{60}$.

4. Results

Table (3) shows the weights in three relatively extreme scenarios in the context of the BLM as well as those obtained using the CRM and the standard approach in specifying **Ω**. The weights of the portfolio with no views are equal to the market capitalization weights. This is because of equation (19), when using the Alternative Reference Model, the model will return the market capitalization weights when no views are specified.

Table 3. Final Weights Based on No Views, 0% Confidence, 100% Confidence and the CRM.

Asset:	No Views*:	0% Confidence*:	100% Confidence*:	CRM**:
Japan	9.15%	9.15%	9.15%	9.00%
USA	69.10%	69.10%	-6.60%	29.66%
UK	5.07%	5.07%	5.07%	4.98%
France	3.58%	3.58%	3.58%	3.53%
Germany	3.21%	3.21%	3.21%	3.16%
Canada	2.97%	2.97%	78.66%	41.22%
Korea	2.22%	2.22%	2.22%	2.19%
Taiwan	2.10%	2.10%	12.80%	9.01%
Australia	2.60%	2.60%	2.60%	2.55%
Sum:	100.00%	100.00%	110.69%	105.30%

Comment: * Denotes Idzorek's method and ** denotes the Canonical Reference Model.

In the portfolio with no views US equities will dominate with 69.1% of the total weight, followed by Japan with 9.15% of the weight with the rest of the countries having between a 5.07% and 2.1% weight. The sum of the portfolio is equal to 100%, suggesting that the investor in this case should be fully invest into the market. The weights of the portfolio with 0% (technically 0.01%) confidence are identical to those of the portfolio with no views. The weights of the assets involved in the views, Canada, Taiwan, and the US are also unchanged. The sum of this portfolio is 100%, once again suggesting that the investor should invest 100% of their budget into the portfolio, according to the BLM. With a 100% confidence level the weights of the assets involved in the views ends up increasing or decreasing both absolutely and proportionally. Whereas the weights of the assets not involved in the views are equal to those of the portfolio based on a 0% level of confidence while being proportionally smaller as the total weight increases to 110.69%.

Taiwan, involved in the absolute view, has a weight of 12.8% in this scenario compared to its original market capitalization weight of 2.1%. meaning that it is around six times as large. In the case of the relative view, Canada outperforming the US, the weights are significantly different from their original values as well. Canada is believed to outperform the US relatively and its weight in the final portfolio at 100% confidence is 78.66% whereas the weight of the US is -6.6%. Suggesting that a short position should be held in the asset representing the US equities market. The absolute difference between the weights in the case of a 100% and a 0% confidence level for the US is 75.7%. The equivalent value for Canada is approximately equal at 75.69% and its weight is around 26 times as large as the market capitalization weight. Finally, the total weight of 110.69% suggests that the investor should be 110.69% invested in the market. This means that the investor should leverage his/her portfolio if the BLM weights are taken at face value.

The portfolio weights computed using the CRM and the standard approach to specifying $\boldsymbol{\Omega}$ are different from those of the three prior extreme scenarios. First, the weights of the assets not involved in the views are equal in the case of no views as well as that of 0% and 100% confidence while using Idzorek's method. Whereas the CRM always regulates the values of the assets not involved in the views. The weight representing the US is 29.66%, less than half of the market capitalization weight and not comparable to any of the extreme scenarios. Canadian equities have a weight of 41.22%, roughly half of the equivalent weight with 100% confidence and substantially increased compared to its market capitalization weight. The weight of Taiwan is 9.01% in this case. The total portfolio weight of 105.3% means that the investor should in theory leverage their portfolio.

The variation in the weights of Canada, the US and Taiwan with respect to C_k is represented in the figures below, with figure (1) representing the first view and figure (2) representing the second view. As table (3) suggests the weights in the assets representing Canada and Taiwan increases whereas the weight of the US decreases with the level of confidence, which can be seen in both figures. Furthermore, there appears to be a non-linear relationship between C_k and the asset weight of Taiwanese equities, the absolute view. This relationship is seemingly exponential as the marginal effect of C_k increases for higher levels of confidence. The relationship between C_k and the assets of the relative view is instead a linear relationship. The weights are of course a discontinuous function of C_k in this computation as the interval can in practice never be fully continuous (you cannot set $I = \infty$ and $C_k = 0$).

