Least-square Monte Carlo based option pricing of European and Bermudan stock index options

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Abstract

On the financial markets, there are a large number of financial instruments. Two of these instruments is the European and Bermudan option, where the Bermudan option can be seen as a discrete version of the American option. Meaning, if one can price the Bermudan option one can also estimate the price of an American option. A method used to estimate the Bermudan option price is the Least-square Monte Carlo approach. It is a numerical approach that uses simulated values of the underlying asset and fits a polynomial for each date exercise is possible. The function is used to estimate the holding value of the option, by which one can determine whether to exercise the option. Using four different price movement models to simulate the value of the underlying asset, European option prices were estimated using the standard Monte Carlo method and Bermudan option prices were estimated using the Least-square Monte Carlo approach. The results show that the pricing of the European options frequently results in options prices outside the ASK/BID-spread. It also shows tendencies towards better estimations using price movement models containing more parameters, but that these models do not always show better results. Probably, it is because of external problems such as parameter fitting. The results also show that the Least-square Monte Carlo approach works sufficiently well when pricing the Bermudan option, but that in some cases incorrect estimations are made stemming from the fitted polynomials. To conclude, the Monte Carlo based option pricing methods are considered to work and result largely in satisfactory estimations, but contain problems such as the choice and fitting of polynomials and parameter calibration.

Keywords: Option, Monte Carlo *, Least-square *, Black-Scholes, Merton, Heston, Bates

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Contents

1 Introduction

In this section the background for the project, its problem formulation, deliminations, and earlier studies conducted on the subject are presented.

1.1 Background

There is a ceaseless development of the financial instrument's prices, and therefore a need for traders to develop superior methods to value these assets and make a profit from them. One instrument whose pricing is particularly complex is options, which is a contract between an option holder and a option issuer. The holder have the right to buy or sell an underlying asset at a future date for a predetermined price. For a trader that want to issue an option, for which there are no current market, there is a need to set a correct price on the option. If prices are already available, there may be a need to control that they are reasonable, regardless of whether the trader intend to issue or buy the option.

One method to price options is through Monte Carlo simulations, which is the approach we will take on in this paper to price European and Bermudan stock index options. Four different models will be used to simulate the underlying asset's price. The simplest is the Black-Scholes-Merton model, in which one assumes the stock index follows a Geometric Brownian motion (a model using constant volatility). We will also consider two extensions of this model, which are the Merton Jump Diffusion (MJD) model and the Heston model. In the MJD model, random jumps in the stock indexes are added, while in the Heston model we take into account that the volatility is stochastic. The last model that is considered in this paper is Bates model, which includes both random jumps and stochastic volatility. A variance reducing technique will also be implemented and tested for the examined types of options.

1.2 Problem formulation

This thesis aims to price European and Bermudan stock index options for the OMXS30 index through Monte Carlo Based option pricing methods, with several models for the underlying asset price movements. Doing this, can one see a difference in performance between methods and models for stock price movements used, and to what extent the European & Bermudan options are correctly priced. Furthermore, do we see any improvement in performance if variance reducing techniques are implemented when performing the Monte Carlo-based option pricing.

1.3 Deliminations

The project conducted is restricted to one underlying asset and one single day, which mainly is due to existing limitations concerning computational power and access to needed data, for instance on Bermudan stock index option prices. The number of configurations and alternations of the methods used in the project was also restricted due to limitations concerning computational power and the scope of the project.

1.4 Earlier studies

Both the fields of Monte Carlo-based statistical methods and financial derivatives are extensively researched. In the extension, we get the field of financial engineering using Monte Carlo-based methods. Since many financial derivatives have no closed solution for their price, more and better methods for pricing these using for example Monte Carlo-based methods will always be sought after. When it comes to pricing European options, the Nobel prize-winning contribution of Robert C. Merton and Myron S. Scholes in collaboration with Fischer Black, was a tremendous leap forward for the world of academics, the world of finance and society at large [16]. A backside of the Black-Scholes-Merton model is that one of its assumptions is that the underlying asset moves according to a geometric brownian motion, and could be considered too crude and imprecise for real-world applications. Later, Peter Carr and Dilip B. Madan developed a way of pricing a European option for any price movement of the underlying asset that is used [10][9]. However, this does not apply to options other than European options. A method that (now) manages to estimate the price of American-style options is the Monte Carlo-based option pricing. Other popular methods used when pricing options include binomial trees and partial differential equation (PDE) methods.

In 1977, the Irish economist Phelim Boyle developed the first Monte Carlo-based method for option pricing [4]. By simulating paths for the underlying asset price, and averaging the discounted payoff for all the paths at the time of maturity, he could estimate a price for the option. The method was at first only applicable for European options, but in 1996, M. Broadie and P Glasserman utilized Monte Carlo methods in the pricing of Asian options. Oppose to American and European options, the price of Asian options depends on the average price of the underlying asset during a certain time period [6]. In the same year, the so-called Least-square Monte Carlo approach was introduced by Jacques F. Carriere to price American-style options, which are contracts that allow the holder to exercise the option continuously during the option's lifetime. Francis Longstaff and Eduardo Schwartz improved on this method by, among other things, excluding paths that are Out of the Money (OTM) in the option pricing algorithm and applying it to derivatives with several underlying factors [21].

2 Theory

In this section, we outline the theories and principles needed and used in this thesis. The overall themes we will discuss are Option theory, Monte Carlo theory, Stock price motion models, and Monte Carlo based option pricing

2.1 Option theory

In the following subsection, we outline the basic principles concerning options and options theory. Furthermore, the theory surrounding asset price movement and the Black-Scholes-Merton differential equation are outlined and explained.

2.1.1 Types of options

Firstly there are two types of options - call and put options. The call option gives the holder the right to buy the underlying asset at one or more specified point-/s in time for a predetermined price to the option issuer. The put option gives the holder the right to sell the underlying asset at one or more specified point-/s in time for a predetermined price to the option issuer. The predetermined price is called the strike price or the exercise price. The last date the holder is permitted to exercise the option is called the expiration date. The remaining time to the expiration date is called Time to Maturity. In European contracts, the option can only be exercised on the expiration date itself, whereas American options can be exercised any time up until and on the date of expiration [15] (p. 23-45). Because of the possibility of early exercise, the price of an American option is greater than or equal to the one of an European option with the same underlying asset, strike price and time to maturity [14]. Furthermore, there are option types called Bermudan options, which are options that can be exercised on predetermined point-/s in time up until and including the expiration date [15] (p. 620-645). A Bermudan option can be viewed as an American option with finite and discrete points in time the option can be exercised. Consequently, the price is less or equal to the one for an American option, but larger than or equal to the price of an European option [14].

A stock index tracks changes in the value of a hypothetical portfolio of stocks. The weight of a stock in the portfolio at a particular time equals the proportion of the hypothetical portfolio invested in the stock at that time. The percentage increase in the value of the stock index over a small interval of time is set equal to the percentage increase in the value of the hypothetical portfolio [15] (p. 71-98). A stock index option is an option with an index as the underlying asset. These are often European type options, where one contract usually is to buy 100 times the index at the specified strike price and settlement is always in cash (rather than receiving the actual asset) [15] (p. 235-255).

The typical appearance of the option price functions for an European put and American put option with the same underlying asset and strike price (which in this case is 10) are demonstrated in figure 1 with the corresponding payoff function. One property to take notice of is that they are convex and non-increasing [17].

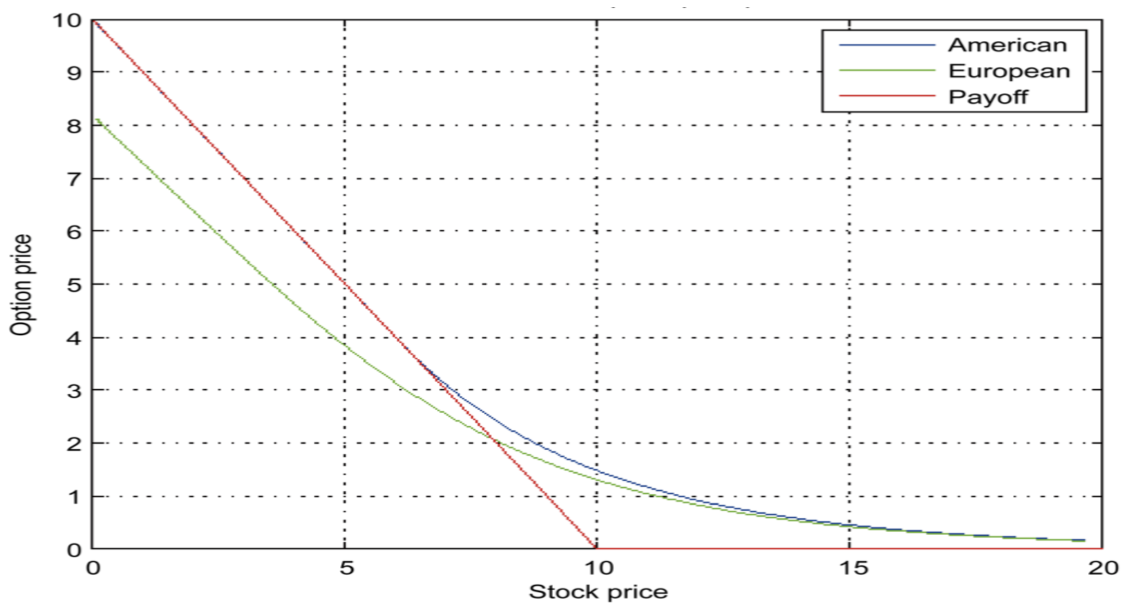

Figure 1: Plot of payoff function and prices for an European and American put option with strike price 10 [17].

2.1.2 Brownian motion

A stochastic process is the name of a family of random variables. We can denote the stochastic process as $\{Z(t), t \geq 0\}$, where the variable's position in the mathematical space is $Z(t)$ at time t, and the change in the variable's position is $\Delta z = \epsilon\sqrt{\Delta}t$, where $\epsilon \sim N(0,1)$, during a small time period Δt . The process is then called a Brownian motion process if the following properties are fulfilled:

(i) $Z(0) = 0$
(ii) Each increment of $\{Z(t), t \geq 0\}$, which we denoted Δz, are stationary and independent.
(iii) $Z(t) \sim N(0, \epsilon^2 t)$ for all $t > 0$ [26]

A standard Brownian motion process is modelled by a basic Wiener process, in which we assume ϵ follow a normal distribution with the mean equal to 0, and variance equal to 1. If we denote a Wiener process with the time step $\Delta t \to 0$ by dz, the Generalized Wiener process can be described as [31]:

$$dx = adt + bdz \tag{1}$$

Consequently, the change of the variable's position is then described by:

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta}t \tag{2}$$

Here, a is referred to as the drift rate and b^2 as the variance rate.

2.1.3 Itô's process & Itô's lemma

Suppose that the value of a variable x follows the process:

$$dx = a(x,t)dt + b(x,t)dz_t \tag{3}$$

, called the Itô's process. Here z is a Wiener process and a and b are functions of x and t, where x has a drift rate of a [31] and a variance rate of b^2 [15] (p. 324-342) [31].

Itô's lemma shows [31][34] that a function $G = G(t, x)$ follows the process [15] (p. 324-342):

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}b\,dz_t, \tag{4}$$

where z is a Wiener process.
This gives that G follows the Itô's process with a drift rate of $(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)$ and a variance rate of $(\frac{\partial G}{\partial x})^2 b^2$.

Replacing equation (3) with the equation:

$$dS = \mu S dt + \sigma S dz_t. \tag{5}$$

It then follows from Itô's lemma that the function $G = G(t, S)$ follows the process [15] (p. 324-342) [31][34]:

$$dG = (\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial G}{\partial S}\sigma S\,dz_t \tag{6}$$

2.1.4 Black-Sholes-Merton

The idea underlying the Black-Scholes-Merton (BSM) differential equation is that the equation must be satisfied by the price of any financial derivative dependent on a non-dividend-paying stock. The BSM differential equation is derived through setting up a riskless portfolio consisting of a position in the derivative and a position in the underlying asset. In absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, which leads to the BSM differential equation [15] (p. 324-375).

The assumptions made when deriving the BSM differential equation are [15] (p. 324-375):
1. The stock price follows a Geometric Brownian motion.
2. The short selling of securities with full use of proceeds is permitted.
3. There are no transaction costs or taxes. All securities are perfectly divisble.
4. There are no dividens during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest, r, is constant and the same for all maturities.

When deriving the BSM model we first consider a derivative's price at a time t, and the time T is the maturity date [15] (p. 343-375). This leads to that the time to maturity is T-t.
We assume the stock price process follow the model [15] (p. 324-342), which is a Geometric Brownian motion:

$$\frac{dS}{S} = \mu dt + \sigma dz_t \tag{7}$$

, which gives:

$$dS = \mu S dt + \sigma S dz_t \tag{8}$$

, where μ is the stock's expected rate of return, σ is the volatility of the stock price, dt is a short period of time, and the variable z follows a Wiener process. Now suppose that f is the price of a call option, or another financial derivative, contignent on S, then equation (6) gives the equation [15] (p. 343-375):

$$df = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S\,dz_t \tag{9}$$

If we discretize (9) we get:

$$\Delta f = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z, \tag{10}$$

where $\Delta z = \epsilon\sqrt{\Delta t}$, with $\epsilon \in N(0,1)$.

If we construct a portfolio where we go short in 1 option and we go long in $\frac{\partial f}{\partial S}$ stock underlying the option the value of the portfolio becomes:

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{11}$$

The change in value of the portfolio for a small time frame then becomes:

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{12}$$

Substitute equation (10) and the discretized version of equation (8) into equation (12) gives:

$$\Delta\Pi = -(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t - \frac{\partial f}{\partial S}\sigma S\,\Delta z + \frac{\partial f}{\partial S}\mu S\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z \tag{13}$$

Simplifying equation (13) gives:

$$\Delta\Pi = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t \tag{14}$$

It follows from equation (6) that since it does not contain Δz it must be riskless during time Δt, and hence it must have the same rate of return as other short-term risk-free securities so not to violate the assumption of no riskless arbitrage opportunities [15] (p. 343-375). This imply that:

$$\Delta\Pi = r\Pi\Delta t, \tag{15}$$

where r is the risk-free interest rate. Substituting equations (11) and (14) into equation (15) and simplify resulting expression gives:

$$rf = \frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S}S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \tag{16}$$

Equation (16) is called BSM differential equation and have many different solutions depending on the different derivatives that is defined and the underlying asset. The different derivatives depend on which boundary conditions that are used, i.e. the specification of the value of the derivative at the boundaries of possible values of S and t [15] (p. 343-375).

A way of solving BSM differential equation is through the of risk-neutral valuation approach. Since BSM differential equation does not contain any variables that are affected by risk preferences (current stock price, time, stock price volatility and the risk-free rate are all independent of risk preference), we can use any risk preferences when evaluating f. Making the assumption of a risk-neutral world, the expected value of a European call option at maturity becomes [15] (p. 343-375):

$$\hat{E}[\max(S_T - K, 0)], \tag{17}$$

and the expected value of a European put option at maturity becomes [15] (p. 343-375):

$$\hat{E}[\max(K - S_T, 0)] \tag{18}$$

Here $\hat{E}$ denotes the expected value in a risk-neutral world. In this risk-neutral world, the Euopean call (c) and put (p) option prices is the expected value discounted at the risk-free rate, which gives [15] (p. 343-375):

$$c = e^{-rT}\hat{E}[\max(S_T - K, 0)] \tag{19}$$

$$p = e^{-rT}\hat{E}[\max(K - S_T, 0)] \tag{20}$$

Solving equation (16) for a non-dividend-paying stock maturing at time T gives [15] (p. 343-375):

$$\begin{aligned} c &= S_0N(d_1) - Ke^{-rT}N(d_2) \\ p &= Ke^{-rT}N(-d_2) - S_0N(-d_1) \\ d_1 &= \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{ln(S_0/K) + (r - \sigma^2/2)}{\sigma\sqrt{T}} == d_1 - \sigma\sqrt{T} \end{aligned} \tag{21}$$

2.1.5 Problems using Black-Scholes-Merton

Problems arise when one wants to either price other options than non-dividend-paying European options, use other models for stock price movements than the Geometric Brownian motion, or both at the same time. We then get pricing formulas that are vastly more complicated than those defined by equation (21), or get no closed formed expression for the price of the option at all [15] (p. 324-342)[15] (p. 646-676) [23]. An alternative way of pricing different types of options with different models for stock price movements is then to use a Monte Carlo based option pricing method.

2.1.6 Continuous dividend yield

If we have dividend payments from the underlying asset, we can derive a no dividend paying model equivalent to the dividend paying one. If we assume that we have continuous dividend yield, one can show that equation (8) instead becomes [5]:

$$dS = (\mu - q)Sdt + \sigma Sdz_t, \tag{22}$$

where q is the dividend yield. Which means that equation (21) becomes:

$$\begin{aligned} c &= S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2) \\ p &= Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \\ d_1 &= \frac{ln(S_0/K) + (r - q + \sigma^2/2)}{\sigma\sqrt{T}} \\ d_2 &= \frac{ln(S_0/K) + (r - q - \sigma^2/2)}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned} \tag{23}$$

2.1.7 Time coherent interest rates model

We define a time coherent interest rate model as:

$$e^{r_{0,i}T_{0,i}}e^{r_{i,j}T_{i,j}} = e^{r_{0,j}T_{0,j}}, \tag{24}$$

where $i < j$

For two options with different time to maturity we define one time of maturity T_i and another T_j. Inserting these times of maturity and corresponding yields in equation (24) allows us to calculate a yield between time i and j according to:

$$e^{r_{i,j}T_{i,j}} = \frac{e^{r_{0,j}T_{0,j}}}{e^{r_{0,i}T_{0,i}}} \tag{25}$$

The yield between $t = 0$ and $t = i$ composed with the yield between $t = i$ and $t = j$ gives us a coherent yield curve between $t = 0$ and $t = j$.

If one have additional options with time of maturity k, where $k > j$, then using the options with time to maturity j and k and the same calculations as in equation (25) gives us the yield between $t = j$ and $t = k$. Composed with the yield curve calculated earlier we get a yield curve between $t = 0$ and $t = k$. This can be done repeatedly for more options with different time to maturity.

2.2 Monte Carlo theory

In the following subsection we outline and define the mathematical foundation on which the Monte Carlo based option pricing rely on, and that are used in this thesis.

2.2.1 Law of Large numbers

For i.i.d. sequences of one-dimensional random variables $X_1, X_2, ...,$ let $\bar{X_n} = \sum_{i=1}^{n} X_i/n$. The *weak law of large numbers* states that $\bar{X_n}$ converges in probability to $\mu = E\{X_i\}$, if $E\{|X_i|\} < \infty$. The *strong law of large numbers* states that $\bar{X_n}$ converges almost surely to μ if $E\{|X_i|\} < \infty$. Both results hold under the more stringent but easily checked condition that $var\{X_i\} = \sigma^2 < \infty$ [12] (p. 1-20).

2.2.2 Central limit theorem

If $X_1, X_2, ..., X_n$ are n random samples drawn from a population with overall mean μ and finite variance σ^2, and if $\bar{X_n}$ is the sample mean, then the limiting form of the distribution, $Z = \lim_{n\to\infty} \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)$, is a standard normal distribution [22].

2.2.3 Monte Carlo method basic principle

We define the expectation of a function of a random variable as $E\{h(\mathbf{X})\}$. Let f denote the density of $\mathbf{X}$, and μ denote the expectation of $h(\mathbf{X})$ with respect to f. When an i.i.d. random sample $X_1, ..., X_n$ is obtained from f, we can approximate μ by a sample average [12] (p. 151-200):

$$\hat{\mu}_{MC} = \frac{1}{n}\sum_{i=1}^{n} h(\mathbf{X}_i) \rightarrow \int h(\mathbf{X}) f(\mathbf{X}) d\mathbf{x} = \mu \tag{26}$$

as $n \to \infty$, by the strong law of large numbers. Further let $v(\mathbf{x}) = [h(\mathbf{x}) - \mu]^2$, and assume that $h(\mathbf{X})^2$ has finite expectation under f. Then the sampling variance of $\hat{\mu}_{MC}$ is $\sigma^2/n = E\{h(\mathbf{X})/n\}$, where the expectation is taken with respect to f [12] (p. 151-200).

2.2.4 Control variates variance reduction

The control varites variance reduction technique is a technique used improve an estimate by relating the estimate to some correlated estimator of an integral whose value is known. Suppose we want to estimate $\mu = E\{h(X)\}$, and we know the related quantity $\theta = E\{c(Y)\}$ which can be determined analytically. Let $(X_1, Y_1), ..., (X_n, Y_n)$ denote pairs of random variables observed independently as simulation outcomes, which gives $cov(X_i, X_j) = cov(Y_i, Y_j) = cov(X_i, Y_j) = 0$ when $i \neq j$. Then the normal Monte Carlo estimators are $\hat{\mu}_{MC} = \frac{1}{n}\sum_{i=1}^{n} h(\mathbf{X}_i)$ and $\hat{\theta}_{MC} = \frac{1}{n}\sum_{i=1}^{n} c(\mathbf{Y}_i)$. If $\hat{\mu}_{MC}$ is correlated with $\hat{\theta}_{MC}$ one wants to adjust $\hat{\mu}_{MC}$ accordingly . This suggest a control variate estimator looking like [12] (p. 151-200):

$$\hat{\mu}_{CV} = \hat{\mu}_{MC} + \beta(\hat{\theta}_{MC} - \theta), \tag{27}$$

where β is the parameter to be chosen. If we take the variance of equation (27) then we get:

$$var(\hat{\mu}_{CV}) = var(\hat{\mu}_{MC}) + \beta^2 var(\hat{\theta}_{MC}) + 2\beta cov(\hat{\mu}_{MC}, \hat{\theta}_{MC}) \tag{28}$$

Minimizing equation (28) with respect to β gives:

$$\beta = -\frac{cov(\hat{\mu}_{MC}, \hat{\theta}_{MC})}{var(\hat{\theta}_{MC})} \tag{29}$$

One can see it as the control variate estimator is the fitted value on a regression line at the mean value of the predictor, and the standard error of this control variate estimator is the standard error of the fitted value from the regression [12] (p. 151-200).

2.3 Option pricing by Carr and Madan

A general formula for pricing any options in a complete and arbitrage-free market is [10]:

$$V_t(K) = e^{-r(T-t)} E_Q[H(S_t)] \tag{30}$$

$V_t(K)$ = Option value at time t with strike price K
T = Time to maturity
E_Q = Expectation under risk neutrality
$H(S_t)$ = Option payoff

Peter Carr and Dilip B. Madan developed a formula to solve equation (30) for the pricing of European options. The result is presented below [9]:

$$\begin{aligned} C_t(k) &= \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \psi(v) dv \\ \psi_t(v) &= \frac{e^{-r(T-t)} \Phi(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \end{aligned} \tag{31}$$

$C_t(k)$ = European call option price function
$k = ln(K)$, where K is the strike price
α = damping factor
r = interest rate of a risk free asset
Φ = Risk neutral characteristic equation

The risk neutral characteristic function Φ differs depending on the model for stock price movement that is used.

2.4 Stock price motion models

In the following subsection different Stock price motion models are outlined and defined, as well as their specific characteristics, similarities and differences.

2.4.1 Black-Scholes model

What in this project will be referred to as the Black-Scholes model is the price movement model presented in equation (7).

2.4.2 Merton Jump Diffusion model

The Merton Jump Diffusion model, or Merton model, is one extension of the ordinary BSM model. It aims to capture the effect unexpected events may have on the stock prices. New information on the market may cause a stock's price to drastically decline or rise, and these discontinuities does the BSM model not account for. In 1976, Merton introduced a new model to capture this phenomena. It consists of lognormal jumps, driven by a Poisson process, while the diffusion is modelled by a Geometric Brownian motion. These two processes are independent of each other. The equations for the stock price movement is presented below [23]:

$$
\begin{aligned}
\frac{dS(t)}{S(t)} &= (r - \lambda \bar{j})dt + \sigma dW(t) + jdq(t) \\
\bar{j} &= (e^{\mu_j + \frac{\sigma_j^2}{2}} - 1)
\end{aligned}
\tag{32}
$$

$S(t)$ = Stock price at time t.
r = interest rate of a risk free asset.
$dW(t)$ = Geometric Brownian motion
$dq(t)$ = Poisson process.
λ = intensity of the jumps.
j = coefficient for the jump size.
$\bar{j}$ = mean of log jump.
σ_j = standard deviation of log jump.

The value of λ affect the volatility and kurtosis, which is a measure of the tail-size in the probability distribution. The jump parameter k affect the skewness, which is a measure of the steepness of the probability distribution [18]. The model can be utilized to price options by the use of the method developed by Carr and Madan, which is presented in (31). The characteristic function for the MJD model is [23]:

$$
\begin{aligned}
\Phi_{\text{Merton}}(\phi) &= \Phi_{BS}(\phi)\exp(\lambda(T-t)(e^{i\phi\mu_J - \phi^2\sigma_J^2/2} - 1 - i\phi(e^{\mu_J + \sigma_J^2/2} - 1))) \\
\Phi_{BS}(\phi) &= \exp\left(i\phi\left[\ln(S(t)) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] - \frac{\phi^2\sigma^2}{2}(T-t)\right) \\
\phi(u) &= E[exp(iu \cdot log(S(T)))]
\end{aligned}
\tag{33}
$$

2.4.3 Heston model

Unlike the BSM model, in which the volatility of the stock price is assumed to be constant over time, does the Heston model utilize stochastic volatility. The model's equation for the movement of the stock price is described as follows [29][1]:

$$
\begin{aligned}
\frac{dS(t)}{S(t)} &= rdt + \sqrt{V(t)}dW_1(t) \\
dV(t) &= \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2(t) \\
\rho dt &= dW_1(t)dW_2(t)
\end{aligned}
\tag{34}
$$

$S(t)$ = price of the underlying asset at time t
r = interest rate of a risk free asset
$\sqrt{V(t)}$ = asset price volatility at time t
σ = volatility of $\sqrt{V(t)}$
θ = mean of the long term squared volatility
κ = mean-reversion rate back to θ
dt = indefinitely small time step $(0 \leftarrow \Delta t)$
$W_1(t)$ = Brownian motion for asset price
$W_2(t)$ = Brownian motion for asset's price variance
ρ = correlation coefficient for the two Brownian motions

One major advantage with the Heston model is that it captures the so called volatility smiles, which is the phenomena where options further Out of the Money (OTM) or In the Money (ITM) is accompanied with higher implied volatility. In this case, implied volatility is an estimate of an option's volatility based on the volatility of other options on the market. If we were to compare implied volatility against strike price, where the options' expiration date and underlying asset is the same, the graph would be concave. This is unlike the BSM model, for which the graph would be a straight line due to the assumption of constant volatility [28]. The correlation coefficient ρ affects the skewness of the smile, while the volatility parameter θ influence the kurtosis [18].

