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Comparing Risk Parity Portfolios

Does a Tail-Risk Parity strategy provide better downside protection than the Risk Parity strategy during economic crisis?

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Abstract

This thesis evaluates the risk parity and tail-risk parity approach against conventional weight budgeting approach. The risk parity and tail-risk parity approach, in contrast to weight budgeting approach, is about distributing the risk between the asset classes in the portfolio. The risk, is traditionally measured in terms of volatility together with the assumption that returns follows a Gaussian distribution. In these thesis, different risk measure will be introduced together with an Empirical distribution. The main objective of this thesis is to evaluate and improve the tail-risk parity approach by applying Expected Shortfall as risk measure, to capture if it will provide a better protection against downside risk during the economic crises; Dot-com bubble, Global Financial crisis and Covid-19 recession, as opposed to the risk parity approach with volatility as risk measure. Our results suggest, based on the performance measured in risk-adjusted returns, that the tail-risk adjusted approach had a superior performance in relation to risk parity and the weight budgeting approach during the full period. When considering the downside protection during the economic crisis, the tail-risk parity and risk parity approach performed fairly even, where tail-risk parity approach showed slightly higher risk-adjusted returns during Dot-com bubble and Covid-19 recession.

Keywords: Tail-risk parity, Risk parity, Expected Shortfall, Weight budgeting, Risk budgeting, Capital Allocation, Risk Contribution

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1 Introduction

Portfolio selection theory has a long history starting with the influential work by Markowitz (1952) introducing the mean-variance framework where one maximizes the return given a level of volatility, this subsequently shaped much of what is modern portfolio theory. The technique of performance budgeting requires considerable computational work, including estimating expected returns, variance-covariance matrices, and an optimization technique. Chopra, Hensel & Turner (1993) and Best & Grauer (1991) show that this form of a mean-variance optimization is sensitive to the estimation and assumptions about the model inputs.

A different perspective on portfolio selection theory that does not require expected returns or optimization techniques, i.e., it does not suffer from the above-mentioned issues, is the weight budgeting approach. This approach is intuitive as it prescribes assigning investment capital per pre-defined portfolio weights (Roncalli, 2014). This includes the equally weighted portfolio, or as Benartzi & Thaler (2001) refer to as the naïve diversification strategy of $1/N$, as well as the traditional 60/40 portfolio popularized by John C. Bogle of Vanguard, among others. This technique seemingly derives diversification by spreading the investment capital across different assets, and some authors have shown promising results using the equally weighted portfolio (DeMiguel, Garlappi & Uppal, 2009). Despite this, it is not hard to imagine a situation where this selection method fails to diversify, for instance, in a crisis or a situation where assets have distinctly different intrinsic risks. Indeed Qian (2011) shows that the equity component of a 60/40 portfolio contributes to 90% of the portfolio risk.

Furthermore, the failure of the weight budgeting portfolios to properly diversify risk during the subprime mortgage crisis in 2008 increased the effort to obtain a different portfolio selection method; one such proposed method is the risk budgeting approach (Qian, 2011). First introduced in practice by Bridgewater Associates in 1996 with the introduction of their *All-Weather* asset allocation strategy (Steward, 2010) as a strategy that performs independently of economic circumstances. This strategy was later coined the risk parity approach by Qian (2005) as a particular case of the risk budgeting approach, where the strategy is focused on equalizing asset risk allocation in the portfolio. Hence instead of allocating capital per $1/N$, the assets are allocated such that risk contribution is equalized. This approach can be viewed as a compromise between the performance, weight budgeting approaches, still demanding estimations of risk, risk contribution, and an optimization technique using Euler's decomposition of a homogenous risk function but foregoing the calculation of expected returns (Cesarone & Colucci, 2017).

Traditionally, the risk budgeting approach has used asset and portfolio volatility as the risk measure (Bai, Steinberg, & Tutuncu, 2016, Maillard, Roncalli, & Telitche, 2010, Roncalli, 2014). Others have expanded on the risk parity approach using expected shortfall to measure tail risk (Cesarone & Colucci, 2017).

This thesis aims to empirically evaluate the performance and downside protection of the risk and tail risk parity approach together with the weight budgeting approaches, the equally weighted portfolio, and the 60/40 portfolio. The methods will be evaluated through several performance measures in the long-run setting of 1999-2021. In addition, the method will be applied using short runt portfolios during three distinct economic crises; the Dot-com bubble, the Global Financial crises, and the Covid-19 recession by the performance measures.

The remainder of the thesis is organized as follows. Section 2 presents the background of the weight and risk budgeting approaches and previous research. Section 3 presents the theoretical framework including estimations of risk, risk contributions and performance measurements. Section 4 presents the data and methodology and the modeling of the tests performed in the thesis. Section 5, presents the findings. Section 6 contains an analysis and reflection of the portfolios and comparisons against theoretical models and previous studies.

2. Background

In this section, a brief background on necessary and relevant information for this thesis will be provided. Knowledge regarding the different portfolio approaches will be discussed, together with previous work related to the subject and the additional risk measures used in each approach.

2.1 Weight Budgeting Approach

An essential question for portfolio managers and investors alike is how the investment capital should be allocated in the portfolio. One method for resolving this problem is the versatile and manageable weight budgeting approach. This method does not suffer from the computational difficulties of the mean-variance framework and appears relatively intuitive as it prescribes assigning investment capital per pre-defined portfolio weights (Roncalli, 2014). Hence while a mean-variance framework allows some discretionary power concerning which assets are included in the portfolio, the weight budgeting approach consists of the additional freedom to decide how the assets in the portfolio should be allocated.

A classic example of a weight budgeting strategy is the 60/40 portfolio. This allocation approach was popularized with the foundation of the balanced index fund in 1990 by John C. Bogle, the founder of Vanguard. The philosophy of the portfolio is that a balanced passive portfolio of 60% risky equities and 40% less risky bonds is a viable investment strategy that does not rely on beating the markets. Furthermore, John C. Bogle (1990) mentions the importance of only rebalancing during exceptional situations since one could end up with taxable gains and incurring trading costs. The 60/40 portfolio posits that diversification and risk budgeting occurs from the portfolio's weighting among these asset classes.

Another example of a widely used weight budgeting strategy is the equally weighted portfolio (Windcliff & Boyle, 2004) or, as Benartzi & Thaler (2001) refer to, the naïve diversification strategy. This strategy proposes that investment capital is allocated equally across asset classes. Hence, this strategy is more universal than the 60/40 portfolio because it considers assets beyond equities and bonds. Furthermore, as shown by Choueifaty & Coignard (2008), the diversification of this portfolio comes closest to the *Most diversified portfolio*. While this allocation method appears simple, it has a proven track record of showing good out-of-sample results (DeMiguel, Garlappi & Uppal, 2009).

The combination of possible portfolio compositions and the discretionary nature of the weight budgeting approach and, as Qian (2005) mentioned, a well-understood investment axiom on investing is that weights will matter, implying that this method is still relevant today. However, the downside of this method is that diversification is derived from capital allocation into a wide variety of different assets per weight. Hence, while the 60/40 portfolio might appear balanced in portfolio weights, it is highly concentrated from the perspective of risk allocation (Qian, 2005). Indeed, multiple authors have shown that weight budgeting often fails to provide proper diversification since assets risk contribution to the portfolio risk is not determined by the weight of the assets in the portfolio but rather with the intrinsic risk and cross-asset correlation of the assets. For example, Qian (2011) and Bai, Scheinberg & Tutuncu (2016) show that the equity component of the 60/40 portfolio contribute to 90% of the portfolio risk, while multiple authors have shown similar results for the equally weighted portfolio (Maillard, Telietche & Roncalli, 2010, Bai, Scheinberg & Tutuncu (2016). Furthermore, the failure of the weight budgeting approach to provide risk diversification during times of financial stress led many investors to question the diversification benefits of the weight budgeting approach and an increased effort to obtain a different portfolio selection method, where the risk budgeting approach is one proposed method (Qian, 2011).

2.2 Risk Parity Approach

Failure breeds innovation, and the global financial crisis 2007-2009 prompted renewed interest in the prospect of a truly diversified portfolio. In the aftermath of the crisis, a portfolio allocation strategy that garnered interest was the idea of risk budgeting with particular consideration for the special case of risk budgeting; equal risk contribution (ERC), or commonly referred to as risk parity (Qian, 2011).

As previously discussed, the weight budgeting strategies suffer from portfolio risk being concentrated in the risker proportion of the portfolio. This lack of risk diversification makes the portfolio more sensitive to significant drawdowns. The risk budgeting approach, first suggested by Qian (2005), instead focus on estimating and diversifying the asset's individual risk contributions in the portfolio. In the risk budgeting approach, the particular case of ERC has garnered significant academic interest (Maillard, Telietche & Roncalli, 2010, Bai, Scheinberg & Tutuncu, 2016). The ERC is compelling as it parallels the equally weighted portfolio but where risk contributions supplant portfolio weights as the equalized variable. Indeed, the ERC also shares similarities to the minimum-variance portfolio in the mean-variance framework, which minimizes risk contributions but only on a marginal basis (Maillard, Telietche & Roncalli, 2009).

Multiple methods have been proposed for the estimation of ERC. Maillard, Telietche & Roncalli (2009) derive a risk parity portfolio using Euler’s theorem of homogenous functions and a sequential quadratic programming solution that states the problem as a numerical problem to be solved by an algorithm. The portfolios are empirically benchmarked against the equally weighted and mean-variance portfolios with the result that the equally weighted portfolio dominates the ERC portfolio concerning returns. However, the ERC dominates when adjusting for risk. Stefanovits (2010) finds similar results using both a simple approach of equalizing risk contributions using the harmonic means and a similar numerical algorithm that solves the risk parity portfolio using historical data.

Bai, Scheinberg & Tutuncu (2016) instead opts for a least square’s solution to the risk parity problem using both a convex and non-convex approach. The former approach sometimes fails to deliver a unique solution while the latter, in this case, approximates a risk parity portfolio. Cesarone & Colucci (2017) use a non-parametric historical bootstrap method, and Cagna & Casuccio (2014) extends this by using historical resampling to minimize estimation errors to derive the ERC portfolio using both volatility and expected shortfall as risk measures. The empirical conclusion finds dominance of the ERC portfolios when benchmarked against the equally weighted approach.

