Thompson's Groups

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Abstract

In this thesis a fairly self-contained introduction to Thompson's groups and a few of its related groups are given. The groups F, T and $PLF(\mathbb{R})$ and some of their subgroups are discussed extensively. Finally, some interesting topics one could study after reading this thesis are summarized.

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1 Introduction

In 1965 Richard Thompson discovered the groups F, T and V in connection with his research in logic. Part of the initial interest in these groups was the fact that T and V gave the first examples of infinite simple finitely presented groups.

Since then the groups in question have developed a life of its own being studied using tools from a wide range of areas in mathematics. The main reason as to why we can study Thompson's groups using a wide variety of tools is that these groups can be viewed naturally in multiple ways.

First, the groups can be studied purely algebraically, and has both finite and infinite presentations which we can use as we desire. Second, the elements of the groups can be viewed as different types of functions mapping the real line to itself. This view allows us to study the properties of these groups using purely analytic and dynamic tools. Third, we can study these groups as maps between finite binary rooted trees. This view allows us to use combinatorial and topological results to study this group.

In this thesis we will first introduce Thompson's group F studying its algebraic, analytic and combinatorial aspects extensively. We will give both the finite and infinite presentations and proceed to show many interesting results regarding the commutator subgroup of F as well.

Next we will discuss Thompson's group T. Our primary topic on T will be its finite presentation and simplicity. Following this discussion we will study a family of groups which were motivated by Thompson's groups F and T. In the final section we will discuss some results which we could not include here but would be interesting to study for anyone who reads this thesis.

The prerequisites required to read this thesis are minimal. A basic understanding of analysis and group theory are sufficient to understand the core material covered. In order to understand a few of the remarks a little topology could also help, however none of this will be central to the thesis.

2 Preliminaries

2.1 Free groups and group presentations

Central to our discussion on Thompson's groups is the idea of free groups and group presentations. We discuss this idea in the following sections.

Definition 2.1. Given a group G and a subset S of G, the subgroup generated by S is the smallest subgroup of G that contains S. This is denoted as $\langle S \rangle_G$. We say that S generates G if $\langle S \rangle_G = G$.

Definition 2.2. A *free group* F with *basis* X is a group F with subset X such that for every group G and map $f : X \to G$, there exists a unique group homomorphism $\bar{f} : F \to G$ which extends f.

This definition is summarized by the following diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & G \\ F \arrow[ur, dashed] & \end{tikzcd}$$

This definition does not show that such an object actually exists. So in the following sections we will give an explicit construction for a group and show that it is indeed a free group.

Definition 2.3. Given a set X we define the set X^{-1} to be the set which contains for all $x \in X$ an element x^{-1}. A *word* on X is a sequence $(x_1, x_2, \dots)$ where $x_i \in X \cup X^{-1} \cup \{1\}$ and $x_i \neq 1$ for only finitely many terms. The word where all $x_i = 1$ is called the empty word.

We typically denote the empty word by e and any arbitrary word as

$$x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$$

where all $x_i \neq 1$, and $\epsilon_i = \pm 1$. We also define concatenation of words $a = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$, $b = y_1^{\mu_1} y_2^{\mu_2} \cdots y_m^{\mu_m}$ to be the word

$$ab = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots x_n^{\epsilon_n} y_1^{\mu_1} y_2^{\mu_2} y_3^{\mu_3} \cdots y_m^{\mu_m}.$$

Definition 2.4. Let A be the set of all words on X. We define $F(X)$ to be $A/\sim$ where $\sim$ is the equivalence relation generated by $xss^{-1}y \sim xy$ and $xs^{-1}sy \sim xy$ for all words x, y and all letters s.

This set of equivalence classes along with the operation induced by concatenation forms a group where $[e]$ is the identity, and $[x_n^{-\epsilon_n} \cdots x_3^{-\epsilon_3} x_2^{-\epsilon_2} x_1^{-\epsilon_1}]$ is the inverse of $[x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots x_n^{\epsilon_n}]$. Associativity is inherited from the associativity of concatenation.

Theorem 2.5. [18, Theorem 2.2.7] *For a given set X the group $F(X)$ is a free group with basis X.*

Proof. We first let i be the map from X to $F(X)$ such that $x \mapsto [x]$. Let G be an arbitrary group and f a map from X to G. Define $f^* : A \to G$ inductively as follows:

$$\begin{aligned} e &\mapsto 1_G \\ xw &\mapsto f(x)f^*(w) \\ x^{-1}w &\mapsto f(x)^{-1}f^*(w) \end{aligned}$$

where x is an arbitrary letter and w a word. This induces a map $\bar{f} : F(X) \to G$ which is well-defined as f^* is compatible with the equivalence relation $\sim$. The map $\bar{f}$ is also clearly a group homomorphism such that $\bar{f} \circ i = f$. As $i(X)$ generates $F(X)$ there can be no other such homomorphism.