In Equation (34), the correlation coefficient ρ varies between -1 and 1, but in reality it tends to be negative due to the leverage effect. The negative relationship between the asset return and its volatility implies that declining prices tends to be accompanied with increasing volatility and vice versa. Another phenomena the model captures is the tendency for the asset price volatility to revert back to the long run variance with the rate k [23].

When pricing option based on the stock price movement suggested by Heston, one can utilize the method developed by Carr and Madan. The formulas (31) are used with the risk neutral characteristic equation for the Heston model, which is presented below [23]:

$$\begin{aligned}
\Phi_{\text{Heston}}(\phi) &= \exp\left(C + DV_0 + i\phi\ln(S(t))\right) \\
C = ri\phi(T-t) + \frac{\kappa\theta}{\sigma^2}&\left[\left(\kappa + \lambda_{VolRisk} - \rho\sigma i\phi + d\right)(T-t) - 2\ln\left(\frac{1-ge^{d(T-t)}}{1-g}\right)\right] \\
D &= \frac{\kappa + \lambda_{\text{VolRisk}} - \rho\sigma i\phi + d}{\sigma^2}\left(\frac{1-e^{d(T-t)}}{1-ge^{d(T-t)}}\right) \\
g &= \frac{\kappa + \lambda_{\text{VolRisk}} - \rho\sigma i\phi + d}{\kappa + \lambda_{\text{VolRisk}} - \rho\sigma i\phi - d} \\
d &= \sqrt{\left(\kappa + \lambda_{\text{VolRisk}} - \rho\sigma i\phi\right)^2 + \sigma^2\left(i\phi + \phi^2\right)} \\
\phi(u) &= E[exp(iu \cdot log(S(T))), V(t)]
\end{aligned} \tag{35}$$

$\lambda_{VolRisk}$ in the equation is the risk premium the stakeholders demand as compensation for the volatility of the asset price, but in a risk-neutral world it is equal to zero.

2.4.4 Bates Model

A combination between the Heston model and MJD model was introduced by David Bates in 1996, which takes both stochastic volatility and random jumps into account in the modeling of the stock price movement. The stock price movement is described by the following equations [18]:

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= (r - \lambda\bar{j})dt + \sqrt{V(t)}dW_1(t) + jdq(t) \\
dV(t) &= \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2(t) \\
\rho dt &= Cov(W_1(t)W_2(t))
\end{aligned} \tag{36}$$

$S(t)$ = price of the underlying asset at time t
r = interest rate of a risk free asset
λ = intensity of the jumps
j = coefficient for the jump size.
$\bar{j}$ = mean of log jump.
$\sqrt{V(t)}$ = asset price volatility at time t
σ = volatility of $\sqrt{V(t)}$
θ = mean of the long term squared volatility
κ = mean-reversion rate back to θ
dt = indefinitely small time step $(0 \leftarrow \Delta t)$
$W_1(t)$ = Brownian motion for asset price
$W_2(t)$ = Brownian motion for asset's price variance
$q(t)$ = Poisson process
ρ = correlation coefficient for the two Brownian motions

The characteristic equation for the Bates model is [23]:

$$\begin{aligned}\Phi_{\text{Bates}}(\phi) &= \exp\left(C + DV_0 + i\phi\ln(S(t))\right)\cdot A \\ A &= \exp(\lambda(T-t)(e^{i\phi\mu_J - \phi^2\sigma_J^2/2} - 1 - i\phi(e^{\mu_J+\sigma_J^2/2} - 1))) \\ \phi(u) &= E[exp(iu\cdot log(S(T))), V(t)]\end{aligned} \tag{37}$$

The formulas for C and D are given in (35)

2.5 Parameter calibration

To use the models mentioned in section 3.3, the parameters need to be calibrated as they are usually unknown beforehand. The idea is to use the Carr and Madan method (31) and the characteristic equation for each model based on some parameters that we estimate. We then compare these prices with the market option prices. If the price evolution corresponds well, we use the estimated parameters in the models when pricing options with Monte Carlo methods.

To determine the parameters, one may use the weighted least square method. The idea is to determine the parameters that minimize the squared distance between the option prices calculated with the model's closed-form formula and the observed prices on the market. We introduce the following equations and notations to explain the problem mathematically [20]:

$$\begin{aligned}L_t^{\text{WLS}}(\theta) &= \sum_{s=1}^{t}\sum_{i=1}^{N_s}\lambda_{s,i}\left(c_s^\star(K_i,\tau_i) - c_s^{\text{Model}}(K_i,\tau_i;\theta)\right)^2 \\ \hat{\theta}_t &= \arg\min_{\theta\in\Theta} L_t^{\text{WLS}}(\theta)\end{aligned} \tag{38}$$

L_t^{WLS} = Objective function, i.e. sum of squared residuals.
$c_s^\star$ = Market price.
c_s^{Model} = Predicted price by the model.
θ = Vector of parameters in each model.
$\hat{\theta}$ = Vector of parameters that minimizes the problem function.
$\lambda_{s,i}$ = Weights

The Market price is approximated as the middle-value between the bid and ask price. The choice of the weight λ can vary, but to receive minimal variance one should set it as the inverse of the residuals' variance. One can also choose λ as to be proportional to the inverse of the bid-ask spread, or simply use a constant [20].

Oftentimes, the problem function may not be convex and inherit several local minimum points. In this case, it is not assured that a certain solution $\hat{\theta}$ is the optimal one, as it will depend on which parameters we start with. To get closer to the global minimum, one can utilize sequential parameter calibration. The idea is to start with a set of parameters at time $t = 1$ and minimize the objective function for the observations at this time. The resulted vector-matrix $\hat{\theta}_1$ is then used as a start point for the optimization problem when including observations of the option prices at both time $t = 1$ and $t = 2$. The resulting vector $\hat{\theta}_2$ is then the starting point at time $t = 3$ and so on. The goal is to get closer to a global minimum by sequentially updating the parameter vector.

2.6 Monte Carlo based option pricing

In the following section the basic principles used, when options are priced by the Monte Carlo method, are outlined and discussed.

2.6.1 Monte Carlo based option pricing methodology

When using Monte Carlo-based option pricing we start by looking at equations (19) and (20) again. These equations tell us that the present value of a call and put option, under the assumption of a risk-neutral world, is equivalent to its discounted expected pay-off. This is the basis for option pricing through Monte Carlo methods. A payoff is received by sampling price paths of the underlying asset, and the expected payoff is estimated by discounting the value with the risk-free rate. This is finally used to obtain the value of a derivative dependent on the underlying asset [15] (p. 472-515).

The basic principle when pricing options with the Monte Carlo method is as follows [15] (p. 472-515) [15] (p. 646-676):

1. Sample a random path for the underlying asset in a risk-neutral world.
2. Calculate the payoff from the derivative.
3. Repeat step 1 and 2.
4. Discount the payoffs to present time.
5. Calculate the mean of the sampled discounted payoffs to get an estimate of the value of the derivate.

By combining equations (30) and (26), with the *Law of large numbers* and the basic principle when pricing options with the Monte Carlo method, this can be written as [15] (p. 472-515) [15] (p. 646-676):

1. Sample a random path for the underlying asset in a risk-neutral world.
2. Calculate if the is any early exercise of the option, and then calculate the payoff from the same. The payoff from the option at exercise time T for iteration i can be written as:
   i. $c_{T,i} = \max(S_{T,i} - K, 0)$ for a call option
   ii. $p_{T,i} = \max(K - S_{T,i}, 0)$ for a put option.
3. Repeat step 1 and 2, n times.
4. Discount the payoff to present time gives the discounted payoff:
   i. $c_{t,i} = e^{-r(T-t)} \max(S_{T,i} - K, 0)$ for a call option
   ii. $p_{t,i} = e^{-r(T-t)} \max(K - S_{T,i}, 0)$ for a put option.
5. estimate the value of a:
   i. call option as $C_t = \frac{1}{n}\sum_{i=1}^{n} c_{t,i}$
   ii. put option as $P_t = \frac{1}{n}\sum_{i=1}^{n} p_{t,i}$

2.6.2 The Least-square approach

When estimating the price of an option, for which there are several dates for possible exercise, one need to estimate the value of being able to perform early exercise. There are several possible methods of doing this, one of which is the Least-square Monte Carlo (LSM) approach.

The idea behind the LSM approach is at each time point when the holder can exercise the option, compare the payoff for immediate exercise with the expected payoff for continued holding. This is done for all simulated price paths. There are multiple versions of the LSM approach, but it is the version developed by E. Schwartz and F. Longstaff that will be utilized and somewhat expanded in this paper. We start by defining the value of an American option [8]:

$$\begin{aligned} F &= \max(u(S,t), \tilde{F}) \\ \tilde{F} &= \max_{t<\tau<T} E'\left[e^{-r(\tau-t)} u(S(\tau),\tau) \mid S(t) = S\right] \end{aligned} \tag{39}$$

F = Value of the American option
$u(S,t)$ = Payoff for exercise at time t and stock price S
$\tilde{F}$ = Payoff for postponed exercise
$E'(...)$ = Risk neutral expectation

r = risk free interest rate
τ = The time of future payoff
T = Time of maturity

The first step of the LSM algorithm by Longstaff and Schwartz is to simulate N random paths for the stock price (S_n^k, t_n), where $1 \leq k \leq N$ and $t_n = ndt$. k is the price trajectory index and n denotes the time index. The first iteration is done at the second to last time point the holder can exercise the option, which is the time point before the time of maturity. Therefore, we can assume that $F_{n+1}^k = F(S_{n+1}^k, t_{n+1})$ is known. We define the current asset value as $X = S_n^k$ and the expected future value as $Y^k = e^{-rdt}F(S_{n+1}^k, t_{n+1})$. To increase the accuracy of the conditional expectation function near the stock prices for which exercise is relevant, only ITM trajectories are included. The regression of Y is performed conditional on a finite number of basis functions $p_1(X), p_2(X), ..., p_m(X)$ such that the squared residuals is minimized, yielding the following expression for the conditional expectation function:

$$E(Y|X) = \sum_{j=0}^{m} a_j p_j(X) \tag{40}$$

For each ITM path, $E(Y|X)$ is compared with the payoff received by early exercise $u(S,t)$. The holder should exercise the option immediately if $E(Y|X) < u(S,t)$, and keep the option alive to a future state if $E(Y|X) > u(S,t)$. If the option is decided to be exercised at a certain time point, the possible future payoffs will be set to zero. Proceed recursively, we now consider the third last time point and examine whether the holder should exercise the option early or keep it alive, then the forth and so on. The option price is estimated by the average of the discounted future payoffs $\tilde{F}$ [21]. Note, it is not the conditional expected payoffs $E(Y|X)$ that are discounted. These values are simply used in the decision of keeping the option alive or exercise it at another point in time. Since the LSM algorithm utilizes the stock prices from the simulated stock price paths (denoted from both simulated price and volatility movements when working with stochastic volatility models), rather than the stock prices and corresponding volatilities at each time point, we can apply the same algorithm when working with stochastic volatility models [7].

There are several alternative basis functions to use in the regression of the conditional expectation function. Longstaff and Schwartz mention a few proper alternatives in their paper, among which this paper focuses on regressions based on polynomial of power functions (41), Hermite polynomial (42), weighted Laguerre polynomial (43) [2] [21], and Non-weighted Laguerre polynomial (44).

$$p_i(X) = X^i \tag{41}$$

$$p_n(X) = (-1)^n e^{x^2} \frac{d^n}{dX^n} e^{-x^2} \tag{42}$$

$$p_n(X) = \exp(-X/2) \frac{e^X}{n!} \frac{d^n}{dX^n} \left(X^n e^{-X}\right) \tag{43}$$

$$p_n(X) = \frac{e^X}{n!} \frac{d^n}{dX^n} \left(X^n e^{-X}\right) \tag{44}$$

As (43) shows does each component of the weighted Laguerre polynomial include a negative exponent of the stock price. Unless a proper change of variables is implemented, there might arise computational scaling problems and consequently inaccuracy in the fitted expectation function. An appropriate choice for the put option case is to divide all cash flows and prices with the strike price.

Then, the objective of fitting a function is limited to the area of $L = \frac{S}{K} \in [0, 1]$, in which the basis function decreases with S. This is a desired property, since the conditional expectation function for a put option should be decreasing in S. However, for the call option, another variable change is needed since its expectation function increases as S increases. It would be a bad fit to represent an increasing function with decreasing components. Therefore, we introduce the variable $L = \frac{K}{S}$ and divide the payoffs with the stock price, thus transforming the problem to the same area as in the put-option case. More precisely, we fit the function on a compact area for which it is decreasing in L.

For an American put option, if the conditional expectation function is fitted correctly, it should behave like the price function plotted in figure 1. The price is based on future realized cash flows, which is why it is a proper representation of the continuation values. A similar appearance of the function is expected for a Bermudan put option as well, since it is simply an American-style option with discrete time points where the holder can exercise the option.

2.6.3 Exercise Boundary

A property of the time constant volatility models' payoff functions is quasi-linearity. A consequence is that it is possible to predetermine an exercise boundary, which can be used in the comparison procedure at each time step for any option we wish to price. A problem of the ordinary LSM algorithm is that the estimated expectation functions tend to be bad representations of the continuation values when it is based on only a few numbers of ITM trajectories. To get a better fit and decision basis on whether to exercise the option early or at a later point in time, we can determine criteria for when to exercise the option beforehand. To begin with, instead of expressing the prices and payoffs in absolute terms, they are put as a relative value. More specifically, the payoff function for a put option can be rewritten as $u(S, K) = K \cdot u(\frac{S}{K}, 1)$. For a call option, it is instead transformed in the following way $u(S, K) = S \cdot u(1, \frac{K}{S})$, to get the desired properties for the regression of the expectation function (see section 3.5.2). These transformations also mean the definition set, let us call it D, is on the form $D \in [0.1]$. The start values for the price simulation are spread out around the strike price to get more trajectories ITM. Start values for a put option should be below its true stock price and the opposite for a call option. Then, at each point in time, a conditional expectation function is fitted. By simulating many trajectories at proper start points, these functions are (hopefully) good estimates of the continuation values. The stock price for which the expectation function intersects the payoff function in each time point forms the so-called optimal exercise boundary. If there are several intersects, the one that is ITM and closest to the strike price is the one of interest. In the pricing procedure, new trajectories are simulated for the price of the underlying asset, where the start value is set to the initial price of the underlying. Then, starting from the first point in time, the underlying asset's price is compared with the corresponding critical price in the exercise boundary. For a put, if the underlying asset's price is below the corresponding critical price, the holder should exercise the option at this point. For a call option, the opposite criteria apply. Since the values in the exercise boundary are in relative terms, it is possible to use it as a stopping criterion for options with different strike prices and time to maturity, as long as the dynamics of the underlying asset price has constant volatility.

For the Black-Scholes model, the exercise boundary precisely before the option expires approaches the limits $\max(r/q, 1) \cdot K$ for the call option and $\max(\min(r/q, 1), 0) \cdot K$ for the put option. These limits are true when the underlying price movement model of the underlying asset follows the Black-Scholes model. Another way of determining the exercise boundary precisely before the option expires, and a way that holds true regardless of the price movement model of the underlying asset, is to think of the limit as the price of the European option at this point in time. Imagine a discretized timeline with the possibility to exercise the option at each discrete point. At the second to last point, the value of the option can be represented by the European option price since there is no point at which one can exercise the option between the second to last and the last point in time. This in turn must lead to that the exercise boundary precisely before the option expires can be calculated by finding the value for which the pay-off function equals the value of the European option. Another property of the exercise boundary is that it is a concave and monotonically increasing function of time to maturity for the call option, and a convex and monotonically decreasing function of time to maturity for the put option [32].

For stochastic volatility models, the exercise boundary needs to include both the price and volatility of the underlying asset. This means that a multinomial base function needs to be fitted, which depends on the asset price and the volatility. When this multinomial base function is fitted, one can determine an exercise boundary that is dependent on both the price and volatility of the underlying asset.

2.6.4 Option pricing with control variates

When pricing European options, the underlying assets provide a source of control variates. Suppose we are working with the risk-neutral measure and that the interest rate is r, and dividend yield q. If $S(t)$ is an asset price, then $e^{-(r-q)t}S(t)$ and $E[e^{-(r-q)t}S(t)] = S(0)$ [13]. Suppose we are pricing options with S as the underlying asset and the discounted pay-off $Y_i = \phi(S_i(T)) = e^{-rT}\max(S_i(T) - K, 0)$ for a call option and $Y_i = \phi(S_i(T)) = e^{-rT}\max(K - S_i(T), 0)$ for a put option. From independent replications of S_i we can form the control variate estimator of the European option price as [13][19]:

$$\begin{aligned} \hat{V}_{CV} &= \frac{1}{N}\sum_{i=1}^{N}(\phi(S_i(T)) - \beta^*(S_i(T) - S(0)e^{(r-q)T})) \\ \beta^* &= \frac{Cov(\phi(S_i(T)), S_i(T))}{var(S_i(T))} \end{aligned} \tag{45}$$

The estimator for the Bermudan option, where t_i is the time of exercise, becomes:

$$\begin{aligned} \hat{V}_{CV} &= \frac{1}{N}\sum_{i=1}^{N}(\phi(S_i(t_i)) - \beta^*(S_i(T) - S(0)e^{(r-q)T})) \\ \beta^* &= \frac{Cov(\phi(S_i(t_i)), S_i(T))}{var(S_i(T))} \end{aligned} \tag{46}$$

This is a control variate that is usable for all contracts but not always efficient [19]. When pricing Bermudan options, an alternative way of reasoning is that the price of the European option can be used. The European options mean value can analytically be calculated and can therefore be seen as known, and the estimated values of the Bermudan and European options is correlated (see chapter 3.2.7) [11][24]. Then, if $\hat{V}^{Eu}$ is the estimated value of the European option and V^{Eu} is the value of the European option derived from the closed form solution we can form the control variate estimator, where t_i is the time of exercise, of the Bermudan option price as:

$$\begin{aligned} \hat{V}_{CV} &= \frac{1}{N}\sum_{i=1}^{N}(\phi(S_i(t_i)) - \beta^*(\hat{V}^{Eu} - V^{Eu}) \\ \beta^* &= \frac{Cov(\phi(S_i(t_i)), \hat{V}^{Eu})}{var(\hat{V}^{Eu})} \end{aligned} \tag{47}$$

Here the call option can be written as:

$$\beta^* = \frac{Cov(\phi(S_i(t_i)), e^{-rT}\max(S_i - K, 0))}{var(e^{-rT}\max(S_i - K, 0))}, \tag{48}$$

and the put option can be written as:

$$\beta^* = \frac{Cov(\phi(S_i(t_i)), e^{-rT}\max(K - S_i, 0))}{var(e^{-rT}\max(K - S_i, 0))} \tag{49}$$

3 Methodology

In this section, the implementations and approaches, as well as assumptions and delimitations, are explained.

3.1 Implementation

To be able to price European and Bermudan stock index options, and going forward be able to answer the questions posed in the problem formulation, a literature study was initially conducted. Here suitable models and approaches were chosen and further researched. The result of this literature study is outlined in the theory part of this thesis (see chapter 2). During this phase, suitable data and tools were also collected. The data includes index prices and European stock index option data for OMXS30 between the dates 1992-May-24 - 2010-August-31. Tools used in this thesis include MATLAB and Excel, where all code that was used was written by the authors, except opt_price.m which was provided by Magnus Wiktorsson. The file opt_price.m is a file used to analytically calculate the European option prices and is implementable for all the price movement models used in the project.

Estimations of option prices for the European and Bermudan stock index option prices were done for four different price movement models. These were the Black-Scholes model, Merton model, Heston model, and Bates model (see chapter 2.4). For all of these, we used models adjusted for continuous dividend yield (see chapter 2.1.6), and a time coherent interest rate model (see chapter 2.1.7). Initially, the parameters of the models were calibrated with help of collected data according to the theory on parameter calibration (see chapter 2.5). Here, the sequential calibration was done over one day (effectively not making it sequential, or sequential with a length of one). This to be able to have the same number of days that the parameters are calibrated for for all models, as the same time one does not need to handle that in the Heston and Bates model the variance is as a process and therefor can take on different values for different days. The weights λ were set to one for one of the group of simulations, and to the inverse of the ASK/BID-spread for another. The option prices for both the European and Bermudan-style options where calculated according to Monte Carlo based option pricing methodology (see chapter 2.6.1). For the Bermudan, the LSM approach was implemented to calculate possible early exercises of options (see chapter 2.6.2). The basis functions examined where a third degree polynomial of power functions (41), the first four Hermite polynomials (42), the first four Laguerre polynomials (44) and the first four weighted Laguerre polynomials (43), from which one or several where chosen to conduct simulations with. The discretized time jumps used in the calculations were one day, and the Bermudan style option were assumed to be able to be exercised at each of these with exception for when $t = 0$.

In addition to the standard LSM approach, the normalized exercise boundary approach was studied and implemented in the pricing of Bermudan options (see chapter 2.6.3). When calculating the normalized exercise boundaries for the models with deterministic volatility (Black-Scholes and Merton), for each day, a polynomial is fitted and the intersection between the polynomial and the pay-off function is calculated to determine the normalized exercise boundary. This is done with the first four weighted Laguerre polynomial and parameter weights set to one. When the intersection is calculated the non-linear least square function (lsqnonlin in MATLAB) is used, and the limit of the first intersection is set to 1. The limits of the following intersections is set to the value of the previous intersection. When calculating the normalized exercise boundaries for the models with stochastic volatility (Heston and Bates), for each day, a multinomial is first fitted. Then for 100 evenly spread volatilities between 0 and 1 the intersection between the multinomial and the pay-off function is calculated to determine the normalized exercise boundary. This is done with a multinomial of second degree and parameter weights set to one. The non-linear least square function (lsqnonlin in MATLAB) is used to calculate the intersections. Here, no transformation is done to get the same definition set for the call and put options. The limit (lower limit for the call option and upper limit for the put option) of the first intersection is set to 1. For the call option, the limit of the following intersections is the one which is the greatest of the last intersection calculated the same day or the intersection calculated for the corresponding volatility the previous day. For the put option, the limit of the following intersections will be the smallest value of the last intersection calculated the same day or the intersection calculated for the corresponding volatility the previous day.

Simulations were also replicated using control variates as a variance reducing technique (see chapter 2.2.4 & 2.6.4), for the Black Scholes model to be able to evaluate the effect of the implementation of control variates. Here, MATLAB was used to implement above described methodologies, and the resulting data was collected, plotted and analyzed with MATLAB and Excel.

The results were presented (see chapter 4) and analyzed (see chapter 5), to try to answer the questions posed in the problem formulation (see chapter 1.2). Finally a discussion were conducted and conclusions were drawn (see chapter 6) based on the presented results and the analysis.

3.2 Assumptions and limitations

There are two major limitations primarily affecting this project. Firstly, access to data is restricted when it comes to market data concerning options. The data available to the public provided by Nasdaq Nordic concerning options are only daily data and not historical data. Accessing the data from elsewhere has proven to be difficult, and often very costly. Consequently, the project has been restricted to center around data provided by the authors' supervisor. Secondly, computational power and access to computer units have inhibited the project's progress and success during the course. The restricted computer power has restricted the accuracy of the calculations, while the number of computer units accessible has put a limit on how many calculations, simulations, and other tasks, could be performed simultaneously.

There are several assumptions made when deriving the theory concerning both the options and the price movements of the underlying asset, which all can be found in chapter 2. Furthermore, we have assumed a Bermudan option that can be exercised at the end of each day, i.e. at $t = 1$, $t = 2$,..., $t = T$.

4 Results

In this section, the results are presented, as well as the meaning of the results and how they were obtained.

4.1 Basis functions

The choice of basis functions was examined by comparing the appearance of the conditional expectation functions for put options with varying numbers of ITM paths the regressions are based on. At first, the functions are plotted with about 100 ITM trajectories and then increased to a number between approximately 1000. This was done for the following sets of basis functions: Power polynomials, Hermite polynomials, non-weighted Laguerre polynomials, and the weighted Laguerre polynomials, all with four components. The plots of the expectation function fitted on a set of weighted Laguerre polynomial is shown in figure 2 and 3, while the rest can be found in appendix (figure 12 - 19). The plots are taken at random points in time for arbitrary put options.

