Study of Generalizations of the Discrete Bak-Sneppen Model

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Abstract

In 1993, Per Bak and Kim Sneppen proposed a model of co-evolution between species, where survival of a particular species affects the survival of its neighbouring species. In the discrete case of the model, each species, or an entry in a set with periodic boundary conditions, is an element $x_i \in \{0, 1\}$, in the set of size N, where x_i represents the fitness. An entry of the least fitness is chosen and replaced together with its two neighbours each with $Bernoulli(p), p \in (0, 1)$ random variables. If the parameter p is larger than some p_{cr} [1], the whole set is consumed with 1.

In this paper, we study the generalizations of the discrete case of Bak-Sneppen model and evaluate p_{cr} both analytically and numerically. For that end, we first examine the case where in each iteration a vertex x_i and its both neighbours x_{i-1}, x_{i+1} are replaced by the same Bernoulli(p) variable. Then, we study the case where the type of the model - whether the entry is replaced alone or with its neighbours - is determined by a Bernoulli(r) variable. Finally, we find a non-trivial p_{cr} for a 2-dimentional set of entries.

Introduction

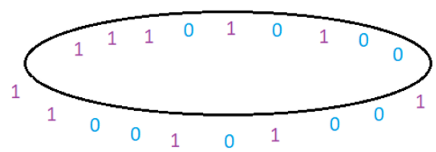
Historical Background

The term ”self-organised criticality” was coined by the physicist Per Bak in his 1987 paper ”Self-organized criticality: an explanation of 1/f noise”, which describes a property of scale-invariance in dynamic systems as they move towards the critical point of a phase transitions. There he shows that the power laws of pink noise can be modelled by the dynamic of self-organised critical state of minimally stable clusters. [**5**]

This phenomenon is later explored in the paper ”Punctuated equilibrium and criticality in a simple model of evolution” [**6**], where Per Bak and Kim Sneppen apply self-organised criticality to explain the theory of punctuated equilibrium. This theory says that that species stay in stasis until a major rapid change occurs, which invokes a change in species - either mutations or appearance of new ones. Each species is represented by a particular fitness, which is a parameter describing its ability to survive. Imitating the real-life setting, each species fitness depends on some other species fitness, and whenever a rapid change in the system occurs, the change of ”connected” species is likely to change together.
Motivated by the complexity of the original Bak-Sneppen model, multiple researches explored its discrete case. The main interest for the studies is the phase transition between the two states of the system - when it is consumed by species with fitness 0, by species with fitness 1. In this paper, we look at several generalizations of the discrete Bak-Sneppen model and try to evaluate numerically and analytically the moment of the phase transition for each of them.

Overview

The discrete case of the model is represented by the following design.
In the set N of n Bernoulli(p) variables $x_i \in \{0,1\}$ with periodic boundary conditions, each vertex represents a fitness of a species. At a discrete time step t_i, a vertex with the least fitness, typically a zero, is chosen, and together with its two neighbours replaced by three new $Bernoulli(p)$ variables. A neighbour can be a zero or a one, it must not necessarily be a zero. If there are no zeros in the set, then the least fitness vertex will be one. The same algorithm is repeated at t_{i+1}, for the updated setting. The setting xi_i for $i \to \infty$ is a Markov Chain that converges to the stationary distribution $\pi^{(n)}(p)$ [**1**], where p is the parameter of a Bernoulli distribution, equal to a probability of replacing the least fitness vertex with a "1".



Definition 0.0.1 *A stationary distribution is a distribution that remains unchanged after a shift of the time scale.*

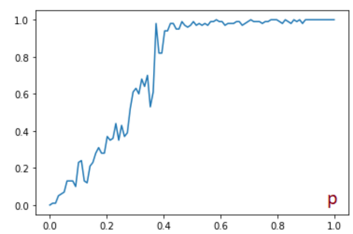
This definition is partially taken from [**9**].
Then by [**1**], the probability of one randomly chosen vertex being a "1" is

$$\nu^n(p) = \sum_{x_i \in \{0,1\}} \pi^n(p) \cdot \mathbf{1}_{x_i=1}$$

Furthermore, for a particular parameter p of $Bernoulli(p)$, after multiple repetitions, the zeros in the set will not survive, and the model will implode to ones. That is, the probability $\nu(p)$ of a randomly chosen vertex in the set being one will tend to 1 as the number n of repetitions will tend to infinity.

$$\lim_{n\to\infty} \nu^n(p) = 1$$

Intuitively, such a critical value would be somewhere between 0.6 and 1. But the simulations indicate that the phase transition is happening at the value $p \approx 0.36$.