All that remains is to show that i is injective. Let $x_1, x_2 \in X$ and $f : X \to \mathbb{Z}$ be the map such that $x_1 \mapsto 1$ and $x_2 \mapsto -1$. Further, let $\bar{f} : F(X) \to \mathbb{Z}$ be the corresponding homomorphism. Then,

$$\begin{aligned} \bar{f}(i(x_1)) &= f(x_1) = 1 \\ \bar{f}(i(x_2)) &= f(x_2) = -1 \end{aligned}$$

This implies that $i(x_1) \neq i(x_2)$. Hence, we can identify X with $i(X)$, and we get that $F(X)$ is a free group. $\square$

Note that a free group of a particular basis is unique up to isomorphism as if there were two different free groups F, F',we could find two homomorphisms from F to F' and vice-versa using the definition of free groups which would act as inverses to each other.

Theorem 2.6. [21, Corollary 11.3] *Every group is isomorphic to the quotient of a free group.*

Proof. Let G be a group. Let $X = \{x_g \mid g \in G\}$ and $F(X)$ its free group. Let $f : X \to G$ be the map such that $x_g \mapsto g$. Then the definition of free groups imply that there exists a homomorphism from $F(X)$ to G. This is surjective as f is, and the first isomorphism theorem gives us the result. $\square$

Definition 2.7. Given a group G and set S, we define the normal subgroup generated by S to be the smallest normal subgroup that contains S. This group is denoted by $\langle\langle S \rangle\rangle$.

Definition 2.8. Given a set X and a set R be a subset of all words on X. Let $F(X)$ be the free group generated by X. We define the group generated by S with relations R, denoted by $\langle X \mid R \rangle$ to be defined as follows.

$$\langle X \mid R \rangle := F(X)/\langle\langle R \rangle\rangle$$

If $G \cong \langle X \mid R \rangle$, we say that $\langle X \mid R \rangle$ is a presentation of G.

2.2 Groups of bijections

Given any set S it is clear that the set of bijections of S forms a group. By a group of bijections we will mean a subgroup of any such group. Given a group G we let G' denote the commutator subgroup of G. In this section G will always be a group of bijections

Definition 2.9. Let G be the group of bijections of a set S. We then define the *support* of an element $g \in G$ as follows:

$$\operatorname{supp}(g) = \{s \in S \mid g(s) \neq s\}.$$

Now we will discuss some results that we will require later on in this thesis. The first of these is a remarkable theorem due to Graham Higman. We follow the proof and discussion of this theorem as done by Burillo [9, pp. 42–44] with slight changes in presentation. The following hypothesis will be repeatedly used in proving the theorem and for this reason we define it separately as follows.

Definition 2.10. Let α and β be two elements of G and let $S = \operatorname{supp}(\alpha) \cup \operatorname{supp}(\beta)$ and let $\gamma \neq 1$ be a third element in G. If we can find an element $\rho \in G$ such that $\gamma(\rho(S)) \cap \rho(S) = \emptyset$ then we say group G satisfies the *Higman condition*.

Our goal is to prove that if G satisfies the Higman condition then G' is simple. In order to do this we first need the following lemma.

Lemma 2.11. *Suppose G satisfies the Higman condition, and let N be a non-trivial normal subgroup of G. Then $G' \leq N$.*

Proof. There is a non-trivial element γ in N as N is non-trivial. Let α and β be arbitrary elements in G. Using the Higman condition let ρ be an element such that $\gamma(\rho(S)) \cap \rho(S) = \emptyset$. This implies that $\gamma(\rho(\operatorname{supp}(\alpha))) \cap \rho(\operatorname{supp}(\beta)) = \emptyset$. Let $\delta = \rho^{-1}\gamma\rho \in N$. Then we get that $\delta(\operatorname{supp}(\alpha)) \cap \operatorname{supp}(\beta) = \emptyset$. As $\delta(\operatorname{supp}(\alpha)) = \operatorname{supp}(\delta\alpha\delta^{-1})$ we have that $\delta\alpha\delta^{-1}$ and β commute. Hence, we get that,

$$\begin{aligned}
\alpha^{-1}\beta^{-1}\alpha\beta &= \alpha^{-1}(\delta\alpha\delta^{-1})\beta^{-1}(\delta\alpha^{-1}\delta^{-1})\alpha\beta \\
&= (\alpha^{-1}\delta\alpha)\delta^{-1}(\beta^{-1}\delta\beta)(\beta^{-1}\alpha^{-1}\delta^{-1}\alpha\beta) \in N.
\end{aligned}$$