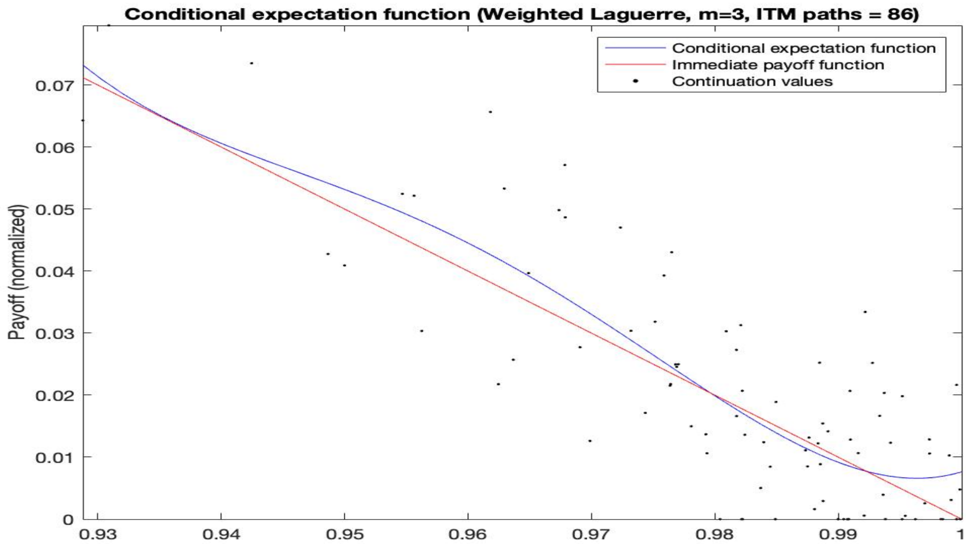

Figure 2: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four weighted Laguerre polynomials. The regression is based on 86 ITM trajectories.

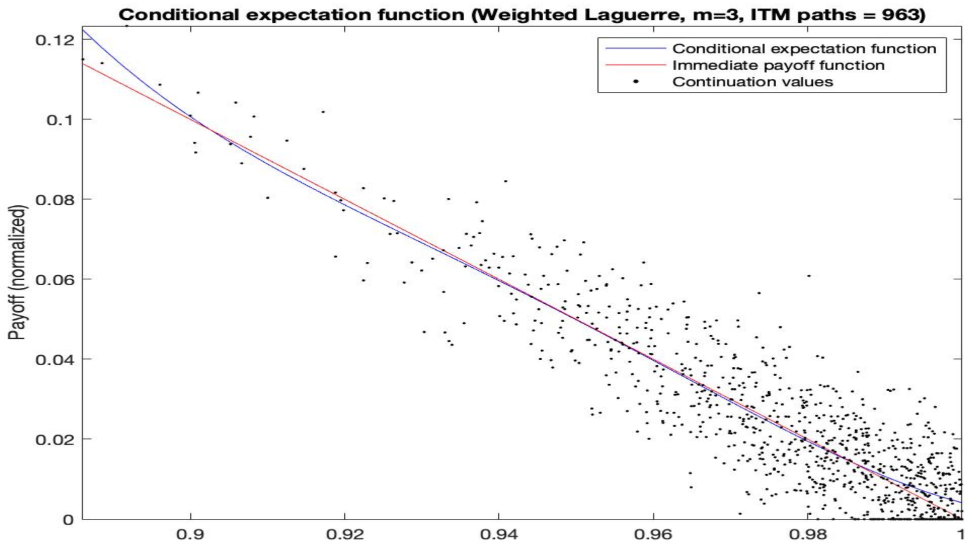

Figure 3: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four weighted Laguerre polynomials. The regression is based on 963 ITM trajectories.

4.2 Exercise boundary

When the behavior of the exercise boundaries was examined, it was found that the limits of the exercise boundary precisely before maturity were not met. For example if we set $r = 0.03$, $q = 0.01$ and $K = 1$ we expect that the exercise boundary for the call option should approach $\max(0.03/0.01, 1) \cdot 1 = 3$ and for the put option $\max(\min(0.03/0.01, 1), 0) \cdot 1 = 1$, precisely before maturity. When the exercise boundary were calculated for a call and put option with 365 days to maturity, with $r = 0.03$ and $q = 0.01$, with the weighted Laguerre polynomials as basis functions, exercise boundaries according to figure 4 and 5 where acquired. When $r < q$, the limit of the exercise boundary precisely before maturity were met for the call option but instead not for the put option. It was found that due to the behavior of the Monte Carlo simulation, it is highly unlikely to get points in the desired area to get the right limit precisely before the maturity of the call option when $r >> q$ or of the put option when $r << q$. For $r = 0.03$ and $q = 0.01$ we expect the exercise boundary for the call option to be 3 precisely before maturity of the option.

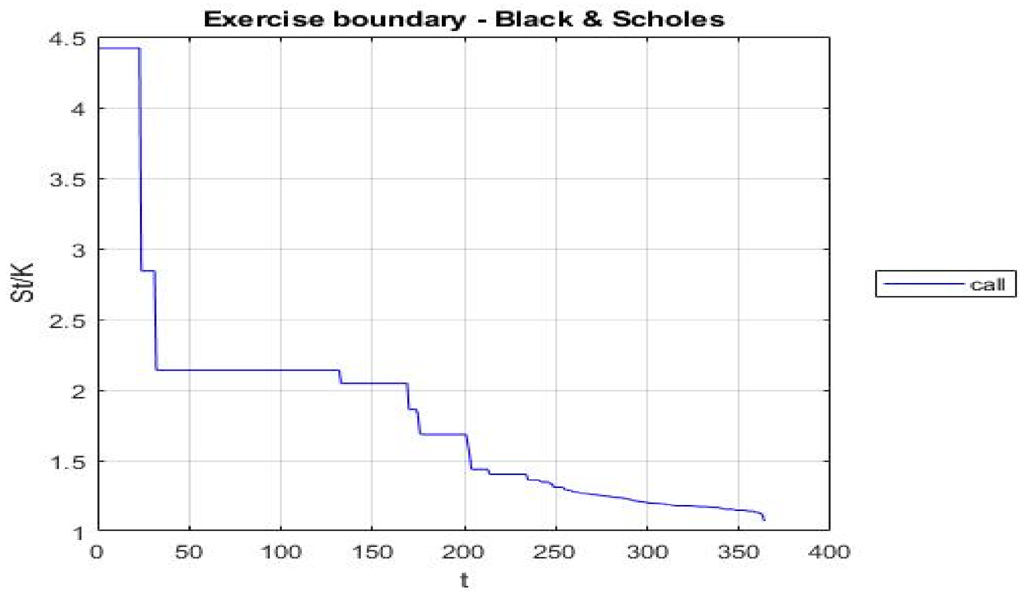

Figure 4: Plot of estimated exercise boundary for a Bermudan call option with 365 days to maturity, where r = 0.03 and q = 0.01 - implemented with weighted Laguerre polynomial basis functions

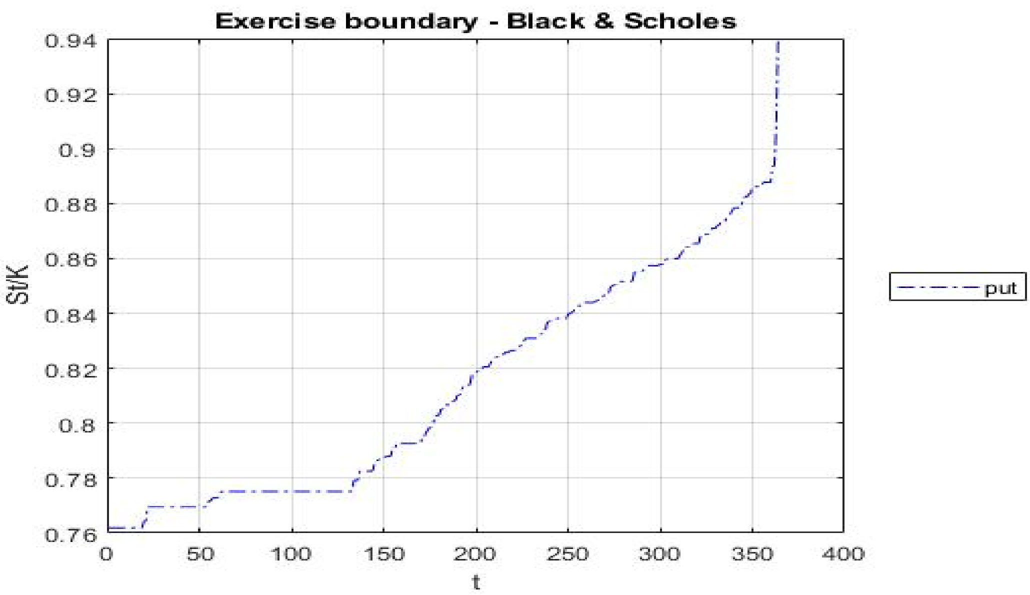

Figure 5: Plot of estimated exercise boundary for a Bermudan put option with 365 days to maturity, where r = 0.03 and q = 0.01 - implemented with weighted Laguerre polynomial basis functions

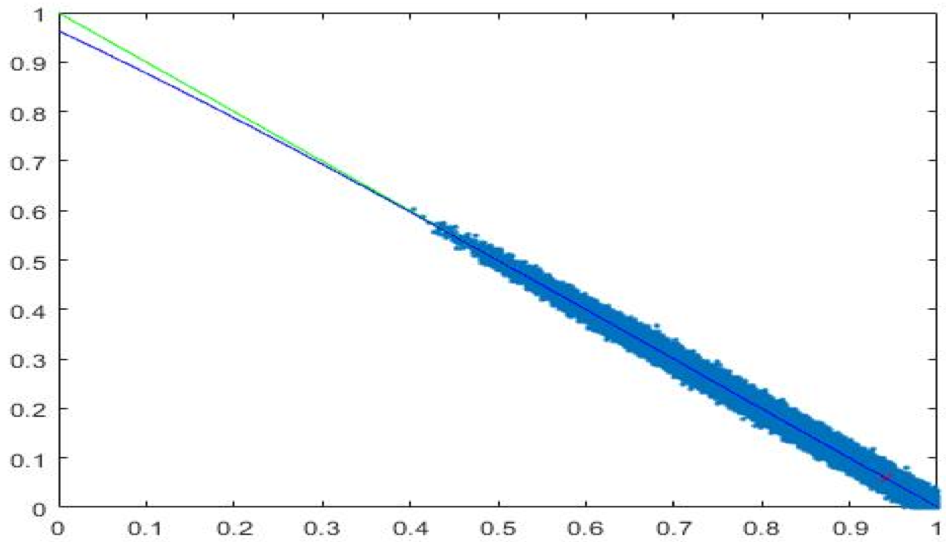

Figure 6: Plot of the pay-off function (green line), estimated pay-offs (blue dots), the fitted polynomial (blue line) estimating the continuation value for a Bermudan call option, and the root where the pay-off function equals the fitted polynomial (red x) - precisely before maturity of the option

4.3 Option pricing

The European and Bermudan options for June 25, 2007, were priced using four different price movement models (Black-Scholes, Merton, Heston and Bates), Laguerre polynomials as basis functions and the parameter weight set to one, yielding option prices according to figure 43-50 (see appendix). The percentages of estimated European option prices that are observed inside the ASK/BID-spread can be seen in table 1. The percentages of Bermudan option prices that are estimated to be larger or equal to the corresponding European option's price, and the mean percentage deviation in prices between the Bermudan and European options can be seen in table 2.

Table 1: Table of the proportion (%) of European option prices (analytical pricing) that are inside the ASK/BID-spread when calibrating the parameters, the proportion (%) for estimated European option prices (Monte Carlo pricing) that are inside the ASK/BID-spread, the mean (absolute) deviation (%) from the market mean prices for the estimated prices (Monte Carlo pricing), and the mean (absolute) deviation (%) from the nearest price of the market ASK/BID-prices for the estimated prices (Monte Carlo pricing), for June 25, 2007 - implemented with parameter weight set to one

Statistic \ Price model	Black-Scholes	Merton	Heston	Bates
Proportion (%) of options inside ASK/BID (analytical pricing)	9.02 %	65.57 %	88.52 %	89.34 %
Proportion (%) of options inside ASK/BID (Monte Carlo pricing)	9.02 %	2.46 %	77.87 %	61.48 %
Mean deviation (%) from market mean (all options)	26.93 %	38.20 %	11.14 %	11.37 %
Mean deviation (%) from market mean (options outside ASK/BID)	28.97 %	39.09 %	25.79 %	12.85 %
Mean deviation (%) from market mean (options inside ASK/BID)	6.26 %	2.54 %	6.97 %	10.43 %
Mean deviation (%) from ASK/BID (options outside ASK/BID)	21.28 %	38.89 %	13.80 %	9.26 %

Table 2: Table of the proportion (%) of estimated Bermudan option prices that are larger or equal to estimated European option prices and the mean (absolute) deviation (%) from the European option prices, for June 25, 2007 - implemented with weighted Laguerre polynomial basis functions and the parameter weight set to one

Statistics \ Price model	Black-Scholes	Merton	Heston	Bates
Proportion (%) of options for which $V_{Ber} \geq V_{EU}$	54.92 %	59.02 %	61.48 %	59.02 %
Proportion (%) of options for which $C_{Ber} \geq C_{EU}$	3.51 %	12.28 %	17.54 %	12.28 %
Proportion (%) of options for which $P_{Ber} \geq P_{EU}$	100.00 %	100.00 %	100.00 %	100.00 %
Mean deviation (%) between V_{Ber} and V_{EU}	7.58 %	7.50 %	7.04 %	6.95 %
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} \geq C_{EU}$	0.06 %	0.57 %	0.39 %	0.57 %
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} < C_{EU}$	0.57 %	0.32 %	0.28 %	0.32 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} \geq P_{EU}$	13.52 %	13.18 %	12.94 %	12.74 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} < P_{EU}$	-	-	-	-

The Least-square approach was used to estimate normalized exercise boundaries for the Bermudan options and four different price movement models (Black-Scholes, Merton, Heston and Bates), implemented according to the methodology outlined in chapter 3.1, yielding the normalized exercise boundaries according to figure 23-42 (see appendix). The European and Bermudan options for June 25, 2007, was priced using four different price movement models (Black-Scholes, Merton, Heston and Bates) and the previously calculated exercise boundaries, yielding option prices according to figure 59-62 (see appendix). The percentages of Bermudan option prices that are estimated to be larger or equal to the corresponding European option's price, and the mean percentage deviation in prices between the Bermudan and European options can be seen in table 3.

Table 3: Table of the proportion (%) of estimated Bermudan option prices that are larger or equal to estimated European option prices and the mean (absolute) deviation (%) from the European option prices, for June 25, 2007 - implemented with exercise boundaries and the parameter weight set to one

Statistics \ Price model	Black-Scholes	Merton	Heston	Bates
Proportion (%) of options for which $V_{Ber} \geq V_{EU}$	49.18 %	46.72 %	8.20 %	8.20 %
Proportion (%) of options for which $C_{Ber} \geq C_{EU}$	0.00 %	0.00 %	0.00 %	0.00 %
Proportion (%) of options for which $P_{Ber} \geq P_{EU}$	92.31 %	87.69 %	15.38 %	15.38 %
Mean deviation (%) between V_{Ber} and V_{EU}	4.05 %	1.76 %	5.29 %	0.59 %
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} \geq C_{EU}$	-	-	-	-
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} < C_{EU}$	0.91 %	0.77 %	0.82 %	0.79 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} \geq P_{EU}$	7.13 %	2.73 %	2.19 %	0.01 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} < P_{EU}$	0.56 %	0.55 %	10.50 %	0.48 %

The European and Bermudan options for June 25, 2007, was priced using four different price movement models (Black-Scholes, Merton, Heston and Bates), Laguerre polynomials as basis functions and the parameter weight set to the inverse of the ASK/BID-spread, yielding option prices according to figure 51-58 (see appendix) The percentages of estimated European option prices that are observed inside the ASK/BID-spread can be seen in table 4. The percentages of Bermudan option prices that are estimated to be larger or equal to the corresponding European option's price, and the mean percentage deviation in prices between the Bermudan and European options can be seen in table 5.

Table 4: Table of the proportion (%) of European option prices (analytical pricing) that are inside the ASK/BID-spread when calibrating the parameters, the proportion (%) for estimated European option prices (Monte Carlo pricing) that are inside the ASK/BID-spread, the mean (absolute) deviation (%) from the market mean prices for the estimated prices (Monte Carlo pricing), and the mean (absolute) deviation (%) from the nearest price of the market ASK/BID-prices for the estimated prices (Monte Carlo pricing), for June 25, 2007 - implemented with the parameter weight set to the inverse of the ASK/BID-spread

Statistic \ Price model	Black-Scholes	Merton	Heston	Bates
Proportion (%) of options inside ASK/BID (analytical pricing)	8.20 %	56.56 %	87.70 %	95.08 %
Proportion (%) of options inside ASK/BID (Monte Carlo pricing)	8.20 %	3.28 %	80.33 %	49.18 %
Mean deviation (%) from market mean (all options)	27.03 %	39.10 %	10.85 %	11.24 %
Mean deviation (%) from market mean (options outside ASK/BID)	28.87 %	40.34 %	24.76 %	11.94 %
Mean deviation (%) from market mean (options inside ASK/BID)	6.45 %	2.74 %	7.45 %	10.51 %
Mean deviation (%) from ASK/BID (options outside ASK/BID)	21.80 %	40.39 %	14.29 %	9.76 %

Table 5: Table of the proportion (%) of estimated Bermudan option prices that are larger or equal to estimated European option prices and the mean (absolute) deviation (%) from the European option prices, for June 25, 2007 - implemented with weighted Laguerre polynomial basis functions and the parameter weight set to the inverse of the ASK/BID-spread

Statistics \ Price model	Black-Scholes	Merton	Heston	Bates
Proportion (%) of options for which $V_{Ber} \geq V_{EU}$	54.92 %	59.84 %	56.56 %	59.84 %
Proportion (%) of options for which $C_{Ber} \geq C_{EU}$	3.51 %	14.04 %	7.02 %	14.04 %
Proportion (%) of options for which $P_{Ber} \geq P_{EU}$	100.00 %	100.00 %	100.00 %	100.00 %
Mean deviation (%) between V_{Ber} and V_{EU}	8.02 %	8.15 %	7.42 %	7.49 %
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} \geq C_{EU}$	0.20 %	0.44 %	0.46 %	0.63 %
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} < C_{EU}$	0.57 %	0.36 %	0.31 %	0.34 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} \geq P_{EU}$	14.19 %	14.21 %	13.64 %	13.73 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} < P_{EU}$	-	-	-	-

4.4 Control variates

The European and Bermudan options for June 25, 2007, were priced using the Black-Scholes price movement model, Laguerre polynomials as basis functions and implementing two different control variates (the underlying asset and the European option), yielding option prices according to figure 63-65 (see appendix). Variances for the European and Bermudan option prices using different control variates can be seen in table 6. The percentage of estimated European option prices that were observed inside the ASK/BID-spread and the mean deviation from the market mean prices, using the underlying asset as control variate, is 13.11% and 25.40% respectively. The percentage of Bermudan option prices that are estimated to be larger or equal to the corresponding European option's price, and the mean percentage deviation in prices between the Bermudan and European options can be seen in table 7.

Table 6: Table of variances of estimated option prices for the European and Bermudan options for June 25, 2007 - implemented with Black-Scholes price movement model, Laguerre polynomial basis functions, the parameter weight set to one, the underlying asset as control variate for the European options and two different control variates for the Bermudan options

Option \ Control variate	No control variate	Underlying asset	European option
European option	59.67	25.99	-
Bermudan option	47.95	27.26	44.59

Table 7: Table of the proportion (%) of estimated Bermudan option prices that are larger or equal to estimated European option prices and the mean (absolute) deviation (%) from the European option prices, for June 25, 2007 - implemented with Black-Scholes price movement model, Laguerre polynomial basis functions, the parameter weight set to one, the underlying asset as control variate for the European options and two different control variates for the Bermudan options

Statistics \ Control variate for Bermudan options	Underlying asset	European option
Proportion (%) of options for which $V_{Ber} \geq V_{EU}$	64.75 %	59.02 %
Proportion (%) of options for which $C_{Ber} \geq C_{EU}$	24.56 %	12.28 %
Proportion (%) of options for which $P_{Ber} \geq P_{EU}$	100.00 %	100.00 %
Mean deviation (%) deviation between V_{Ber} and V_{EU}	7.91 %	8.39 %
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} \geq C_{EU}$	0.93 %	0.36 %
Mean deviation (%) between C_{Ber} and C_{EU}, for $C_{Ber} < C_{EU}$	0.43 %	0.82 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} \geq P_{EU}$	13.76 %	14.43 %
Mean deviation (%) between P_{Ber} and P_{EU}, for $P_{Ber} < P_{EU}$	-	-

5 Analysis and discussion

In this section the results are analyzed and discussions concerning the results and the analysis of the same are held.

5.1 Least-square Monte Carlo approach

When simulating the value of the underlying asset as a GBM, we noticed that the choice of basis functions including four terms did not have a major effect on the expectation functions. It had a larger impact, although small, when the regression was made with only a few ITM trajectories, but that the resulting prices barely varied across the different basis functions. On this account, only one set of basis functions were used for the subsequent results when varying the models for the underlying asset price movement. The first four terms of the Weighted Laguerre polynomials were selected, which is aligned with Longstaff and Schwartz proposal of basis functions. Since they also found that the numerical results do not change when including more than three basis functions [21], we expect that this choice is sufficient.

At some points in time, there arose problems concerning the regressed expectation functions. First of all, for some options with a short time to maturity and/or an initial underlying asset price that lies far OTM, their price was estimated to be zero. The explanation is that none of the trajectories became ITM during the options lifetime, thus yielding no payoffs the price could be estimated from. In reality, no one would issue an option with the price zero, so these results are not trustworthy. The problem lies in the methodology of Monte Carlo rather than the LSM approach itself. Although the expectation function in most cases could be fitted, they did frequently not comply with the appearance of the payoff function for an American-style option (see figure 1). Problems of larger scale tend to occur when the number of ITM trajectories were few, which is exemplified in figure 2, 12, 13 and 14. They are plots of the conditional expectation functions regressed on a set of Power polynomials, Hermite polynomials, Laguerre polynomials, and weighted Laguerre polynomials respectively, together with the 86 realized discounted cash flows the regressions are based on. For some prices of the underlying, the functions misjudge the continuation values. For instance, in the situations illustrated in figure 2 and figure 12 and 14 in the appendix, the value of the expectation function is larger than the immediate payoff for many stock prices for which the opposite should apply. One can see that problems in the curvature appear in the areas where there only are a few realised discounted cash flows. In these areas, the amount of points the expectation function is be regressed on are not sufficient to fit a function according to the theory. In 2, the graph is increasing in a small section where the underlying asset price approaches 1, but the theory suggests that the graph is monotonically decreasing. The regressed Hermite polynomial in figure 13 lacks intersects with the payoff function, yielding larger function values for all stock prices that are ITM at this point in time. In these cases, the function will overvalue the continuation values for all stock prices that lie below the point where the intersect should have been. A significant improvement are seen when the number of ITM trajectories is increased, which is demonstrated in figure 3, and the figures 16 - 19 in appendix. As seen are the differences in the graphs' appearance not as noticeable, and they behave more like what is expected from the theory. However, even for a large number of ITM trajectories, the expectation functions occasionally appeared with unwanted properties. For instance, all of the expectation functions have several intersects and consequently misjudge some of the continuation values. The desired property is to only have one intersection, which should be for the stock price from which the American option price is larger than the payoff (see figure 1). The problem of a nonexistent intersection could also arise for many iterations, which then result in overvalued continuation values for all stock prices.

5.2 Exercise Boundary

The Exercise Boundary approach was implemented to solve some of the problems with the LSM approach, one of which is the potential lack of ITM trajectories. By running many simulations of the value of the underlying asset from proper start values for put and call options respectively, we received more trajectories on which the conditional expectation function could be regressed. Consequently, better expectation functions were fitted. In the one variable-case, a set of Weighted Laguerre polynomials were used as a basis function, and for the two-variable case a second-order power multinomial. The idea of the Exercise Boundary approach is to use the value of the stock price for which the payoff function intersects the expectation to evaluate when the holder should exercise the option. With the Matlab command lsqnonlin, the stock price that minimizes the distance between the function lines is returned. Therefore, even when there is a lack of intersection, the stock price for which we believe there should be an intersection is selected anyway. If there are several intersects, the one of interest is the one closest to the strike price. Another advantage is that the method is not concerned with the polynomial's behavior far from the intersect, since the estimate of when to exercise the option is solely dependent on the chosen breakpoints. As seen in the examination of the expectation functions, they do frequently appear with unwanted properties. All of these defects do not necessarily need to be a problem in the exercise boundary approach, since we only care about calculating a proper breakpoint.

The exercise boundaries are obviously very dependent on the selected breakpoints between the payoff functions and the expectation functions at the points in time where exercise is possible, since it is these points the boundary consists of. Which points that are selected depends on several things, first of all the method used to localize them. The one used in this paper is the non-linear least square method (in MATLAB lsqnonlin), but other methods, such as fminbnd, can also be used. These may yield another point on the curve depending on the behavior of the expectation function. In addition, the selected value will also depend on the previous one. This is because the methods start at some given point in the search for the intersect, which we have chosen to be the previous intersect. Also, the realized discounted cash flows the expectation function is regressed on will be dependent on the previous intersect, since it will determine where the payoffs are located in the payoff matrix from which the continuation values are calculated. Therefore, since all the breakpoints are strongly dependent on one another, the error from one poorly chosen intersect will spread to the subsequent ones. In this manner, the method is very sensitive to the regression of the expectation functions near the breaking point of interest, and the method used to calculate this point.