Sokoloff & Zaytsev (2014) developed a model aiming to replicate the risk parity strategy. The author concluded that one of the main advantages of the Risk Parity portfolio is that it achieves high Sharpe ratios using only beta risk. These returns can be achieved through investments in ETFs that replicate the index performance of one given asset class index. Secondly, the portfolio will not be as dependent on the returns of the equities, which implies that the sensitivity to equity market tail risk events will be lower. Moreover, the naïve approach of portfolio construction is somehow developed. In the naïve approach of constructing risk parity portfolios, one uses standard deviation measures of asset classes and their covariance matrix. After that, the risk contribution of each asset class is allocated to the portfolio risk in equal proportions.

Historically the most used risk measure in the risk budgeting approach is the asset- and portfolio volatility (Bai, Scheinberg, & Tutuncu, 2016., Maillard, Telietche & Roncalli, 2009). However, volatility as a risk measure has also been criticized as a risk measure in the risk parity framework. Inker (2010) warns that volatility does not adequately assess an asset's inherent risk. Hence other risk measures have been suggested, Alankar, Depalma, Scholes (2012) suggests using the downside risk measure Expected Shortfall, while Bellini *et al* (2021) suggests using expectiles.

2.3 Tail Risk Parity Approach

The usage of volatility as a risk measure has its share of critics. Keppler (1990) points out the disconnect between investors' perception of risk and the risk of losing money and volatility, which measures the mean assets' squared deviation. Furthermore, Inker (2010) argues that since volatility treats deviations symmetrically as a risk measure, it fails to properly account for risk when risks are asymmetric. In addition, Boudt, Carl, Peterson (2013) criticize the implied assumption of gaussian normally distributed asset returns in the volatility risk measure. Therefore, risk budgeting with alternative risk measures have been suggested with particular consideration for the downside risk measure expected shortfall due to its desirable properties as a coherent and convex risk measure (Colucci, 2011)

Furthermore, Bhansali (2011) studied the importance of and practical implementation of portfolio tail risk management. The authors stress the significance of taking uncertainty and fatter tails into account and controlling for severe events, and prescribes tail risk hedging as a viable tool when protecting the portfolio from drawdowns. However, given that a tail hedging strategy ensures against drawdowns means abandoning some possible upside. The author specifies the importance of active and dynamic tail-risk management when allocating assets to the portfolio. Tail-risk hedging is seen as offensive risk management, as it allows access to liquidity to increase exposure to risk assets as the asset has the highest prospective returns. Furthermore, the authors argue that conventional tail risk parity scales exposure against volatility in the portfolio composition while it could be that one might over-over-ensure the portfolios as it only could be noise.

Alankar, Depalma, Scholes (2012) coined the term tail risk parity (TRP) as the risk parity method where expected shortfall supplant volatility as the risk measure. One of the key advantages of the tail risk parity is that it offers assets from different TRP buckets, including mostly non-overlapping drawdowns. This then implies that the weighting would ensure the portfolio manager that the depth of drawdowns in all components included in the TRP portfolio would be pretty similar. This shift from using volatility to expected shortfall means the risk budgeting approach retains its intuitive nature, and the application of a pure downside risk measure should ideally provide better protection against downside risk. Furthermore, if assets are gaussian normally distributed and the volatility adequately captures the tail risk, then the risk parity approach is a subset of tail risk parity (Alankar, Depalma, Scholes, 2012) Hence, if the risk of tail events would be fractional, one could expect the TRP allocations to resemble the Risk Parity allocations.

While TRP and its benefits are relatively new, multiple studies have been published on estimation methods and how it compares with both risk parity, performance budgeting, and weight budgeting portfolios. For instance, Alankar, Depalma, Scholes (2012) derive implied expected tail loss, a measure of expected shortfall derived from options market information, and finds that TRP dominates the RP approach in protecting against the downside. In addition, the protection is of such magnitude that it is more cost-efficient downside protection than purchasing such protection through options markets, as suggested by Bhansali (2011).

Colucci (2011) estimates ES risk contributions utilizing both the convexity property of ES and a filtered bootstrap method. They find no significant outperformance between ERC utilizing ES or volatility. Furthermore, they find no such outperformance when compared with the equally weighted portfolio. However, accounting for costs related to rebalancing the ERC portfolio using ES incurs fewer transaction costs.

Cagna & Casuccio (2014) use a non-parametric approach with a bootstrap resampling procedure to estimate the ERC for ES. They observe that levered TRP portfolios exhibit higher risk-adjusted returns and lower drawdowns than performance budgeting and the equally weighted portfolio. Cesarone & Colucci (2017) argue that a solution to TRP might not be attainable and instead approximate TRP using a similar non-parametric bootstrap method. The empirical results contradict Cagna & Casuccio (2014). TRP displays smaller returns than the equally weighted portfolio but dominates after adjusting for risk. However, TRP is subsequently dominated by the minimum variance approach after risk adjustment.

Boudt, Carl, Peterson (2013) instead approximate the minimum-variance portfolio by minimizing the most considerable ES contribution using Cornish-Fisher expansion to minimize portfolio ES and offer downside risk diversification. Since they are not equalizing risk contributions, this is not a TRP portfolio. Similar to Cagna & Casuccio (2014), the empirical results place it between the minimum variance portfolio and the equally weighted portfolio when it comes to performance. Hence this portfolio appears to be a more conservative minimum variance portfolio.

In addition to supplanting volatility, Bruder, Kostyuchyk,& Roncalli (2016) introduce a model that incorporates skewness risk in the risk budgeting framework. By implementing a gaussian mixture model distribution to model returns that also include volatility jumps. Empirically, this jump-induced TRP model displays similar performance as Colucci (2011), with no significant outperformance compared with the risk parity approach and smaller rebalancing costs. Vu (2018) argues that the relationship between diversifying skewness and relying on the correlation parameter as volatility diversification is a weak form of diversification. Therefore, the author used a similar method as Roncalli et al. (2016), implementing the expected shortfall as a risk measure alongside the Gaussian Mixture Model to estimate the parameters in the model. The idea of formulating prior information as convex constraints either on the source of the information parameters and using the constraints when solving constrained convex optimization problems. The author models an allocation algorithm of risk, which is then minimized based on the constraints. Based on this algorithm, they overcome the shortcomings of volatility-based risk parity by higher estimated risk-adjusted returns and lower maximum drawdown.

3 Theory

In this section, the theoretical background will be presented. Section 3.1 covers distributions including probability-, Gaussian-, and Empirical distributions. 3.2 covers the coherency of risk measures and the risk measures utilized for the risk budgeting portfolios. 3.3 describes risk budgeting and an approach for attaining the portfolios. In conclusion, 3.4 covers the performance measures used for evaluating the portfolios.

3.1 Probability Distributions

The use of probability distributions helps one understand how likely it is for different events to occur. Assume a random variable X, the distribution function of this variable will be defined as $P(X \leq x)$And provides the probability that X will take on a value less than or equal to x. If X is a continuous random variable, it will apply that there exists a density function $f_X(t)$:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt. \quad (1)$$

3.1.1 Gaussian Distributions

The Gaussian distribution or the normal distribution is applied in the conventional risk budgeting models, with volatility as the risk measurement. The Gaussian distribution is a continuous probability distribution often used when aiming to model natural phenomena. The univariate probability density distribution for the mean μ and the standard deviation σ is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{(x-\mu)^2}{2\sigma^2}\right], x \in \mathbb{R} \quad (2)$$

Furthermore, the Gaussian distribution in a multivariate setting is given by the following density function:

$$f(x) = \frac{1}{2(\pi)^{\frac{n}{2}}} \Sigma^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(x-\mu)' \sum{}^{-1} (x-\mu)\right] \quad (3)$$

Here, μ and $\sum$ represents the mean vector and the variance-covariance matrix, respectively.

3.1.2 Empirical Distribution

The empirical distribution function is a non-parametric method for estimating parameters of an unknown distribution function. Hence, parameter estimates are derived from observations instead of parametric methods like the gaussian, making distributional assumptions (Wellner, Shorack, 1986). Assume an unknown distribution function $F(x) = P(X \leq x)$For the observations $x_1, x_2, \dots, x_n$ of the identically and independently distributed (IID) random variables $X_1, X_2, \dots, X_n$. The empirical distribution $F_n(x)$ is defined by:

$$F_n(x) = \frac{1}{n}\sum_{1}^{n} \mathbb{I}_{\{x_i \leq x\}} \quad s.t \ \mathbb{I} = \begin{cases} 1 \ if \ x_i \leq x \\ 0 \ \ otw \end{cases} \quad (4)$$

The estimated parameters used in the empirical distribution function are the mean, standard deviation, covariance, and correlation is given by the sample- mean, $\bar{X}$, standard deviation, s_x, covariance, s_{xy} and correlation, r_{xy} respectively:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s_x = \sqrt{s_x^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$s_{xy} = \frac{1}{n-1}\sum\nolimits_{i=1}^{n}[(x_i - \bar{x})(y_i - \bar{y})]$$

$$r_{xy} = \frac{s_{xy}}{s_x s_y} = \frac{\sum_{i=1}^{n}[(x_i - \bar{x})(y_i - \bar{y})]}{[\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2]} \quad (5)$$

Since the empirical distribution only relies on actual observations, it can estimate risk measures, including volatility, VaR, and ES. The advantage of this method is that it is simple and widely applicable. However, the disadvantage is that since the method relies on observed information, the estimation is sensitive to extreme values. An extreme variation still in the estimation window will affect estimates beyond what might seem reasonable, contributing to more extreme estimates than would be the case in a well-fitted parametric distribution.

3.2 Risk Measures

3.2.1 Coherent Risk Measurements

Following Artzner *et al.* (2009), A risk measure $\mathcal{R}$, is said to be a coherent risk measure if it satisfies the following properties:

1. **Subadditivity**

$$\mathcal{R}(x_1 + x_2) \leq \mathcal{R}(x_1) + \mathcal{R}(x_2) \quad (6)$$

The risk of a portfolio of assets cannot be greater than the risk of combining the separate assets. Hence a coherent risk measure will encourage diversification.

2. **Homogeneity**

$$\mathcal{R}(\lambda x) = \lambda \mathcal{R}(x) \; if \; \lambda \geq 0 \quad (7)$$

Leveraging (deleveraging) a portfolio contributes to an equivalent increase (decrease) in the risk measure of the same magnitude.

3. **Monotonicity**

$$if \; x_1 \prec x_2 \; then \; \mathcal{R}(x_1) \geq \mathcal{R}(x_2) \; (8)$$

If x_2 first-order stochastically dominates x_1 in all possible scenarios than the risk measure $\mathcal{R}(x_1)$ should be higher than the risk measure $\mathcal{R}(x_2)$.