On this graph, we see the fraction of "1" in the set of length 100, after 10000 repetitions of the algorithm described above. Before some value p_c of p, the fraction $\nu(p)$ is increasing, but after p_c it is approximately 1.
Multiple researches attempt to pinpoint the critical value for this classical discrete case, and the closest approximation found so far is 0.41 by S. Volkov [**1**]. In this paper, we attempt to find the upper bound p_* for the critical value for the following variations of the model:

- **Solidarity.** We assume that the least fitness species and its neighbours are *all* replaced by the same *Bernoulli(p)* variable. The simulations show that the upper bound p_*in this case should be around 0.2. We attempt to find it analytically.

- **Sometimes, just itself**. We add a level to the algorithm. Now, we introduce a new parameter, $r \in [0, 1]$. At each time step, we either replace the least fit species according to the original Bak-Sneppen model with probability $1 - r$, or we replace it solely without its neigbours, with probability r. The goal is to find a $p_{cr}(r)$ as a function of r, if that is possible.

- **Two particular cases**. We find the upper bound for the cases where $n = 4, n = 5$.

- **Ladder.** We extend the discrete Bak-Sneppen model to a two-dimensional case.

The reason why these generalizations are interesting is because to my knowledge, no one has done that before, and since the upper bound for the classical discrete Bak-Sneppen model is counter-intuitively small, one would possibly expect the generalizations to also behave in a curious manner.

Mechanism

Let $\xi(t)$ denote the configuration of a set of size n at a time step t, and let $Z(t)$ be a set of indices that satisfies the following properties:

1. $\xi_k(t) = 1$ for all $k \notin Z(t)$
2. $Z(t)$ has the smallest number of elements among such sets
3. $Z(t)$ has periodic boundary conditions

In every case above, we define $D_t =$card$(Z(t))$ as the largest distance between the leftmost and the rightmost zero.

Definition 0.0.2 (Cardinality) *A cardinality of a set is the number of entries in that set.*

For example,

$$\xi(t_1) = (1, 0, 1, 0, 0, 0), D_{t_1} = 5$$
$$\xi(t_2) = (1, 0, 0, 0, 0, 0), D_{t_2} = 5$$
$$\xi(t_3) = (1, 1, 1, 0, 0, 0), D_{t_1} = 3$$

In [**1**] and [**2**], the following method was used. At each time step, when the configuration changes from $\xi(t)$ to $\xi(t + 1)$, we find the shift $D_{t+1} - D_t$ in the cardinality of the set .

If at the time step $t : t+1$ the distance between the left- and the rightmost zero increases by 1, we say that $D_{t+1} - D_t = 1$. We then try to bound D_{t+1} from above by the function M_{t+1}. For all tasks in this thesis except for the last one we define M_{t+1} as follows:

$$M_{t+1} = D_{t+1} - \beta(1_R + 1_L),$$

where R means that the entry to the right of the leftmost zero is also zero, and L means that the entry to the left of the rightmost zero is zero. The role of $\beta \in [0, \frac{1}{2}]$ is to "tune" M_{t+1}. When there are a lot of zeros at the border, the chances of these zeros being replaced by ones are higher, so we subtract β to counteract this notion.

Definition 0.0.3 (Martingale) *A discrete-time martingale is a stochastic process for which the conditional expectation of the next state given all previous to it observations is equal to its current state.*

$$E(X_{n+1}|X_1, X_2, ..., X_n) = X_n$$

Definition 0.0.4 (Supermartingale) *A discrete-time supermartingale is a stochastic process for which the conditional expectation of the next state given all previous to it observations is less or equal than its current state.*

$$E(X_{n+1}|X_1, X_2, ..., X_n) \leq X_n$$

Let $I(t) \in Z(t)$ denote the set of indices $[1, .., k]$ that satisfy the following properties:

1. $I_1(t) = Z_3(t)$
2. $I_k(t) = Z_{n-2}(t)$

Suppose that $D_t \leq 6$, Then by [**1**], $M_{t+1} \leq M_t$ on $I(t)$, while on $i_t \notin I(t)$, for $p > p_{cr}$ and some $\epsilon(p) > 0$, it holds that

$$\Delta_{t+1} = \mathbb{E}(M_{t+1} - M_t|\mathcal{F}) < -\epsilon$$

In this equation, M is a supermartingale. We are aiming at finding such p_{cr} that ensures that $\Delta_{t+1} < 0$, that is, the smallest p_{cr} for which $I(t)$ will decrease in size.

Solidarity

In this task, we assume that instead of drawing 3 different variables from the *Bernoulli*(p) distribution and replacing the entries with them, the new variable is drawn once for all 3 elements.

First we look at the case where $D_i < n$.
When choosing a zero that is close to the left corner, there exist 4 possible outcomes ξ, listed below. The reason they look like this is because the leftmost element of $I(t)$ should be 0 to satisfy the conditions, and the two elements that follow it can be any configuration of 0 and 1.