The weight of the US asset starts at 69.1% for $C_k \approx 0$ and then ends up being equal to that of Canada at roughly 43% confidence. At a confidence of approximately 92% the model instead assigns it a negative weight. The weights of the assets involved in the views in the CRM portfolio are as previously described 29.66% for the US, 41.22% for Canada, and 9.01% for Taiwan. Using Idzorek's method, similar weights for US and Canadian equities occur around a 50-51% confidence level. Whereas Taiwanese equities have an equal weight at 53.7% confidence.

Figure 1. Weights of Canadian and US Equities for Each Level of Confidence.

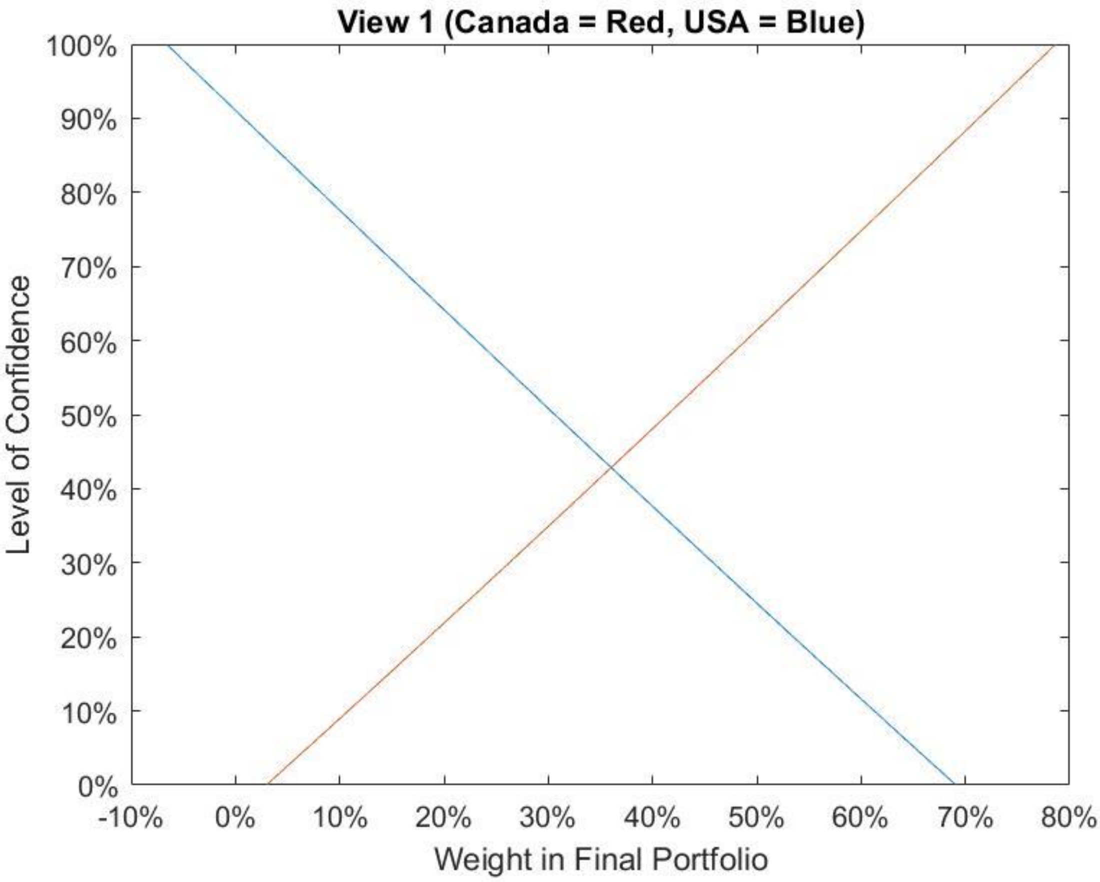
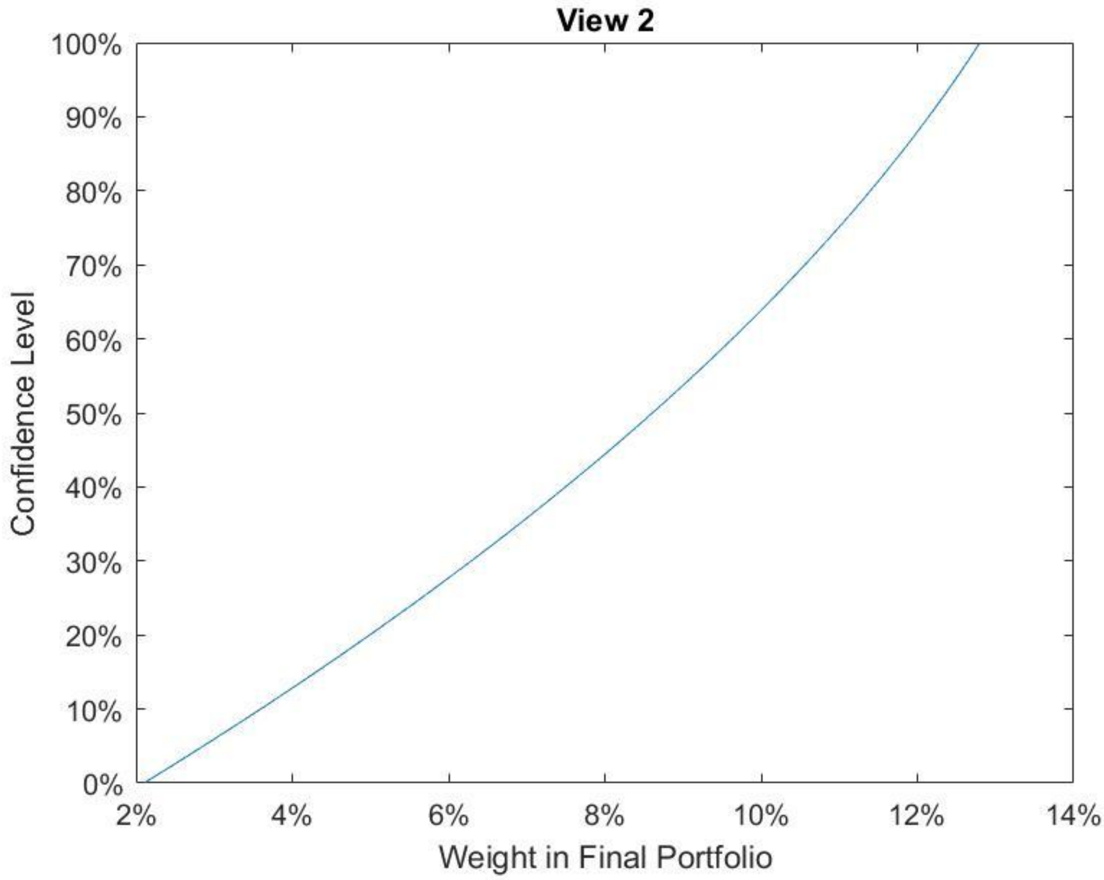


