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How Do Traditional Models for Option Valuation Perform When Applied to Cryptocurrency Options?

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Abstract

The market for cryptocurrencies has been known to be volatile with an asymmetrical return distribution where occasional extreme returns appear. In later years options have been introduced on the asset; but due to the characteristics of cryptocurrency returns, researchers have found it troublesome to value these options. Primarily, research has been advocating for advanced pricing models and suggested the elemental Black-Scholes model to perform inferior to these models. The market for cryptocurrency options have grown rapidly, which puts into question whether the Black-Scholes model in today's more liquid market can reliably predict option prices? If not, can the Heston model, which allows for stochastic volatility, be a better fit? This study is, to my knowledge, the first to evaluate the Heston model on options with Ethereum as underlying asset; wherefore it contributes to research by incorporating a broader analysis of the model's performance on cryptocurrency options. Based on real market prices I calibrate and evaluate the two models on puts and calls with Bitcoin and Ethereum as underlying assets. My findings indicate that the Black-Scholes model outperforms the Heston model for all the four considered option types and that the models are better suited for valuing options on Ethereum than Bitcoin.

Acknowledgements

I would like to thank my supervisor Anders Vilhelmsson for guiding and supporting me throughout the process of writing this essay. Without your expertise in MatLab this study would have ended before it even started. I would also like to express my gratitude to my family who, despite not understanding the subject, took their time to proofread this paper.

Contents

Notations

K	Strike price
S	Spot price
c	Call option price
p	Put option price
r	Interest rate
σ	Volatility
BTC	Bitcoin
ETH	Ethereum
B-S	Black-Scholes
TTM	Time to maturity
M	Moneyness
ITM	In-the-money
ATM	At-the-money
OTM	Out-of-the-money
DOTM	Deep-out-of-the-money
DITM	Deep-in-the-money

1. Introduction

In the recent decade cryptocurrencies have risen in popularity and the market is constantly expanding with more cryptocurrencies. As for any other investment asset, the demand for hedging tools increases along with the demand for the underlying asset. This has become apparent, as the trade of future and option contracts on cryptocurrencies has intensified. In comparison to the futures market, the market for cryptocurrency options is still in its infancy and researchers have found it difficult to reliably value the options due to the characteristics of cryptocurrencies' return distributions. Studies have consistently discovered a high level of excess kurtosis and skewness of the return distribution (Baur, Hong & Lee, 2018; Bianchi, 2020), which do not fully coincide with traditional asset classes (Baur, Hong & Lee, 2018; Bianchi, 2020; Ram, 2019). When pricing options, researchers have tried bringing the issue to bay by using advanced pricing models, allowing for different types of jump processes. Hou et al. (2020) proposed jumps in the return and variance processes as well as cojumps to be of importance when valuing cryptocurrency options. Notwithstanding, Madan, Reyners and Schoutens (2019) suggested that stochastic volatility of the asset returns is of foremost importance since they found models with and without jumps to perform well as long as they allowed for stochastic volatility.

When considering less advanced pricing models which assume a constant volatility of returns, such as the Black-Scholes (B-S) model, the research on cryptocurrency options is scarce. Madan, Reyners and Schoutens (2019) covered the B-S model when they studied options on Bitcoin (BTC) and concluded the model to perform poorly. Yet, since year 2019 the market for cryptocurrency options has become more liquid, and Ethereum (ETH) options have become increasingly popular with a quadrupled open interest during the year of 2021 (Coinglass, n.d.). This puts into question whether the findings of Madan, Reyners and Schoutens (2019) regarding the B-S model are valid on today's more liquid market and if the results are valid for ETH options.

In practice, the B-S model systematically misvalues options (Black, 1975; Dupoyet, 2006; MacBeth & Merveille, 1979; Rubinstein, 1994), due to its crude assumption of constant volatility and log-normal distributed asset prices (Karlsson, 2009; Wu, 2019). For that reason, it is of interest to compare the B-S model's performance on cryptocurrency options with a model which relaxes these assumptions. Hence the Heston model, allowing for stochastic volatility, will also be evaluated in this paper. The Heston model has, to my knowledge, only been evaluated on BTC options in previous research, which makes this study the first to extend the analysis to ETH options.

Both the B-S model and Heston model will be calibrated and evaluated based on stated market prices, with the aim to obtain an answer as to whether the B-S model reliably can estimate option values for cryptocurrencies and if not, is the Heston model a better fit?

2. Theory

2.1 Theoretical Background

A call (put) option gives the holder the right to buy (sell) the underlying asset for an agreed-upon price at some prespecified future time. If the investor is exposed to an asset, the option can be used to hedge the price risk of the exposure. That is, if the spot price of the underlying asset at maturity date is disadvantageous in comparison to the strike price, the holder will not exercise the option.

When the option's contract is entered, the holder must pay a premium, which is commonly known as the price of the option. For the investor it is usually of interest to find a theoretical value of the option, based on the probability of the option being exercised.

2.2 The Black-Scholes Model

Black and Scholes (1973) constructed an option pricing model based on the no arbitrage argument, which suggests that an investor cannot make a risk-free profit from a portfolio of short and long positions in the underlying asset and the option. In the absence of arbitrage, there can only exist one fair price for an option (Black and Scholes, 1973). To derive this price, Black and Scholes (1973) assumed the following:

1. The price of the underlying asset follows a random walk in continuous time with a constant variance rate. The stock price distribution at the end of any finite interval is log-normal
2. There are no transaction costs in buying or selling the underlying asset or the option

3. It is possible to borrow at the short-term interest rate, which is known and constant through time
4. Short selling is allowed
5. The option is European, and its underlying asset does not pay any dividends

Based on these assumptions, Black and Scholes (1973) developed a closed form solution of options pricing, where the option value only depends on time, the price of the underlying asset and known constants:

$$c = S_t * N(d_1) - K * e^{-r(T-t)} * N(d_2)$$
$$p = K * e^{-r(T-t)} * N(-d_2) - S_t * N(-d_1) \quad (1)$$

where

$$d_1 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) * (T-t)}{\sigma * \sqrt{(T-t)}}$$

$$d_2 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) * (T-t)}{\sigma * \sqrt{(T-t)}}$$

and N() is the standard normal density function.

Furthermore, the volatility in formula 1 cannot be directly observed and thus must be estimated. Hull (2018) described there to be multiple methods for estimating the volatility, but that the most common approach is to utilize the volatility implied by the market. This volatility should, according to assumption number one above, be constant and therefore not vary depending on time to maturity or strike price.