Therefore, we get that $[\alpha, \beta] \in N$ for all $\alpha, \beta \in G$. $\square$

Lemma 2.12. *If a group G satisfies the Higman condition then $G' = G''$.*

Proof. We first begin by noting that G'' is normal in G as $a[g', h']a^{-1} = [ag'a^{-1}, ah'a^{-1}] \in G''$ if g', h' are in G'. We will now show that $G' = G''$. If G' is trivial then this statement is obvious. Hence, suppose G' is not trivial. Then if we show that G'' is non-trivial we are done as the previous lemma gives us that $G' \leq G''$.

To do this we consider a non-trivial element a of G', and let $\alpha = \beta = \gamma = a$. Using the same method that we used in the previous lemma we get an element

$\delta \in G'$ such that $\mathrm{supp}(\delta a \delta^{-1}) \cap \mathrm{supp}(a)$ is empty. This means that G' has two different elements with disjoint supports. This implies that $\delta a \delta^{-1} a^{-1} \neq 1$. Hence, we have found a non-trivial element in G'' and the fact that $G'' = G'$ follows. $\square$

Theorem 2.13. *If a group G satisfies the Higman condition then G' is simple.*

Proof. In order to prove this we will apply Lemma 2.11 to G' in order to show that $G'' \leq N$ for all non-trivial normal subgroups N of G'. As $G' = G''$ by Lemma 2.12, it will follow that G' is simple. In order to prove this we have to show that G' satisfies the Higman condition.

Let $\alpha, \beta \in G'$ and γ a non-trivial element in G'. Then the Higman condition gives us an element $\rho \in G$, such that $\rho(S) \cap \gamma(\rho(S)) = \emptyset$ where $S = \mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$.

Now we let $S' = \mathrm{supp}(\gamma) \cup \mathrm{supp}(\rho)$ and use the Higman condition again on the elements γ, ρ and γ to find $\sigma \in G$ such that $\sigma(S') \cap \gamma(\sigma(S')) = \emptyset$.

Let $\nu = \sigma\gamma\sigma^{-1}$ and use the argument used in Lemma 2.11 in order to get $\mathrm{supp}(\gamma) \cap \mathrm{supp}(\nu^{-1}\rho\nu) = \emptyset$. We get that $\nu^{-1}\rho\nu$ is the identity on $\rho(S)$ as $\rho(S) \subseteq \mathrm{supp}(\gamma)$ and this implies that $\rho(S) = \nu^{-1}\rho^{-1}\nu\rho(S)$. As $\epsilon = \nu^{-1}\rho^{-1}\nu\rho \in G'$ we have found an element $\epsilon \in G'$ such that $\epsilon(S) \cap \gamma(\epsilon(S)) = \emptyset$. Hence, we have that G' also satisfies the Higman condition, and we are done. $\square$

Next we give one final lemma which will be useful in discussing groups that occur naturally as subgroups of Thompson's group F.

Lemma 2.14. [7, Lemma 1.2] *Given a set A, let S_A be the group of all permutations of A. Let X be a subset of S_A such that all $f \in X$ have infinite order, and any two distinct elements f and g have disjoint supports. Then X generates a free abelian subgroup of S_A.*

Proof. As distinct elements of X have disjoint supports, it is clear that X generates an abelian group.

Let $F(X)$ be the group generated by X. Every $f \in F(X)$ can be written as $g_1^{a_i} \cdots g_n^{a_n}$, where $g_i \in X$ and $g_i \neq g_j$ if $i \neq j$. If $f = 1$, we have that each $a_i = 0$ due to the fact that g_i has infinite order for each i. Hence $F(X)$ is a free abelian group. $\square$

2.3 Rewrite systems

Definition 2.15. A *rewrite system* $\mathcal{T}$ consists of a set $X = \mathrm{obj}(\mathcal{T})$, a set $Y = \mathrm{moves}(\mathcal{T})$ and two maps,

$$\begin{aligned} Y &\to X \times X, \quad y \mapsto (\mathfrak{i}(y), \tau(y)) \\ Y &\to Y, \qquad\quad y \mapsto y^{-1}, \end{aligned}$$

where for all $y \in Y$ we have that $(y^{-1})^{-1} = y$, $y^{-1} \neq y$ and $\mathfrak{i}(y) = \tau(y^{-1})$. Here $\mathfrak{i}$ and τ are called the *initial* and *terminal* objects respectively.