To test whether the exercise boundaries we formed were reasonable or not, the result from varying values of r and q were compared with the theory of how the boundary should behave for an American option (see chapter 2.6.3). Since Bermuda options are American-styled options with finite dates of early exercise, the limits of the exercise boundaries should be close to the limits of the exercise boundaries of the corresponding American option. In our testing (see chapter 4.2), the limits precisely before maturity were not always met. The reason is that the simulation of the stock prices does not reach the area of interest with our initial choice of starting point, which is demonstrated in figure 6. The simulation can be forced into the desired area, which is by choosing the start point dependent on the values of r and q. Otherwise, not enough trajectories are in the area in which we wish to regress a good polynomial in. However, it was also found that problems related to the degeneration of the fitted polynomial and to the lack of points in the area of interest could be amplified, resulting in the exercise boundary taking incorrect values far from the date of maturity to a greater extent than before. The initial exercise point calculated (the point in time closest to the time of expiration) can also be forced to follow the theoretical limit of this point (see chapter 2.6.3). This however does not solve the problem of lack of simulated points in the area of interest, for which one needs to regress a good polynomial. The same problems were also found when using other basis functions, for instance, the unweighted Laguerre polynomial basis functions. It was found that when using unweighted Laguerre polynomials and weighted Laguerre polynomials as basis functions, the fitted polynomial estimating the continuation value behaves similarly to one another for days close to maturity of the option. So, just like for the ordinary LSM approach, the choice of basis functions does not have a major impact on the breakpoint when there are many ITM trajectories the expectation function is regressed on. It was found that when calculating the normalized exercise boundaries for the models with deterministic volatility there were signs of these problems (see figure 7).

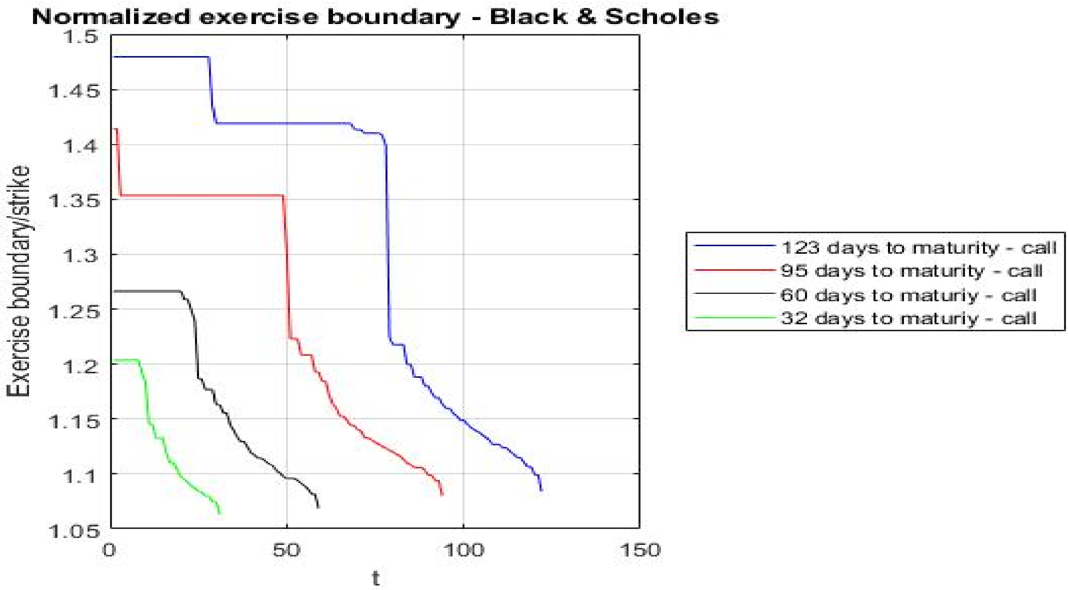

Figure 7: Plot of estimated call option normalized exercise boundaries for June 25, 2007 - implemented with the Black-Scholes price model, weighted Laguerre polynomial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

When we simulate the underlying value with a model using stochastic volatility we expect the boundary to decrease with the volatility for the put option, and the opposite for the call option. This behavior is clearly noticeable for only some of the exercise boundaries calculated with the Heston model (see figure 8). For the Bates model, this behavior is noticeable for even fewer calculated exercise boundaries. For a call option, the exercise boundary should decrease as we get closer to the expiration date, and the opposite for the put option. However, all options' exercise boundaries exhibit constant values for some periods as the time to maturity decreases (see figure 9). In summary, Bates model seems to perform worse than Heston when trying to create an exercise boundary. As mentioned in the previous section, there are complications already in the one-variable case, which are amplified when the expectation function is represented by a multinomial. Factors that might be the cause of the problems are incorrect parameters for the underlying asset price movement, improper basis functions for the expectation function, and methods to find the breakpoints.

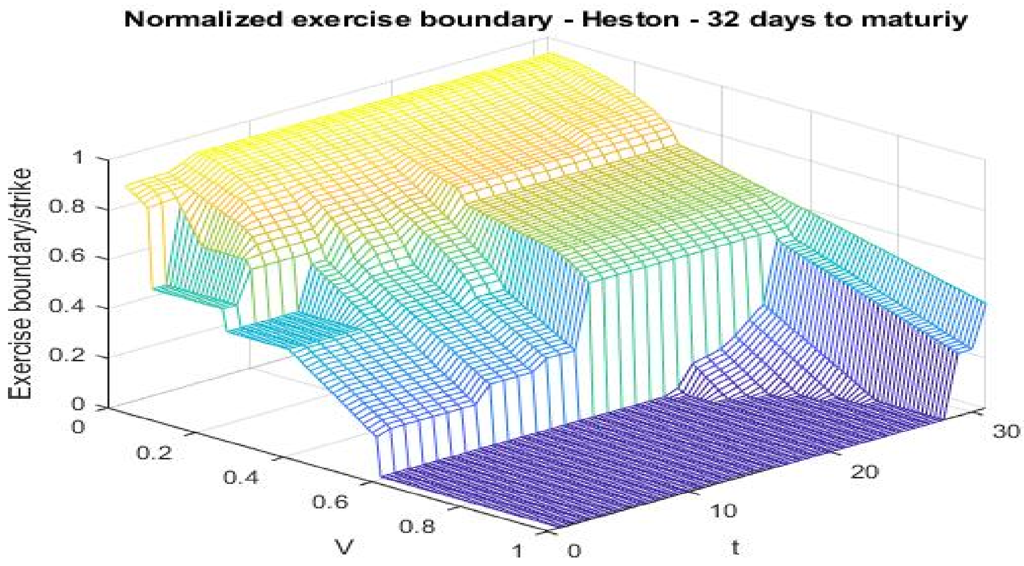

Figure 8: Plot of estimated put option normalized exercise boundary, with time to maturity 32 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

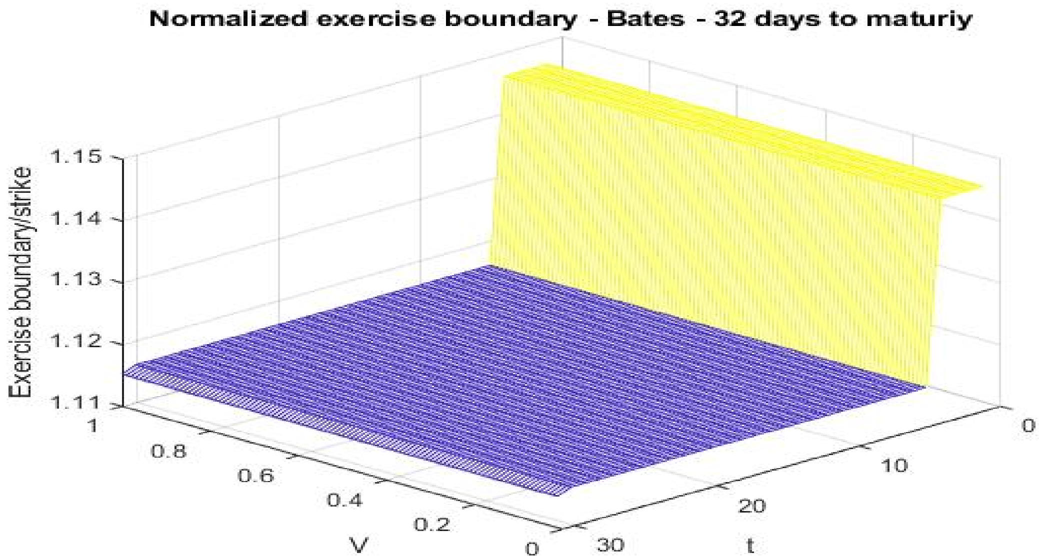

Figure 9: Plot of estimated call option normalized exercise boundary, with time to maturity 32 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

5.3 Option pricing

When pricing European options with the different models using Laguerre polynomials as basis functions and the parameter weight set to one according to our simulations the Heston model performs the best, in terms of the proportion of estimated European option values that are inside the ASK/BID-spread (see figure 10). The Heston model is then followed by Bates model, Black-Scholes model, and Merton model. This is in contrast to when looking at the proportion of the European options that are inside the ASK/BID-spread when calibrating the parameters. Here we can see that Bates model performs the best, followed by Heston model, Merton model, and Black-Scholes model. The same ordering of which model performs the best does not hold if one looks at how much the estimated option prices deviates from market mean prices or the ASK/BID-spread. If we look at the mean deviation from the market mean, the performance of the Bates and Heston models are the best and quite similar, followed by Black-Scholes model and Merton model. We can also see that, for instance, Bates model, deviates less from the market mean prices than the Heston model when only looking at option prices outside the ASK/BID-spread, but vice versa holds for option prices inside the ASK/BID-spread. It also holds that for the Bates model, options outside the ASK/BID-spread deviate less from the ASK/BID-spread than the options outside the ASK/BID-spread for the Heston model. It is worth noting here that when calibrating the parameters, the mean deviation of the analytically calculated option price from the nearest edge of the ASK/BID-spread is smallest for the Bates model, followed by the Merton model, Heston model, and Black-Scholes model. This could be part of the explanation of why the above-mentioned results look like they do. If we get analytical values close to the edge of the ASK/BID-spread when calibrating the parameters, it is also more likely to get estimated values outside the same spread when simulating the values. The behavior of these models holds in these regards, even when one sets the parameter calibration weight λ to the inverse of the ASK/BID-spread. Finally, we note that there seem to be a problem when we use a model that include price jumps. This can be seen when one compares the proportion of options inside the ASK/BID-spread when simulating option prices with the same when the parameters are calibrated, respectively for the Merton and Bates model. The difference between these values is significantly larger for Merton and Bates model than they are for Black-Scholes and Heston model. One possible explanation to why this is the way could be that the parameter calibration is harder for these models.

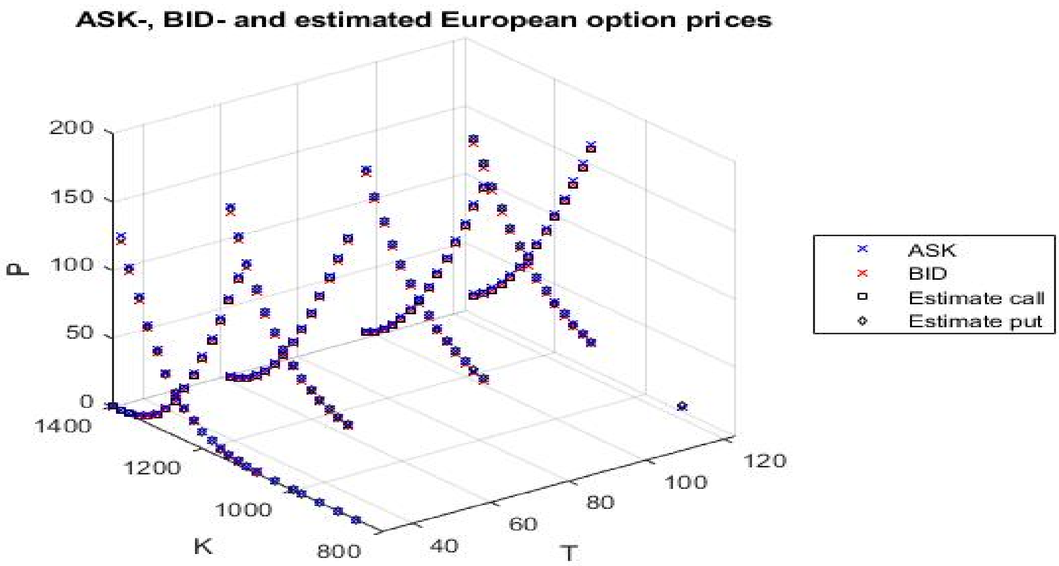

Figure 10: Plot of estimated European option prices and the markets ASK- & BID-prices for June 25, 2007 - implemented with the Heston price model and the parameter weight set to one

When pricing the Bermudan options, we barely have any way of controlling whether the prices are correct, and therefore no way of comparing the models' performance. This is due to the lack of market data on the Bermudan option prices with this particular underlying asset. What we can do is compare the different models and see if any of their price estimations clearly deviates from the others' estimations. We can also make sure that the estimated Bermudan option prices follow the theory concerning how the prices should behave in relation to the European option prices. Since we have chosen to estimate option prices for which $r > q$ every day up until maturity, we expect that the Bermudan call option price should be approximately equal to the European call option price. When pricing the Bermudan options with the different models using weighted Laguerre polynomials as basis functions and the parameter weight set to one, we can see that the Bermudan put option prices are larger or equal to the corresponding put option prices for all models. The mean deviation from the European option prices lies between $6.95 - 7.58\%$ for the four models. The big difference between the models can be seen if we look at the proportions of Bermudan call options that are estimated to be larger or equal to the corresponding European option price, ranging between $3.51 - 17.58\%$. Although, if we look at how much the estimated Bermudan call option prices deviate from corresponding European call option prices we can see that a mean absolute deviation range between $0.06 - 0.57\%$. This leads us to think that it is likely that the Bermudan call option price should be equal to the European call option prices, and that the differences stem from the Monte Carlo method and the LSM approach. When pricing the Bermudan options with the parameter weight instead set to the ASK/BID-spread, we get results leading us to the same conclusions. This with some differences in the exact values compared with when the parameter was set to be one, but not large enough to draw any other major conclusions. In the end, we want to say that the Heston model performs the best (see figure 11), with a small margin, followed by Bates and Merton model and finally Black-Scholes model. Although, it is worth noticing that the differences are so small that they could be considered to be neglectable.

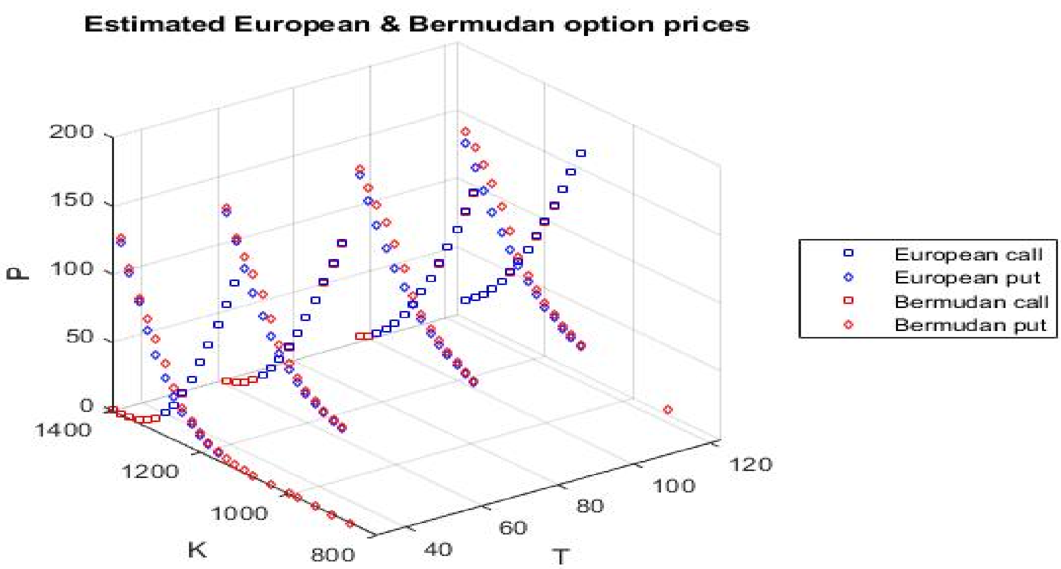

Figure 11: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Heston price model, weighted Laguerre polynomial basis functions and the parameter weight set to one

When pricing options using exercise boundaries (calculated as described in chapter 4.3), we can see that the results are worse than when using the standard LSM approach. The thought when using this method was that the results should be better since we can eliminate incorrect decisions regarding early exercises due to the behavior of the fitted polynomial. The results indicate that the approach works but, although worse than the LSM approach. The determination of the exercise boundaries poses a whole new set of problems, especially when it comes to models with a stochastic variance, which need to be solved for the approach to work better than the standard LSM approach.

Some of these problems have already been discussed (see chapter 5.2).

A general problem with the more advanced models, or the models with more parameters, is that the more parameters one have the harder it gets to correctly calibrate the model. For one thing, there are more starting values when calibrating the model that can result in different parameter choices. We are not ensured to find the global minimum of the least square problem, but only a local minimum. This leads to that the more parameters you have, the harder it gets to find ”the best minimum”. It also holds that with more parameters more data is required to fit the data, and therefore one needs more data to correctly estimate the parameters for the more advanced models. Since we only use data from one day when calibrating our parameters this could result in non-optimal calibrations.

5.4 Control variates

Using control variates as a variance reducing technique simulated with the Black-Scholes model, the results show that when using the underlying asset as control variate the variance is reduced for the estimation of both the European options and the Bermudan options. It also shows that when using the European option as control variate for the Bermudan option, one sees no significant reduction in the variance of the estimated Bermudan option price. When we look at the proportions of the Bermudan option prices that are estimated to be larger or equal to the European option prices, the results show that usage of both the control variates results in that a larger proportion of Bermudan option prices follow this equality. Although, there is a problem when using the European option as a control variate for the Bermudan option. When the Bermudan option is estimated to be worth less than the corresponding European option the deviation is larger than when not using any control variate. This may be explained by that if we use a control variate that doesn’t reduce the variance significantly the error could be amplified when controlling each estimation. One could also implement the usage of multiple control variates when pricing options [33], but in the interest of limiting this thesis somewhat this is not researched further.

6 Conclusion

In this section, conclusions that can be drawn from the analysis of the results are presented. This is also compared to the problem formulation which here is answered.

The aim to successfully price European and Bermudan stock index options is judged to be fulfilled. Whether we can see a difference in performance between methods and models is another question, even though the use of any of them works. Regarding the performance, if we look at the proportion of estimated European option prices inside the ASK/BID-spread, we can conclude that the Heston model outperforms the rest. If we instead look at the estimated Bermudan option prices, it is not known which of the models perform the best due to the lack of price data. In a comparison of the two different methods, it is clear to say that the ordinary LSM approach performs better. With that said, the usage of the exercise boundaries has the theoretical potential of performing sufficiently good, if not better, in case one could solve the added problems faced with using this method. The models perform differently, but the ordering of performance was not as predicted. The probable reason is external problems, such as the calibration of parameters, and not the model itself. Finally, implementing control variates have proven to work as a variance reducing technique. However, how well it works highly depends on the choice of control variate and how one calculates the same.

All in all, the questions posed in the problem formulation are considered to be answered in the affirmative for the most part. When this not is the case, the general conclusion is that it is most likely due to external factors that cause added problems, not sourcing from the main model or method in itself.

7 Future work

In this section the authors' thoughts on what future research and work that needs to be conducted are presented. This is based on the project results, analysis, discussion and conclusions.

The two major problems faced in this project have been the finding and fitting of polynomials, and the calibration of parameters, including the choice of parameter weight. These two subjects are considered two have had a major impact on the results even though they have not been the main focus of the project. Focus on these two subjects in future work is considered to be of importance for the work in option pricing. Future work that focuses on the calculation of the exercise boundaries is also considered of importance and could help forward the methodology of option pricing. Another interesting area for further research could be how one could combine the usage of the LSM approach and the exercise boundary approach when pricing options. Research about how to simultaneously implement these criteria for early exercise may result in an option pricing approach superior to both approaches on their own. Further research concerning the choice and implementation of control variates, or other variance reducing techniques, could also be of interest.

Other areas of the subject that have had an impact on the results and that could be interesting areas of future work, are mainly computational power and programming efficiency. Increasing the computational power would increase and enhance the possibilities of what could be done in all numerical research, and therefore in the research concerning Monte Carlo-based option pricing. The same goes for future work with programming efficiency, in which we include both types of research concerning efficient code writing and the development of efficient programming languages.

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9 Appendix

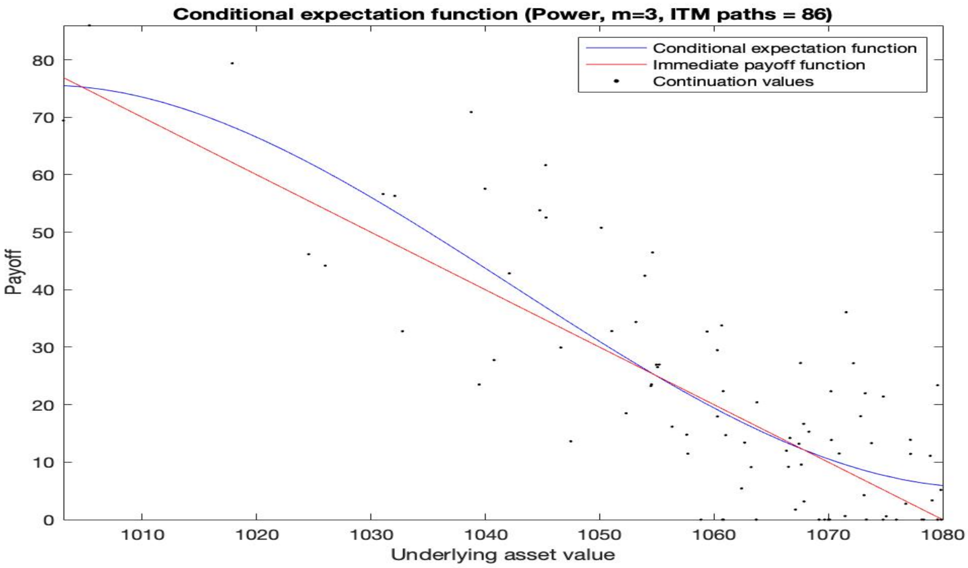


Figure 12: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a power polynomial of third degree with 93 ITM trajectories.

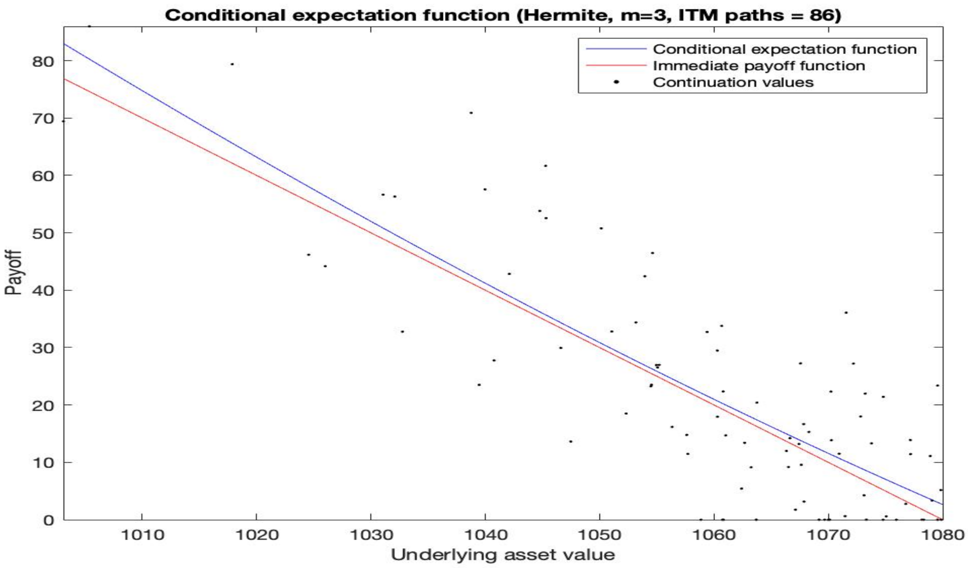


Figure 13: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four Hermite polynomials. The regression is based on 81 ITM trajectories.

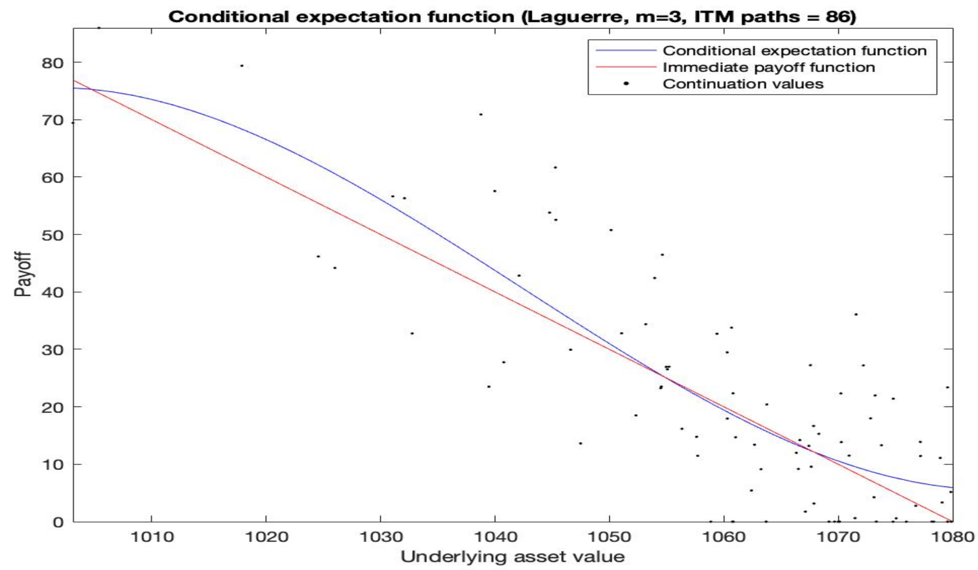


Figure 14: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four Laguerre polynomials. The regression is based on 127 ITM trajectories.