2.3 The Heston Model

Heston (1993) called attention to the shortfalls of the B-S pricing model and suggested that its return skewness and strike price biases can be explained by an arbitrary correlation between the underlying asset's returns and a non-constant variance. Heston (1993) presented the following processes for the underlying asset price and variance, where $dz_1(t)$ and $dz_2(t)$ are correlated Weiner processes:

$$\begin{cases} dS = \mu * S * dt + \sqrt{v(t)} * S * dz_1(t) \\ dv(t) = k[\theta - v(t)]dt + \sigma * \sqrt{v(t)} * dz_2(t) \end{cases} \quad (2)$$

As shown in the formula, the variance of the underlying asset's return is determined by five parameters: The long-run mean variance θ, the mean reversion speed *k*, the correlation of the Weiner processes δ, the current variance *v(t)* and the volatility of the volatility parameter σ (Heston, 1993).

To obtain the price formula for a European call option, Heston (1993) assumed all investors to require the same return independently of his/her risk-exposure and that it is not possible to conduct arbitrage. Given these assumptions, the option value is a function of the spot price of the underlying asset, its variance and time:

$$c(S, v, t) = S * P_1 - K * e^{r(T-t)} * P_2 \quad (3)$$

where P1 and P2 can be viewed as risk-neutral probabilities and *r* is the constant interest rate (Heston, 1993). To obtain the value for a put option, put-call-parity can be applied:

$$p = c + K * e^{-r*T} - S \quad (4)$$

The full derivation of the Heston model is beyond the scope of this paper.

3. Previous Research

3.1 Option Pricing Models and Traditional Asset Classes

The Black-Scholes model has been frequently used for pricing options; although, some researchers assert that it is an inferior model.

In the years following the creation of the B-S model, researchers observed some systematic biases associated with the model. Black (1975) suggested that the model undervalued OTM options and overvalued ITM options. MacBeth and Merville (1979) found that the B-S values generally deviated from the market prices in the opposite direction: the model undervalued ITM calls, and this bias grew along with the time to maturity. Rubinstein (1985) observed both stock option pricing biases introduced by Black (1975) and by MacBeth and Merveille

(1979), but at different time periods. Rubinstein (1985) concluded that the B-S model generally over- or underestimates non-ATM stock call option prices, and that this bias's direction changes over time.

Rubinstein (1994) applied the B-S model on stock index options and found that, based on data from the late 1980's market crash, OTM put options were undervalued by the B-S formula. The author explained that these price discrepancies were related to higher implied volatility for options with lower strikes, in comparison to options with high strike prices. Hull (2018) described that this relationship between implied volatility and strike price for stock options, known as 'volatility skew', is still used by traders decades later. The skew can be explained by the asset price distribution, where the lognormal distribution assumed by the B-S model has a left tail which is too light and a right tail which is too heavy in comparison to the implied distribution (Hull, 2018).

Dupoyet (2006) applied the B-S model on foreign currency options and observed that short term call options further from being ATM revealed higher pricing errors and implied volatilities than those closer to being ATM. Hull (2018) explained that this relationship between moneyness and implied volatility is a common occurrence for foreign currency options and can be explained by the asset price distribution where the lognormal distribution assumed by the B-S model is too heavy tailed in comparison to the implied distribution.

The aforementioned studies show that the B-S model is associated with pricing biases whose direction depend on the asset class and the period of which the data represents. The causes for the pricing biases have been frequently discussed in financial literature and an abundance of pricing models have been developed to improve the prediction of option prices. Bakshi, Cao and Chen (1997) conducted a horserace amongst pricing models relaxing different assumptions of the B-S model. When comparing models with stochastic volatility, interest rates and jumps they concluded that "taking stochastic volatility into account is of the first-order importance in improving upon the BS formula" (Bakshi, Cao & Chen, 1997, p. 2042-2043).

Heston (1993) presented an option pricing model allowing for stochastic volatility correlated with the spot returns and argued that it, in comparison to the B-S model, is a more appropriate pricing model for multiple asset classes. Karlsson (2009) compared the B-S and Heston models and found the Heston model to perform the best overall, due to its disregard of log-normal distributed asset prices. Although, for ITM calls and calls with long maturities, the

Heston model performed inferior to the B-S model (Karlsson, 2009). By contrast, Zhang and Shu (2003) argued that the Heston model mispriced short-term call options severely and that the observed mispricing of ITM and OTM options decreased as the maturity increased. Wu (2019) found similar results when applying the two models on index options, although in contrast to Zhang and Shu (2003), the Heston model appeared to perform poorly for OTM and DOTM calls.

Bhat (2019) applied the Heston model on foreign currency call options and discovered that for DOTM, OTM and ATM options, the pricing errors were larger than those for the B-S model. Furthermore, the author observed that the Heston model's pricing errors for short maturity options were more notable than the B-S model's. For options with longer maturities the Heston model performed better than the B-S model (Bhat, 2019).

Prior research on the Heston model is equivocal regarding the Heston model's performance for options at different levels of moneyness and maturity (Bhat, 2019; Karlsson, 2009; Wu, 2019; Zhang & Shu, 2003). Nevertheless, the research suggests the Heston model to generally perform better than the B-S model.

3.2 Cryptocurrency Options

I have chosen to divide the previous research into traditional asset classes and cryptocurrencies because it appears to be uncertain what to categorize cryptocurrencies as. Baur, Hong and Lee (2018) identified the return distribution for BTC to have a high level of excess kurtosis and a considerable negative skewness, which appeared to be similar to precious metals, high yield corporate bonds as well as some currencies. By contrast, Bianchi (2020) found BTC and ETH return distributions to be positively skewed and discovered a moderate correlation between the return distribution of precious metals and cryptocurrencies, yet did not find any significant relationship between cryptocurrencies and other traditional asset classes. Ram (2019) suggested BTC to be an asset class of its own, with a vague correlation to traditional asset classes.

Previous research appears to have agreed upon a high level of excess kurtosis in the return distribution for cryptocurrencies, but is dubious regarding the skewness (Baur, Hong & Lee, 2018; Bianchi, 2020). Additionally, the classification of cryptocurrencies seems to be an ongoing discussion (Baur, Hong & Lee, 2018; Bianchi, 2020; Ram, 2019).

Madan, Reyners and Schoutens (2019) conducted an empirical study where they compared multiple pricing models' performances in calculating the price of call options with BTC index as the underlying asset. The authors estimated option values using simple models, one of them being the B-S model, as well as more complex models with stochastic volatility (Heston) and models with jumps. When comparing the pricing errors, Madan, Reyners and Schoutens (2019) found that the models assuming constant volatility (B-S being one of them) performed the worst and that the Heston model along with one of the models allowing for jumps performed the best overall.

The options market for cryptocurrencies has grown considerably since Madan, Reyners and Schoutens (2019) presented their research, which puts into question whether their findings are still valid and if they can be extended to ETH options.