4. **Translation Invariance**

$$if \; c \in \mathbb{R}, then \; \mathcal{R}(x + c) = \mathcal{R}(x) + c \; (9)$$

Adding a pure cash position of size c to a portfolio $\mathcal{R}(x + c)$ reduces the portfolio's risk by the same magnitude.

Acerbi & Tasche (2002) show that expected shortfall is a coherent risk measure while Value-at-Risk violates the axiom of subadditivity (Artzner *et al.*, 1999). In addition, volatility is not a coherent risk measure since it violates both the monotonicity and the hypothesis of translation invariance.

3.2.3 Value-at-Risk

Value-at-risk (VaR) is a commonly used statistical measure of the risk and magnitude of a loss for an investment over a specific time. Precisely, VaR measures the loss at the a-quantile of a profit & loss distribution (P&L), for example, what is the worst possible loss at a 5% probability. The definition for VaR is as follows:

$$\mathcal{R}(x) = VaR_a(x) = \min\{\, x \mid \Pr\{\, X > x \,\} \leq 1 - a \,\} \; (10)$$

VaR has the advantage of being intuitive and easy to use. However, VaR is not a coherent risk measure since it does not satisfy the axiom of subadditivity and hence does not encourage diversification (Acerbi & Tasche, 2002). Another limitation of VaR is that it does not capture the information in the tail of the $P\&L$ distribution; hence it does not provide investors with all the necessary information for managing risk (Linsmeier & Pearson, 2000). If financial data were normally distributed, this would not be a huge problem (Roncalli, T., 2014); however, it is a stylized fact that financial data exhibits, among other things, fat tails and excess skewness. While this inherent flaw was well established, it was the failure of VaR models during the financial crisis 2007-2009 (Oanea, & Angelache, 2015) that accelerated the regulatory- and financial institutions move towards expected shortfall. Since VaR ignores tail information, it is not used as a tail measure but instead used as part of the calculation of ES.

3.2.4 Expected Shortfall

Expected shortfall (ES) or conditional value-at-risk (CVaR) introduced by Rappoport (1993) is a risk measure that incorporates all the information in the tail. By design, it is a more conservative risk measure than VaR. The formal definition of ES for confidence level a is given by:

$$\mathcal{R}(x) = ES_a(x) = \frac{1}{a}\int_0^a VaR_a(x)du \quad (11)$$

$$ES_a(x) = E[\, x \mid x \leq VaR_a(x)] \quad (12)$$

A reasonable interpretation of the first definition is that ES is the average of all VaR estimates beyond and including value at the a-quantile. The second definition, while equivalent, is even more intuitive, ES is the expected value of losses conditional on losses being more significant than or equal to VaR. ES contrasts with VaR as a coherent risk measure. A graphical explanation of the differences between the ES (CVaR in the figure) and VaR is shown in the figure below.

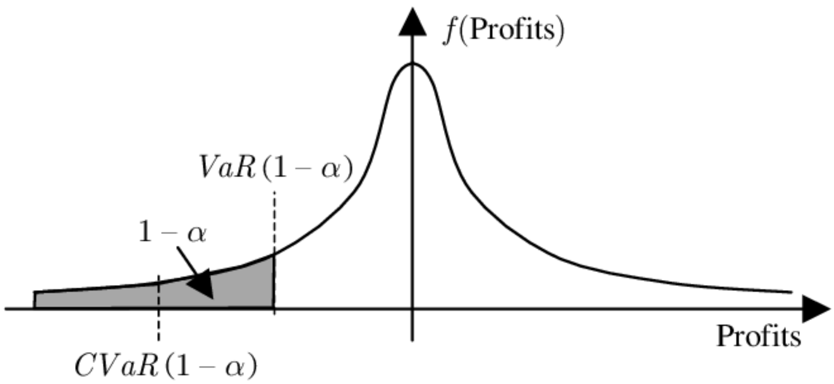

Figure 1. ES (CVaR) is the darkened density & Value-at-risk quantile at $(1 - a)$

Like (Cesarone & Colucci 2017), ES is estimated using historical simulation and is utilized to measure tail risk and subsequently risk budgeting measure in the tail risk parity portfolio.

3.3 Risk Budgeting and Capital Allocation Portfolio Construction

3.3.1 Risk Budgeting General

The risk budgeting approach is one of many portfolio allocation methods; two other notable methods are the weight allocation approach and performance-based budgeting. In short, the weight allocation method allocates capital per some weight-based preference (60% stock and 40% bond in the 60/40 portfolio). In contrast, the performance-based allocation method allocates capital per some pre-defined risk-adjusted return goal. The risk budgeting approach instead allocates capital following the desired risk allocation. The most intuitive form of risk budgeting is the equal risk contribution (ERC) approach, which allocates capital to equalized risk contributions across assets.

3.3.2 Euler Allocation Principle

The first step to risk budgeting is the estimation of the risk measure. The second step is to decompose the portfolio risk into a sum of individual risk contributions by assets. Denault (2001) illustrates the widely used Euler principle of risk allocation.

Assume the portfolio profit and loss distribution, X. This can be decomposed as the sum of n asset X_i:

$$X = \sum_{i=1}^{n} X_i$$

Consider the risk measure $\mathcal{R}(P\&L)$. Following Tasche (2008), a general risk-adjusted performance measure, Θ for asset i of portfolio $P\&L$ can be defined as:

$$\Theta(X_i|X) = \frac{E(X_i)}{\mathcal{R}(X_i \mid X)}$$

Tasche (2008) states two desirable properties of risk contribution:

1. The sum of individual risk contributions satisfies the entire allocation property if:
$$\sum_{i=1}^{n} \mathcal{R}(X_i \mid X) = \mathcal{R}(X)$$
2. Risk contributions $\mathcal{R}(X_i \,|X)$ are risk-adjusted-performance compatible if there are some $\varepsilon_i > 0\ s.t.$:
$$\Theta(X_i|X) > \Theta(X) \rightarrow \Theta(X + hX_i) > \Theta(X)\ \forall\ 0 < h < \varepsilon_i$$
This can be interpreted as assets with better risk-adjusted performance than the portfolio retains this edge if the allocation increases in small increments.

Tasche (2008) shows that if these two properties are satisfied, the individual risk contribution can be uniquely determined, and the risk measure is homogenous of degree 1.

This implies that the risk contribution of asset i is uniquely defined as (Roncalli, 2014):

$$\mathcal{R}C_i = x_i \frac{\partial \mathcal{R}(X)}{\partial x_i}$$

And this satisfies the Euler decomposition and results in the Euler allocation principle:

$$\mathcal{R}(x) = \sum_{i=1}^{n} x_i \frac{\partial \mathcal{R}(x)}{\partial x_i} = \sum_{i=1}^{n} \mathcal{R}C_i \quad (14)$$

3.3.3 Risk Parity or Equal Risk Contribution

The conventional risk measure used in risk parity portfolios has historically been volatility, assuming that asset returns are gaussian normally distributed. The portfolio risk is defined as:

$$\mathcal{R}(x) = \sigma(x) = \sqrt{x'\Sigma x}$$

Using Euler's allocation principle, the individual risk contribution of asset i in the portfolio and the proof that the sum of risk contributions equals the portfolio risk can be shown as:

$$\mathcal{R}C_i = x_i \frac{(\Sigma x)}{\sqrt{x'\Sigma x}}, \qquad \mathcal{R}(x) = \sigma(x) = \sum_{i=1}^{n} x_i \frac{(\Sigma x)}{\sqrt{x'\Sigma x}} = \frac{x'\Sigma x}{\sqrt{x'\Sigma x}} = \sqrt{x'\Sigma x} \quad (15)$$

For the estimation part of this thesis, the target of the risk budgeting approach is to attain a portfolio where the individual assets have equal risk contribution, i.e., a risk parity portfolio or equal risk contribution portfolio. How this portfolio is reached is shown in section 4.

$$\mathcal{R}C_i = \frac{\sigma(x)}{n} \quad (16)$$

3.3.4 Tail Risk Parity or Equal Tail Risk Contribution

The risk budgeting approach can be extended to include risk measures beyond volatility. To capture the tail risk of the portfolio, consider expected shortfall as a risk measure. As the case with risk parity, Roncalli (2014) show that per the Euler allocation principle, the portfolio tail risk can be decomposed into the sum of the asset tail risks $i, \dots, n$:

$$\mathcal{R}(x) = ES_a(x) = \sum_{i=1}^{n} x_i \frac{\partial ES_i}{\partial x_i} = \sum_{i=1}^{n} \mathcal{R}C_i(x)$$

Using the expression for expected shortfall defined in equation (13), the individual risk contribution of assets i can be defined as:

$$\mathcal{R}C_i(x) = x_i \frac{\partial ES_a}{\partial x_i} = x_i \frac{1}{1-a} \int_a^1 VaR_u du = x_i ES_i \quad (17)$$

Like the risk parity portfolio, an equal risk contribution approach target is set, and the method for which this is attained is expanded upon in section 4.

3.3.5 Capital Allocation Portfolios

Two portfolios using a weight budgeting approach are considered to benchmark the risk parity and tail risk parity portfolio against a more conventional weight budgeting approach. The weights considered are the classical 60/40 portfolio and an equally weighted portfolio. The construction of these portfolios is expanded upon in section 4.

3.4 Performance Measurements

3.4.1 Sharpe Ratio

The Sharpe Ratio (SR) (Sharpe, 1966) is one of the financial industry's most prevalent performance measures of assets and portfolio. The ratio provides investors with better information and comparability by relating investment return to the associated risk measured by volatility. The following expression defines SR:

$$SR = \frac{r_p - r_f}{\sigma(x)} \quad (18)$$

Where r_p, r_f And $\sigma(x)$ is the return for the portfolio, the risk-free rate of return, and portfolio volatility, respectively.

3.4.2 Tail Risk-Adjusted Returns

To capture the returns relative to tail risk, the Tail Risk-Adjusted Returns (TRAR) are considered. TRAR (Maillard, 2018) is an adjusted Sharpe ratio where a measure of tail risk replaces the volatility. We can define the TRAR as:

$$TRAR = \frac{r_p - r_f}{ES_a(x)} \quad (19)$$

Where r_p, r_f and $ES_a(x)$ are the portfolio return, risk-free rate of return, and expected shortfall for the portfolio at alpha a respectively.