An element in X is called an *object* and an element in Y is called a *move*. A *derivation* is either an object or a sequence of moves $e_1e_2e_3\cdots e_n$ such that $\tau(e_i) = \mathfrak{i}(e_{i+1})$. An *orientation* is a subset of moves Y_0 such that $Y = Y_0 \cup Y_0^{-1}$ and $Y_0 \cap Y_0^{-1} = \emptyset$. Moves of Y_0 are called positive moves and moves of Y_0^{-1} are called negative moves.

If we have a positive move e with $\mathfrak{i}(e) = a$ and $\tau(e) = b$ we say that a can be positively rewritten to b and denote it as $a \rightarrowtail b$. If there is a derivation with only positive moves from a to b it is called a positive derivation and denoted $a \to b$. If there is a derivation with either negative or positive moves from a to b (and hence from b to a as well) we write $a \leftrightarrow b$.

A rewrite system is *terminating* if every positive derivation

$$a_1 \to a_2 \to a_3 \to \dots$$

stabilizes after a finite number of moves. Terminating systems are also referred to as well-founded systems. This is due to the fact that a relation is terminating if and only if well-founded induction holds [1, pp. 13–14].

A rewrite system is *confluent* if for all a, b, c such that $a \to b$ and $a \to c$ there is an element d such that $b \to d$ and $c \to d$. A rewrite system is *locally confluent* if for all a, b, c such that $a \rightarrowtail b$ and $a \rightarrowtail c$ there is an element d such that $b \to d$ and $c \to d$.

Clearly every confluent system is locally confluent, but a locally confluent system is not necessarily confluent. Nevertheless, we have the following.

Lemma 2.16. *If we have that $a \leftrightarrow b$ in a confluent rewrite system, there exists an object c such that $a \to c$ and $b \to c$.*

Proof. We give a sketch of a proof in the form of Figure 1 following that given by Sapir [22, Lemma 1.7.8]. We can consider negative moves as going up a slope and positive moves as going down a slope. Then as there are a finite number of moves from a to b we can use the local peaks and confluence to create elements at lower levels than the elements at troughs. Each colour of Figure 1 represents one such layer of moving to a lower level. In this way we can reduce each of the peaks and make one deep canyon. ☐

In a terminating confluent system, Lemma 2.16 combined with the terminating nature implies that there is a unique terminal object.

Unfortunately it is not always easy to show that a rewrite system is confluent directly. Nevertheless, showing local confluence tends to be much easier to show, and the following theorem helps us in showing that our rewrite system is confluent if we have a terminating, locally confluent rewrite system.

Theorem 2.17. [1, Lemma 2.7.2] ***(Newman's diamond lemma)*** *A terminating locally confluent system is confluent.*

Proof. We prove this statement by well-founded induction on the following statement $P(x)$. We need the system to be terminating in order for well-founded induction to hold.

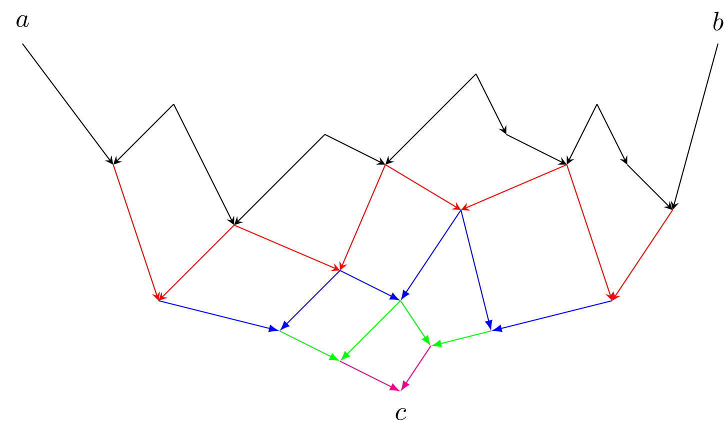

Figure 1: Reducing peaks

Let $P(x)$ be the statement that given objects x_1 and x_2 such that $x \to x_1$ and $x \to x_2$ there is an element y such that $x_1 \to y$ and $x_2 \to y$.