4. Data

4.1 Data Collection

It is, to my knowledge, not possible to download current or recent historical option data on BTC or ETH without being sponsored by or pay for a membership at the leading market players. Hence, the BTC and ETH option data was collected manually at 08:00 UTC every day for 14 days (11/10/2021-11/24/2021), from one of the leading market platforms. All the stated data on the platform was copied and pasted into an excel sheet. The format was not suited for calculations and contained multiple parameters not relevant to my research. Therefore, all data had to be reorganized into another spreadsheet and edited. Then the original spreadsheet was compared with the edited one, to minimize the risk of errors. After collection, all gathered data had to be united into yet another spreadsheet. The integrated spreadsheet was compared to the individual ones. This process was repeated for all the considered options: BTC calls, BTC puts, ETH calls and ETH puts.

The data collection was cumbersome and therefore limited data could be collected within the time-frame of the research. The final data sets constitute a very small fraction of the population, considering that the cryptocurrency option contracts trade around-the-clock for 365 days yearly. Furthermore, the collected data was restricted to European "vanilla" options, to facilitate comparison with prior research.

The historical and current values on the underlying indexes were publicly available for download on the same leading market platform from which the option data was collected. The spot index prices were gathered in tandem and the hourly spot prices for each date at 08:00 UTC were matched with the collected options data.

As described previously, the B-S model assumes a constant interest rate. The Heston model, however, may consider a varying interest rate. For the purpose of model comparison, the short-term interest rate was assumed to be constant at 5%.

4.2 Variables and Data Frame

The integrated spreadsheets of data contained time to maturity (days), strike price (USD) and market price for all options. The underlying asset for the BTC option contracts was a BTC index tracking the BTC to USD spot exchange. Similarly, the underlying asset for the ETH option contracts was an ETH index tracking the ETH to USD spot exchange. The market prices for the options were stated in BTC or ETH, therefore these prices were multiplied with the spot price of the underlying index; making all prices quoted in USD. An additional variable 'moneyness' was added to the data set, by dividing the strike price with the spot price (K/S). Hull (2018) described that moneyness, which is if often calculated this way, creates a more stable relationship to implied volatility than the strike price alone. Furthermore, all maturities were divided by 365 days, as TTM is quoted in days/year in the option pricing models.

The revised variables were added to the final data sets and all data was listed by date. Additionally, observations with no stated market prices were removed from the data frame.

5. Method

The method to be described in this section was repeated four times, to obtain results for all the considered types of options contracts. The calibrations were performed in MatLab, and further calculations were done in R studio.

5.1 Black-Scholes Calibration

To calculate the B-S option value from formula 1, the volatility of the underlying asset returns had to be estimated. A common approach for estimating the volatility is to utilize the volatility implied by the market's option price (Hull, 2018). In accordance with this approach, an arbitrary value of the volatility parameter was set into formula 1. Then an algorithm was used to minimize the squared price difference by altering the guessed volatility.

$$for\ call\ option: \quad min\ f(\sigma) = min\ [\ c_{market} - c_{B-S}(\sigma)\ ]^2$$

$$for\ put\ option: \quad min\ f(\sigma) = min\ [\ p_{market} - p_{B-S}(\sigma)\ ]^2$$

This optimization problem was repeated for each option's contract in the data set. The equation's outputs were added to the data frame as implied volatilities. By inserting the implied volatilities into formula 1, the B-S option values were calculated and added to the data frame.

5.2 Black-Scholes Assumption Evaluation

The B-S model's assumption of constant volatility means that the volatility shall not depend on either time to maturity or moneyness. To analyze whether the assumption was valid, both visual and statistical tools were utilized. A volatility surface was created by plotting the implied volatility against moneyness and time to maturity. For a constant volatility, the surface shall be flat. Additionally, a multiple regression model was arranged where the implied volatility was set as response variable, meanwhile moneyness and time to maturity were set as explanatory variables.

$$\sigma_i = \alpha + \beta_1 * TTM_i + \beta_2 * M_i + \varepsilon_i \qquad for\ i = 1,..,n$$

Before further analyzing the regression models, a Breusch-Pagan test was performed to determine if the error terms were homoscedastic (H0) or heteroscedastic (H1).

$$H_0: \ Var\ (\varepsilon_i |\ X) = \sigma(\varepsilon)^2$$

$$H_1: \ Var\ (\varepsilon_i |\ X) = \sigma(\varepsilon)_i^{\ 2}$$

If the null hypothesis could be rejected at a 1%-level, White's robust standard errors were utilized to obtain approximately correct estimates. If the null hypothesis could not be rejected, the ordinary least square estimates were computed.

To test whether the explanatory variables were significant, a Wald Chi-squared test with robust standard errors was computed with the following hypothesis:

$$H_0 : \ \beta_1 = \beta_2 = 0$$

$$H_1 : \beta_1 \neq 0 \ and/or \ \beta_2 \neq 0$$

A rejection of the null hypothesis would mean that at least one of the explanatory variables is significant and ought to be included in the regression model. In other words, that the implied volatility does depend on maturity and/or moneyness.

5.3 Heston Calibration

To calculate the Heston model's option values, the five unknowns parameters in the variance process (see section 2.3) had to be estimated. Therefore, a vector (**Ω**) constituting of the five parameters was set with starting values.

$$\Omega = \{ \theta, v(t), k, \sigma, \delta \}$$

Then, an algorithm minimized the squared difference between the market price and the Heston model's value by altering the values in the vector.

$$for\ call\ option: \ min\, f(\Omega) = min\, [\, c_{market} - c_{Heston}(\Omega)\,]^2$$

$$for\ put\ option: \ min\, f(\Omega) = min\, [\, p_{market} - p_{Heston}(\Omega)\,]^2$$

Due to the optimization problem's complexity, it required a considerable amount of computer power to be solved from a large data set. Hence, a subsample had to be assembled for the calculations. I adopted the 'trial-and-error' method combining different number of observations at different strike prices and maturities, searching for values on the parameters that would generate the least pricing errors. As this approach was highly time consuming when applying it to four option types, I settled with five different time to maturities: 1, 2, 16, 51 and 135 days. Another obstacle was that the different option types (BTC put/call, ETH put/call) did not have the same number of stated market prices for the selected maturities. Therefore, a distinction was made between calls and puts, using different number of observations from the different maturities yet the same number of observations for the whole subset.

Table 1 *Number of observations for calibration of the Heston model*

TTM	BTC call	BTC put	ETH call	ETH put
1	9	7	9	7
2	14	17	14	17
16	17	17	17	17
51	17	16	17	16
135	11	11	11	11
Total	68	68	68	68

This table reports the number of observations, for the selected maturities, that were used for each options type when calibrating the Heston model.