Figure 2. Weight of Taiwanese Equities for Each Level of Confidence.



5. Analysis and Discussion

The results indicate that C_k basically shifts the weight of Taiwan, the absolute view, non-linearly. For higher values of C_k its effect is greater whereas for lower values it is lower and so its marginal effect is increasing. The precise economic intuition behind this is not clear and Idzorek (2005) does not explain it in his paper, neither does Walter (2014). It could simply be a byproduct of the way that the model is set up. However, taken at face value it would basically imply that as an investor gets more confident, further confidence has a greater effect. If, for example, an investor is already 80% confident and then their confidence increases to say 85%, this will have a relatively large impact on the assets weight. Compared to an equivalent scenario but instead going from for example 20% to 25% confidence. This non-linear relationship can however not be observed for the relative view. The relationship between C_k and the weights of Canada and the US instead appears to be linear. Unlike the absolute view this means that the marginal effect is constant. It does not matter at which level of confidence that the investor has, increasing or decreasing it will have the same marginal effect. This obvious discrepancy between the absolute and relative view is rather illogical. Why should it make any difference if a view is relative or not? In practice this means that the investor cannot treat C_k in the same manner for the two types of views. As adjusting them in the same way will have different effects on the model outputs. Using the exact inputs and views of this paper the investor would get different effects when adjusting C_1 and C_2. The non-linearity of view 2 is relatively small and the weights it generates are also relatively small. However, the weight of Taiwan is still more sensitive to changes in C_2 for values closer to 1 and less sensitive to changes in C_2 for values closer to 0. Meaning that the sensitivity of the weight is not constant with respect to C_k. When adjusting C_1 the investor would not need to take this into account as the relationship appears to be essentially linear. Whether C_1 is close to 1 or 0 will have a negligible impact when making further adjustments.