Let a be an arbitrary object in our rewrite system, and suppose that the above statement is true for all objects that can be obtained by positive paths from a. Let b and c be elements such that $a \to b$ and $a \to c$. Then let b' and c' be the elements obtained by one positive move on the derivations from a to b and c respectively ($a \rightarrowtail b' \to b$ and $a \rightarrowtail c' \to c$). Then local confluence implies that there exists a u such that $b' \to u$ and $c' \to u$. Our induction hypothesis implies that there is an element v such that $c \to v$ and $u \to v$. This gives us derivations $b' \to v$ and $b' \to b$. Again we use the induction hypothesis to get an element d such that $v \to d$ and $b \to d$. As $c \to v$ we have proven the statement for a. As the statement is trivially true for any terminal object we are done. □

2.4 Cayley graphs

Definition 2.18. Let G be a group given as the quotient of a free group on a set S. The word length $l_s(\gamma)$ of a word γ is the smallest n such that $\gamma = s_1 s_2 \cdots s_n$ where $s_i \in S$. The word metric $d_S(\gamma_1, \gamma_2)$ is defined as $l_S(\gamma_1^{-1}\gamma_2)$. In this way the group G can be given a metric structure.

Note that in thinking of a group as a metric space in this way the distance between two elements is the number of elements from the set of generators required to move from one group element to the next. Motivated by this we get the following definition for the Cayley graph of a group whose metric coincides with the above metric.

Definition 2.19. Given a group G and a generating set S of G define the *Cayley graph* denoted by $\mathrm{Cay}(G, S)$ to be the graph whose set of vertices are the elements of G and whose edges are the elements (g, gs) where $g \in G$ and $s \in S$.

Definition 2.20. The *growth function* of a Cayley graph maps $k \in \mathbb{N}$ to the number of elements at a distance less than or equal to k. That is $\beta_{(G,S)} : \mathbb{N} \to \mathbb{N}$ such that $k \mapsto |B_r^{(G,d_S)}(e)|$, where $B_r^{(G,d_S)}(e) = \{g \in G \mid d_S(g,e) \leq r\}$. A *generalised growth function* is a non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$.

A growth function β of a group can be studied as a generalised growth function by defining $\alpha(t) = \beta(\lfloor t \rfloor)$.

Definition 2.21. We say that a generalised growth function α_1 *weakly dominates* α_2 if there is a $\lambda \geq 1$ and $C \geq 0$ such that

$$\alpha_1(t) \leq \lambda \alpha_2(\lambda t + C) + C$$

for all $t \in \mathbb{R}^+$.

Two growth functions are weakly equivalent if they weakly dominate each other. Weak equivalence can be shown to be equivalence relation on all growth functions.

Definition 2.22. Let $(X,d), (X',d')$ be two metric spaces then we say that $f : X \to X'$ is a *quasi-isometric embedding* if there is a $\lambda \geq 1$ and $C \geq 0$ such that

$$\frac{1}{\lambda} d(x,y) - C \leq d'(f(x), f(y)) \leq \lambda d(x,y) + C$$

for all $x, y \in X$.

A map $f' : X \to X'$ is said to be a *finite distance* from f if there is a $C \geq 0$ for all $x \in X$ we have that $d'(f(x), f'(x)) \leq C$.

We say that f is a *quasi-isometry* if there is a quasi-isometric embedding $g : X' \to X$ such that $f \circ g$ and $g \circ f$ are finite distance from the respective identity maps on the metric spaces.

Being quasi-isometric can be shown to be an equivalence relation on metric spaces.

Lemma 2.23. [18, Proposition 6.2.4] *Let G, H be groups and S, T be finite generating sets for them respectively. Then if there exists a quasi-isometric embedding $(G, d_S) \to (H, d_T)$ we get that $\beta_{(G,S)}$ is weakly dominated by $\beta_{(H,T)}$.*

Proof. Let $f : G \to H$ be a quasi-isometric embedding then we have the following for some $\lambda \geq 1$ and $c \geq 0$.

$$\frac{1}{\lambda} d_S(g, g') - c \leq d_T(f(g), f(g')) \leq \lambda d_S(g, g') + c.$$

Then we get that if $g \in B_r^{(G,d_S)}(e)$,

$$d_T(f(g), e') \leq \lambda d_S(g, e) + c \leq \lambda r + c.$$

This implies that $f(B_r^{(G,d_S)}(e)) \subseteq B_{\lambda r + c}^{(H,d_T)}(e')$.