3.4.3 Maximum Drawdown

Maximum Drawdown (MDD) (Acar & James, 1997) is a technical analysis tool used to analyze movements from a peak to troughs in the asset price before another peak is achieved. Note that MDD only considers the magnitude of the losses and not frequency. The measure can be used as an indicator of relative riskiness when comparing assets concerning capital preservation. MDD is defined using the following formula:

$$MDD = \frac{Trough\,Value - Peak\,Value}{Peak\,Value} \quad (20)$$

An intuitive interpretation is that MDD measures the maximum loss of an investor who buys an asset at the peak price and sells it at the bottom following the peak. This can be shown graphically as follows:

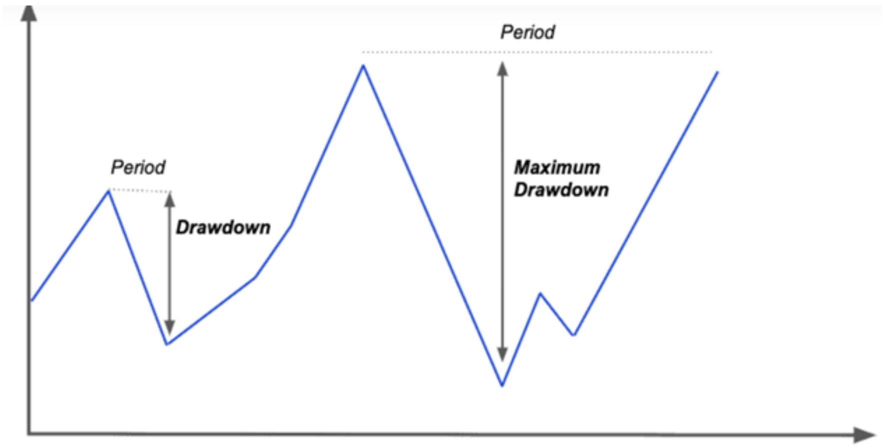

Figure 2. Maximum drawdown (FinanceTrain,2019) where Maximum drawdown defined as the maximum loss from a peak-to-trough during an investment period

4. Data and Methodology

Section 4 consists of two subsections data and methodology. Section 4.1 covers the data, sources, and how portfolios of asset classes are constructed. 4.2 provides a brief overview and then describes the estimation method of risk measures, risk contributions, the portfolio rebalancing approach, and the numerical problem a solver algorithm solves to find portfolio weights for the risk budgeting portfolios.

4.1 Data

To capture the portfolio during periods of economic hardship, including the Dot-com bubble, the Global Financial Crisis, and the Covid-19 recession, a daily total return set ranging from 1998 – 2021 was chosen. Assets were chosen based on retail and institutional availability, different return-risk profiles, and differing correlations. This includes equity, bonds, commodities, real estate investment trusts (REITs), and credit indexes. To approximate a buy, reinvest, and hold strategy, the total returns indices of the asset classes are used. To increase compatibility between data sets of different starting dates, asset class the total return indexes are combined into the five categories stocks, bonds, commodities, REITs, and credit. The assets used, including time lengths, data sources, and weights in the portfolios above, are presented in the table.

Asset Class	INDEX	Ticker	Length	Source
Stocks	S&P500 Total Return Index	SPXT	1999-2021	Datastream
	MSCI World Total Return Index	MSWO	2001-2021	Datastream
Bonds	Bloomberg Barclays Global Aggregate index	LEGATRUU	1999-2021	Bloomberg
	Bloomberg Barclays Corporate Investment Grade Index	LGCPTRUU	2001-2021	Bloomberg
	Bloomberg Barclays US corporate High-Yield	LF98TRUU	2000-2021	Bloomberg
Commodities	Bloomberg Commodity Total Return Index	DJUBSTR	1999-2021	Datastream
REIT	Wilshere Real Estate Investment Trust index	-	1999-2021	FRED*
Credit	Markit Investment Grade Credit Index	CDXTIL15	2007-2021	Bloomberg
	Market high-yield credit index	CDXTHL15	2007-2021	Bloomberg

* ST. Louis Federal Reserve Economic Data

Table 1. Description of Asset Classes.

The commodities and REIT asset classes are available from the start of the period and simulate an exposure towards the commodities market and real estate investment trusts. The Bloomberg commodities index is liquidity and dollar-weighted to affect a diversified exposure towards commodities as an asset class. Wilshere US REIT index measures the performance of publicly-traded real estate securities and acts as a proxy for direct real estate investments (Wilshere, 2020).

To construct a stock asset class with a broad passive investment strategy in the US stock market with a cross-border exposure, the total return indices for the S&P500 (SPXT) and MSCI World (MSWO) were used. MSWO is only available from 2001, and onwards, the index is added to the stock asset class portfolio when available and weighted following the table above.

To simulate exposure to broad investment in the bond market, portfolio bonds of the LEGATRUU, LGCPTRUU and LF98TRUU indices are constructed. The three indices capture different risk-return profiles in the bond sector, where LEGATRUU is a global measure of investment-grade, treasury, government-related, corporate, and securities bonds. LGCPTRUU captures global investment-grade bonds while LF98TRUU captures the high yield junk bond market.

An exposure towards debt markets is simulated by including the total return indices for CDXTHL15 and CDXTIL15. The indices capture exposure to high-yield and investment-grade credit risk products, including credit default swaps and exchange-listed products.

4.2 Methodology

Methodology overview:

To evaluate the long-term performance of the risk budgeting and weight budgeting approaches, portfolios with quarterly rebalancing frequency are run between 1999-2021. In addition, to assess whether the risk budgeting approach protects against downside risk during economic stress, short-term portfolios are constructed with approximately monthly rebalancing frequency during the Dot-com bubble, Financial crisis, and Covid-19 recession. Since the life expectancy of the short-term and long-term portfolios differ, the estimation window for risk and risk contribution differs in estimation length. For both the long-term and short-term portfolios, the period in between rebalancing dates is a period where the allocations are allowed to fluctuate following the movements in the market. For both the risk budgeting and weight budgeting approach, the initial value of the portfolios is equivalent to an initial investment of $100.

In line with the (Basel committee, 1996), the long-term portfolios use a historical window of the previous 252 trading days for estimations of the risk measures, variance-covariance matrix, and subsequent risk contributions. The short-term portfolios initiate and end only months before and after the crisis, respectively. To better capture changes in the risk measures, the estimation window is limited only to capture the most recent 100 days. This gives a slightly more variable risk measure while also retaining a sufficiently large sample of historical observations. However, the smaller sample necessitated a slight adjustment in the estimation of ES. The alpha at which ES is measured was increased from 2.5% to 5% to increase the number of expected observations in the tail and to increase comparability with the expected tail observations in the long-term portfolios.

Estimation of risk measures and risk contributions:

To construct the risk budgeting portfolios, the risk measures and the asset risk contributions are calculated from historical data using a rolling window technique. The risk parity approach with volatility as a risk measure requires forecasting the variance-covariance matrix by estimating the historical variance-covariance matrix:

$$\Sigma_t^* = \Sigma_{t-1-n}$$

Where Σ_t^* and Σ_{t-1-n} is the variance-covariance forecast at time t and the historical variance-covariance matrix for the estimation window $t-1, \dots, t-n$, where n is the size of the estimation window. Using the forecasted variance-covariance matrix Σ_t^*, the portfolio risk at time t for the vector of weights x is defined as:

$$\mathcal{R}_t^{\sigma}(x) = \sqrt{x' \Sigma_t^* x}$$

This process is subsequently repeated for each day using a rolling window technique. Similarly, the risk contribution of asset i at time t is forecasted using the forecasted variance-covariance matrix Σ_t^*. The individual risk contribution of asset i at time t is estimated using the following formula:

$$\mathcal{R}C_{it}^{\sigma} = x_{it} \frac{\partial \mathcal{R}(x)}{\partial x_{it}} = x_{it} \frac{\Sigma_{it}^* x_{it}}{\sqrt{x' \Sigma_t^* x}} \quad (21)$$

Where x_{it} is asset i's weight in the portfolio and Σ_{it}^* Is the forecasted variance-covariance matrix of asset i by the rest of the portfolio.

The tail risk parity approach requires forecasting of ES. Like the risk parity approach, the data set is split into estimation and a forecasting part. There are both parametric and non-parametric methods for estimating ES (Taylor, 2008). During testing, the results derived from the parametric method were deemed unsatisfactory due to its reliance on the volatility. Using volatility in the RP and TRP would have resulted in identical portfolios.

The non-parametric historical simulation method is utilized. Forecasting ES using this method requires the forecasting of VaR as a threshold value. For the profit and loss distribution X^i of asset i at time $t-1, \dots, t-n$, The forecast of $VaR_{a,t}(x)$ and $ES_{a,t}(x)$ at time t is given by:

$${VaR_{a,t}^{i}}^{*}(x) = \min\{\, x \mid \Pr\{\, X_{t-1-n} > x\}\,\} \leq 1 - a \,\} \quad (22)$$

$$ES_{a,t}^{i}{}^{*}(x) = E[\, X_{t-1-n} \mid X_{t-1-n} \geq VaR_{a,t}(x)] \quad (23)$$

The forecasted $VaR_{a,t}(x)$ is the a-quantile of X at time $t-1, \dots, t-n$ and the forecasted $ES_{a,t}(x)$ is the expected value of losses larger than or equal to this threshold value in X. With a rolling window technique, this estimation is carried out each day to generate estimates of ES for the rebalancing days.

As was shown in *section 3.3.4* and *equation 17,* the individual tail risk contribution of asset i can be decomposed into the asset weight at time t, x_{it} times the forecasted ES, $ES_{a,t}^{i}{}^{*}(x_{i,t})$:

$$\mathcal{R}C_{it}^{ES} = x_{i,t} ES_{a,t}^{i}{}^{*}\left(x_{i,t}\right) \quad (24)$$

Portfolio rebalancing approach:

The risk budgeting approach lacks a clear-cut analytical method for estimation (Roncalli, 2014). However, portfolios that satisfy the risk budgeting target can be approximated. By stating the problem as a minimization problem and allowing an algorithm to numerically minimize the target function by varying the portfolio weights subject to some constraints, a unique solution can be found. A general definition of the minimization problem is given by:

$$x = \min \sum_{i=1}^{n} \left[\frac{\mathcal{R}C_i}{b_i \mathcal{R}(x)} - 1\right]^2 \quad s.t. \sum_{i=1}^{n} x_i = 1, \; \sum_{i=1}^{n} \frac{1}{b_i} = 1\,, \mathcal{R}C_i \geq 0$$

Where x, b_i, $\mathcal{R}(x)$ and RC_i Is the resulting vector of portfolio weights, the risk budget target of asset i, the portfolio risk, and the individual risk contribution of asset i respectively. The first and second constraint guarantees that the portfolio is fully invested and that each asset attains the desired risk weight in the portfolio risk. The last constraint restricts assets with a negative risk contribution. This constraint is crucial since a negative risk contribution implies an infinite amount of possible portfolio weight solutions. Imagine two assets with identical but opposite risk contributions; a position in these two assets creates a risk-neutral position that is infinitely scalable.