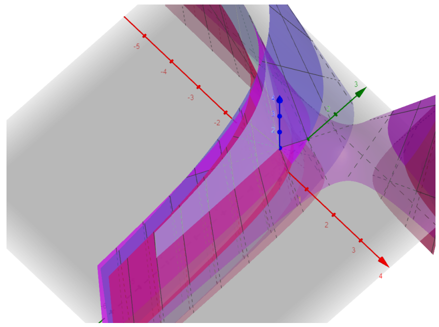
$$\xi_{00} = (1, 1, 1, 0, 0, 0...)$$

$$\xi_{11} = (1, 1, 1, 0, 1, 1...)$$
$$\xi_{01} = (1, 1, 1, 0, 0, 1...)$$
$$\xi_{10} = (1, 1, 1, 0, 1, 0...)$$

Then the drift $\Delta_{t+1} = \mathbb{E}(M_{t+1} - M_t|\mathcal{F})$ in each of these cases will be bounded from above by:

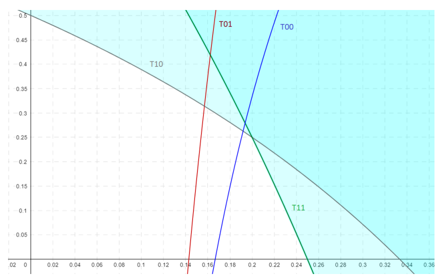
$$T_{00} = (1 - p) + p(-2 + \beta) + p(-3 + \beta) + p\beta$$
$$T_{11} = (1 - \beta)(1 - p) - 3p$$
$$T_{01} = (1 - p) - +p(-3 + \beta) + p(-3 + \beta)$$
$$T_{10} = (1 - \beta)(1 - p) - 2p - \beta(1 - p)]$$

The calculations above are illustrated in the figures in the end of the section.
In order to compute the value of p that will be the largest possible so that T's are non-positive, the latter are represented as functions of $p \in [0, 1]$ and $\beta \in [0, \frac{1}{2}]$.



Then we take the projections of T's on the XY plane (where p is represented by x, and β is represented by y), and find the largest p on the intersection of the areas that correspond to the negative values of T.

We separate the area for which *all* T's will be non-positive:



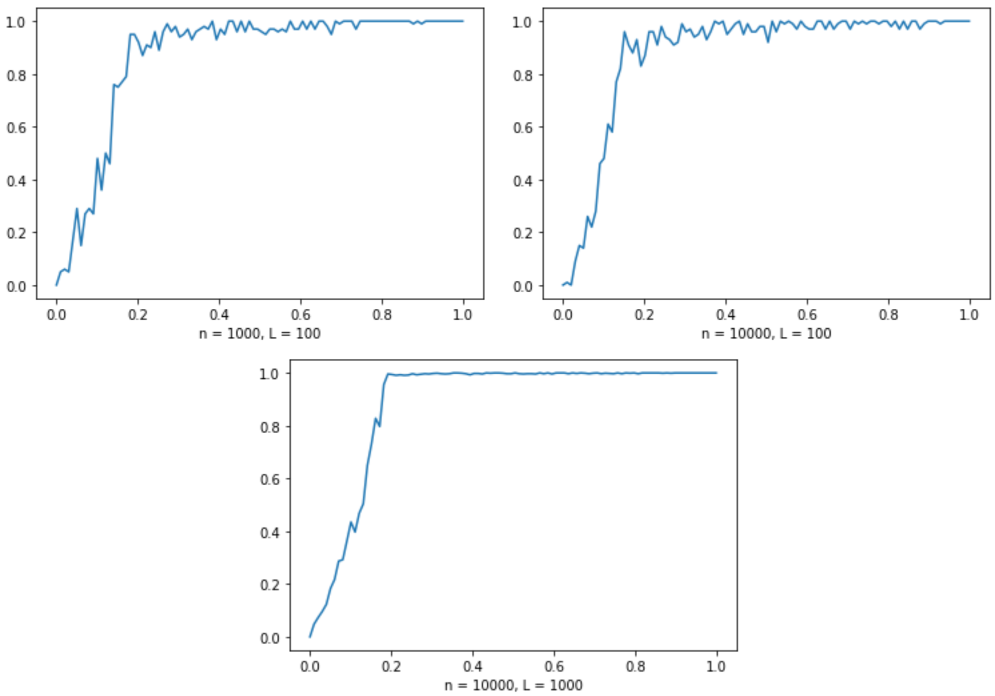
From the plot it is visible that the point of intersection between T_{00} and T_{11} contains the smallest possible p so that all T's are non-positive. Solving $T_{00} = 0$ and $T_{11} = 0$ for (p, β) gives: $p = 0.1937, \beta = 0.2792$.