Further if $f(g) = f(g')$, we get that

$$d_S(g, g') \leq \lambda d_T(f(g), f(g')) + \lambda c = \lambda c.$$

We use these two estimates to get,

$$\beta_{(G,S)}(r) = |B_r^{(G,d_S)(e)}| \leq \beta_{(G,S)}(\lambda c)\beta_{(H,T)}(\lambda r + c).$$

Hence, the theorem follows. $\square$

This implies that quasi-isometric groups are weakly equivalent.

Lemma 2.24. [18, Proposition 5.2.5] *Let G be a finitely generated group and S, S' be two generating sets for it. Then the two Cayley graphs viewed as metric spaces (G, d_S) and (G, d'_S) are quasi-isometric.*

Proof. We will show that the identity map gives us a quasi-isometry from (G, d_S) to $(G, d_{S'})$.

Let $c = \max\{d'_S(e, s) \mid s \in S\}$ and let g, h be two elements in G. Then $g^{-1}h = s_1 s_2 \cdots s_n$. From this we obtain

$$\begin{aligned}
d_{S'}(g, h) &= d_S(g, gs_1 \cdots s_n) \\
&\leq d_{S'}(g, gs_1) + d_{S'}(gs_1, gs_1s_2) + d_{S'}(gs_1s_2, gs_1s_2s_3) + \cdots \\
&\quad + d_{S'}(gs_1 \cdots s_{n-1}, gs_1 \cdots s_n) \\
&= d_{S'}(e, s_1) + d_{S'}(e, s_2) + \cdots + d_{S'}(e, s_n) \\
&\leq cn = cd_S(g, h).
\end{aligned}$$

Similarly, we can show the reverse as well. Hence, we get that it is a quasi-isometry. $\square$

Definition 2.25. We define the growth rate of a group G with finite generating set S and growth function β as,

$$w(G, S) = \limsup_{k \to \infty} \beta^{\frac{1}{k}}.$$

If $w(G, S) > 1$, we say that the group has *exponential growth.* If $w(G, S) = 1$, we say that it has *sub-exponential growth.* If a group has sub-exponential growth and its growth function $\beta(k)$ is weakly dominated by a function k^d, we say that the group has *polynomial growth.* If the group has sub-exponential growth but is not polynomial we say that it has *intermediate growth.*

It is clear that if two growth functions are weakly equivalent they will have the same growth rate. Therefore, we get that growth type is a quasi-isometric invariant, and we can ignore the generating set when talking about the growth type of finitely generated groups. Recall that a semigroup is an algebraic structure similar to that of a group but where we relax the conditions on existence of inverse and identity elements. A monoid is a semigroup which has an identity element.

Theorem 2.26. [15, Proposition p.187] *A finitely generated group which contains a free semigroup on two generators is of exponential growth.*

Proof. We can talk about growth functions of semigroups similar to the way we do in groups. Then it is clear that the growth function β of a semigroup of 2 generators is as follows:

$$\beta(k) = 1 + 2 + \cdots + 2^k \geq 2^k$$

If a group contains a subsemigroup of 2 generators, we can get a generating set of the group that contains the generating set of the subsemigroup. Then the growth function of the group will be greater than $\beta(k)$, as the distance between two elements of a subsemigroup will always be greater than the distance between the elements in the group. Hence, we get that the growth rate is exponential. $\square$

3 Thompson's group F

In this chapter we begin our discussion on Thompson's groups by talking about Thompson's group F. We first give a few different descriptions of the group before proceeding to discuss some of its interesting properties. Further, we look at the commutator subgroup of F, and discuss some of its interesting properties. The material in this chapter can be found in the paper by Cannon, Floyd and Parry [11, pp. 2–10], as well as in the book by Burillo [9, pp. 9–46].

3.1 Different descriptions of F

From this section onwards, we write function composition as $fg(x) = g(f(x))$ as it is more natural when talking about groups. A dyadic rational number is a rational of the form $\frac{k}{2^n}$, and a dyadic interval is an interval of the form $[\frac{a-1}{2^n}, \frac{a}{2^n}]$, where $a \leq 2^n$. A dyadic partition is one where all its dividing partitions are dyadic intervals.

Definition 3.1. We define Thompson's group F as the subgroup of homeomorphisms of $[0,1]$ which have the following properties:

- The elements are piecewise linear.
- When the elements are differentiable the derivatives are powers of 2.
- There are finitely many breakpoints, and they are dyadic rationals in $[0,1]$.

We will now verify that this is indeed a group. Let the break points be $0 < x_1 < x_2 < \cdots < x_n < 1$.