For the risk parity and tail risk parity portfolios, the target risk allocation is equally distributed risk. Hence, $b_i = 1/n$ the following minimization problem is defined:

$$\min \sum_{i=1}^{n} \left[\frac{\mathcal{R}C_{it}}{b_i \mathcal{R}(x)} - 1\right]^2 = \min \sum_{i=1}^{n} \left[\frac{\mathcal{R}C_{it}}{1/n\, \mathcal{R}(x)} - 1\right]^2 \ s.t. \ \sum_{i=1}^{n} x_{i,t} = 1\,, \mathcal{R}C_i \geq 0 \quad (25)$$

This minimization problem is subsequently repeated at each rebalancing date.

4.2 Weight Budgeting Portfolios

While the risk budgeting approach requires an algorithm to solve each rebalancing day's optimal weights, the weight budgeting portfolios only require rebalancing to satisfy adherence to the pre-specified weight budgeting target. The initial capital is allocated per weight budgeting target, and at rebalancing, the portfolio is rebalanced following:

$$P_t^{\frac{60}{40}} = 0{,}6\, P_{t-1} + 0.4\, P_{t-1}, \qquad P_t^{EW} = \sum_{i=1}^{N} \frac{1}{N} P_{i,t-1} \quad (26)$$

Where P_t and P_{t-1} is the sum of asset values at time t and $t-1$. More intuitively, at rebalancing time t, the weight budgeting portfolios rebalance the portfolio capital available the day before at time $t-1$ following either the 60/40 or equally weighted depending on desired weight specification.

4.3 Performance Evaluation

To evaluate the performance of the risk-, tail risk parity and weight budgeting portfolios the performance measures described in *section 3.4* including SR, TRAR and MDD is calculated.

5 Results

In this section, the results of the study will be presented. The portfolios will be evaluated by the above specified performance measure. Moreover, the risk contribution of the asset classes together with the capital allocations will be presented. thereafter, the portfolios will be evaluated during the Dot-com bubble, Global Financial crises and Covid-19 recession in smaller sub-periods.

5.1 Portfolio Development Over Time

In figure 3, the performance of the portfolios for the period 1999 – 2021 with a starting value of 100 are displayed.

Figure 3. Portfolio development over time.

Given that all portfolios except the 60/40 portfolio contain the same assets but in different weight they display similar developments over time. A surprising find is the almost identical paths of the RP and TRP portfolio. However already here one can observe that the RP and TRP portfolio show signs of providing better downside protection compared with the weight budgeted portfolios. While the equally weighted portfolio tracked the RP and TRP closely prior to the financial crisis, the portfolio never recovered from the drawdown associated with this period, illustrating the need for downside protection. The 60/40 Portfolio while also affected by the crisis, recovers and overtakes the equally weighted portfolio due to its exposure to the post crisis bull-market.

5.2 Capital allocation And Risk Contribution

5.2.1 60/40 Portfolio

In figure 3, the capital allocation for the 60/40 portfolio is presented. Here, the portfolios are rebalanced quarterly with 60% in stocks and 40% in bonds. The portfolio retains its 60/40 allocation well even in the periods between rebalancing dates where it is allowed to fluctuate with the development of the market.

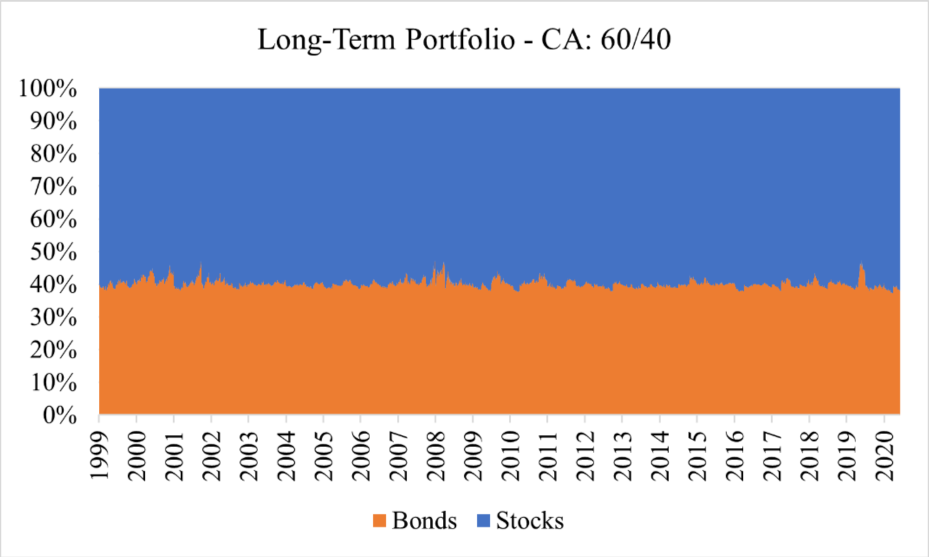

Figure 4. Capital allocation for the 60/40 Portfolio

In figure 5, the risk contribution of the 60/40 portfolio is presented. As expected, the majority of the risk contribution in the 60/40 portfolio originates from the larger exposure to stocks. This result is in line with what previous studies found (Qian, 2011) that the 60/40 portfolios exposure to the higher volatility of the stock market contributes to a larger than portfolio weights risk contribution.

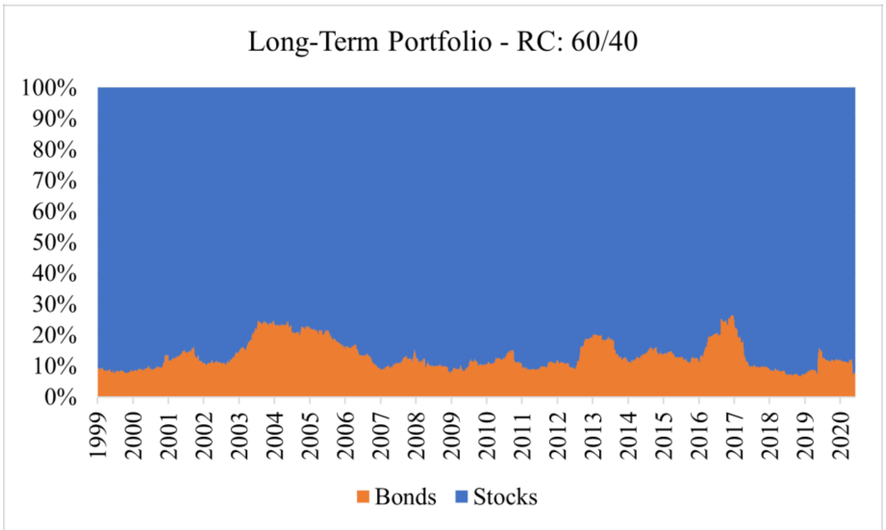

Figure 5. Risk contribution for all asset classes for the 60/40 Portfolio.

5.2.2 Equally Weighted Portfolio

In figure 6 the capital allocation of the equally weighted portfolio is displayed. The assets are allocated equally in the portfolio and the jump at 2007 is due to the inclusion of the credit asset class which was not available prior to this.

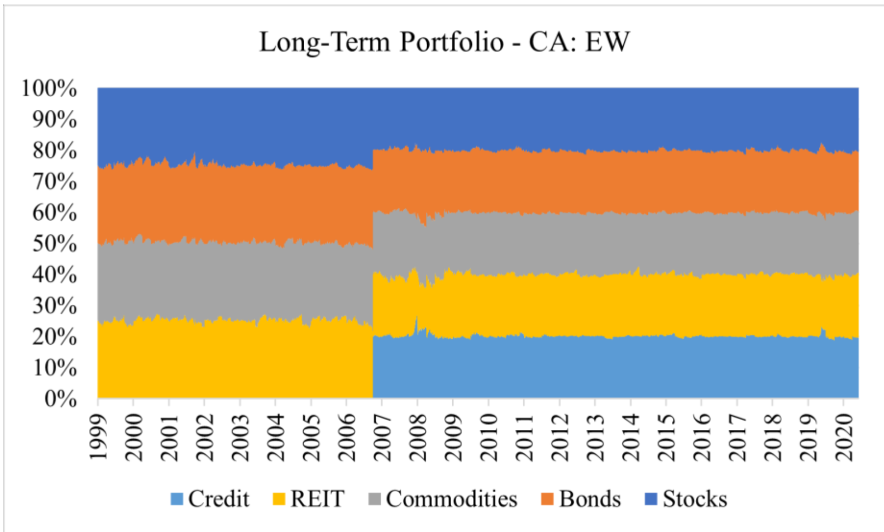

Figure 6. Capital Allocation for Equally Weighted Portfolio.

The risk contribution of the EW portfolio is illustrated in figure 7. In line with (Bai, Scheinberg & Tutuncu, 2016) the results indicate that while the portfolio is diversified in terms of portfolio weights, the risk contribution is far from equalized. For instance, stocks, REITs and commodities dominate bonds and credit with respect to risk contribution. Furthermore, one can observe the higher volatility in stocks and REITs attributable to the Dot-com and financial crisis respectively.

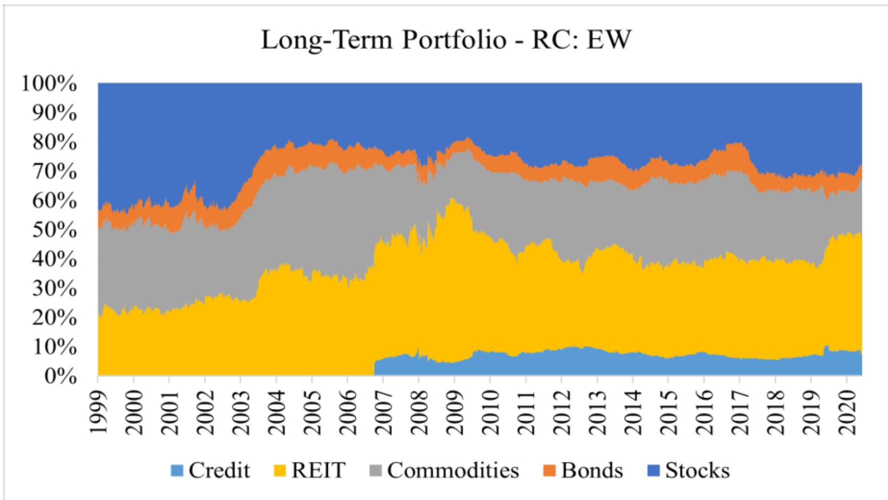

Figure 7. Risk Contribution for all asset classes for Equally Weighted Portfolio.