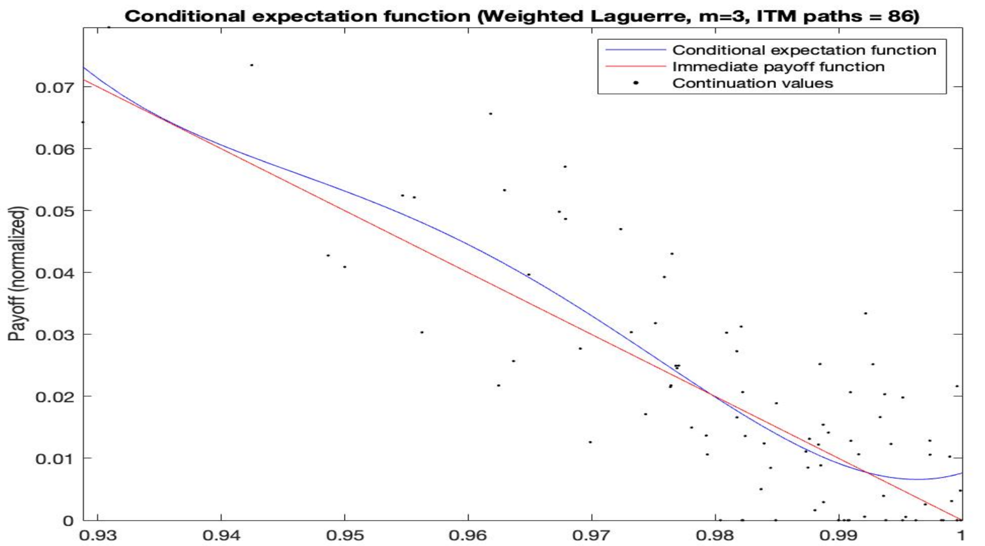


Figure 15: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four weighted Laguerre polynomials. The regression is based on 103 ITM trajectories.

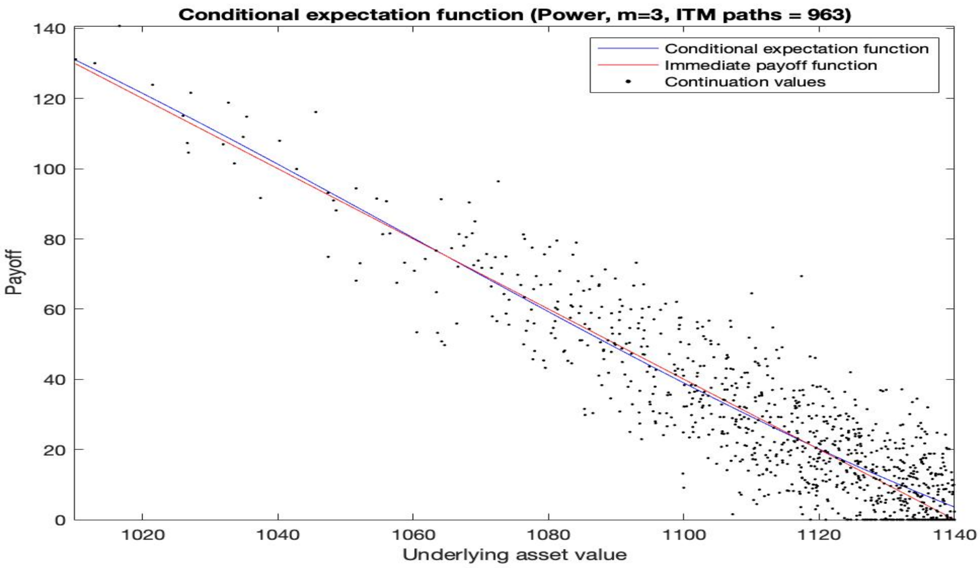


Figure 16: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a power polynomial of third degree. The regression is based on 548 ITM trajectories.

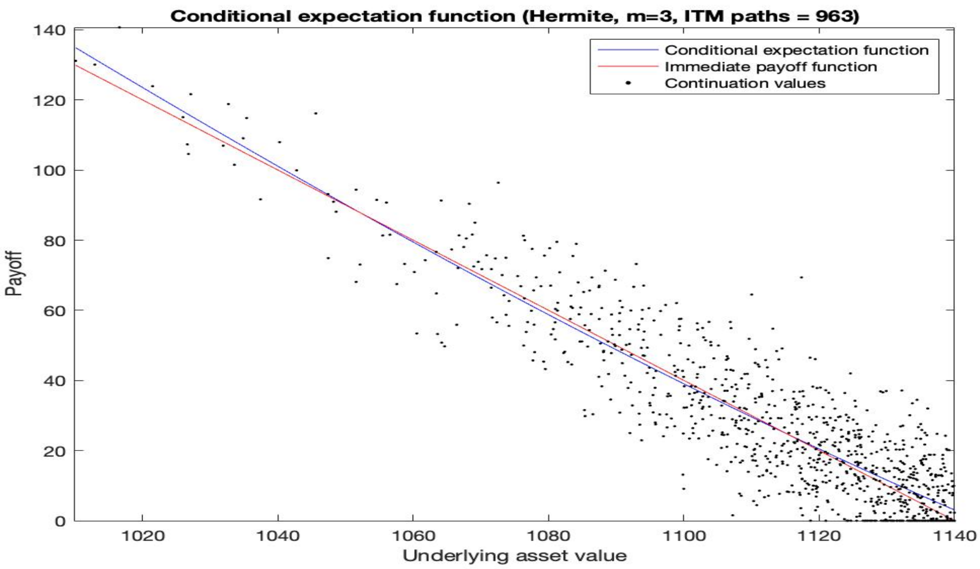


Figure 17: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four Hermite polynomials. The regression is based on 564 ITM. trajectories.

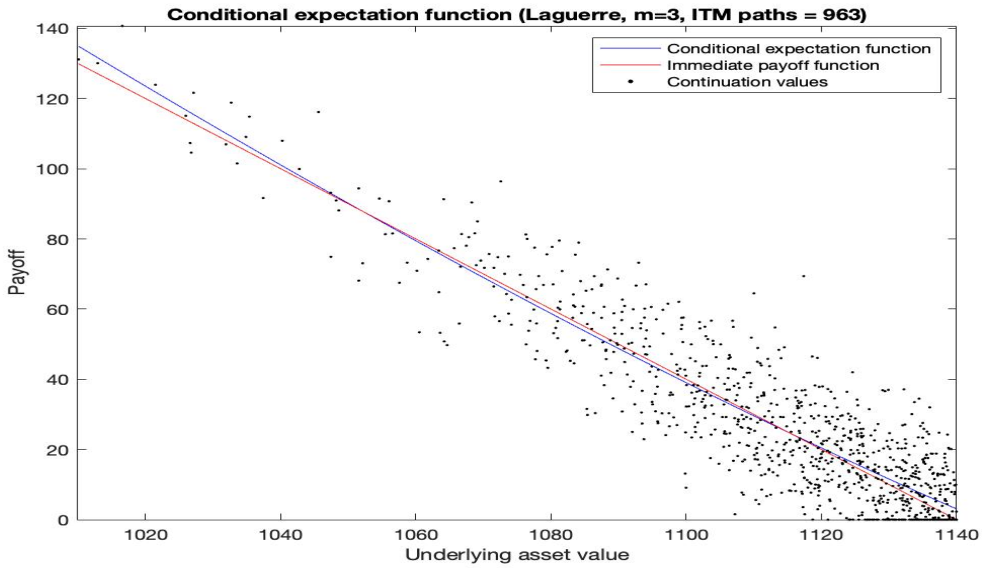


Figure 18: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four Laguerre polynomials. The regression is based on 598 ITM trajectories.

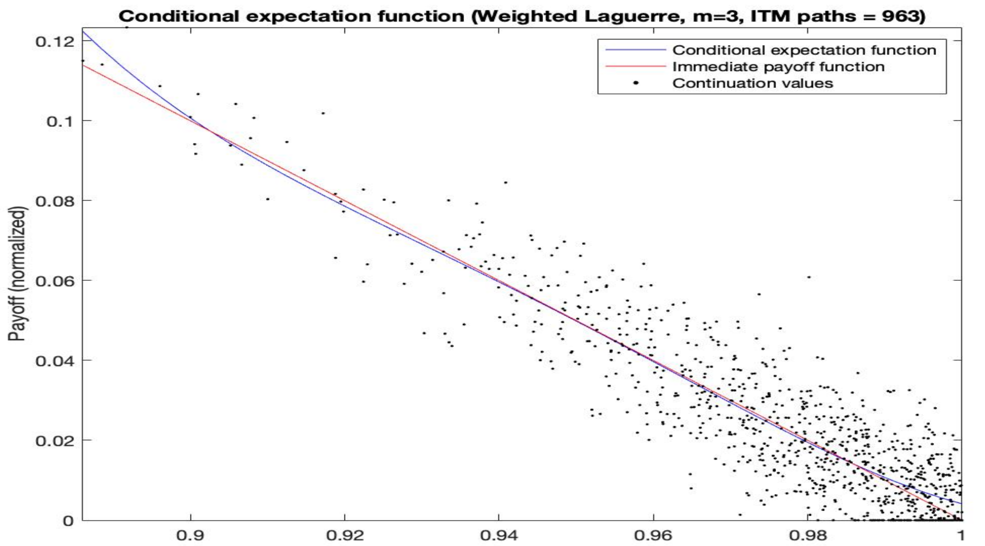


Figure 19: Plot of realized continuation values, function for immediate payoff and the conditional expectation function regressed on a set of the first four weighted Laguerre polynomials. The regression is based on 608 ITM trajectories.

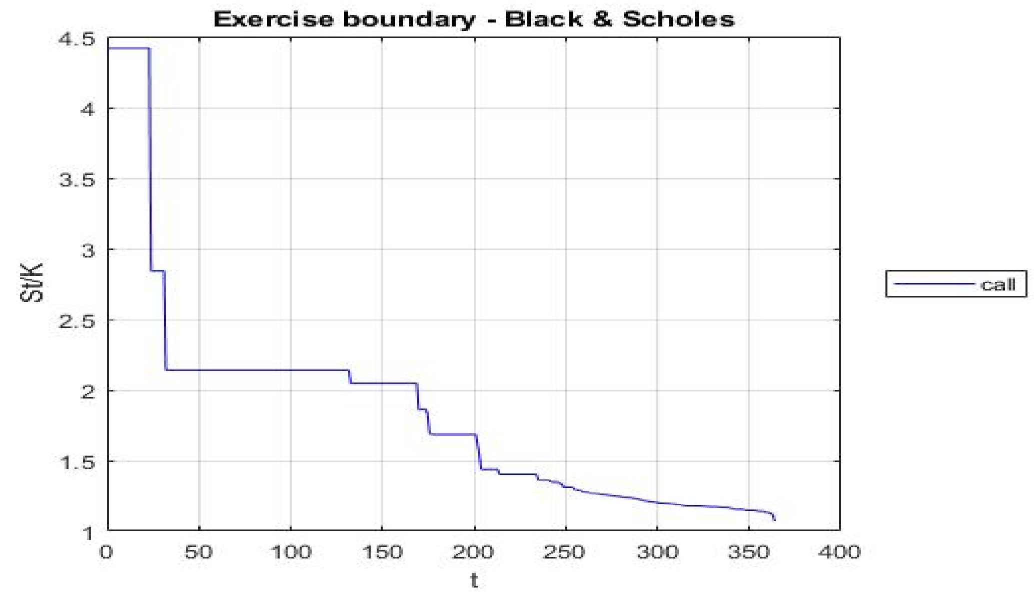


Figure 20: Plot of estimated exercise boundary for a Bermudan call option with 365 days to maturity, where r = 0.03 and q = 0.01 - implemented with weighted Laguerre polynomial basis functions

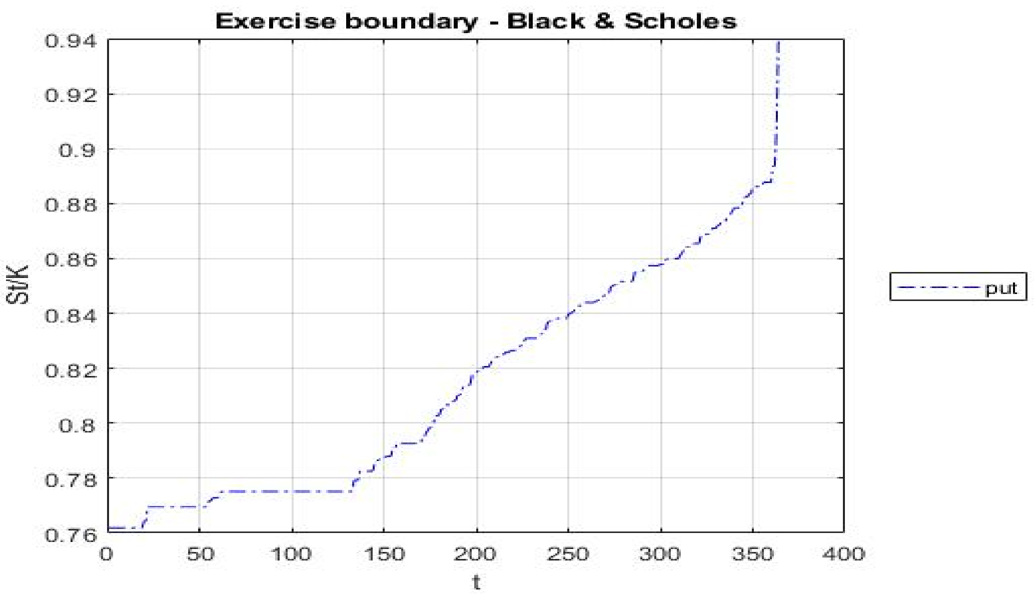


Figure 21: Plot of estimated exercise boundary for a Bermudan put option with 365 days to maturity, where r = 0.03 and q = 0.01 - implemented with weighted Laguerre polynomial basis functions

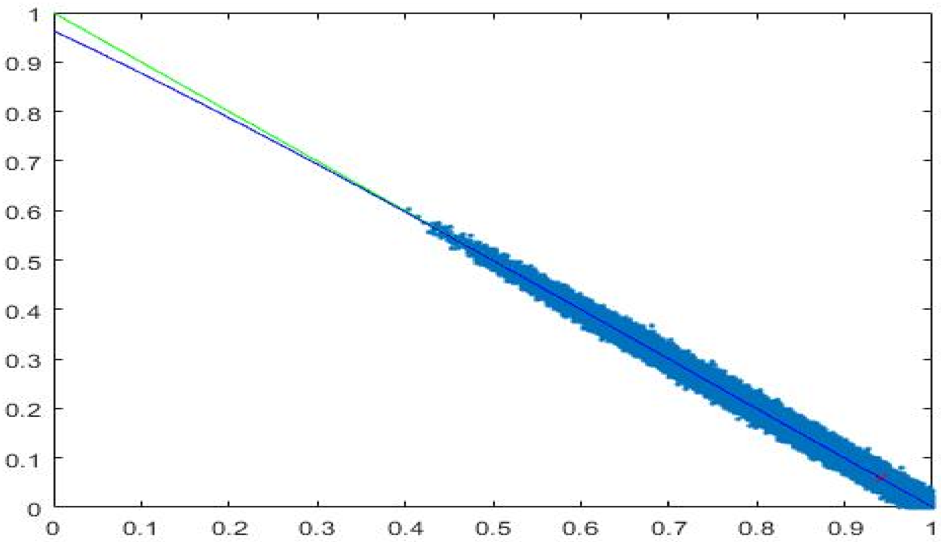


Figure 22: Plot of the pay-off function (green line), estimated pay-offs (blue dots), the fitted polynomial (blue line) estimating the continuation value for a Bermudan call option, and the root where the pay-off function equals the fitted polynomial (red x) - precisely before maturity of the option

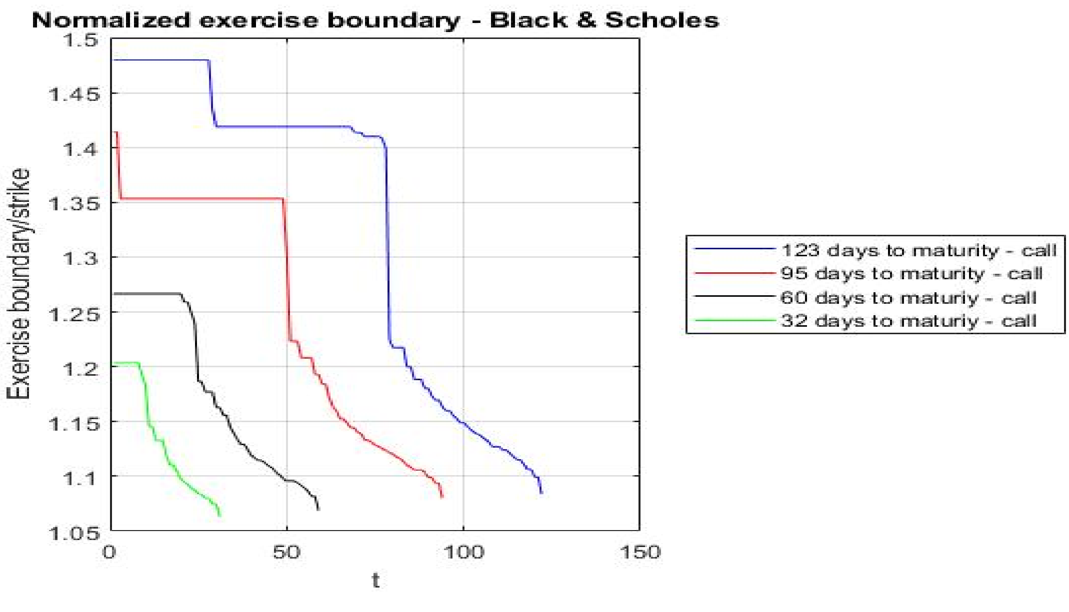


Figure 23: Plot of estimated call option normalized exercise boundaries for June 25, 2007 - implemented with the Black-Scholes price model, weighted Laguerre polynomial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

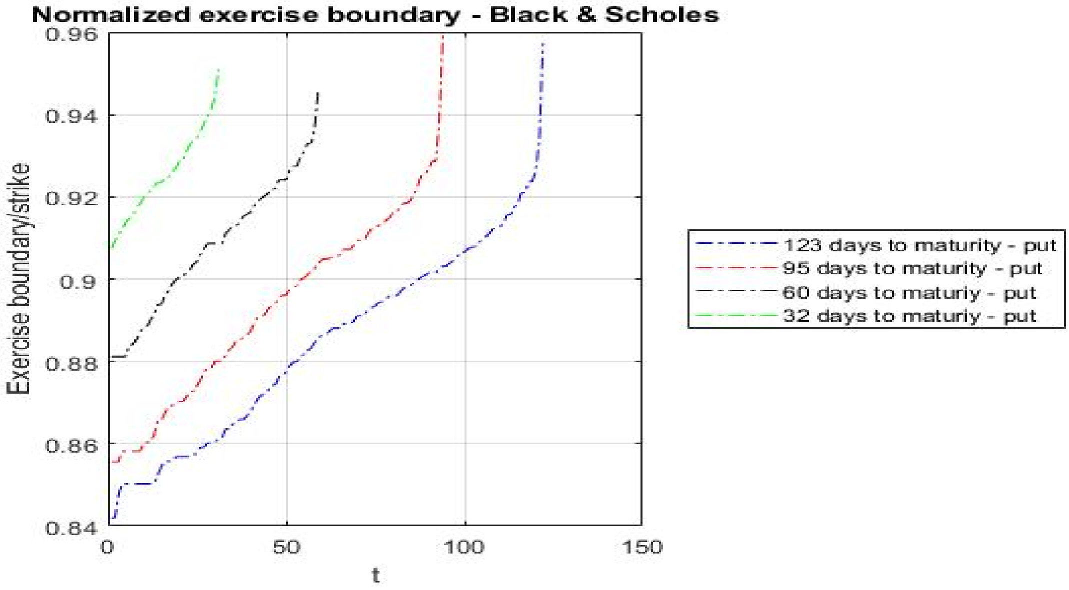


Figure 24: Plot of estimated put option normalized exercise boundaries for June 25, 2007 - implemented with the Black-Scholes price model, weighted Laguerre polynomial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

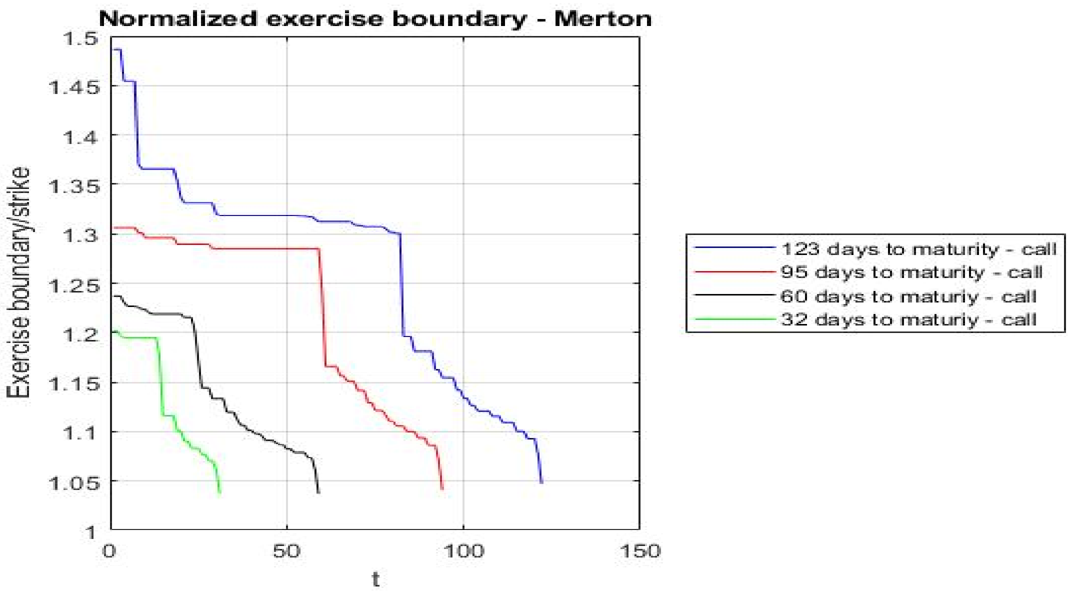


Figure 25: Plot of estimated call option normalized exercise boundaries for June 25, 2007 - implemented with the Merton price model, weighted Laguerre polynomial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

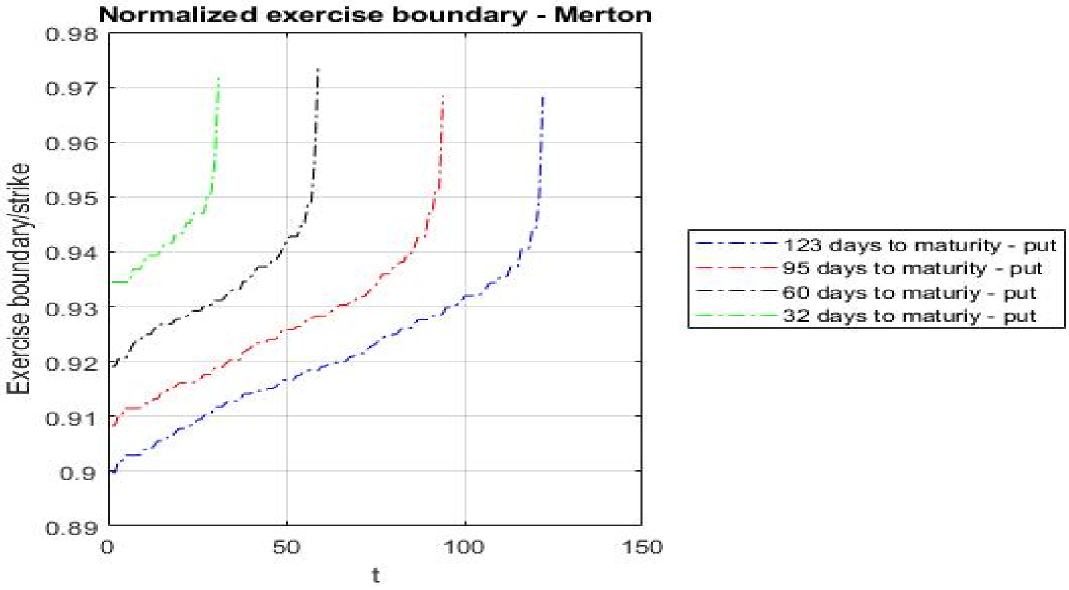


Figure 26: Plot of estimated put option normalized exercise boundaries for June 25, 2007 - implemented with the Merton price model, weighted Laguerre polynomial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

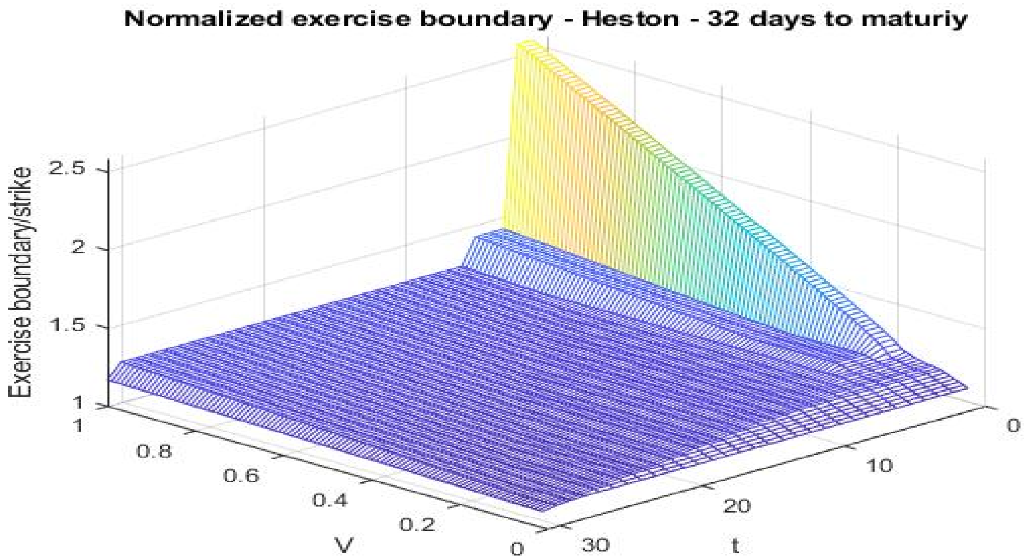


Figure 27: Plot of estimated call option normalized exercise boundary, with time to maturity 32 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