When $x \in [0, x_1]$, we have that $f(x) = 2^{a_1}x$ which implies that $f(x_1)$ is a dyadic rational. Similarly, if $x \in [x_1, x_2]$, we have that $f(x) = 2^{a_2}x - 2^{a_2}x_1 + f(x_1)$. This implies that $f(x_2)$ is a dyadic rational as well. Inductively it follows that $f(x_i)$ is a dyadic rational for all i. From this it is clear that compositions of such maps, and inverses of such maps will also be in F. Associativity is inherited from that of function composition.

Remark 3.2. We can use the above to represent any element of f by listing their break points in the form

$$\{(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_{n-1}, f(x_{n-1}))\}.$$

Note that giving pairs of dyadic rationals which preserve order is not sufficient to define elements of F as the gradients may not always be a power of 2.

Example 3.3. Consider the function given by the breakpoints $\{(\frac{1}{2}, \frac{3}{4})\}$. This will not be an element of F as the derivative when $x \in (0, \frac{1}{2})$ is $\frac{3}{2}$.

Theorem 3.4. [9, Theorem 1.1.6] *The group F is torsion free.*

Proof. Let f be any non-identity element, and x_1 the first breakpoint such that the right derivative is not 1 (the right derivative here will be 2^k where $k \neq 0$). Then at x_1 we have that f^n will have right derivative 2^{nk}. Which means it cannot be the identity unless $n = 0$. Hence, the group F is torsion free. □

Before we give any examples for elements of this group we will give an alternate description for Thompson's group F in terms of binary rooted trees. Naturally, in order to do this we first need to discuss the idea of binary rooted trees.

Definition 3.5. A *binary rooted tree* is a tree such that:

- It has a root v_0.
- If v is a vertex then there are exactly two edges, the left edge $e_{v,l}$ and the right $e_{v,r}$, which contain v but are not in the geodesic from v_0 to v.
- Vertices with valence 0 or 1 are called leaves.

We give such a tree's leaves a natural ordering from left to right starting from 0.

Definition 3.6. We now define the *standard tree of dyadic intervals* (Figure 2) as follows:

- The vertices are the standard dyadic intervals in $[0, 1]$.
- An edge is a pair of dyadic intervals (I, J) such that I is either the left half or the right half of J. We then call the edges the left or right edges respectively.

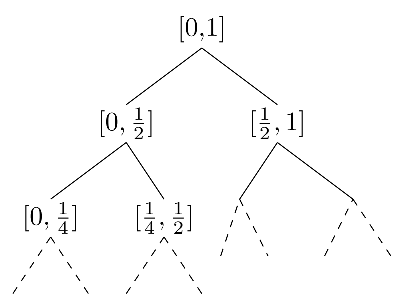

Figure 2: The standard tree of dyadic intervals

Any finite subtree of this tree which contains the root $[0, 1]$ and where all other non-leaf vertices has three edges gives us a dyadic partition of $[0, 1]$. In practice, we will draw any finite subtree as seen in Figure 3.

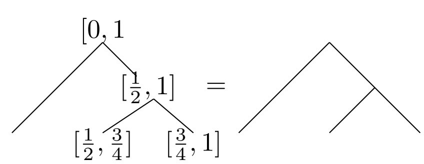

Figure 3: How we draw finite trees.

We also define a caret to be a subtree which has a root and just two edges.

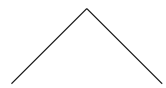

Figure 4: A caret

Theorem 3.7. [11, Lemma 2.2] *For every $f \in F$ there is a dyadic partition $0 = x_0 < x_1 < \cdots < x_n = 1$ such that f is linear on each $[x_i, x_{i+1}]$ and $0 = f(x_0) < f(x_1) < \cdots < f(x_n) = 1$ is a dyadic partition of $[0,1]$.*

Proof. Let P be a partition of $[0,1]$ whose breakpoints are dyadic rationals and f is linear on each interval of P. Let $[a,b]$ be one such interval and suppose that the derivative of f there is 2^{-k}. Then $f(x) = 2^{-k}x + c$ on $[a,b]$, and let $b = a + \frac{r}{2^n}$ for some r. We then have that $f(a) = 2^{-k}a + c$ and $f(b) = 2^{-k}b + c = f(a) + \frac{r}{2^{n+k}}$. Hence, we can get dyadic partitions $a < a + \frac{1}{2^n} < \cdots < b$ and $f(a) < f(a) + \frac{1}{2^{n+k}} < \cdots < f(b)$. As we can do this for all intervals we get a dyadic partition for $[0,1]$. $\square$

As each standard dyadic partition has a corresponding tree diagram we can describe each f as a tree diagram (R,S), where R and S have the same number of leaves. We call R the domain tree and S the target tree.