What caused this discrepancy? It could partially be because of the way that $\boldsymbol{P}$ is specified. In the case of the absolute view $\boldsymbol{p_2}$ contains a single 1 in the eight column (matching the Taiwanese asset), with the rest of the values being 0. Whereas for the relative view $\boldsymbol{p_1}$ will weight the two assets against each other. $\boldsymbol{p_1}$ contains a -1 in the second column representing the US and a 1 in the sixth column representing Canada.

What this in effect means is that any increase in the weight of the Canadian asset will be matched by an equal decrease in the weight of the US asset. This can be observed in the results, the combined weight of Canada and the US is 72.07% for 0% confidence and 72.06% with a complete confidence. What C_k does in this case is basically just shift weight from US equities to Canadian equities linearly with an increased level of confidence. Whereas in the case of the absolute view, represented by Taiwan, there is no such limitation. This could affect the relationship by making it less susceptible to becoming a non-linear relationship since the weights regulate each other. Another thing to take into consideration is τ as it scales $\mathbf{\Omega}$ through equation (24). This could have a different effect on the relative view of Canada and the US compared to the absolute view. The scaling effect of τ could have driven what was a non-linear relationship towards a linear relationship. Because $\tau = \frac{1}{60}$ it would essentially regulate the relationship by decreasing the impact of α and thereby C_k. This suggests that whether a view is linear or not depends on the inputs of the model as well as Idzorek's method itself. As a smaller sample size for example would increase the size of τ and so on. Walter (2014) suggests that τ can be dropped from equation (24). Clearly this would alter the effect of C_k. This oversight could be because τ does not matter at a confidence of 0% and 100%. At 100% confidence τ has no effect on the relationship because the diagonal value of $\mathbf{\Omega}$ is 0 as $\alpha = 0$. At 0% confidence $\alpha = \infty$ (technically 0.01% confidence in this case) and so $\Omega \rightarrow \infty$ meaning that τ will have no effect either. Further suggesting that the relationship between C_k and the final portfolio weights is often overlooked. Because the two views have a different relationship to C_k it is not unreasonable to assume that this could happen again with different inputs. Basically, if an investor uses Idzorek's method the marginal effect of C_k could be different each time and for different views.

The Taiwanese asset starts of at the market capitalization weight of 2.1% at 0% confidence and then increases to 12.8% at 100% confidence. Its weight has nothing counterbalancing it and simply increases or decreases with C_k without affecting the other assets. This shift in the Taiwanese asset represents the entire shift in the total weight of the portfolio. The sum of the asset weights at 0% confidence is 100% and then gradually increases to 110.7% which is equivalent to the entire shift in the weight of the Taiwanese asset. This is because the combined weight of the assets in the relative view remains unchanged and the weights of the assets not involved in any view also remain unchanged. What this means is that the absolute view is the only thing driving the total weight of the portfolio. And in practice whether the model tells an investor whether to leverage their portfolio.

Given that this result is so precise, it should be the case for any portfolio put together using this method. Basically, when using Idzorek's method, any portfolio including absolute views should result in a leveraged portfolio unless 0% confidence is specified. This would essentially make Idzorek's extension pointless to any investor that does not wish to or cannot leverage their portfolio. The method outlined by Idzorek does not consider any budget constraints although his method could be altered by adding them. Even though a budget constraint would go against the logic of the model. In a scenario with a budget constraint the weights of the assets not involved in any views should in theory be shifted towards Taiwan for higher degrees of confidence. Instead of driving the total portfolio weight above 100% and suggesting a leveraged portfolio. This could however have some conceivable disadvantages. For example, in a relatively extreme scenario it could cause some of the final weights to receive exceptionally low values. To the extent where it might not be worth considering the asset when construction the portfolio in practice because its weight is negligible. This is of course purely speculative and there could be other possible advantages and disadvantages.