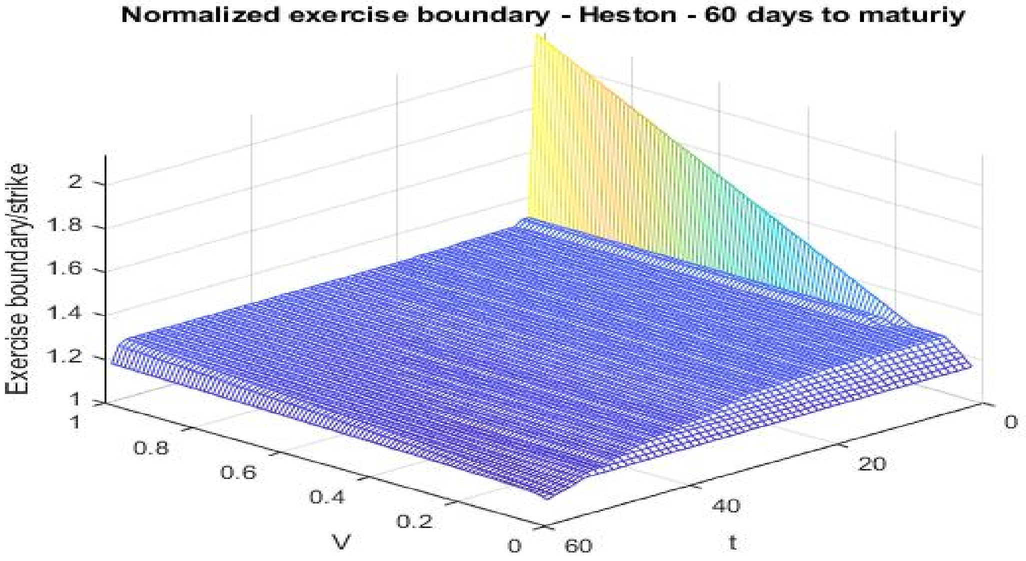


Figure 28: Plot of estimated call option normalized exercise boundary, with time to maturity 60 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

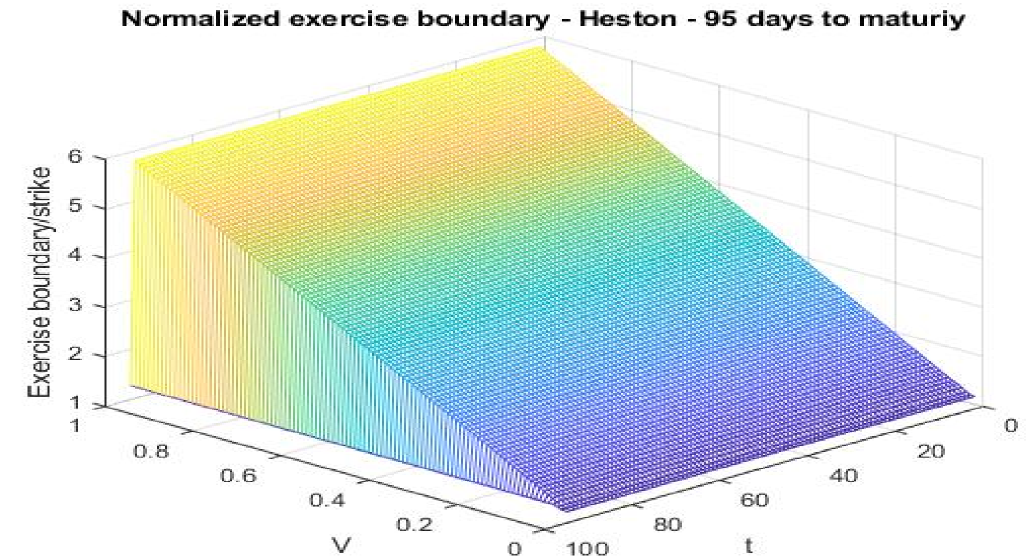


Figure 29: Plot of estimated call option normalized exercise boundary, with time to maturity 95 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

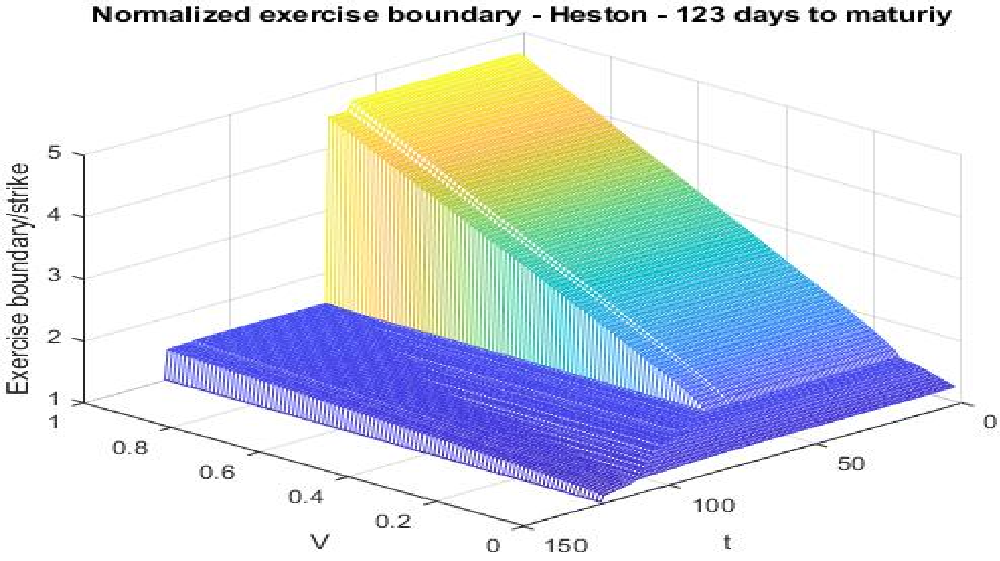


Figure 30: Plot of estimated call option normalized exercise boundary, with time to maturity 123 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

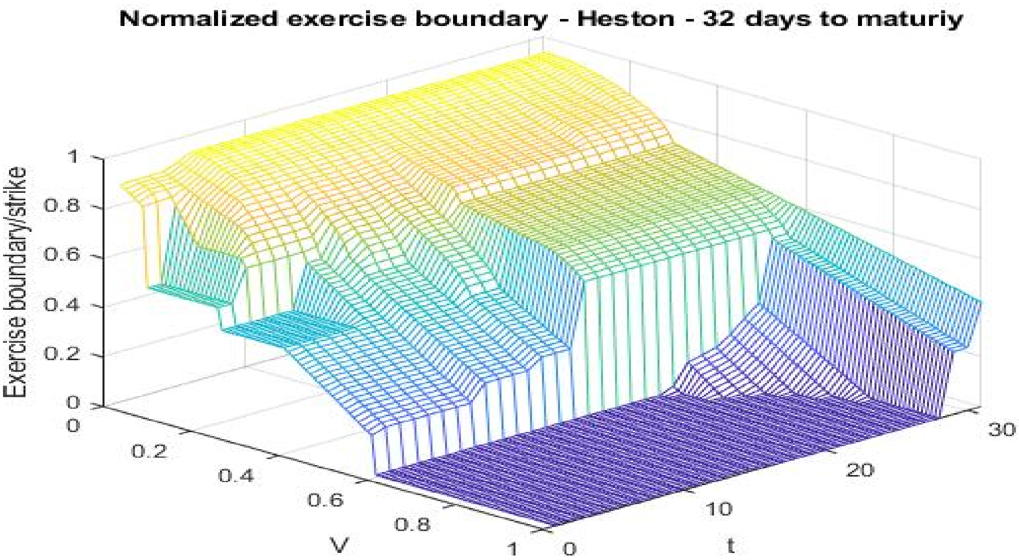


Figure 31: Plot of estimated put option normalized exercise boundary, with time to maturity 32 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

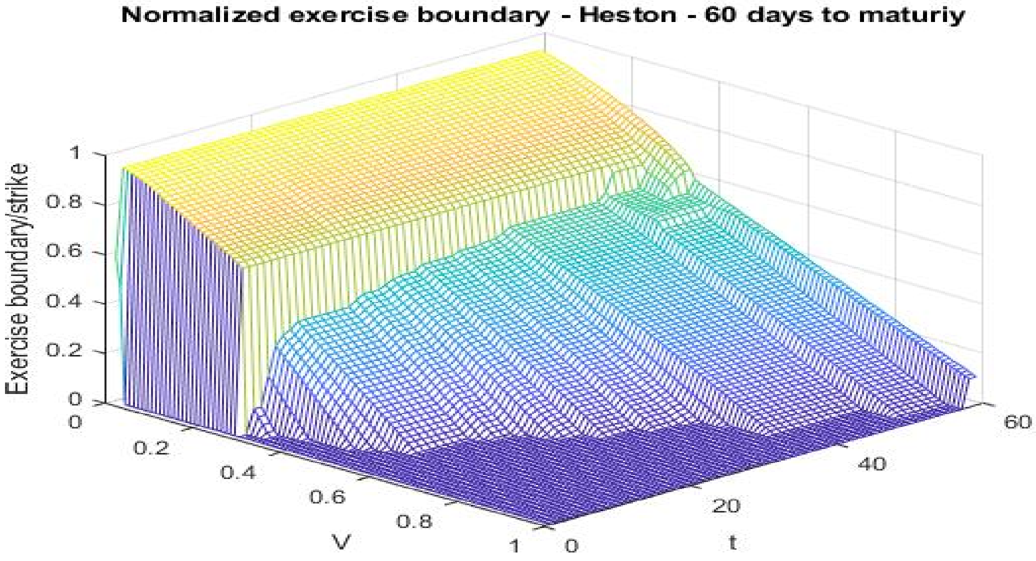


Figure 32: Plot of estimated put option normalized exercise boundary, with time to maturity 60 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

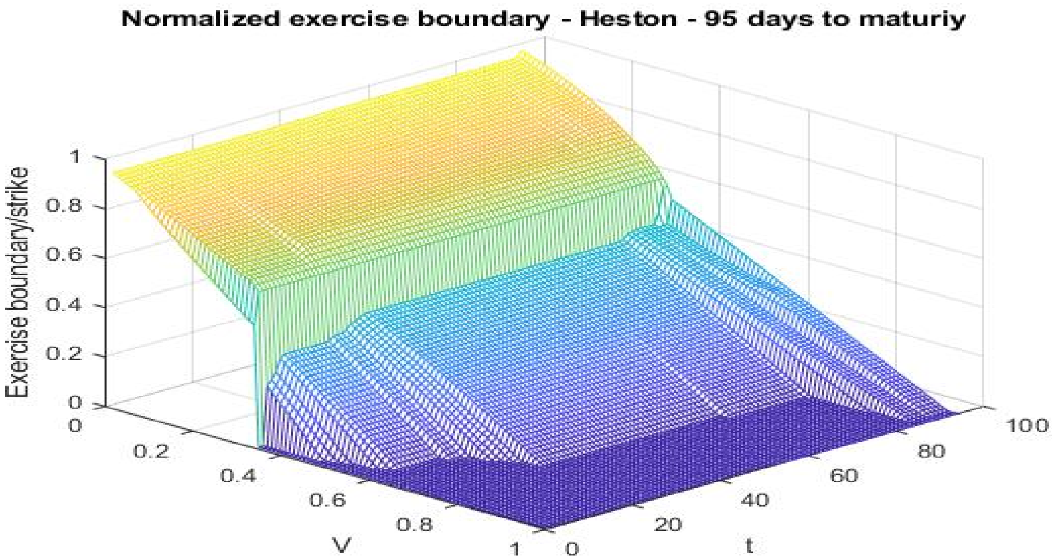


Figure 33: Plot of estimated put option normalized exercise boundary, with time to maturity 95 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

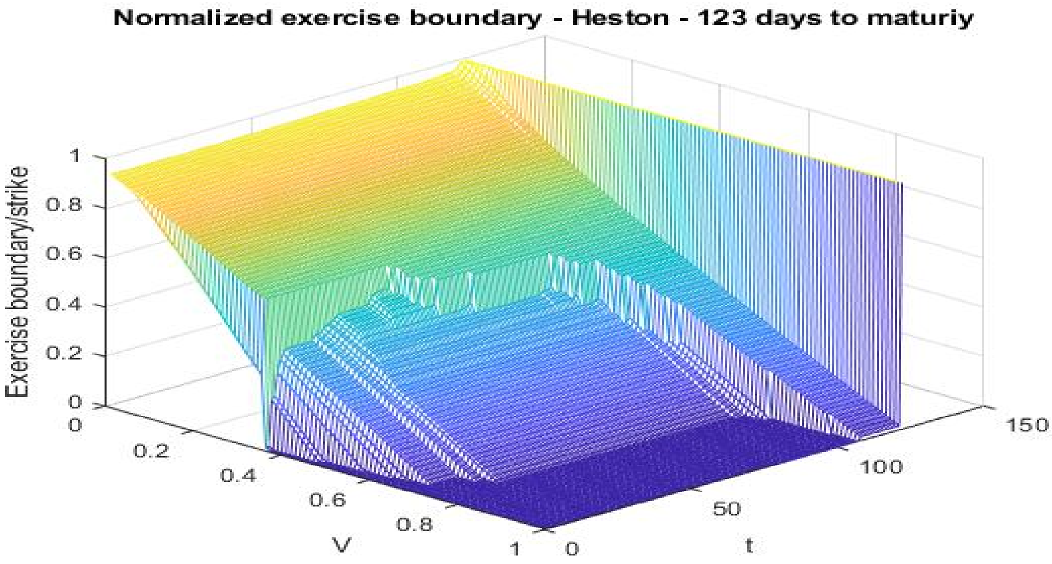


Figure 34: Plot of estimated put option normalized exercise boundary, with time to maturity 123 days, for June 25, 2007 - implemented with the Heston price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

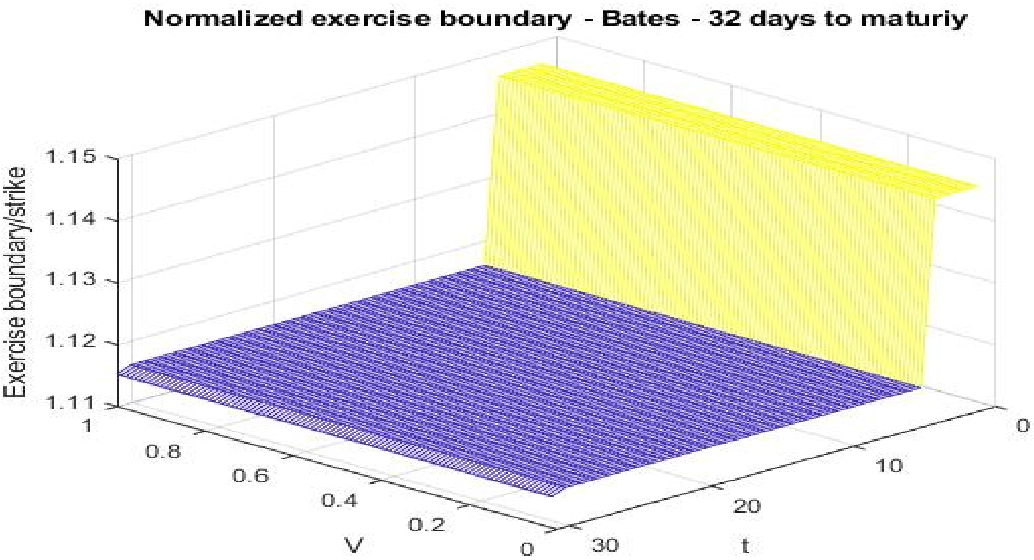


Figure 35: Plot of estimated call option normalized exercise boundary, with time to maturity 32 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

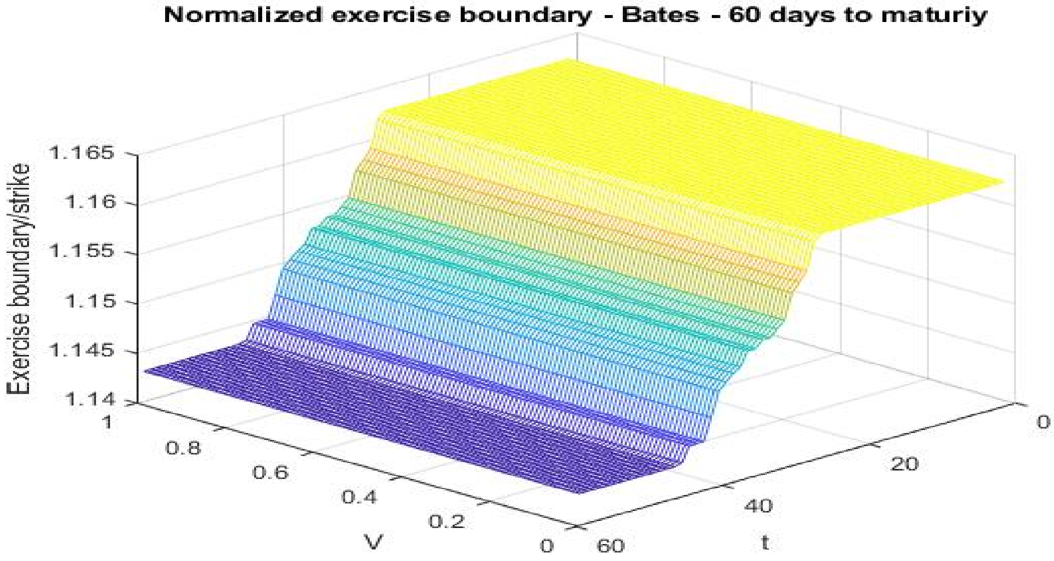
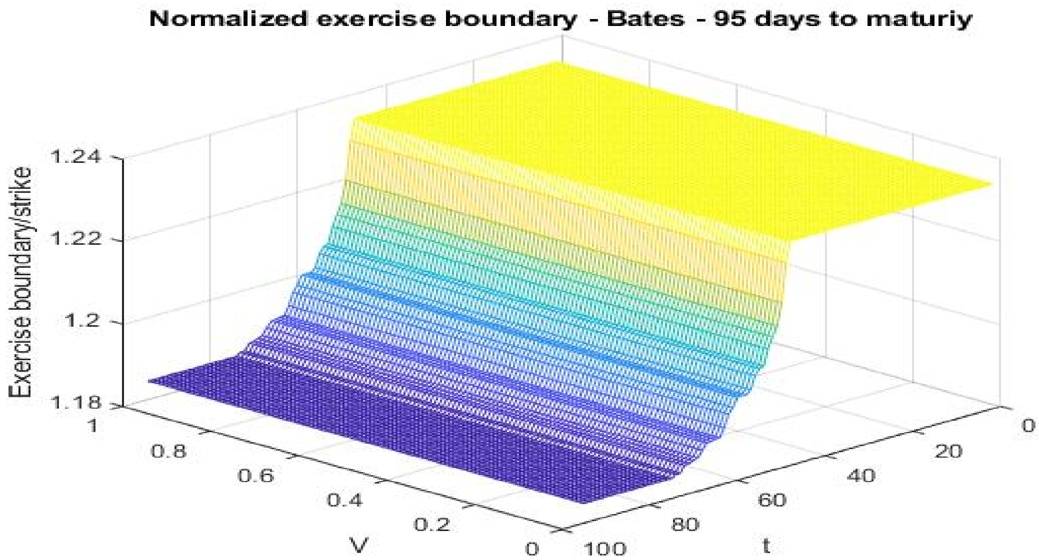


Figure 36: Plot of estimated call option normalized exercise boundary, with time to maturity 60 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations



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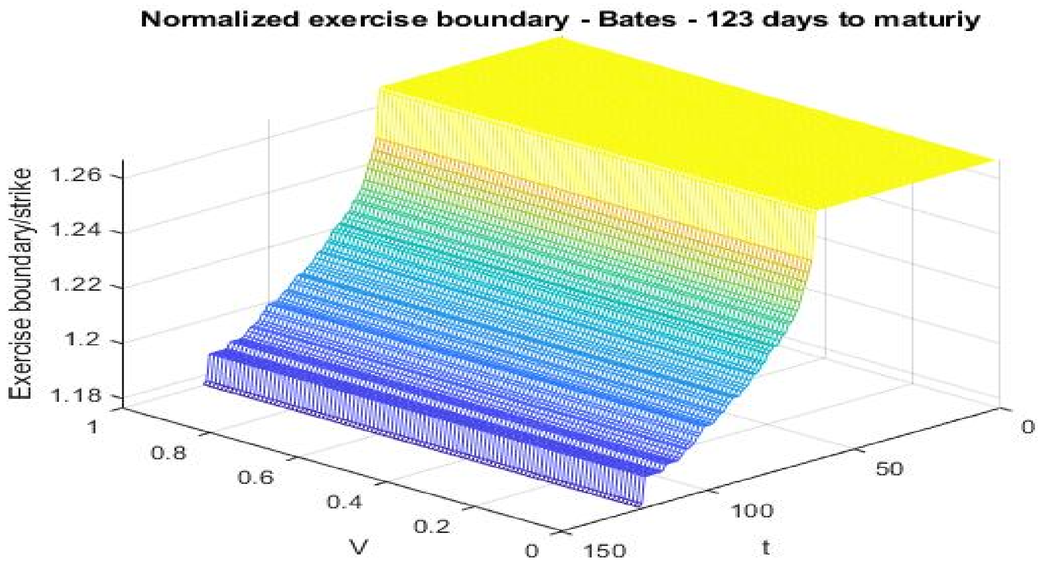

Figure 38: Plot of estimated call option normalized exercise boundary, with time to maturity 123 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

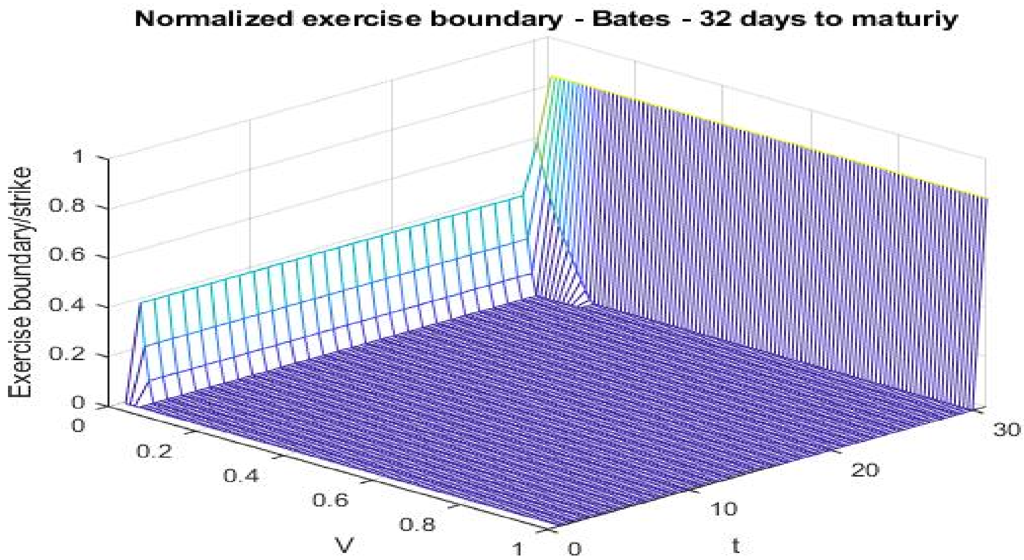

Figure 39: Plot of estimated put option normalized exercise boundary, with time to maturity 32 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

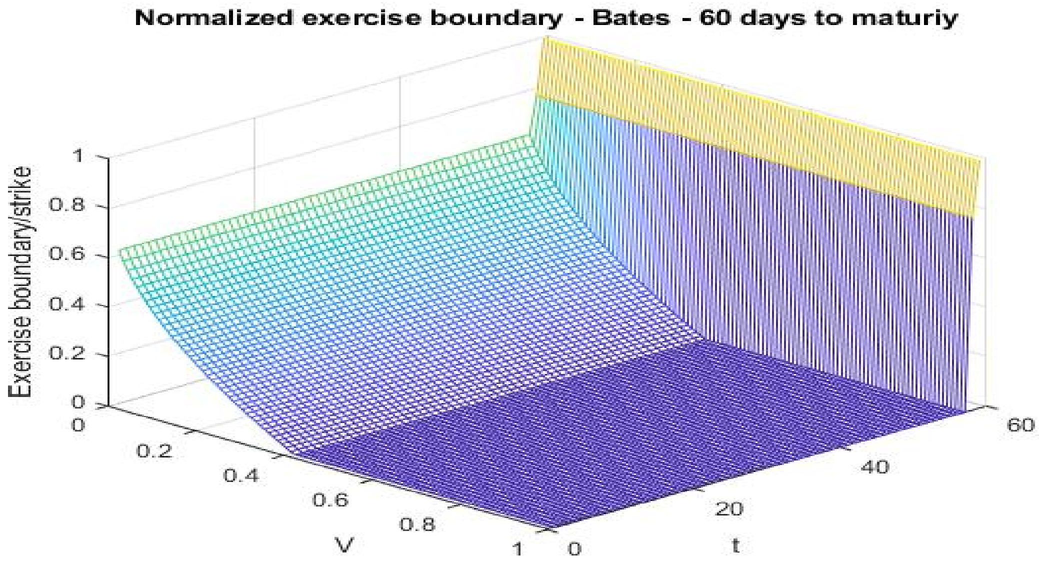

Figure 40: Plot of estimated put option normalized exercise boundary, with time to maturity 60 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