$$\begin{cases} (1-p) + p(-2+\beta) + p(-3+\beta) + p\beta = 0 \\ (1-\beta)(1-p) - 3p = 0 \end{cases} \iff \begin{cases} 1 - 6p + 3p\beta = 0 \\ \beta = 1 - \frac{3p}{1-p} \end{cases}$$

$$\iff \begin{cases} 1 - 3p - \frac{9p^2}{1-p} = 0 \\ \beta = 1 - \frac{3p}{1-p} \end{cases} \iff \begin{cases} 1 - 4p - 6p^2 = 0 \\ \beta = 1 - \frac{3p}{1-p} \end{cases}$$

For the case where $D_t = n : \delta_{t+1} = \mathbb{E}(M_{t+1} - M_t | \mathcal{F}) = (-3-\beta)p - \beta(1-p) = -3p - \beta < 0$. The drift for this case is negative for any pair (p, β) .

The simulations show the fraction of 1's in several sequences of length L after n iterations. It is easy to notice that as the number of iteration increase, the value $p = p_*$ is more visible to be around 0.2.


0.0
0.2
0.4
0.6
0.8
1.0
n = 1000, L = 100
n = 10000, L = 100
n = 10000, L = 1000

0.1 Calculations for the Solidarity model

T00

Probability	x = (11000*...)	D = M(t+1)-M(t)
1st 0: (1-p)	(10000*...)	1
p	(11110*...)	-2 + b
2nd 0: (1-p)	(11000*...)	0
p	(11111*...)	<= -3 + b
3d 0: (1-p)	(110000...)	0
p	(110111...)	beta

T01

Probability	x = (11001*...)	D = M(t+1)-M(t)
1st 0: (1-p)	(10001*...)	1
p	(11110*...)	-2 + b
2nd 0: (1-p)	(11000*...)	0
p	(11111*...)	<= -3 + b

T11

Probability	x = (11011*...)	D = M(t+1)-M(t)
1st 0: (1-p)	(10001*...)	1 - b
p	(11111*...)	<= -3

T10

Probability	x = (11010*...)	D = M(t+1)-M(t)
1st 0: (1-p)	(10000*...)	1 - b
p	(11110*...)	-2
2nd 0: (1-p)	(110000*...)	-b
p	(110111...)	0

Sometimes, just itself

In this section, we combine the classical Bak-Sneppen model with a new model, called «just itself», where only a single zero is replaced by 0 or 1.
First, we draw a random Bernoulli(r) variable to determine which algorithm to use. With probability r we replace just the chosen zero with a new Bernoulli(p) variable, and with probability (1-r) we replace the chosen zero together with its two neighbours, each is replaced independently by a Bernoulli(p) random variable.
The objective now is to study how p_{cr} depends on p and r, and to estimate the upper bound p_* for the critical value p_{cr} as a function of r.
First we compute the boundaries for the drift Δ_{t+1}. Since for every configuration of 0 and 1 there are four possible settings for the beginning of the sequence, starting from the left, we consider four possible upper boundaries, for each of the cases. We denote them $T_{i,j}$, where $i, j \in (0, 1)$, for the sequences starting with $(110\mathbf{00}), (110\mathbf{01}), (110\mathbf{10}), (110\mathbf{11})$, just as in the previous section.
Since this section deals with the conditional expectation, the general formula for $T_{i,j}$ will be of the form:

$$T_{i,j} = (1-r) \cdot T_{original} + r \cdot T_{just-itself}$$

The new $T_{i,j}$ are now:

$$T_{10} = \frac{1}{2}(1-r)[(1-p)^3(1-2\beta)+p(1-p)^2(2-4\beta)+p^2(1-p)(-2\beta)+p^3(-2)]+\frac{1}{2}r(-2p),$$

$$T_{01} = \frac{1}{2}(1-r)[(1-p)^3+(1-p)^2p(1+2\beta)+(1-p)p^2(6\beta-3)+p^3(-6+3\beta)]+\frac{1}{2}r(-p+2p\beta),$$

$$T_{00} = \frac{1}{3}(1-r)[(1-p)^3+(1-p)^2p(1+3\beta)+(1-p)p^2(7\beta-3)+p^3(-5+3\beta)]+\frac{1}{3}r(p\beta-p),$$

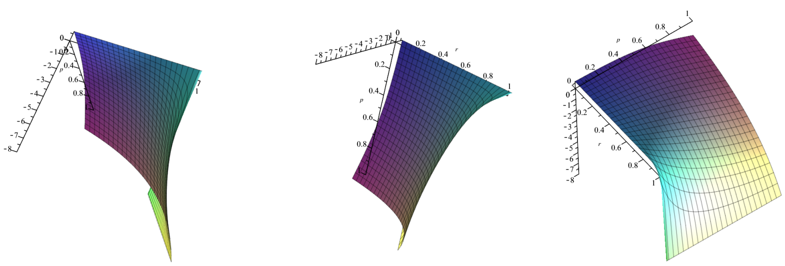
$$T_{11} = (1-r)[(1-p)^3(1-\beta) + (1-p)^2p(2-2\beta) - p^3] - 3pr,$$

see the calculations on the bottom of this section.