Furthermore, the starting values for the parameters had to be selected carefully. Crisóstomo (2014) described that the local minima which the algorithm finds highly depends on one's initial guessed values for the parameters. Crisóstomo (2014) suggested that creating a frame of reasonable values for the parameters would be of advantage when searching for the optimal solution. I therefore set up the following bounds:

- Long-run mean variance: $0 < \theta \leq 2$
- Initial variance: $0 < v(t) \leq 2$
- Mean reversion speed: $0 < k < 10$
- Volatility of the volatility parameter: $0 < \sigma \leq 10$
- Correlation: $-1 \leq \delta \leq 1$

Again, the 'trial-and-error' approach was used to test different values (within the bounds) for the parameters, to find the combination generating the smallest pricing errors. The guessed errors were altered by the algorithm in the optimization problem and the output values were used in formula 3 to solve for Heston's option values. The estimated option values were then added to the data frame.

5.4 Model Performance Evaluation

In the calibration of the models, the optimization problems were solved by a root-finding algorithm which is based on the Newton-Raphson method. This method sometimes does not converge to a solution, due to the starting value being too far from the root. Consequently, the calculations of the B-S and Heston models constituted of different number of missing values. For a more accurate comparison of the models, only options for which both models had an estimated value were used in further calculations.

To evaluate the B-S and Heston models' performances in pricing BTC and ETH options, the estimated prices were compared to the market prices by utilizing two measurements: mean absolute error (MAE) and mean squared error (MSE).

$$MAE = \frac{1}{n}\sum_{i=1}^{n} |market\ price - model\ price|$$

$$MSE = \frac{1}{n}\sum_{i=1}^{n} (market\ price - model\ price)^2$$

Both measurements give information on the goodness-of-fit and they complement each other due to their mathematical traits. MSE is a commonly used measurement, however it weights outliers heavily and thus may give a poor prediction when the data contains a lot of noise. Therefore, MAE will also be used, since it detects how close on average the model values are to the actual prices without punishing for outliers as much as MSE does.

To detect if and how the pricing errors differed depending on maturities and levels of moneyness, the data was split into subsets in accordance with table 2 and 3.

Table 2 *Groups of time to maturity*

Short TTM	Medium TTM	Long TTM
TTM < 21	21 ≤ TTM ≤ 60	TTM > 60

This table reports the boundaries, expressed in days, which were used to define three levels of maturity: short, medium and long.

Table 3 *Groups of moneyness*

	DOTM	OTM	ATM	ITM	DITM
Call	M > 1.09	1.09 ≥ M > 1.03	1.03 ≥ M ≥ 0.97	0.97 < M ≥0.91	M < 0.91
Put	M < 0.91	0.91≤ M < 0.97	0.97 ≤ M ≤ 1.03	1.03 < M ≤ 1.09	M > 1.09

This table reports the boundaries of moneyness (M=K/S) which were used to categorize the five levels of moneyness: DOTM, OTM, ATM, ITM and DITM. The boundaries are different for puts and calls, to match their different payoff functions.

6. Results and Discussion

The number of options with an estimated value by B-S and by Heston differed, due to the algorithm not being able to compute the exact same number of estimates for both models. Hence, the data frame was narrowed down by omitting all N/A, resulting in a data frame of options where both the B-S and Heston models had an estimated option value. The number of option values that both models could estimate are presented in table 4.

Table 4 *Number of observations used to compute the results*

	BTC call	BTC put	ETH call	ETH put
Observations	2378	1991	2470	2135

This table reports the number of observations for each options' type that were used when evaluating the pricing models.

As shown in table 3, there is a distinction between puts and calls regarding the levels of moneyness. The results are therefore divided into groups of puts and calls, in the purpose of reducing the risk of misconception.

6.1 BTC and ETH put options

6.1.1 Black-Scholes Implied Volatility

Table 5 shows the results from the multiple linear regression models performed on the put options. The regression models have moneyness and time to maturity as explanatory variables and implied volatility as response variable. For both the BTC and ETH options, the null hypothesis of homoscedastic error terms could be indisputably rejected. Therefore, the regression models were computed by utilizing White's robust standard errors. The coefficients for BTC were all highly significant and in addition, the Wald chi-square test reported that explanatory variables were jointly different from zero. In comparison to BTC, the coefficient for time to maturity was not as significant for ETH. The performed Wald test reported that at least one explanatory variable was significant. From the results presented in table 5, it appears that both the BTC and ETH put options' implied volatilities depend on time to maturity and moneyness, which is in opposition to the B-S assumption of constant volatility.

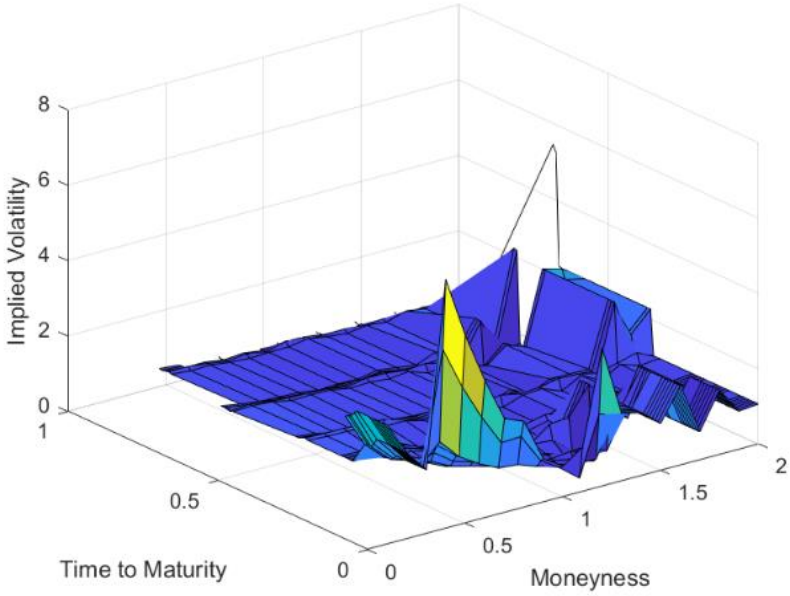
Table 5 *Regression outputs for BTC and ETH put options*

Response variable: Implied volatility	BTC put	ETH put
Constant	0.613 ***	1.096***
	(0.000)	(0.000)
Time to Maturity	-0.392 ***	-0.176**
	(0.000)	(0.009)
Moneyness	0.623***	0.348***
	(0.000)	(0.000)
Breusch-Pagan test	0.000	0.000
Wald X^2-test	0.000	0.000

This table reports the outputs from regressing implied volatility on moneyness and time to maturity, for BTC and ETH puts separately. The coefficients were computed with White's standard errors. P-value in parentheses. Breusch-Pagan and Wald X^2 statistics are stated as P-values, for P-values above 0.01 the null hypotheses are not rejected. Significance levels: *p<.05 **p<.01 ***p<.001.