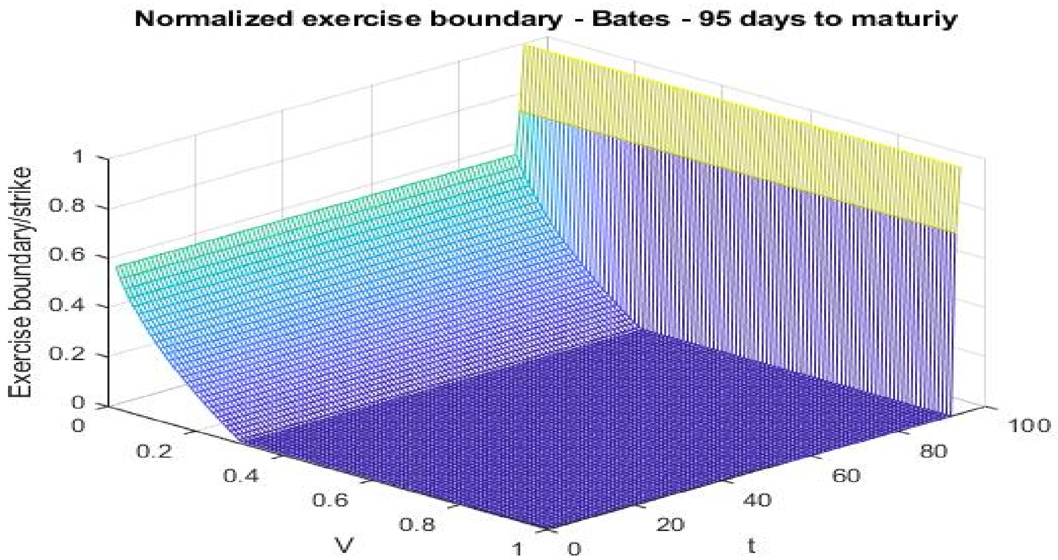

Figure 41: Plot of estimated put option normalized exercise boundary, with time to maturity 95 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

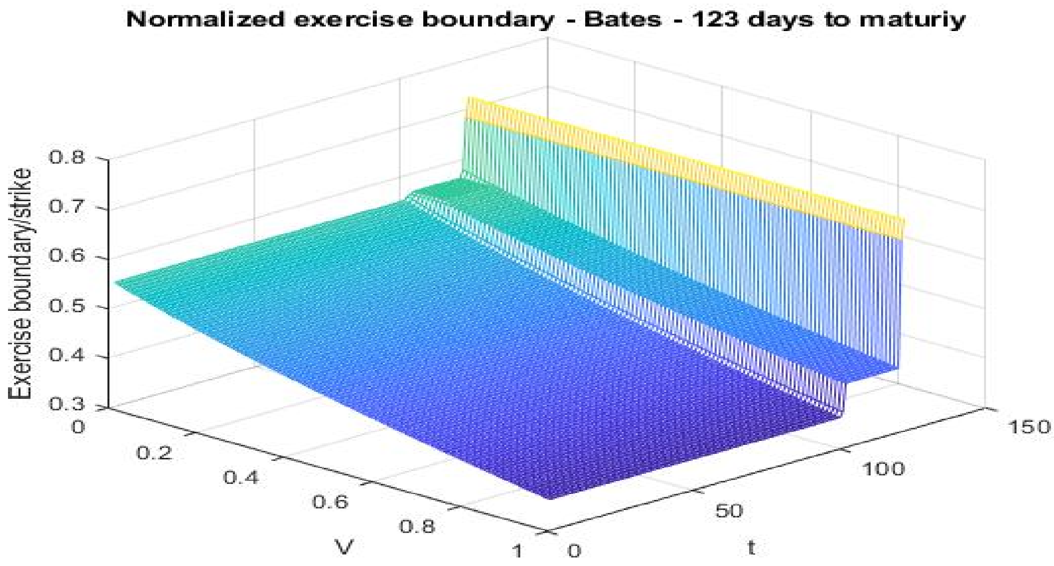

Figure 42: Plot of estimated put option normalized exercise boundary, with time to maturity 123 days, for June 25, 2007 - implemented with the Bates price model, multimonial basis functions, the parameter weight set to one, and estimated with 500,000 iterations

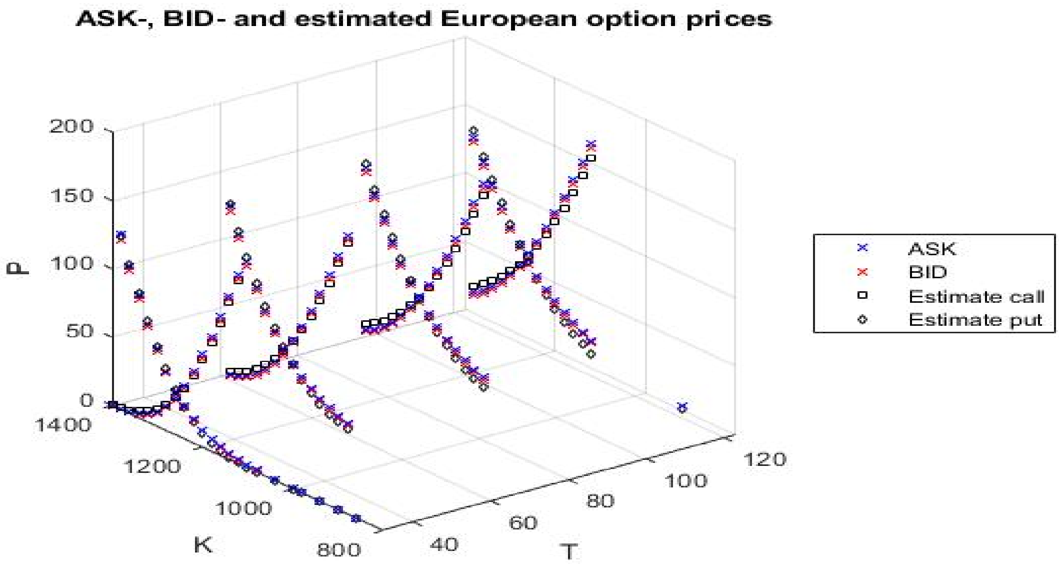

Figure 43: Plot of estimated European option prices and the markets ASK- & BID-prices for June 25, 2007 - implemented with the Black-Scholes price model and the parameter weight set to one

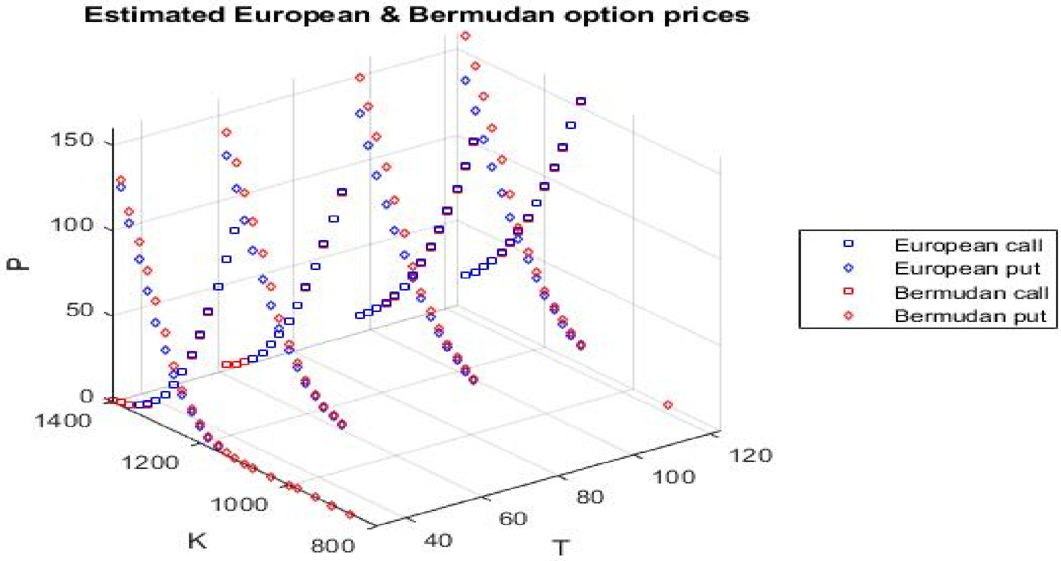

Figure 44: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Black-Scholes price model, weighted Laguerre polynomial basis functions and the parameter weight set to one

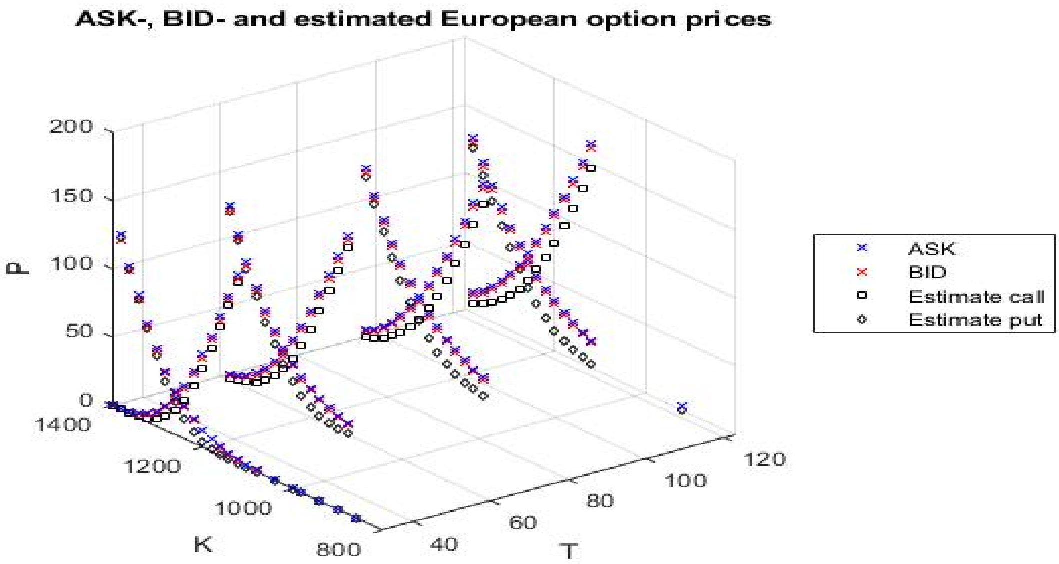

Figure 45: Plot of estimated European option prices and the markets ASK- & BID-prices for June 25, 2007 - implemented with the Merton price model and the parameter weight set to one

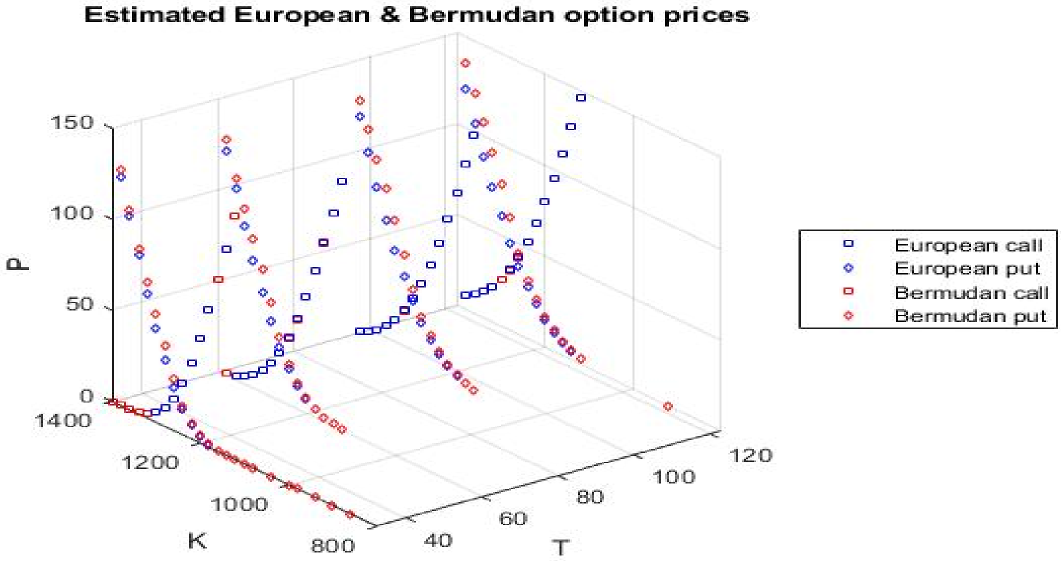

Figure 46: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Merton price model, weighted Laguerre polynomial basis functions and the parameter weight set to one

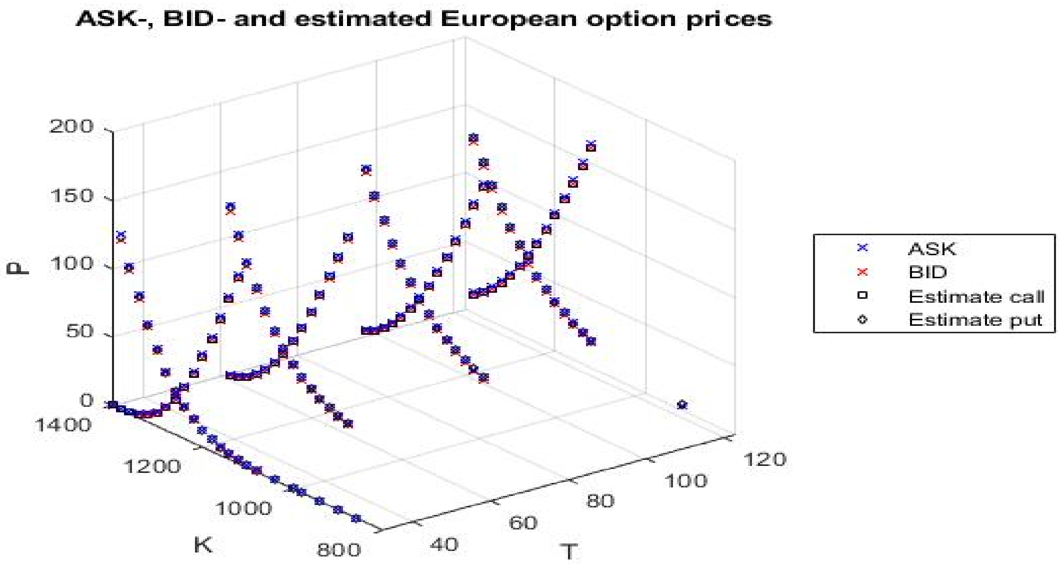

Figure 47: Plot of estimated European option prices and the markets ASK- & BID-prices for June 25, 2007 - implemented with the Heston price model and the parameter weight set to one

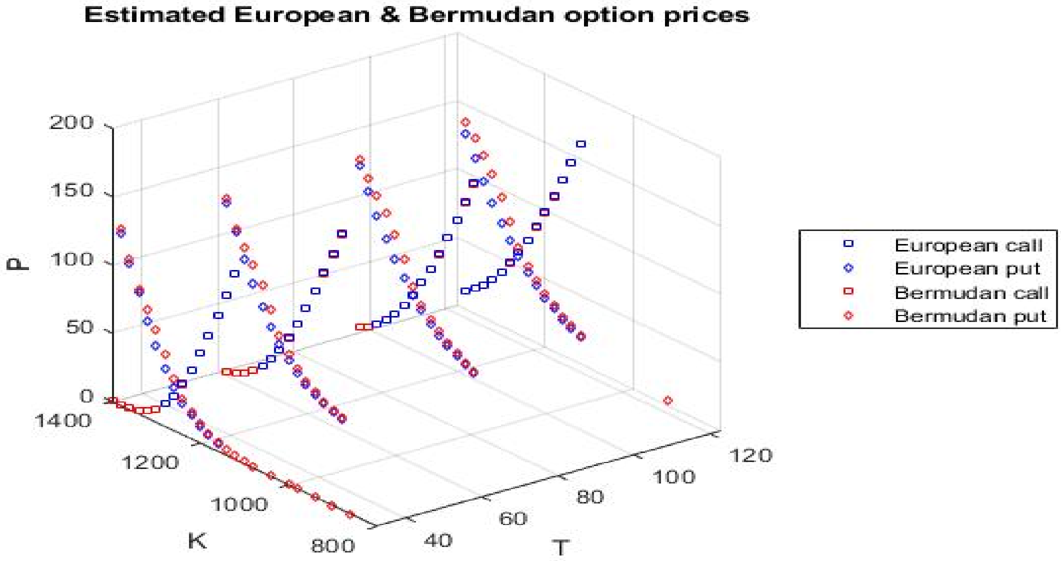

Figure 48: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Heston price model, weighted Laguerre polynomial basis functions and the parameter weight set to one

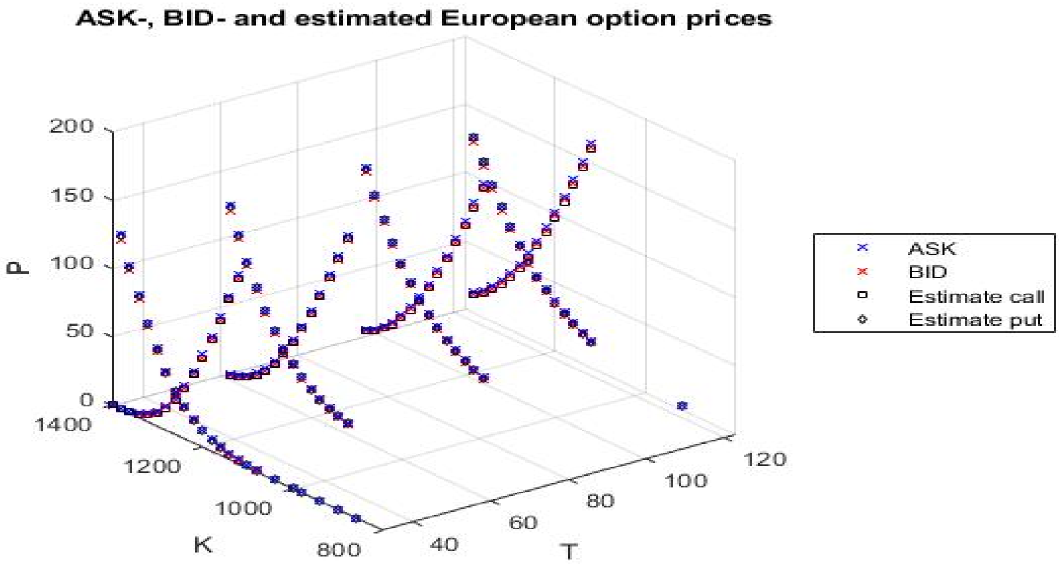

Figure 49: Plot of estimated European option prices and the markets ASK- & BID-prices for June 25, 2007 - implemented with the Bates price model and the parameter weight set to one

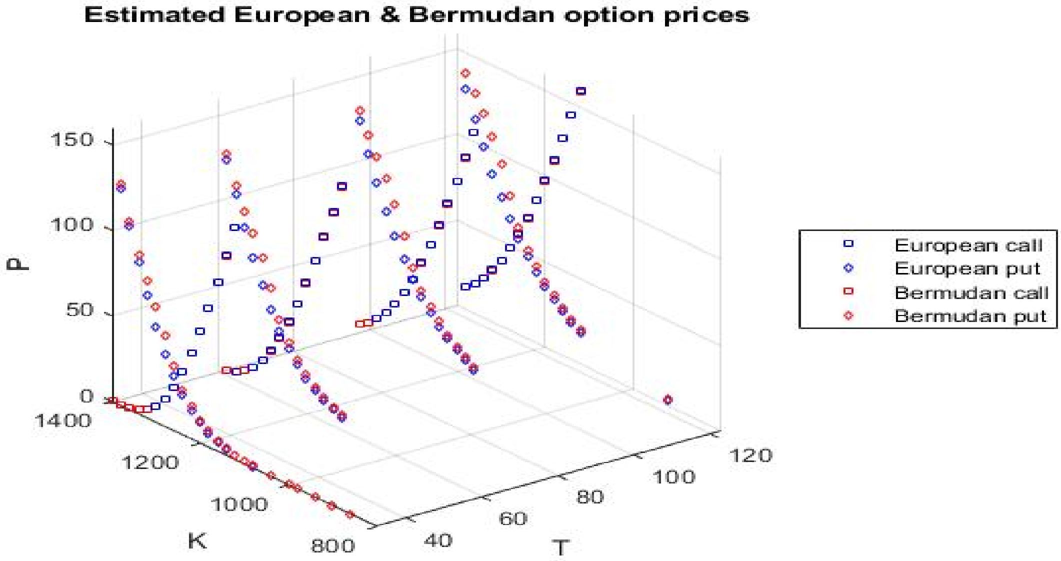

Figure 50: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Bates price model, weighted Laguerre polynomial basis functions and the parameter weight set to one

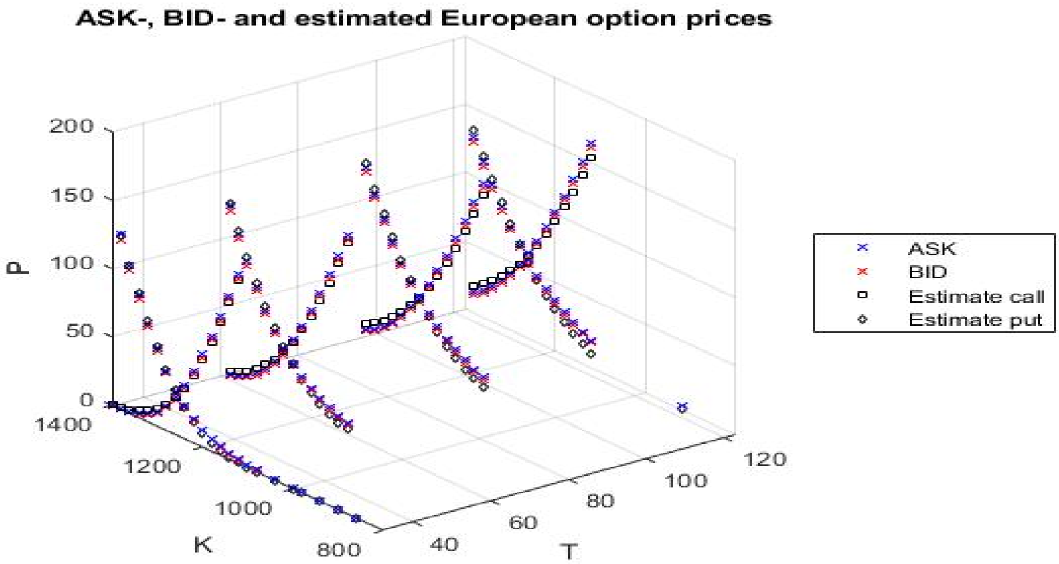

Figure 51: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Black-Scholes price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

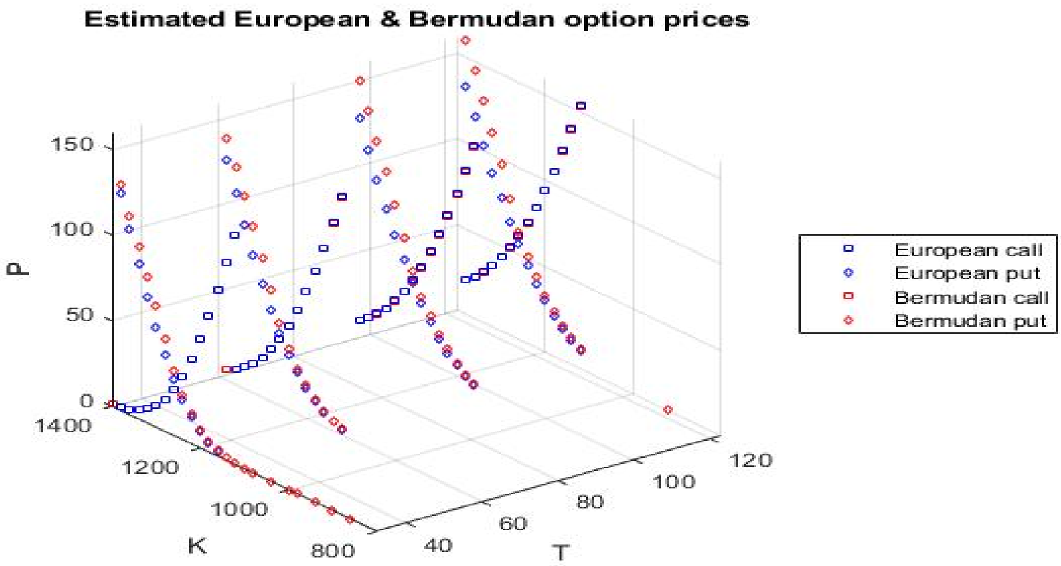

Figure 52: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Black-Scholes price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