Now $T_{i,j}$ are the functions that bound the drift for every of the four cases from above, so we set them equal to zero and find the smallest $p_*(r)$ such that for every $p(r) > p_*(r)$ it holds that $T(p(r), \beta) < 0$ on $p \in [0,1], r \in [0,1], \beta \in [0, \frac{1}{2}]$

T_{00}, T_{11} are the two largest functions which projections on the (p, β) plane intersect at the point p_{cr} which is the smallest p for which T_{00}, T_{11} are negative. Solving the equation $T_{00} = T_{11} = 0$ for β gives

$$\beta = -\frac{4p^3r - 4p^3 + p^2r - p^2 - 7pr - p - 2r + 2}{2p^3r - 2p^3 - 2p^2r + 2p^2 - pr + 3r - 3}$$



$\beta(p, r)$ from different angles

Substituting β into T_{00} or T_{11} we get

$$Q = p^5r^2 - 2p^5r + 4p^4r^2 + p^5 - 8p^4r - p^3r^2 + 4p^4 - p^3r - 3p^2r^2 + 2p^3 + 3p^2r + 3p^2 - r^2 + 2r - 1.$$

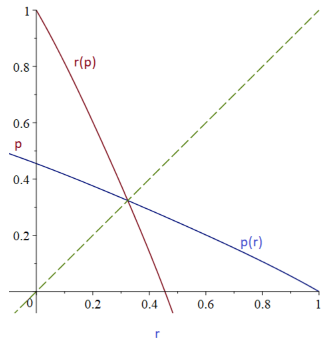
Setting $Q = 0$ and solving for r we get

$$r(p) = \frac{2p^5 + 8p^4 + p^3 - 3p^2 - 2 \pm \sqrt{-12p^7 - 39p^6 + 30p^5 + 45p^4 + 12p^2}}{2(p^5 + 4p^4 - p^3 - 3p^2 - 1)},$$

Since r is the probability that lies withing the range[0,1], we only consider the first solution to the equation, where the square root is positive (otherwise r is negative.)

$$p(r) = \frac{2p^5 + 8p^4 + p^3 - 3p^2 - 2 + \sqrt{-12p^7 - 39p^6 + 30p^5 + 45p^4 + 12p^2}}{2(p^5 + 4p^4 - p^3 - 3p^2 - 1)}$$

From $r(p)$ we can differentiate the upper bound for $p(_cr)$ implicitly by taking its inverse $r^{-1}(p)$ on $p \in [0,1], r \in [0,1]$:



For $r = 0$, p that lies within the range $[0, 1]$ is equal to that of a classical Bak-Sneppen model (0.4549286537), and for $r = 1$, p is equal to 0, meaning that for every p it will hold true that $\lim_{n\to\infty} \nu^{(n)}(p) = 1$

Calculations for the "Sometimes, just itself" model

The calculation for the drifts for the original case were taken from the article [1]. For the «just itself» model:

T_{00}

Index of zero picked	Probability	Setting	Drift
1	$(1-p)$	11000	0
	p	11100	-1
2	$(1-p)$	11000	0
	p	11010	β
3	$(1-p)^2 p$	11000	$-\beta$
	p	11001	0

$$T_{00} = \frac{1}{3}(-p + p\beta)$$

T_{10}

Index of zero picked	Probability	Setting	Drift
1	$(1-p)$	11010	0
	p	11110	-2
2	$(1-p)$	11010	0
	p	11011	0

$T_{10} = \frac{1}{2}(-2p)$

T_{01}

Index of zero picked	Probability	Setting	Drift
1	$(1-p)$	11001	0
	p	11101	$-1+\beta$
2	$(1-p)$	11001	0
	p	11011	β

$T_{01} = \frac{1}{2}(-p + 2p\beta)$

T_{11}

Index of zero picked	Probability	Setting	Drift
1	$(1-p)$	11011	0
	p	11111	-3

$T_{11} = (-3p)$

Two particular cases

Now we are trying to estimate $\nu(p)$, the fraction of "1"'s, as a function of p for the length n = 4, n= 5 of the set with periodic boundary conditions, where the states are changing according to the solidarity model.
First we consider the case where n =4. The possible states are (0000), (0001), (1110), (1111). This is because 3 entries are changed at once so it's possible to only change to the states where there are 3 identical entries in a row. The following table shows the transition probabilities from the states represented in the columns to the states in the rows:

	0000	0001	1110	1111
0000	$1-p$	$\frac{2}{3}(1-p)$	0	0
0001	0	$\frac{1}{3}(1-p)$	$1-p$	$1-p$
1110	p	$\frac{2}{3}p$	0	0
1111	0	$\frac{1}{3}p$	p	p

Solving $\pi P = \pi'$, where P is the transition matrix with an added column $[1111]^T$ for the condition $\pi 1 + \pi 2 + \pi 3 + \pi 4 = 1$ and π is the vector of probabilities, we get

$$[\pi 1 \quad \pi 2 \quad \pi 3 \quad \pi 4] \cdot \begin{pmatrix} 1-p & 0 & p & 0 & 1 \\ \frac{2}{3}(1-p) & \frac{1}{3}(1-p) & \frac{2p}{3} & \frac{p}{3} & 1 \\ 0 & (1-p) & 0 & p & 1 \\ 0 & (1-p) & 0 & p & 1 \end{pmatrix} = [\pi 1 \quad \pi 2 \quad \pi 3 \quad \pi 4 \quad 1] =$$

$$[\frac{2(p^2-2p+1)}{p+2} \quad \frac{-3p(-1+p)}{p+2} \quad \frac{-2p(-1+p)}{p+2} \quad \frac{3p^2}{p+2}]$$

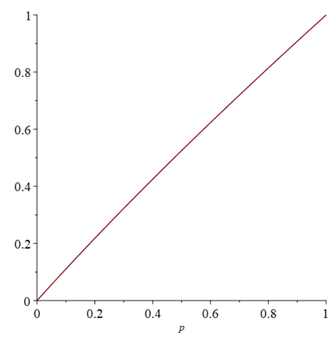
We find the expectation from this distribution, by introducing the new variable x_i that represents the fraction of "1" in state i, and multiplying it with the corresponding probabilities:

$$\nu(p) = \sum_{i=1}^{4} \pi i \cdot x_i = \frac{2(p^2-2p+1)}{p+2} \cdot 0 + \frac{-3p(-1+p)}{p+2} \cdot \frac{1}{4} + \frac{-2p(-1+p)}{p+2} \cdot \frac{3}{4} + \frac{3p^2}{p+2} \cdot \frac{4}{4}$$

$$= \frac{-9p(-1+p)}{4(p+2)} + \frac{3p^2}{p+2}$$

$$= \frac{3p^2+9p}{4(p+2)}.$$

Plotting the graph of this function:



$\nu(p), n = 4.$

Now we consider the case where n=5. The possible states of the process are (00000), (00001), (00011), (00111), (01111), (11111). The transition matrix for these states is:

	00000	00001	00011	00111	01111	11111
00000	$1-p$	$\frac{1}{2}(1-p)$	0	0	0	0
00001	0	$\frac{1}{2}(1-p)$	$\frac{2}{3}(1-p)$	0	0	0
00011	0	0	$\frac{1}{3}(1-p)$	$1-p$	$1-p$	$1-p$
00111	p	$\frac{1}{2}p$	$\frac{1}{3}p$	0	0	0
01111	0	$\frac{1}{2}p$	$\frac{1}{3}p$	0	0	0
11111	0	0	$\frac{1}{3}p$	p	p	p

Solving $\pi P = \pi'$ for π, where P is the transition matrix with an added column $[111111]^T$ for the condition $\pi 1 + \pi 2 + \pi 3 + \pi 4 + \pi 5 + \pi 6 = 1$ and π

is the vector of probabilities, we get

$$[\pi 1 \quad \pi 2 \quad \pi 3 \quad \pi 4 \quad \pi 5 \quad \pi 6 \quad 1]\cdot\begin{pmatrix} 1-p & 0 & 0 & p & 0 & 0 & 1 \\ \frac{1}{2}(1-p) & \frac{1}{2}(1-p) & 0 & \frac{1}{2}p & \frac{1}{2}p & 0 & 1 \\ 0 & \frac{2}{3}(1-p) & \frac{1}{3}(1-p) & \frac{1}{3}p & \frac{1}{3}p & \frac{1}{3}p & 1 \\ 0 & 0 & 1-p & 0 & 0 & p & 1 \\ 0 & 0 & 1-p & 0 & 0 & p & 1 \\ 0 & 0 & 1-p & 0 & 0 & p & 1 \end{pmatrix} =$$

$$[\pi 1 \quad \pi 2 \quad \pi 3 \quad \pi 4 \quad \pi 5 \quad \pi 6] =$$

$$[-\frac{2(-1+p)^3}{p^2+3p+2} \quad \frac{4p(-1+p)^2}{p^2+3p+2} \quad \frac{-3p^3+3p}{p^2+3p+2} \quad \frac{2p(-1+p)^2}{p^2+3p+2} \quad \frac{-4p^2(-1+p)}{p^2+3p+2} \quad \frac{3p^3+3p^2}{p^2+3p+2}]$$