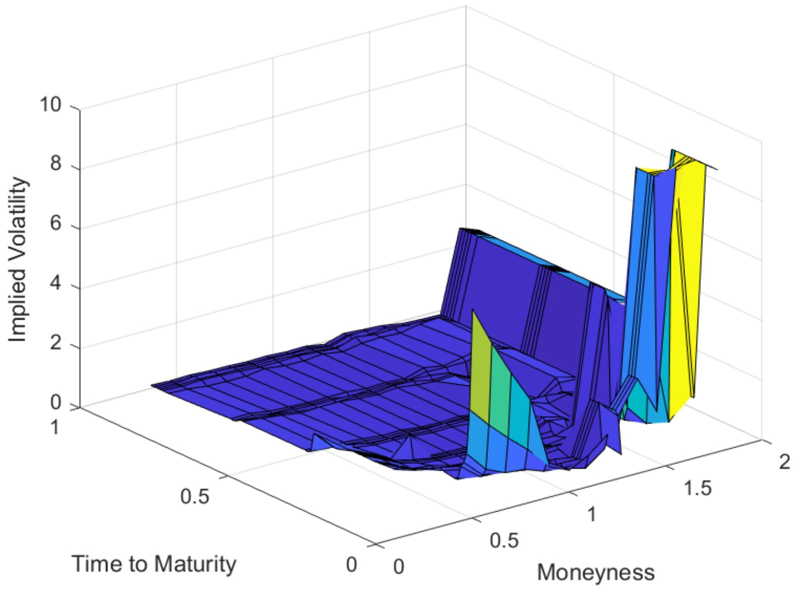
In figure 1 and 2, the volatility surfaces for ETH and BTC put options are presented. The options with moneyness above 2 have been removed, due to them being relatively few and thus resulting in a less visually representative and accurate surface. The volatility surface connects options with various levels of implied volatility, time to maturity and moneyness. For the ETH put options in figure 1, there appears to be a stable level of implied volatility for long maturity options which are ATM, OTM and DOTM. Nevertheless, a large part of the surface consists of peaks and troughs, especially so for ITM and DITM options. The highest peak in implied volatility appears to be reached by short maturity DOTM/OTM options. Similarly, the implied volatility for short maturity DOTM options is relatively high for BTC puts (figure 2), but also short maturity ITM and DITM options have high implied volatilities. For long maturity BTC puts, the implied volatility is quite stable, with the exception for DITM options.

Figure 1 *Volatility surface for ETH put options*



This figure shows the three dimensional relationships between moneyness, time to maturity and implied volatility for ETH puts. Higher implied volatilities are presented with gradually warmer colors. Only options with moneyness (K/S) below 2 are incorporated into the figure.

Figure 2 *Volatility surface for BTC put options*



This figure shows the three dimensional relationships between moneyness, time to maturity and implied volatility for BTC puts. Higher implied volatilities are presented with gradually warmer colors. Only options with moneyness (K/S) below 2 are incorporated into the figure.

6.1.2 Black-Scholes Pricing Errors

In table 6 the Black-Scholes pricing errors for BTC and ETH put options are presented by two metrics. Note that the values have been multiplied by 10^6.

When isolating the analysis to ETH put options, the reported measurements are larger the longer maturity, which implies that the distance between actual prices and estimated values increases along with maturity. When splitting the data on levels of moneyness, both MAE and MSE are the highest for ITM and DITM options. In relation, the MAE and MSE are low for DOTM and ATM options. Furthermore, OTM options appear to score the middle ranking amongst the levels of moneyness.

There seems to be a relationship between pricing errors and implied volatility. The DITM and ITM options are the most misvalued and in section 6.1.1 they were found to have the most varying implied volatility over time. ATM, DOTM and OTM options are less inaccurately valued and in the previous section they were found to have the most stable implied volatility over time, except for short maturities. The relationship between pricing errors and implied volatility appears to be inverse regarding time to maturity; since the pricing errors grew larger for longer maturities, meanwhile the levels of implied volatility decreased.

When instead isolating the analysis of table 6 to BTC put options, the MAE and MSE are larger for options with longer maturity. When splitting the pricing errors on levels of moneyness and observing the MAE and MSE scores, one can conclude that the OTM option values are closest to the true prices. In contrast, ITM and DITM options achieve the highest MAE and MSE values. Both DOTM and ATM options have relatively low pricing errors.

As discussed for ETH, the BTC DITM and ITM options are the most misvalued and in section 6.1.1 they were found to have the highest implied volatilities. The reversed relationship between time to maturity and implied volatility appears to be valid for BTC puts too, since options with longer maturities are relatively highly misvalued, yet they have the steadiest implied volatility.

Table 6 *Black-Scholes pricing errors for ETH and BTC put options*

Groups	**MAE**		**MSE**	
	BTC put	ETH put	BTC put	ETH put
Total	1018	80	11	0.045
Short TTM	468	31	2	0.006
Medium TTM	775	67	7	0.053
Long TTM	2140	156	30	0.091
ITM	834	142	7	0.074
DITM	2985	177	50	0.146
ATM	687	55	2	0.011
OTM	418	68	0.848	0.027
DOTM	530	43	2	0.014

This table reports the Black-Scholes model's pricing errors for ETH and BTC puts in the form of MAE and MSE. The results are based on the full sample (Total) as well as subsamples on maturity and moneyness (K/S). All values have been multiplied by 10^6. Values above 1 are rounded to no decimals. Values below 1 are rounded to three decimals.

6.1.3 Output From Calibration of the Heston Model

As described in the method, a 'trial and error' approach was used to find starting values for the Heston model parameters which would minimize the pricing error. Note that this procedure was executed on a small subsample consisting of 68 observations for each options type. The final guesses I settled for are shown in table 7. In the same table, the algorithm's output at a local minimum is also presented for each options type. For both BTC and ETH put options, the volatility of volatility parameter (σ) was found to be way higher than expected. In addition, the mean reversion speed (k) was quite large for the BTC put options (78.285), and extremely large for ETH put options (144.310). These outputs suggest the B-S assumption of constant volatility to be fallacious. The outputs for the parameters in table 7 were used to estimate the Heston option values for the whole data set.