5.2.3 Risk Parity Portfolio

Capital allocation for the RP portfolio is displayed in figure 8. Given that the weights are determined such that risk is equalized the portfolio weights differs considerably in comparison to the weight budgeting approaches. One can observe that the RP portfolio allocates a significant proportion of the portfolio towards bonds and credit and a smaller proportion to stocks, REITs and commodities. This is due to the lower volatility and cross-asset correlation exhibited by the bond and credit asset class. Note that during the Dot-com and financial crisis, the bond correlation with the rest of the portfolio is negative of such magnitude that that it is excluded from the portfolio. This is due to the constraint that risk contributions cannot be negative.

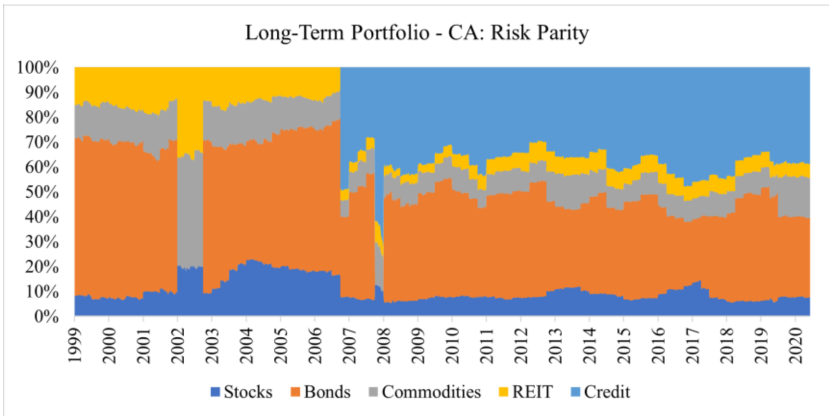

Figure 8. Capital Allocation for the Risk Parity Portfolio

The risk contribution of each asset class in the Risk Parity portfolio is displayed in figure 9. The risk contribution is, as expected, fairly well distributed between the different assets. One can observe that the risk contribution varies between rebalancing dates. This could indicate that the portfolio needs considerable rebalancing similar to what was found in previous studies (Colucci, 2011, Bruder, Kostyuchyk & Roncalli 2016). As mentioned above, bonds are excluded during times of significantly negative risk contribution, whereby the portfolio subsequently equalizes the risk of the assets still in the portfolio.

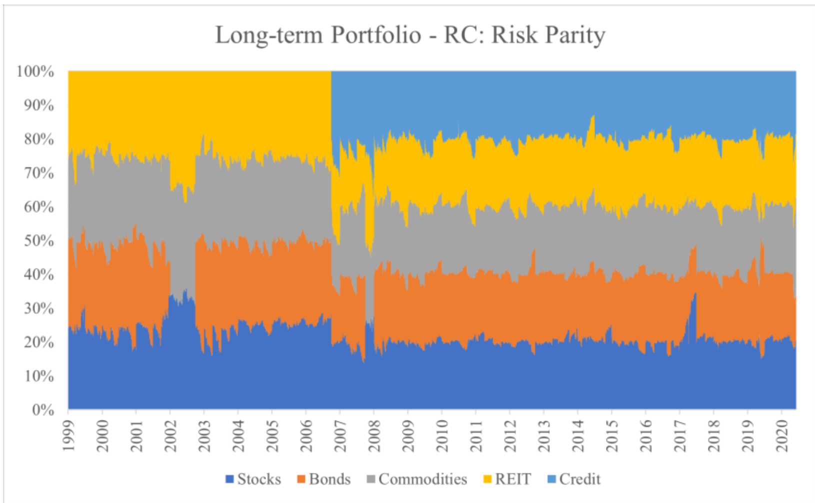

Figure 9. Risk contribution for all asset classes for the Risk Parity Portfolio

5.2.4 Tail Risk Parity Portfolio

In figure 10 the capital allocation of the TRP portfolio is displayed. The weights show similar patterns for the asset classes as in the case for the Risk Parity portfolio, where the bonds and credit assets classes proportions are significant in the portfolio in comparison with the riskier assets.

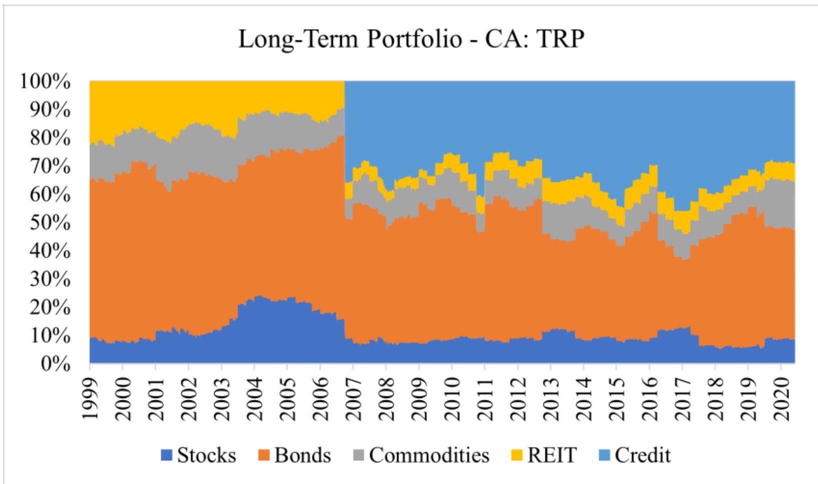

Figure 10. Capital Allocation for the Tail Risk Parity Portfolio.

The risk contributions of the TRP portfolio are displayed in figure 11. Regarding the risk contribution of the asset classes in the Tail Risk Parity portfolio, our results isn't surprising. The risk contribution of each asset class is fairly well distributed in the portfolio and is supported by previous studies, where Cesarone & Colucci (2017) obtains similar results. Similar to (Colucci, 2011, Bruder, Kostyuchyk & Roncalli 2016), the TRP portfolio display less volatility in the risk contributions indicating that the portfolio might require less rebalancing than the RP portfolio.

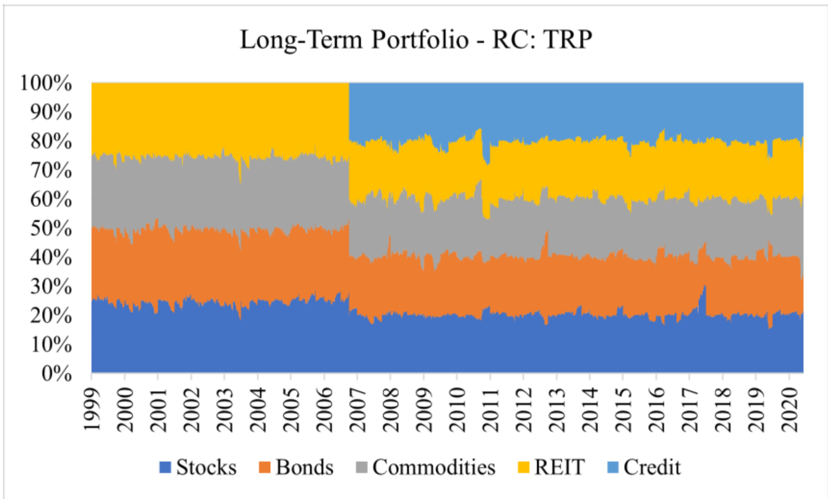

Figure 11. Risk contribution for all asset classes for the Tail Risk Parity Portfolio

5.3 Performance Measurements full period

Table 2 presents the performance measures of the portfolios for the long-term portfolios for the period 1999 – 2021.

	RP	TRP	EW	60/40
Annual Return	4,27%	4,40%	2,84%	3,52%
Annual Volatility	5,2%	5,0%	10,0%	11,2%
Sharpe Ratio	0,83	0,89	0,28	0,31
Tail Risk Adjusted Returns	5,30	5,71	1,74	1,96
Max Drawdown	-29,57%	-29,43%	-51,17%	-47,53%
Max Daily Log return	1,90%	1,82%	4,41%	5,27%
Min Dailty Log Return	-3,22%	-3,30%	-7,26%	-8,13%

Table 2. Performance Measurements for Portfolios during the whole period of study.

For the entire period the RP and TRP portfolio display both higher returns and lower risk than the EW and 60/40 portfolio. Hence, not only does RP and TRP seem to provide a lower risk but also higher return, a result similar to what Alankar, Depalma & Scholes (2012) found. In addition, RP and TRP portfolios provide significantly better protection against downside risk compared to the EW and 60/40 portfolio. However, like Colucci (2011) there is no indication of TRP providing better downside protection then RP. The EW portfolio is the worst performing portfolio across the board possibly due to its large exposure to stocks during Dot-com and REITs during the financial crisis. The 60/40 portfolios allocation towards stocks benefit greatly from the bull market after the financial crisis.

5.4 Portfolio Performance in Different Economic Circumstances

In this section, the RP and TRP portfolios are examined using a shorter estimation window and higher rebalancing frequency to determine whether TRP provide better downside protection than the RP during crisis situations. The three periods of significant uncertainty are the Dot-com bubble 99-02, the Financial crisis 07-09 and, the contemporary Covid-19 recession 19–21. For the first two crises the credit asset is excluded due to a lack of data for estimation during this period.

5.4.1 Risk Parity Portfolio Performance

<u>Dot-com bubble</u>

In the figures 12, the capital allocation and risk contribution of the RP portfolios is presented during the Dot-com bubble. During the bubble, and after it crashed we see that bonds are given the highest amount of weight in the portfolio, where stocks has the lowest amount of weight in the portfolio. This seems reasonable when considering the risk contribution for the portfolio, where the aim is the have a fairly distributed risk contribution among the different asset classes. Similar to the long-term portfolio we observe volatility in risk contributions indicating rebalancing needs.