The expectation is therefore

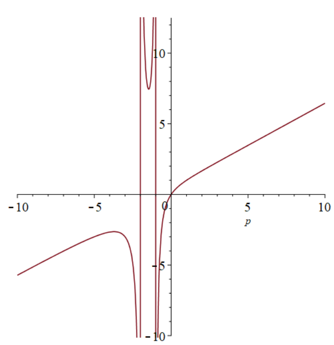
$$\nu(p) = \sum_{i=1}^{4} \pi i \cdot x_i$$

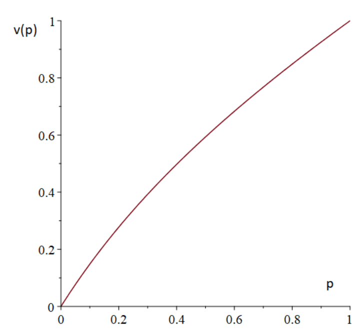
$$= 0 \cdot -\frac{2(-1+p)^3}{p^2+3p+2} + \frac{1}{5}\cdot\frac{4p(-1+p)^2}{p^2+3p+2} + \frac{2}{5}\cdot\frac{-3p^3+3p}{p^2+3p+2} + \frac{3}{5}\cdot\frac{2p(-1+p)^2}{p^2+3p+2}$$

$$+\frac{4}{5}\cdot\frac{-4p^2(-1+p)}{p^2+3p+2} + \frac{5}{5}\cdot\frac{3p^2(1+p)}{p^2+3p+2}$$

$$= \frac{10p(-1+p)^2}{5p^2+15p+10} + \frac{2(-3p^3+3p)}{5p^2+15p+10} - \frac{16p^2(-1+p)}{5p^2+15p+10} + \frac{15p^2(1+p)}{5p^2+15p+10}$$

$$= \frac{35p^3-21p^2+16p}{5p^2+15p+10}.$$



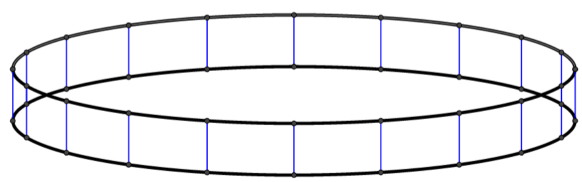


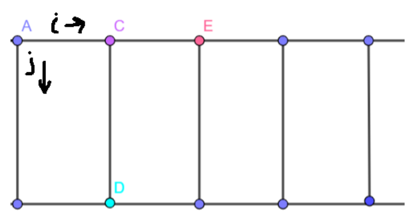
$\nu(p), n = 5.$

We see that for N=5 the dependency looks slightly closer in shape to that in the Solidarity model, because the multiplicity of p is higher.

Ladder

Now we look at a two-dimensional case of the model, called a ladder. A ladder is a graph with the vertices (i, j), where $j = 0, 1$, and $i = 0, 1, .., n$, so that $x_{(i,j)}$ is connected with $x_{(i-1,j)}, x_{(i+1,j)}, x_{(i,1-j)}$





C is connected with A, D and E

Just like before, we use the function T that bounds the drift from above, but this time with no correction β, in order to simplify the calculations. Looking at the left-hand side, there are several possibilities for the initial setting. They are:

$\xi_1 = \begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{smallmatrix}$,

$\xi_2 = \begin{smallmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{smallmatrix}$,

$\xi_3 = \begin{smallmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{smallmatrix},$

$\xi_4 = \begin{smallmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{smallmatrix},$

$\xi_5 = \begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{smallmatrix},$

$\xi_6 = \begin{smallmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{smallmatrix},$

$\xi_7 = \begin{smallmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{smallmatrix}.$

For this case, let $\xi_{(i,j)}(t)$ be a vertex of the set N with coordinates (i, j) at time t, and let $Z(t)$ be a 2-dimensional set of indices that satisfies the following properties:

1. $\xi_{(k,m)} = 1$ for all $i, j \notin Z(t)$,
2. $Z(t)$ has the smallest number of elements among such sets,
3. $Z(t)$ has periodic boundary conditions.