Table 7 *Heston parameters for ETH and BTC put options*

Parameters	**BTC put**		**ETH put**	
	Guess	Output	Guess	Output
θ	1.200	2.515	0.650	0.667
$v(t)$	0.900	2.738	0.800	1.216
k	2.500	10.559	1.000	316.244
σ	5.000	78.285	0.900	144.310
δ	0.800	0.8129	-0.900	0.108

This table reports the guessed and optimized parameter values in the Heston model's variance process for ETH and BTC puts. The parameters are the long-run mean variance θ, the mean reversion speed *k*, the correlation δ of the Weiner processes, the current variance *v(t)* and the volatility of the volatility parameter σ. The optimized parameter values are computed by minimizing the squared difference between market price and model value.

6.1.4 Heston Pricing Errors

Table 8 presents the measures of the Heston model's pricing errors. When isolating the analysis to ETH puts, the pricing error increases along with time to maturity. Both the MAE and MSE suggest DITM and ITM options to be the most mispriced. The least mispriced options are those who are DOTM, followed by OTM and ATM.

A similar pattern in pricing errors for the different levels of moneyness and maturity can be concluded for BTC puts.

Furthermore, all measurements are larger for the BTC puts than for the ETH puts no matter what level of moneyness or maturity. This finding indicates that the Heston model predicts ETH put option values closer to their market prices, compared to the BTC puts.

Table 8 *Heston pricing errors for ETH and BTC put options*

Groups	**MAE**		**MSE**	
	BTC put	ETH put	BTC put	ETH put
Total	2022	120	34396484	205592
Short TTM	673	17	7558871	1292
Medium TTM	1629	138	25926523	231859
Long TTM	4613	244	86446943	455915
ITM	1133	57	1668265	6476
DITM	8415	541	180353533	1064942
ATM	484	23	424548	1126
OTM	530	21	850324	1206
DOTM	499	14	1353290	700

This table reports the Heston model's pricing errors for ETH and BTC puts in the form of MAE and MSE. The results are based on the full sample (Total) as well as subsamples on maturity and moneyness (K/S). All values are rounded to no decimals.

6.2 BTC and ETH call options

6.2.4 Black-Scholes Implied Volatility

Table 9 shows the results from the multiple linear regression models performed on the call options. For both BTC and ETH options, the null hypothesis of homoscedastic error terms could be rejected. Therefore, the regression models were computed by using White's robust standard errors. The coefficient for time to maturity was not significant for the BTC regression. For the regression on ETH calls, the coefficient for moneyness was not significant. On a 1%-level, the null hypothesis in the Wald test could be rejected for both regression models. Therefore, the implied volatility depended on at least one of the explanatory variables.

Table 9 *Regression outputs for BTC and ETH call options*

Response variable: Implied volatility	BTC call	ETH call
Constant	1.230 ***	1.191***
	(0.000)	(0.000)
Time to Maturity	0.024	0.059**
	(0.452)	(0.006)
Moneyness	-0.098***	0.006
	(0.000)	(0.150)
Breusch-Pagan test	0.000	0.000
Wald X^2-test	0.000	0.0017

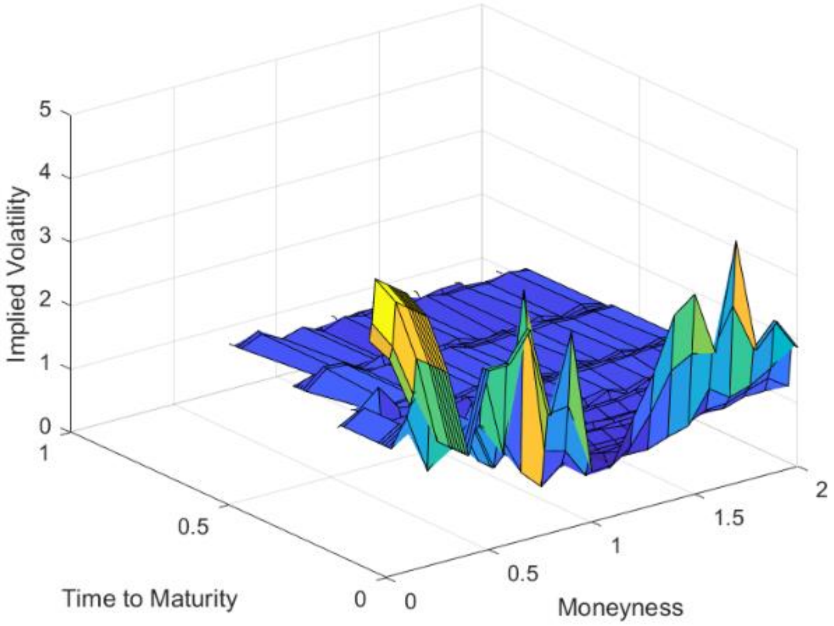
This table reports the outputs from regressing implied volatility on moneyness and time to maturity, for BTC and ETH calls separately. The coefficients were computed with White's standard errors. P-value in parentheses. Breusch-Pagan and Wald X^2 statistics are stated as P-values, for P-values above 0.01 the null hypotheses are not rejected. Significance levels: *p<.05 **p<.01 ***p<.001.

In figure 3 and 4, the volatility surfaces for ETH and BTC call options are presented. Note that options with moneyness above 2 have been removed.

For the ETH call options in figure 3, there appears to be a steady level of implied volatility for long maturity options which are ATM, OTM and DOTM. For short maturity options the implied volatility varies a great deal for all levels of moneyness, with the mere exception for options close to being OTM and ATM. The highest peaks in implied volatility appear to arise from ITM and DITM options.

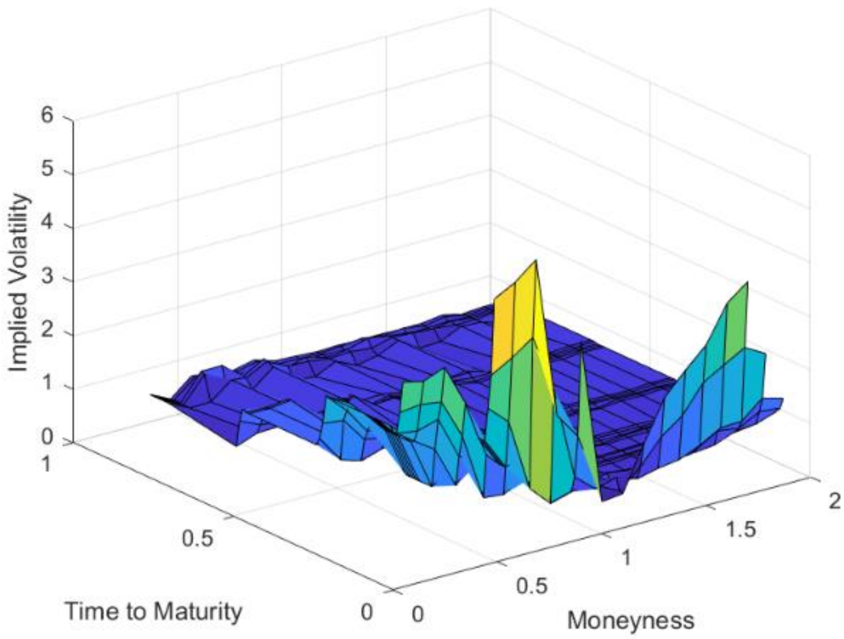
The surface for BTC calls (figure 4) is similar to the one for ETH. The implied volatility is stable for long maturity ATM, OTM and DOTM calls. Short maturity options appear to have the highest implied volatilities, except for ATM options. The highest measured implied volatility surfaces for DITM calls.