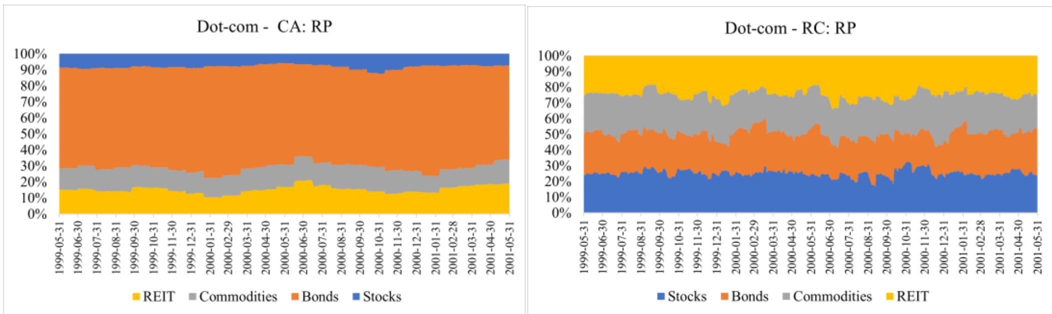

Figure 12 Capital Allocating & Risk Contribution for RP During Dot-com bubble.

<u>Financial Crisis</u>

In figure 13, the capital allocation and risk contribution for the RP portfolio during the financial crisis is presented. The results suggest similar results as for the Dot-com crisis for the RP portfolio. Similar to the long-term portfolio bonds are given a considerable weight in the portfolio while REITs are punished in the aftermath of the subprime mortgage crisis.

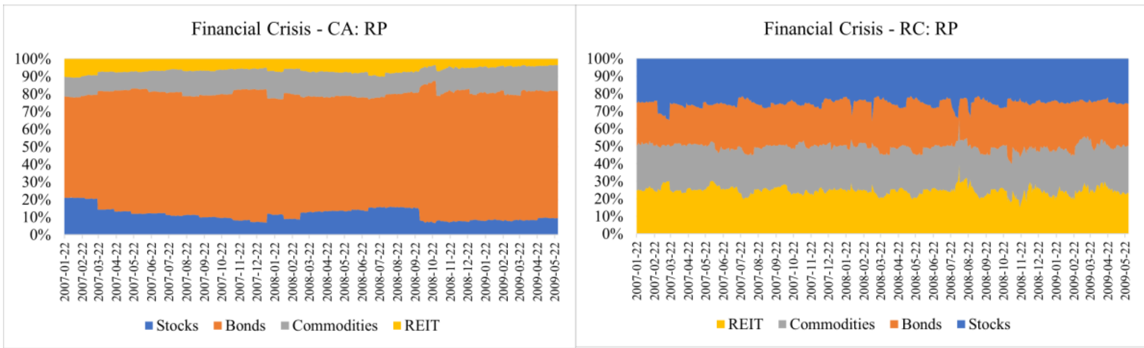

Figure 13 Capital Allocating & Risk Contribution for RP During Financial Crisis.

<u>Covid-19 Recession</u>

In figure 14 the capital allocation and risk contribution of the RP portfolio during the Covid-19 recession is presented. We can observe significant volatility in the risk contribution due to the uncertainty incurred by the outbreak of the pandemic. Similar to what Bruder, Kostyuchyk & Roncalli (2016) found, the RP portfolio is very sensitive to sudden jumps in volatility.

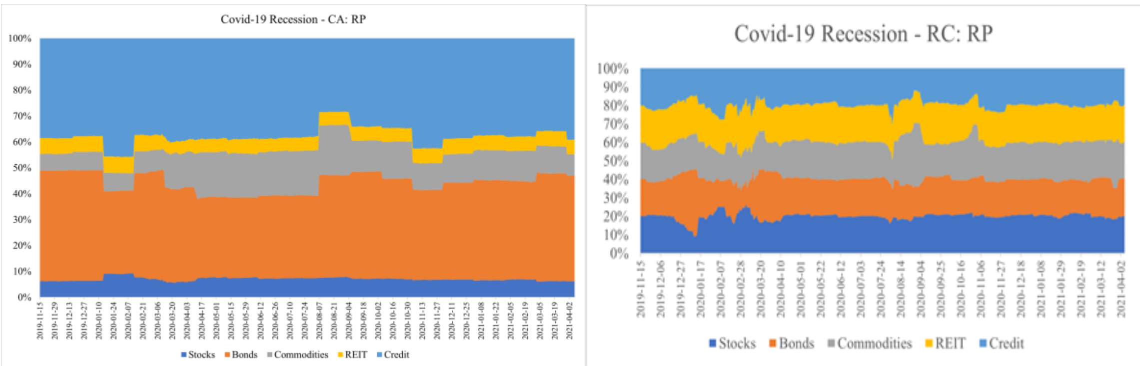

Figure 14 Capital Allocating & Risk Contribution for RP During Covid-19.

5.4.3 Tail Risk Parity Portfolio Performance

<u>Dot-com bubble</u>

In figures 15 the capital allocation and risk contribution of the TRP portfolio during the Dot-com bubble is displayed. We observe the same pattern, a considerable proportion of the portfolio allocated to the less risky assets. In addition, the capital allocation and risk contribution appear less volatile than the RP portfolio.

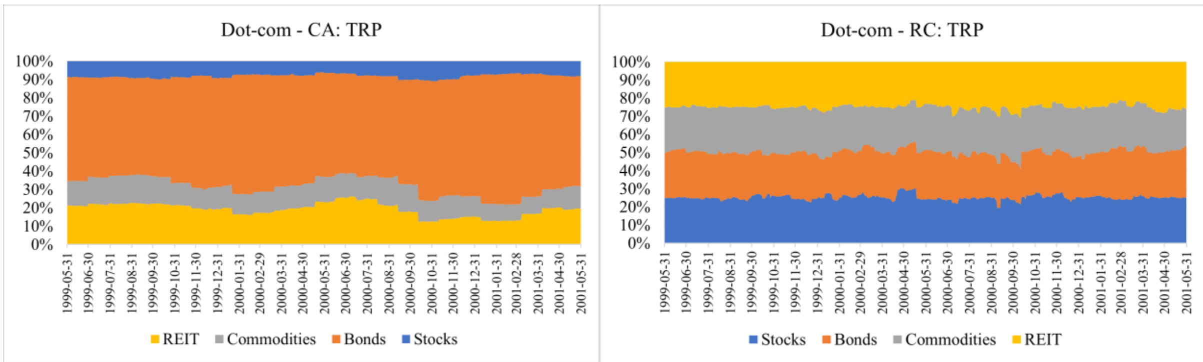

Figure 15 Capital Allocating & Risk Contribution for TRP During Dot-com bubble.

<u>Financial Crisis</u>

In figures 16 the capital allocation and risk contribution of the TRP portfolio during the financial crisis is displayed. We observe that the portfolio shifts out of REITs and stocks as the crisis as a consequence of the subprime mortgage crisis. As above we observe stable capital allocation and risk contributions.

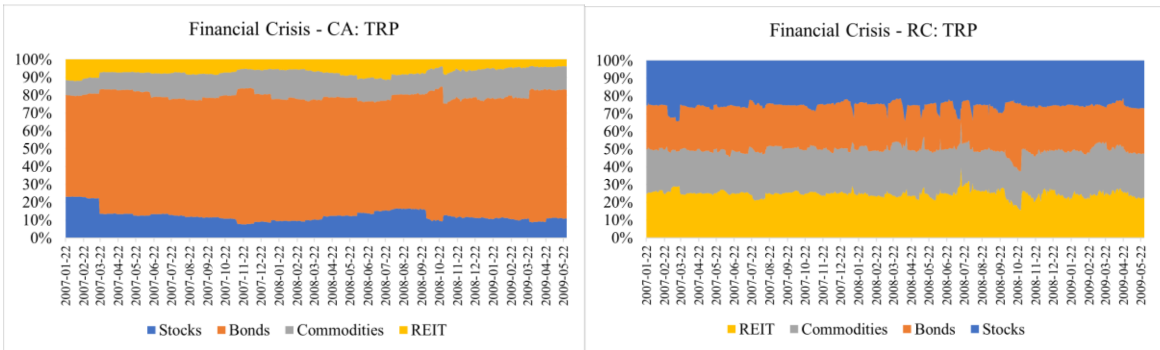

Figure 16 Capital Allocating & Risk Contribution for TRP During Financial Crisis.

<u>Covid-19 Recession</u>

In figures 17 the capital allocation and risk contribution of the TRP portfolio during the Covid-19 recession is displayed. The TRP portfolio during the Covid-19 recession shows some differences when estimating it during a shorter period of time. The differences are found in the capital allocation of the portfolio where commodities are given a higher weight compared to the capital allocation in the full period. Bonds and Credits are still given the highest amount of weight in the portfolio, and when considering the risk contribution of the Tail Risk Parity portfolio the estimated results is desirable as we have an even contribution among the asset classes.

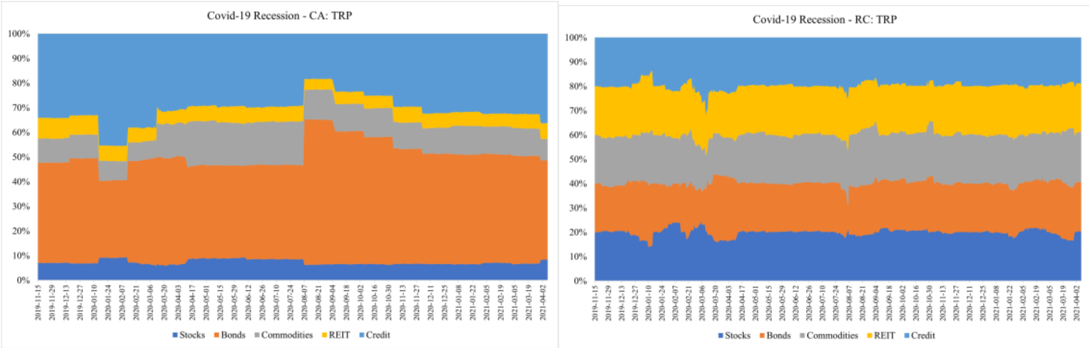

Figure 17 Capital Allocating & Risk Contribution for TRP During Covid-19.

5.4.3 Risk Budgeting Approach During Economic Crises

In table 3, the performance of the portfolios during the three crisis scenarios are presented.