That is, $Z(t)$ constitutes a rectangular array of indices to all columns of the discrete set that have at least one 0. The corresponding one-dimensional subset $I(t) = [1, .., k]$ will be defined as follows:

1. $I_1(t) = Z_{(3,j)}(t)$
2. $I_k(t) = Z_{(n-2,j)}(t)$

Then on any vertex $i \notin I(t)$, for some $\epsilon(p) > 0$, when $p > p_{cr}$, it holds that

$$\Delta_{t+1} = \mathbb{E}(M_{t+1} - M_t|\mathcal{F}) < -\epsilon$$

In each of the cases ξ_i above it is possible to choose the element with the least fitness randomly, and this fact is taken into account while calculating T_i.
Writing out the T_i functions for every of the cases, we get:

$$T_1 = (1-p)^4 + 3p(1-p)^3 + 0 \cdot (1-p)p^3 + 3(1-p)^2p^2 - 2p^4$$

$$T_2 = (1-p)^4 + 3p(1-p)^3 + 0 \cdot (1-p)p^3 + 3(1-p)^2p^2 - 2p^4$$

$$T_3 = \frac{2}{3}(1-p)^4 + 2p(1-p)^3 + 0 \cdot (1-p)p^3 + 2(1-p)^2p^2 - p^4$$

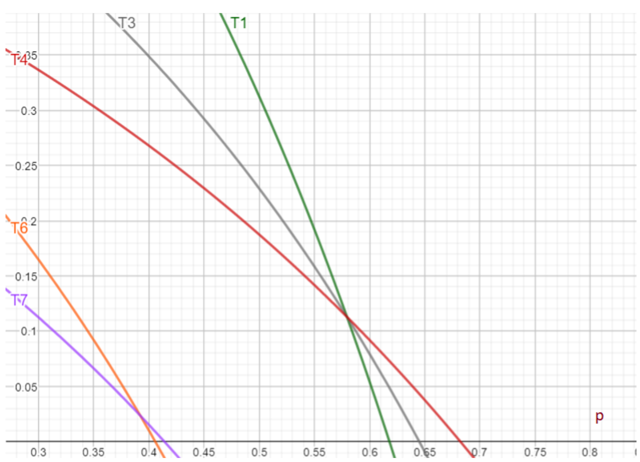
$$T_4 = \frac{1}{2}(1-p)^4 + \frac{3}{2}p(1-p)^3 + 0 \cdot (1-p)p^3 + \frac{3}{2}(1-p)^2p^2 - \frac{1}{2}p^4$$

$$T_5 = \frac{1}{2}(1-p)^4 + \frac{3}{2}p(1-p)^3 + 0 \cdot (1-p)p^3 + \frac{3}{2}(1-p)^2p^2 - \frac{1}{2}p^4$$

$$T_6 = \frac{1}{2}(1-p)^4 + p(1-p)^3 - 2(1-p)p^3 - \frac{5}{2}p^4$$

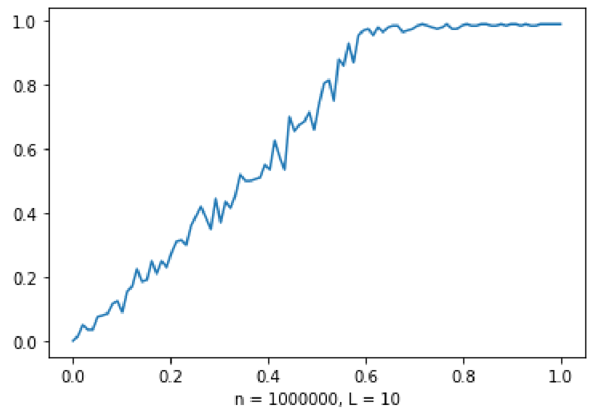
$$T_7 = \frac{1}{3}(1-p)^4 + \frac{2}{3}p(1-p)^3 - 4(1-p)p^3 - 4p^4$$

Now we try to find such value of p that for each $p > p_* \leq p_{cr}$, all $T_i(p)$ will be negative. Observing that $T_1 = T_2$ and $T_4 = T_5$, we plot T_1, T_3, T_4, T_6 and T_7 and see that T_4 is the greatest function that intersects the p axis at the point p_{cr} such that $(T_1(p_{cr}), T_3(p_{cr})) < 0$.



Solving $T_4 = 0$ on $p \in [0, 1]$, we get $p_* \approx 0.68233$.

Running the computer simulation for a ladder of length 1000 with 10000 repetitions for $p = \frac{1}{100}, \frac{2}{100}, .., \frac{100}{100}$, we get the following result:



According to the graph, after p that is approximately equal to our result, the fraction of "1" in the ladder asymptotically approaches 1. It implies that the analytical result is correct. We therefore fond the upper bound $p_* > p_{cr}$ for the parameter p that is smaller than a trivial upper bound.

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