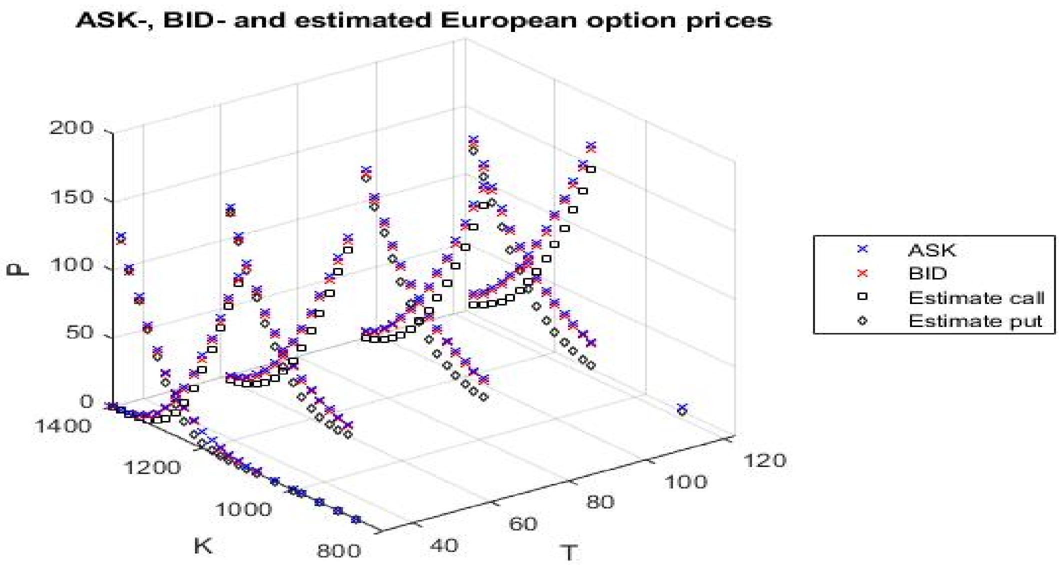

Figure 53: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Merton price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

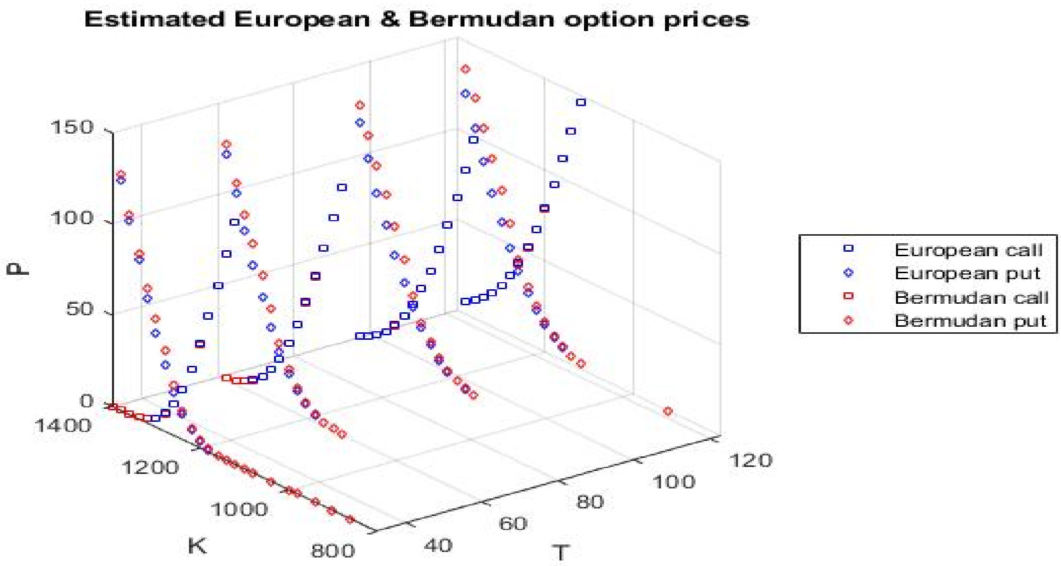

Figure 54: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Merton price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

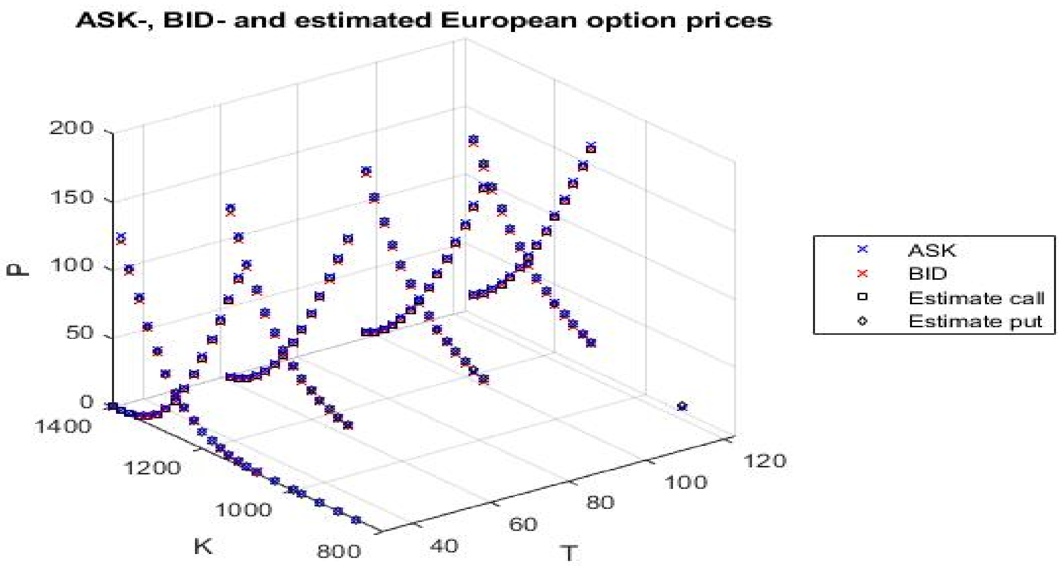

Figure 55: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Heston price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

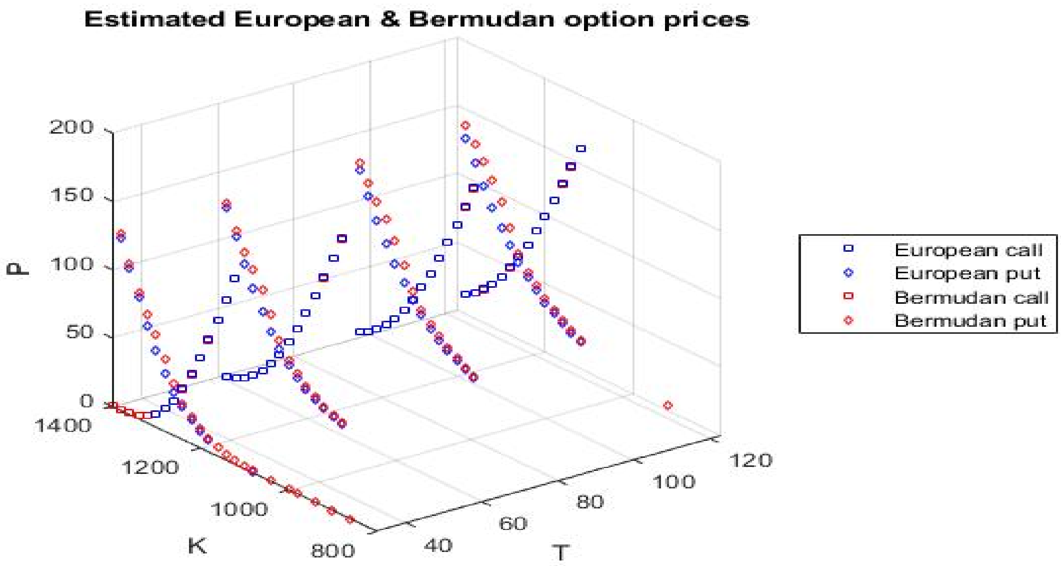

Figure 56: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Heston price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

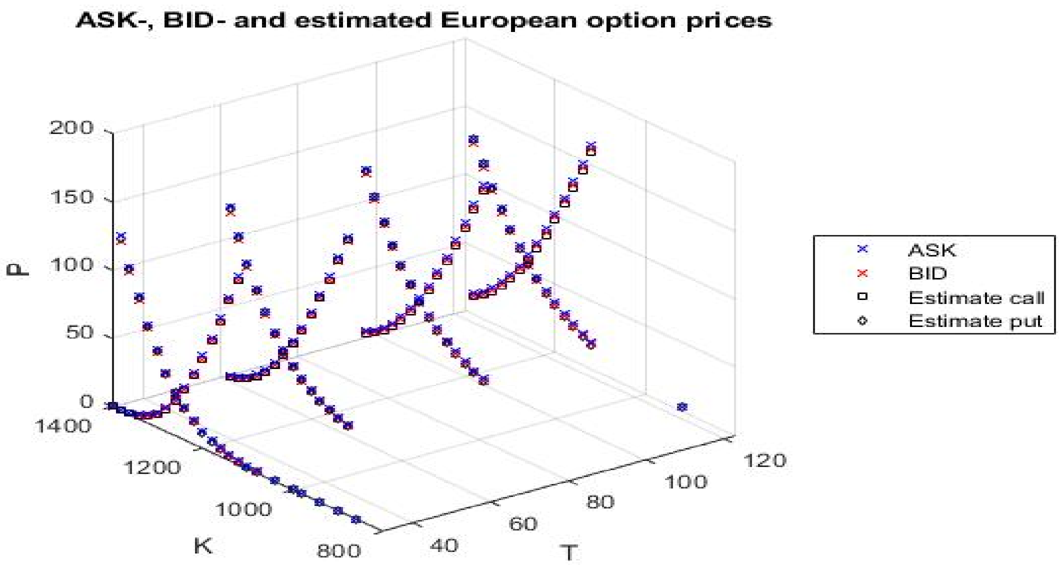

Figure 57: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Bates price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

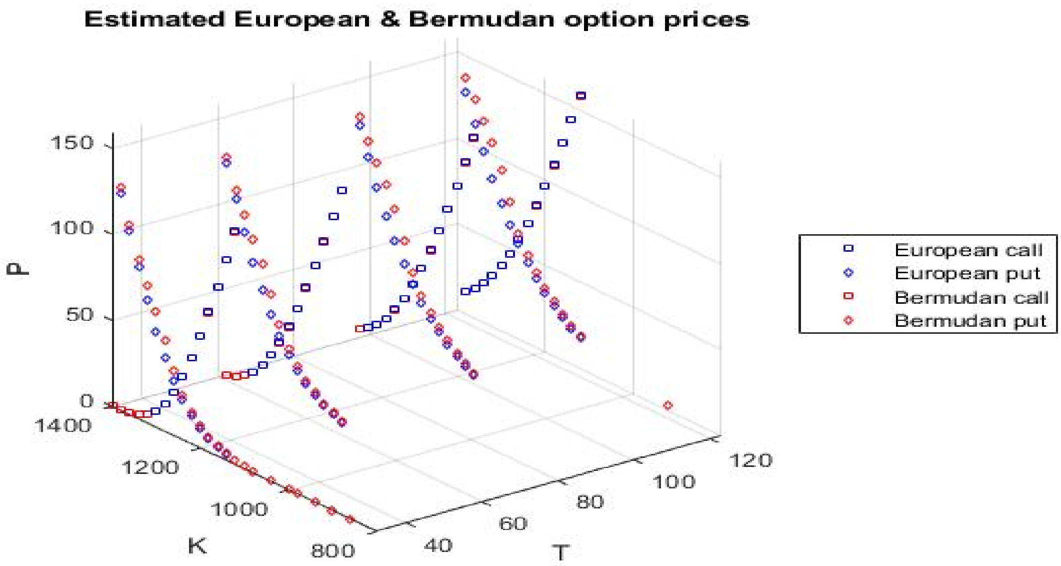

Figure 58: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Bates price model, weighted Laguerre polynomial basis functions and the parameter weight set to the ASK/BID-spread

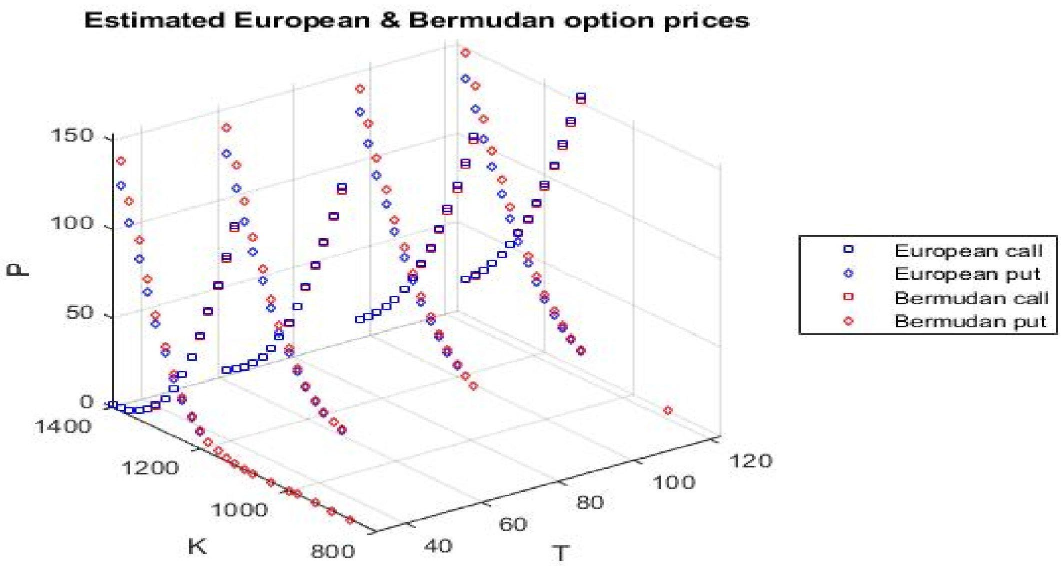

Figure 59: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Black-Scholes price model and normalized exercise boundaries

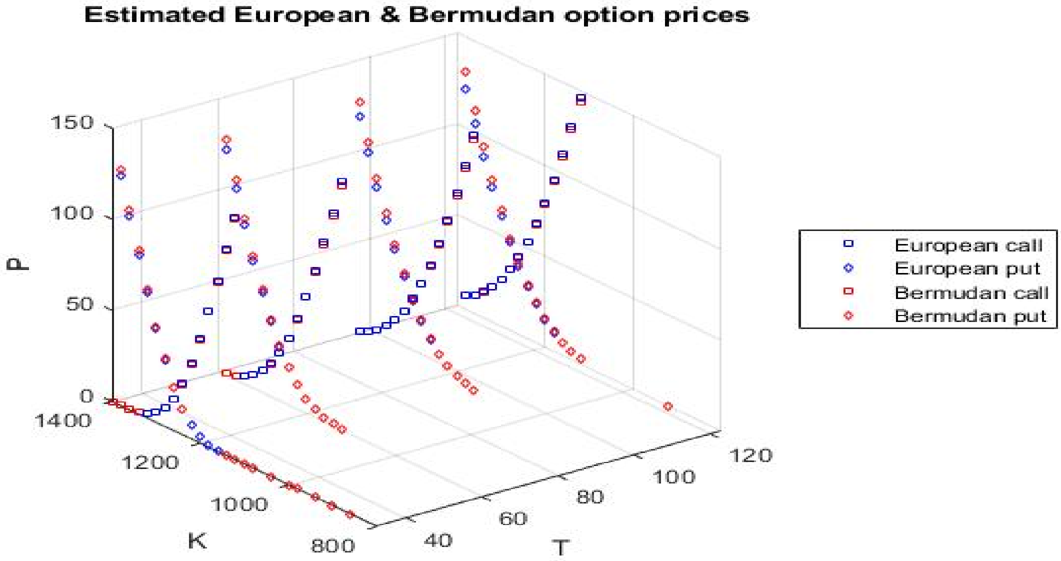

Figure 60: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Merton price model and normalized exercise boundaries

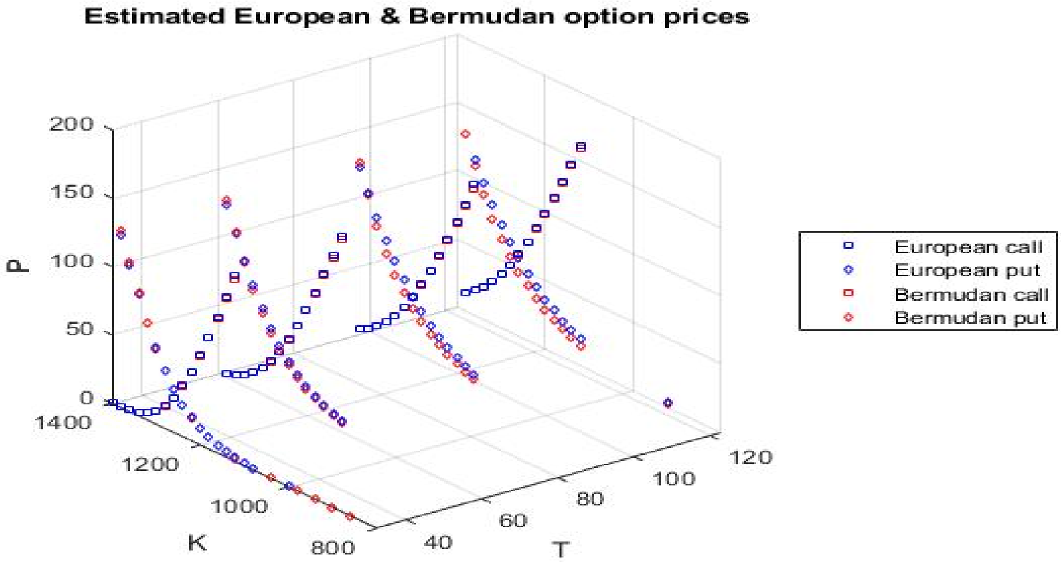

Figure 61: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Heston price model and normalized exercise boundaries

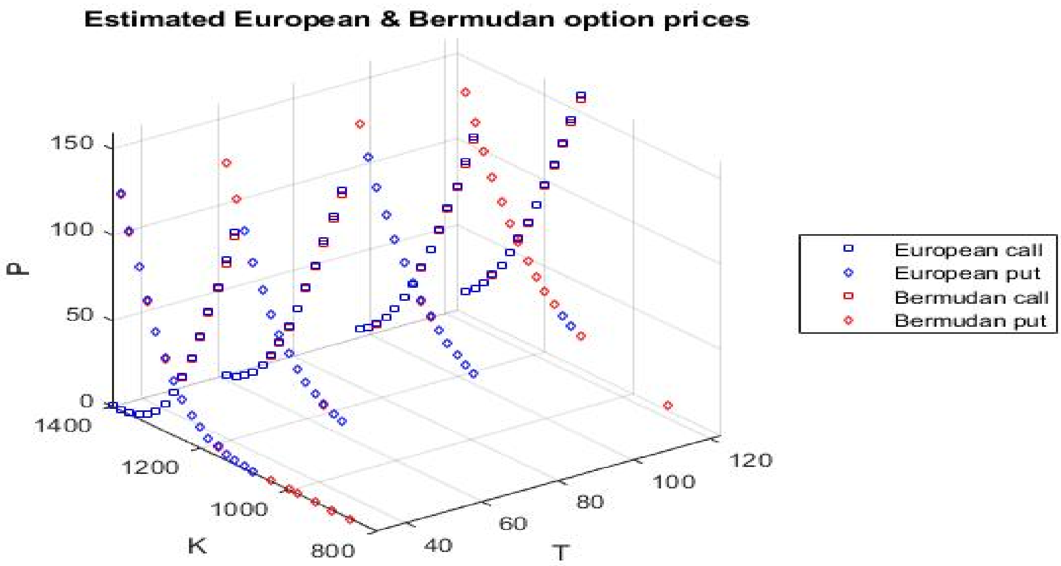

Figure 62: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Bates price model and normalized exercise boundaries

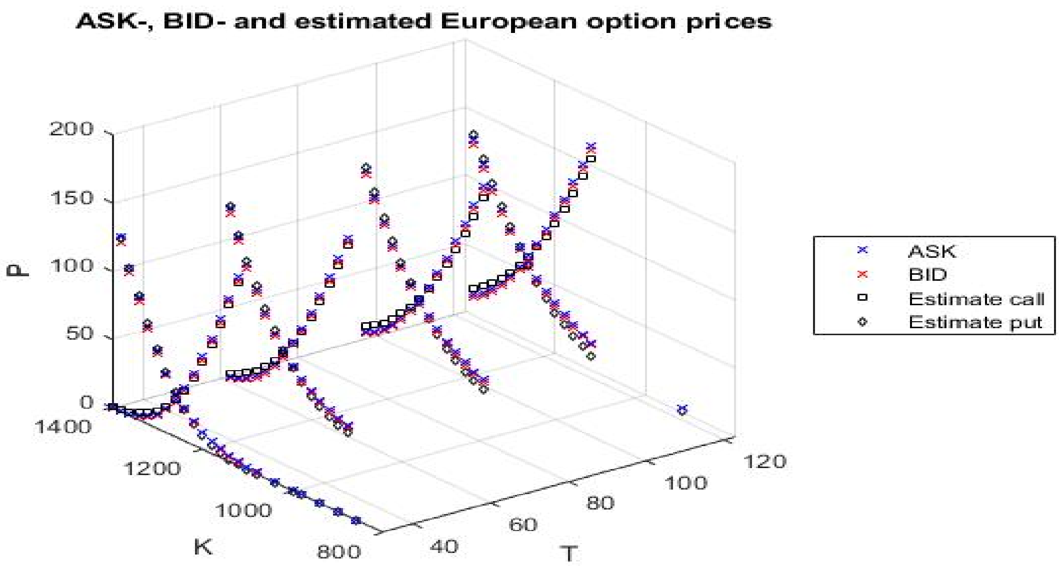

Figure 63: Plot of estimated European option prices for June 25, 2007 - implemented with the Black-Scholes price model, Laguerre polynomials basis functions, the parameter weight set to one, and the underlying asset as control variate for the European and Bermudan options

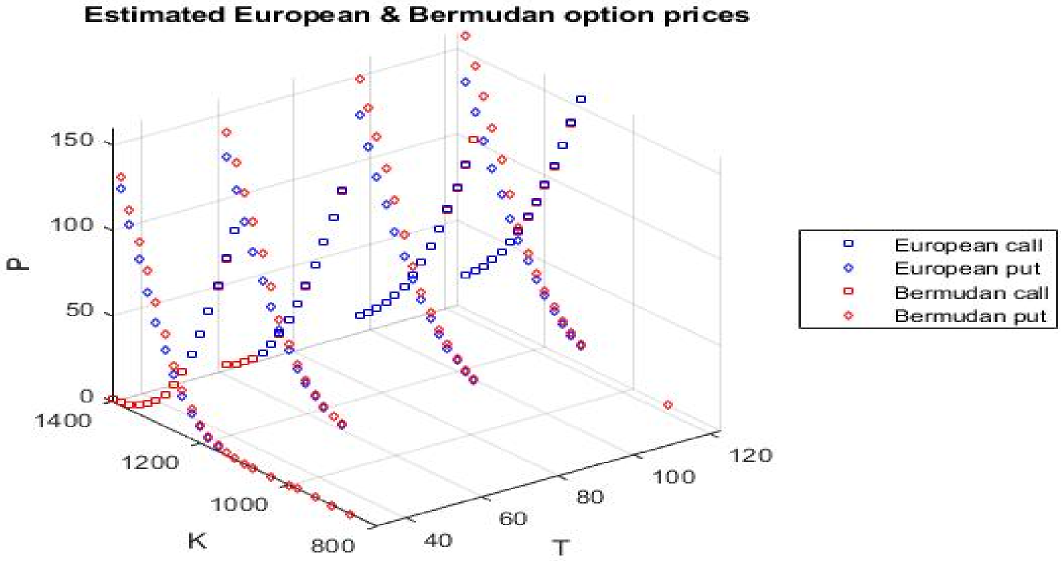

Figure 64: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Black-Scholes price model, Laguerre polynomials basis functions, the parameter weight set to one, and the underlying asset as control variate for the European and Bermudan options

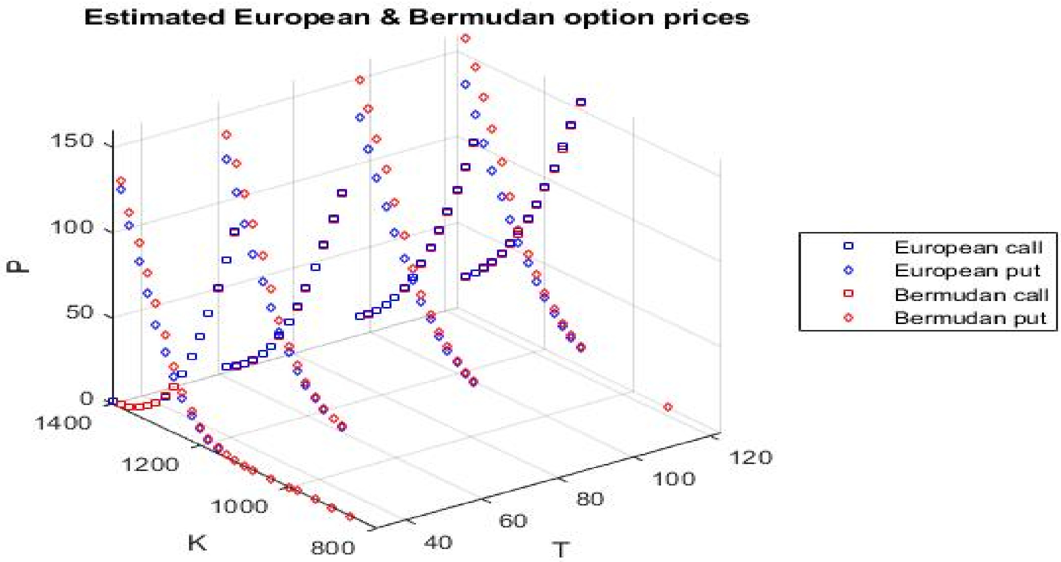

Figure 65: Plot of estimated European and Bermudan option prices for June 25, 2007 - implemented with the Black-Scholes price model, Laguerre polynomials basis functions, the parameter weight set to one, the underlying asset as control variate for the European and the European options as control variate for the Bermudan options