The results of the CRM do not indicate any similar problem with respect to leveraging the portfolio. Using this model, the Taiwanese asset's weight is 9.01%, meaning that its weight has increased by 6.91% compared to the total weight of 105.3%. This is not surprising as τ regulates all assets in the CRM through the posterior covariances, even those not included in the views. Whereas Idzorek's method only uses the historical covariances and therefore returns the original market capitalization weights for any asset not included in a view. Further, the CRM will not return the original market capitalization weights if a complete uncertainty is specified for the same reason, unlike Idzorek's method (Walter, 2014). The $\boldsymbol{P}$ matrix is specified in the exact same manner for both models in this paper and so if the CRM is investigated further some similarities with it and Idzorek's might be easier to spot. Because the $\boldsymbol{P}$ matrix is specified in the same way it will have a similar effect as in Idzorek's method. The main thing driving whether the model suggests a leveraged portfolio should be absolute views as well. But because the default is not a portfolio identical to the market equilibrium portfolio the way that it drives the model will be different. And so, an absolute view with a confidence other than 0% should not necessarily cause a leveraged portfolio for example[9].

[9] Reminder: The CRM does not consider any explicit confidence level in this paper.

The assets of view 1, Canada and the US, are more sensitive to C_k than the asset of view 2, Taiwan. The market capitalization weight in Canadian equities is 2.97% and 2.1% in Taiwanese equities. At 100% confidence they do however end up being weighted entirely differently at 78.66% and 12.80% respectively. This can be seen even at moderate degrees of confidence, for example, at 60% confidence the weights are 48.94% and 9.63% respectively. What the latter example would suggest is that even when an investor is only 60% confident in their view, they should increase the weight of Canadian equities by a factor of 16 compared to its original weight. This could partially be a result of how the views relate to the implied equilibrium returns. When using the CRM and Idzorek's method, views that deviate to a relatively large degree from the implied equilibrium returns will generate more extreme weights (Idzorek, 2005). And so, it is not surprising that the more "extreme" view is also more sensitive to changes in C_k. Because view 1 deviates from the implied equilibrium returns to a larger degree compared to that of view 2 it is more sensitive. The difference is however not very substantial. In the case of Taiwan, the view states that Taiwanese equities will have an absolute excess return of 1%, compared to the implied equilibrium return of 0.79%. The equilibrium return of Canada is 0.07% higher than that of the US whereas the view states a belief that it will in fact be 0.3% higher. Basically, the excess return of view 1 deviates by 0.23% and view 2 deviates by 0.21%. Another important factor that is relevant to view 1 is the original equilibrium weights. Relative views are weighted against each other, meaning that if one of the views has a relatively high market capitalization weight it can cause the weight of the other asset to increase drastically. The market capitalization weight of US equities is well above that of Canada and the other assets. Which means that the relatively high original weight of the US will incrementally be shifted towards the Canadian asset as C_k increases. Both factors are probably relevant to the sensitivity of the weight of the Canadian and US assets in this case. Basically, how $\boldsymbol{Q}$ is specified, and the market capitalization weights should affect the sensitivity with respect to C_k. The market capitalization weights affecting view 1 can also be observed in the CRM where the final weights are 41.22% and 29.66% for Canada and the US, respectively.

An interesting aspect as to how C_k regulates the weights of the assets involved in the views is for what levels of confidence it is comparable to the CRM. Idzorek's method does not produce exactly equivalent weights for view 1, it is however similar at 50-51% confidence. The weight of Taiwan is equal, 9.01%, at approximately 53.7% confidence. The implication of this is basically that the weights of the CRM are roughly equivalent to a 50% confidence in the views.

If the weights of the assets involved in the views of the CRM are seen as “reasonable”, a value of $C_k \approx 0.5$ could be considered as a good starting point for an investor using Idzorek’s method. Any deviation from $C_k \approx 0.5$ would then be considered a deviation from the baseline and decrease or increase the weight of the assets in the views. For example, an investor might not be confident enough in view 1 to warrant investing 41% into the Canadian equities market and 30% in the US equities market. C_1 could then be reduced from 50% to say 30%, instead producing weights of 26.25% and 45.81% respectively. The level of C_k that produces comparable weights will of course vary with the inputs of the model and so it is not necessarily 50% for all views. The fact that it is roughly 50% is not surprising when considering equation (24). If $C_k = 0.5$ then $\alpha = 1$ and $\mathbf{\Omega}$ will be equal in both models. The reason for the divergence in results is the fact that Idzorek’s method uses the historical covariances in the mean-variance framework to derive the final weights. Whereas the CRM uses the posterior covariances. And so, what causes the difference from 50% confidence is how the posterior covariances affect the final weights. Looking at equations (12) and (13), the posterior covariances will always be greater than the historical covariances, everything else being equal. Because the posterior covariances are greater, as it is inverted in equation (19), they will generally decrease the portfolio weights compared to Idzorek’s method. This can be seen when looking at the assets not included in the views of the CRM portfolio. Because the interactions between the covariances also matter this could have slightly increased the weights of the CRM comparably. Leading to slightly above 50% confidence being required when using Idzorek’s method to approximately match the weights in this case.