Figure 3 *Volatility surface for ETH call options*



This figure shows the three dimensional relationships between moneyness, time to maturity and implied volatility for ETH calls. Higher implied volatilities are presented with gradually warmer colors. Only options with moneyness (K/S) below 2 are incorporated into the figure.

Figure 4 *Volatility surface for BTC call options*



This figure shows the three dimensional relationships between moneyness, time to maturity and implied volatility for BTC calls. Higher implied volatilities are presented with gradually warmer colors. Only options with moneyness (K/S) below 2 are incorporated into the figure.

6.2.2 Black-Scholes Pricing Errors

Table 10 contains the measurements of pricing errors for BTC and ETH call options.

When analyzing the ETH calls alone, both MAE and MSE increase along with time to maturity. The metrics are the lowest for ATM options, followed by OTM and DITM options. The measurement scores are the highest for DOTM and ITM calls.

These findings may be traced back to the discussion in section 6.2.1. As it happens, the B-S model misvalues DOTM, ITM and DITM calls the most, and these options are also the ones with the highest peaks in the volatility surface. The reversed relationship between pricing errors and implied volatility seems to occur for time to maturity; for longer maturities the pricing errors are relatively large, yet the implied volatility is relatively stable in comparison to short maturities.

For the BTC call options in table 10, the ATM options are the most correctly valued, followed by ITM calls. Options belonging to the other levels of moneyness have relatively large pricing errors. Additionally, the pricing errors increase along with time to maturity. When comparing these findings with the ones in section 6.2.1 for BTC calls, there appears to be an inverse relationship between time to maturity for pricing errors and implied volatility. However, for the levels of moneyness, the options with the largest pricing errors seem to coincide with high levels of implied volatility found in section 6.2.1.

Table 10 *Black-Scholes pricing errors for ETH and BTC call options*

Groups	**MAE**		**MSE**	
	BTC call	ETH call	BTC call	ETH call
All	1158	109	8	0.070
Short TTM	490	31	1	0.005
Medium TTM	829	43	3	0.009
Long TTM	2036	229	17	0.171
ITM	816	146	3	0.092
DITM	1215	99	6	0.048
ATM	726	58	2	0.014
OTM	1168	80	10	0.034
DOTM	1249	122	10	0.089

This table reports the Black-Scholes model's pricing errors for ETH and BTC calls in the form of MAE and MSE. The results are based on the full sample (Total) as well as subsamples on maturity and moneyness (K/S). All values have been multiplied by 10^6. Values above 1 are rounded to no decimals. Values below 1 are rounded to three decimals.

6.2.3 Output From Calibration of the Heston Model

Table 11 shows the guessed values of the parameters as well as the algorithm's outputs at a local minimum. The guesses were relatively close to the outputs, except for the mean reversion speed (k). Furthermore, the output volatility-of-volatility parameter (σ) value was higher for ETH calls than for BTC calls. The outputs for the parameters in table 11 were used to estimate the Heston option values for the whole data sets of BTC and ETH calls respectively.

Table 11 *Heston parameters for ETH and BTC put options*

Parameters	**BTC call**		**ETH call**	
	Guess	Output	Guess	Output
θ	0.5	0.481	0.5	0.595
$v(t)$	1	1.055	0.98	2.013
k	4	19.757	5	11.299
σ	2	2.182	4	7.250
δ	-0.5	-1	-0.98	-0.030

This table reports the guessed and optimized parameter values in the Heston model's variance process for ETH and BTC calls. The parameters are the long-run mean variance θ, the mean reversion speed k, the correlation δ of the Weiner processes, the current variance *v(t)* and the volatility of the volatility parameter σ. The optimized parameter values are computed by minimizing the difference between market price and model value.

6.2.4 Heston Pricing Errors

Table 12 shows the pricing errors for the Heston model on call options. For ETH calls, the OTM and DOTM options have the highest MAE and MSE scores. The same is true for BTC calls. When comparing the BTC errors to the ones for ETH options, it becomes apparent that the pricing errors are larger for BTC options.

***Table 12** Heston pricing errors for ETH and BTC call options*

	MAE		**MSE**	
Groups	BTC call	ETH call	BTC call	ETH call
All	652	49	1302604	8198
Short TTM	438	18	1297723	1611
Medium TTM	804	28	1565748	2326
Long TTM	796	94	1167728	18391
ITM	1075	58	4251974	9537
DITM	1336	67	3323665	10739
ATM	269	24	149358	1509
OTM	215	24	196223	2919
DOTM	388	52	301538	9372

This table reports the Heston model's pricing errors for ETH and BTC calls in the form of MAE and MSE. The results are based on the full sample (Total) as well as subsamples on maturity and moneyness (K/S). All values are rounded to no decimals.

6.3 Comparative Analysis

6.3.1 Black-Scholes and Heston

The pricing errors for the B-S model (table 6 and 10) have been multiplied by 10^6, while the Heston model's metrics (table 8 and 12) have not. Comparing the values makes it clear that for both puts and calls with the different underlying cryptocurrencies, the B-S model outperforms the Heston model. These results are the complete opposite of what Madan, Reyners and Schoutens (2019) found when comparing the models' performances on cryptocurrency options. Furthermore, when splitting the samples into different maturities and levels of moneyness, the models misvalue options in a similar manner.

In section 2.1 it became clear that the Heston model is not flawless, yet it generally performs superior to the B-S model due to it capturing the non-normality of asset returns (Bhat, 2019; Karlsson, 2009; Wu, 2019; Zhang & Shu, 2003). There are multiple plausible causes as to why the presented results are in contradiction to prior research. One reasonable explanation lies in the optimization problem of the variance process' parameters; namely that the data may be noisy, and if so “the landscape of the objective function is rugged, it is neither convex nor of any regular shape” (Chen, 2007, p. 30). Consequently, the algorithm does not work efficiently or accurately when locating a local minimum and the results highly depend on what minima the algorithm finds (Chen, 2007). Additionally, what solution the algorithm finds highly depends on what input values were used for the parameters (Crisóstomo, 2014).