	DOTCOM Bubble (99-01)				Global Financial Crisis (07-09)				Covid Recession (19-21)			
	RP	**TRP**	**EW**	**60/40**	**RP**	**TRP**	**EW**	**60/40**	**RP**	**TRP**	**EW**	**60/40**
Annual Return	4,16%	4,43%	6,46%	-0,46%	-7,76%	-9,79%	-30,19%	-16,52%	2,44%	2,62%	2,74%	6,65%
Annual Volatility	3,6%	3,7%	7,0%	12,0%	8,78%	9,74%	24,45%	18,06%	7,3%	7,5%	7,4%	17,8%
Sharpe Ratio	1,17	1,20	0,92	-0,04	-0,88	-1,01	-1,24	-0,91	0,33	0,35	0,37	0,37
Tail Risk Adjusted Returns	9,14	9,59	6,70	-0,28	-4,96	-5,49	-6,87	-5,19	1,85	1,94	2,08	2,11
Max Drawdown	-3,89%	-3,76%	-7,91%	-18,48%	-35,31%	-38,15%	-61,85%	-49,42%	-16,89%	-17,23%	-16,89%	-30,82%
Max Daily Log return	0,67%	0,74%	1,65%	2,78%	2,06%	2,55%	5,73%	5,36%	1,87%	1,74%	1,87%	4,88%
Min Daily Log Return	-0,73%	-0,82%	-1,84%	-3,71%	-2,85%	-3,36%	-9,85%	-6,01%	-3,24%	-3,17%	-3,24%	-8,70%

Table 3. Performance Measures for All Portfolios.

During the Dot-com bubble the TRP portfolio has higher risk adjusted returns compared with the RP portfolio using both Sharpe, Tail-risk adjusted returns and the lowest drawdown. This indicates some protection against downside risk. However, the difference between the TRP and RP portfolios is small. Both portfolios outperform the weight budgeting portfolios where the large exposure to stocks in the 60/40 portfolio makes it the worst portfolio during the Dot-com bubble.

During the financial crisis all portfolios fail to deliver positive results. However, the RP and TRP portfolios stand out as delivering the smallest loss with the smallest drawdown. In this case RP seem to edge out TRP slightly with a lower risk adjusted losses and a smaller drawdown. In accordance with both the Sharpe and Tail-risk adjusted return ratios, the 60/40 portfolio is preferred to the TRP portfolio. While unintuitive this result indicates that for the risk exposure in the 60/40 portfolio it delivers less losses than what would be expected. The EW is the worst performer due to its exposure to both the stock market and REITs.

During Covid-19, the 60/40 portfolio delivers fantastic returns while the other portfolios delivers similar results. However, after risk adjustment the 60/40 performs equivalently with the rest of the portfolios with the added bonus of the highest drawdown. Hence, similar to Colucci (2011), we find that neither the RP or TRP outperform the EW portfolio.

Hence, there is evidence that TRP and RP can deliver better risk-adjusted returns and smaller drawdowns then the weight budgeting portfolios. However, we find no significant evidence that RP or TRP perform significantly different or that TRP would protect against downside risk better than the RP approach

6. Conclusion

In this thesis we evaluate how the risk budgeting strategies TRP and RP approach performs in comparison with the conventional weight budgeting strategies over a long-term period and during economic crisis. In addition, we evaluated whether the risk budgeting approach is significantly improved by supplanting the conventional risk budgeting risk measure volatility with downside risk measure ES. By stating the risk budgeting problem as a minimization problem, a numerical algorithm generated portfolio weights that equalized risk in the risk budgeting portfolios.

We are able to show that the Risk Budgeting approach outperforms the weight budgeting approach using long-term portfolios. Both the RP portfolio and TRP portfolio display higher risk adjusted returns and lower drawdowns compared to the weight budgeting portfolios. The higher risk adjusted return echoes the result of Sokoloff & Zaytsev (2014). However, the TRP portfolio had the highest estimated risk adjusted returns as well and as mentioned by Alankar, Depalma & Scholes (2012) the TRP portfolio approach aims to reduce the tail losses while gaining more upside compared to RP approach and traditional portfolio approaches. In the long run the estimated performance measurements indicate that the TRP portfolio achieves this in accordance with Alankar, Depalma & Scholes (2012). The sensitivity of the portfolios, and the results for maximum drawdown during the period shows that the RP and TRP portfolios provide better downside protection when compared with the weight budgeting approaches. However, we find no significant difference between the two approaches in providing downside protection. We also find that in line with Maillard, Roncalli & Teiletche (2010) the weight budgeting approaches are sensitive to significant drawdowns and that the risk budgeting approaches can mitigate this risk. Hence, considering the overall portfolio risk for the portfolios the TRP portfolio has to lowest risk followed by the RP portfolio.

Furthermore, when considering the performance of the portfolios during Dot-com bubble, Global Financial Crisis and Covid-19 recession our results suggests some differences compared to the full period. During the Dot.com bubble and Global Financial Crisis, the 60/40 and equally weighted portfolio is outperformed by Tail Risk Parity and Risk Parity portfolio when considering the risk-adjusted returns. The results aren't surprising based on Maillard, Roncalli & Teiletche (2010) conclusion regarding traditional portfolio approaches sensitivity to drawdowns. The risk-adjusted returns were higher for the portfolios taking tail risk into account during economic distress. Hence, our predicted results for the risk budgeting portfolios during the Dot.com bubble and Global Financial Crisis suggests that the risk budgeting approach provides a protection against downside risk. During the Covid-19 recession, our results suggest slightly higher risk-adjusted returns for both 60/40 and equally weighted portfolios. However, the portfolio risks for the 60/40 and equally weighted portfolio are significantly higher as its volatility and range of log returns exceeds the optimal. For the Tail Risk Parity and Risk Parity portfolio these two measurements are remarkable lower compared to 60/40 and equally weighted portfolios. The results are estimated from 2019-11-15 to 2021-03-15 and when analyzing the performance measurements for the portfolios, the differences between the portfolios are quite small. As the world economy has been in a boom in recent years with a world economy that has performed well, drawing conclusion regarding the impact Covid-19 has have on the world economy could be simplistic. There have been, and is, turbulent times during the Covid-19 pandemic but comparing it with the Dot.com bubble and the Global Financial Crisis we have not reached those levels of affection on the world economy. Based on our estimations on the performance measurements, and the portfolios risk during the full period the Tail Risk Parity portfolio performed best. It had the lowest volatility of all our portfolios and still outperformed the portfolios in the Risk-adjusted returns. Implying that both the Tail Risk Parity portfolio and Risk Parity would provide a better protection against downside risk overall, compared to the 60/40 and equally weighted portfolio. When considering the three different crises, our results suggests that the Tail Risk Parity and Risk Parity portfolio provides a higher protection against downside risk during the Dot.com bubble and Global Financial Crisis. However, during the Covid-19 recession we didn’t see any clear significant differences between the portfolios.

Hence, while we find similar evidence to (Cagna & Casuccio, 2014) that indicate that risk budgeting approach provide higher returns and better downside protection compared with the weight budgeting approach in the long run, and this result holds for both the RP and TRP portfolios. However, we find no evidence that RP or TRP provide better or worse results, since they both display similar results.

7 Further research

Given that ES and its application in the risk budgeting approach is a relatively new innovation, more robust estimation methods for both ES and risk contributions should be considered. In addition, while ES has desirable properties that make it a good risk measure, other risk measures like expectiles could be considered as well.

The application of the Risk Budgeting approach during the three financial crises examined, made us realize the opportunities of developing the Risk Budgeting approach even further. In our study, the Risk Parity portfolio and Tail Risk Parity portfolio allocates the asset classes to the portfolio based on the risk contribution of them, measured by the standard deviation.

Therefore, an extension of the Risk budgeting approach could be done by including inspiration from the ideas of prospect theory as it would be an interesting future research topic. The philosophy of the portfolios will then rest on the results from behavioral economics where a loss looms larger than a gain, meaning that a loss feels worse than an equivalent gain feels good. To achieve this portfolio, one could rescale the returns of the asset classes such that losses would be scaled by the estimated loss parameter. By using the rescaled returns, one could construct the portfolios and measure them during the three financial crises done in this study. As it would be interesting to examine if a further development of the Risk Budgeting approach would imply higher protection against downside risk during these economic circumstances.

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Appendix

	Stocks	Bonds	Commodities	REIT	Credit
Annual Return	5,04%	4,85%	0,75%	5,24%	2,78%
Annual Volatility	18,8%	4,0%	15,7%	28,4%	5,3%
Sharpe Ratio	0,27	1,22	0,05	0,18	0,53
Tail Risk Adjusted Returns	1,34	6,63	0,26	0,83	2,62
Max Drawdown	-59,59%	-20,56%	-77,23%	-83,42%	-16,50%
Max Daily Log return	9,99%	2,00%	5,50%	16,03%	3,07%
Min Dailty Log Return	-13,17%	-3,03%	-6,61%	-24,87%	-3,00%

Table 4 Descriptive statistics assets in portfolio entire time period

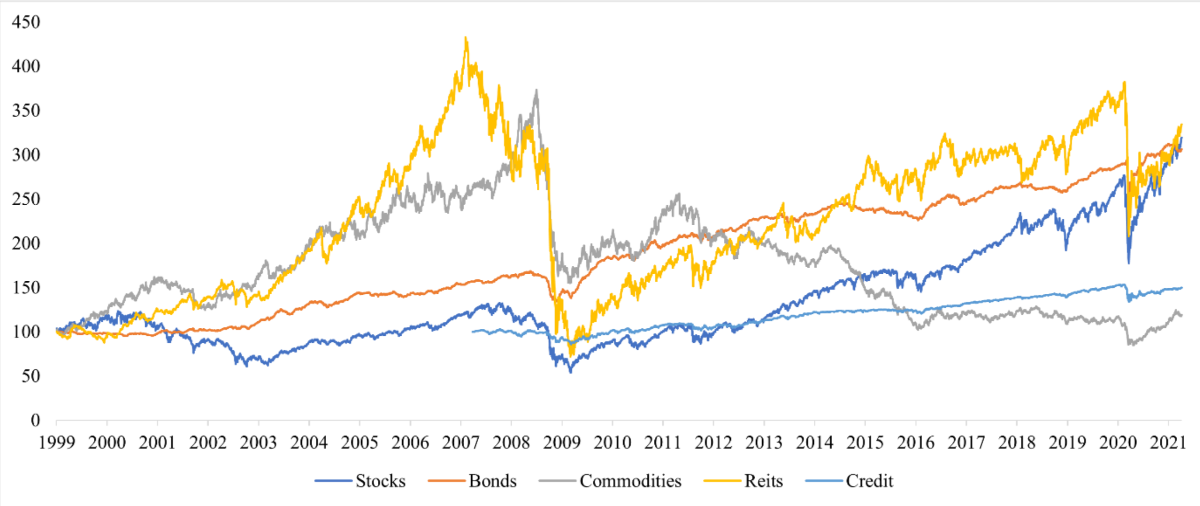

Figure 18 Asset development with a starting value $100