6. Conclusion

In summary, the results show that effect of C_k and its relationship with the estimated asset weights can vary for different views and inputs, and that the CRM is relatively comparable to Idzorek's method at roughly 50% confidence. The fact that the relationship between C_k and the asset weights can be both linear and non-linear has interesting implications. The weight of the absolute view of Taiwanese equities has an exponential relationship with C_k. What this means is that the marginal effect of C_k will increase as opposed to when it is linear. If an investor adjusts his/her confidence it will have different effects depending on their original level of confidence if the marginal effect is not constant. And so, the sensitivity of the weights with respect to C_k can vary for the same view. The relative view of Canadian and US equities is linear, showing that within the same portfolio, the relationship with C_k can differ. An investor would have to consider this fact as the marginal effect of C_k can vary not only between different portfolios, but also within the same portfolio. Idzorek (2005) and Walter (2014) do not show or make any inquiries into this fact. The sensitivity of the weights involved in the two views are different when using Idzorek's method. This is most likely due to a combination of how the views are specified and the market capitalization weights. The second view involving the US and Canada is more sensitive as the market capitalization weight of the US is relatively high compared to the other assets, including Canada. And the fact that it deviates to a higher degree from the implied equilibrium returns than the second view of Taiwan. These two factors, the specified excess return and the market capitalization weights, affect how sensitive a view is to C_k. The weights derived using the CRM are, unsurprisingly, equivalent to those of Idzorek's method at around 50% confidence. The reason as to why it is similar is because of the way that Idzorek's method shifts the weights with the level of confidence. The reason as to why it is not precisely equal is because the CRM uses the posterior covariances whereas Idzorek's approach only uses the prior. Another difference is the weights of the assets not involved in the views. When using Idzorek's method they are unaffected by C_k whereas in the CRM all assets will always be regulated through at least τ. A noteworthy point is that absolute views will always generate a leveraged portfolio while using Idzorek's method for any degree of confidence other than 0%. Finally, when taking all things into consideration, both versions of the BLM in this paper generate relatively stable outputs, which is a testament to the BLM itself. Further research could for example consider ways as to how investors should specify C_k in practice as this question remains unanswered.

7. References

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Appendix

Table 4. the Index Fund Representing each Country.

Country:	Asset:
Japan	iShares JPX-Nikkei 400 ETF
USA	iShares Core S&P 500 UCITS ETF
UK	iShares FTSE 250 UCITS ETF
France	iShares MSCI France UCITS ETF
Germany	iShares MDAX UCITS ETF
Canada	iShares MSCI Canada UCITS ETF
Korea	iShares MSCI Korea UCITS ETF
Taiwan	iShares MSCI Taiwan UCITS ETF
Australia	iShares MSCI Australia UCITS ETF

Table 5. Global Funds used to Calculate the Global Risk Aversion.

Global Funds:	Risk aversion:
Amundi Index Solutions - Amundi Index MSCI World UCITS ETF	4.676
HSBC MSCI World UCITS ETF | HMWD	5.009
HSBC Multi Factor Worldwide Equity UCITS ETF	4.497
Invesco MSCI World UCITS ETF	4.943
iShares Core MSCI World UCITS ETF (Acc)	4.878
Lyxor MSCI World (LUX) UCITS ETF	4.754
SPDR® MSCI ACWI UCITS ETF	4.839
UBS(Lux)Fund Solutions – MSCI World Socially Responsible ETF	5.778
Vanguard FTSE All-World UCITS ETF	4.856
Xtrackers (IE) Plc - Xtrackers MSCI World ESG UCITS ETF	5.229
Average:	4.946