Another cause for the conflicting results is that the thousands of option values were estimated using the output parameter values from a very small sample. These parameters may have been a good fit for the small sample, however when applied to out-of-sample data they generated a poor fit. This issue has been recognized in previous research: “while a more complex model will generally lead to better in-sample fit, it will not necessarily perform better out of sample as any overfitting may be penalized” (Bakshi, Cao & Chen, 1997, p. 2005)

Due to the B-S model’s strong outperformance of the Heston model in this study’s results, the rest of the analysis will be focusing on the B-S model.

6.3.2 Different Underlying Assets (Black-Scholes)

To compare the B-S model’s goodness-of-fit for puts and calls, table 13 was created which summarizes the MAE and MSE results in table 6 and 10. Due to the definition of moneyness being different for puts and calls, table 13 presents the spot price in relation to the strike price and combines DOTM (DITM) scores with OTM (ITM). In table 13, the box contains a B if the BTC pricing error measurements are less than those for ETH. If the pricing errors for ETH are less than those for BTC, the box contains an E. For all levels of moneyness and maturity the B-S model predicts the values for ETH options more accurately than for BTC, since all boxes in table 13 contain an E.

The difference in model performances’ for ETH and BTC options may be due to the data on BTC options containing more noise. For some maturities, the collected data on BTC options included additional small and/or large strikes than for ETH. These contracts may not have been as liquid due to them being very DOTM/DITM; which might have affected their market prices and in turn effected the calibration of the model.

Table 13 *Comparison of pricing errors for different underlying assets*

Groups	Put Options BTC=B, ETH=E	Call Options BTC=B, ETH=E
All	E	E
Short TTM	E	E
Medium TTM	E	E
Long TTM	E	E
S > K	E	E
S ≈ K	E	E
S < K	E	E

This table reports a comparison of the MAE and MSE values for ETH and BTC, which are presented separately in table 6 and 10. The DITM (DOTM) options have been integrated with ITM (OTM) options. The right column compares these pricing error measurements for calls alone. The left column compares the pricing errors for puts. If the box contains B, then BTC's MAE *and* MSE scores are lower than ETH's. If the box contains E, the reversed is true.

6.3.3 Puts and Calls (Black-Scholes)

In table 6 and 10 the MAE and MSE results for the B-S model were presented for puts and calls respectively. Table 14 compares these results for both BTC and ETH options. If the put options had lower MAE and MSE values in comparison to calls, the box contains a P. If the call options had lower scores in comparison to puts, the box contains a C. If the MAE and MSE values were contradictory, the box contains 'N/A'. Table 14 does not include the term moneyness since the definition of moneyness differ for puts and calls.

For the ETH options, the B-S model appears to value puts the most accurately overall. For short and medium maturity options, the B-S mispriced calls less than puts. For options where the spot price was greater than the strike price, calls were the most accurately valued type of ETH option. Options with spot prices close to and below the strike prices, put options were valued closer to their market prices in comparison to calls. When incorporating the term moneyness, these results mean that the B-S model works the best for puts and calls being ITM and DITM. Meanwhile, OTM and DOTM options are the least accurately valued. Additionally, the B-S model's performance on ATM puts is superior to calls.

In the left column of table 14, the results from comparing the pricing errors for BTC puts and calls are presented. The MAE and MSE scores were inconsistent for the first three groups. For long maturity options, the B-S model values BTC calls more accurately than puts. Only when the spot price was much larger than the strike price, calls were less misvalued than puts. For the rest relationships between the spot and strike price, puts were less misvalued than calls. When re-introducing the term moneyness, these results translate to the B-S model performing better for DITM calls than DOTM puts. For all other levels of moneyness, the model values puts more accurately than calls.

Table 14 *Comparison of pricing errors for puts and calls*

Groups	**BTC Options** Put=P, Call=C	**ETH Options** Put=P, Call=C
All	N/A	P
Short TTM	N/A	C
Medium TTM	N/A	C
Long TTM	C	P
S < K	P	P
S << K	P	P
S ≈ K	P	P
S > K	P	C
S >> K	C	C

This table reports a comparison of the MAE and MSE values for calls and puts, which are reported separately in table 6 and 10. The right column compares these pricing error measurements for ETH options alone. The left column compares the pricing errors for BTC options. If the box contains P, then the puts' MAE *and* MSE scores are lower than the calls'. If the box contains C, the reversed is true. If the MAE and MSE scores are contradictory, the box contains N/A.

7. Conclusion

The aim of this paper is to determine whether the B-S model can be used to reliably predict the values for cryptocurrency options; and if not, can the Heston model be a better fit? I used real market prices to calibrate the models and then proceeded to calculate the pricing errors. Based on the results, the conclusion is that the B-S model strikingly outperforms the Heston model for both ETH and BTC calls and puts. This finding is in opposition to previous research performed by Madan, Reyners and Schoutens (2019). Interestingly, Bhat (2019), Karlsson (2009), Wu (2019) and Zhang and Shu (2003) all found the Heston model to generally perform better than the B-S model. This puts into question whether the Heston model's performance would have improved if one were to estimate the model parameters from a larger and less noisy data set and/or calibrate the model with other starting values.

Both models appear to misvalue cryptocurrency options in a similar manner; they perform poorly for long term options, DITM puts and DOTM calls. In addition, the pricing errors generated by both models were higher for BTC than for ETH options. As these tendencies were in common for the two models, the issue may be caused by illiquidity and not model structure. Hence, no certain conclusion can be drawn regarding the described tendencies.

Moreover, the B-S assumption of constant volatility was concluded to be invalid and there appeared to be a relationship between pricing errors and implied volatilities. For the different levels of moneyness, the options with large pricing errors corresponded to the options with high implied volatilities. For the different maturities, however, the options with large pricing errors had the lowest and most stable levels of implied volatilities.

In summary, even though the B-S model's assumption of constant volatility was invalid, the model was shown to be superior to the Heston model. The results brought additional light to the B-S model; it generally appeared to perform better for puts than for calls, and it estimated option values the most accurately for ETH options. All these findings are quite surprising, which creates an opening for future research to test whether the presented results are valid for a larger data set and if the Heston model's performance can be improved.

For an investor it is crucial to evaluate the price of the option since a market price too high may lead to considerable losses. This research however has shown that determining a theoretical value for cryptocurrency options is challenging and thus, it might be troublesome to hedge an exposure to cryptocurrencies by trading options. Additionally, it is difficult to gain access to recent historical data needed for calibration and evaluation of the pricing models. This especially effects small traders who are not backed by an institution/company with access to the data, wherefore a disparity may appear between the leading insiders and the losing outsiders.

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