Example 3.8. Now we give two examples for elements in the group F. Later we will also show how these two elements are sufficient to generate the group F. The tree diagrams for the two elements are given in Figure 5 and 6

$$a(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{4} \\ x + \frac{1}{4}, & \frac{1}{4} \le x \le \frac{1}{2} \\ \frac{x}{2} + \frac{1}{2}, & \frac{1}{2} \le x \le 1 \end{cases} \qquad b(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 2x - \frac{1}{2}, & \frac{1}{2} \le x \le \frac{5}{8} \\ x + \frac{1}{8}, & \frac{5}{8} \le x \le \frac{3}{4} \\ \frac{x}{2} + \frac{1}{2}, & \frac{3}{4} \le x \le 1 \end{cases}$$

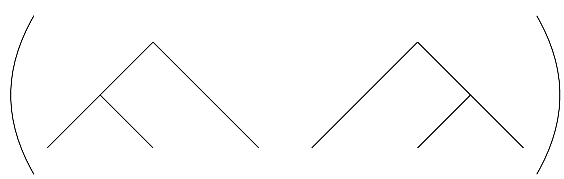

Figure 5: $a(x)$

For a given f this tree diagram (R,S) is not unique. We can take the nth leaves of R and S and add carets to these leaves in order to expand the trees and get new tree diagrams. Conversely, if the nth and $n+1$st leaves of the two trees in a tree diagram have the same parent vertex, we can remove the carets containing the nth and $n+1$st leaves in both the range tree and domain tree in order to get a different tree diagram which correspond to the same element in F

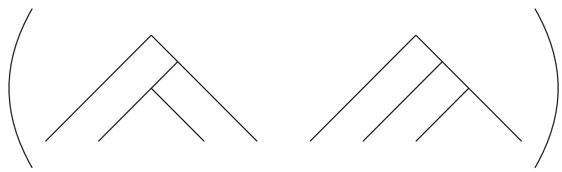

Figure 6: $b(x)$

as before. This process of adding and removing leaves is called *expansion* and *reduction* respectively. If no reductions are possible we call the tree diagram a reduced tree diagram.

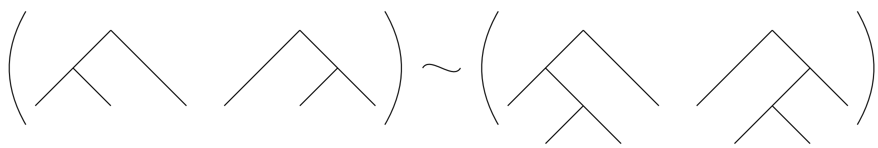

Figure 7: Reduction and Expansion.

It is interesting to note that what we have described here is an example for a rewrite system where the objects are tree diagrams, and expansions and reductions are negative and positive moves respectively.

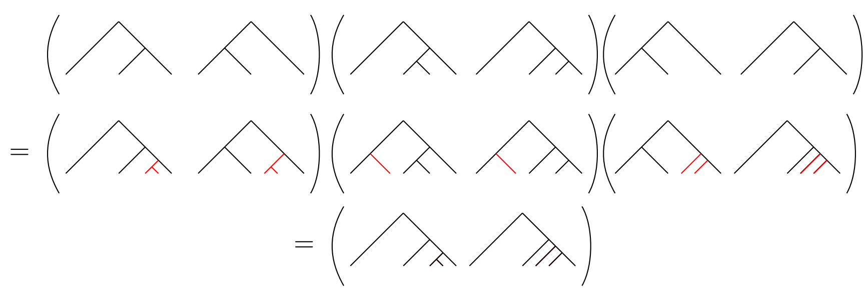

Figure 8: An example for multiplication.

This method of adding or removing leaves is important when multiplying elements of F given as tree diagrams. If we have two elements $f_1 = (R, S)$ and $f_2 = (R', S')$ we would like $f_1 \cdot f_2 = (R, S')$. Unfortunately this does not always make sense as R and S' might not have the same number of leaves. We can rectify this by extending S and R' so that they are the same tree and extending R and S' as required. This can be seen in the example computation of the tree diagram for the element $a^{-1}ba$ (Figure 8).

Lemma 3.9. *Reduced tree diagrams are unique.*

Proof. The idea for this proof closely follows the proof given by Matucci [19, Proposition 2.1.1] for strand diagrams.

We prove this using Newman's diamond lemma (Lemma 2.17). As any tree diagram is built of finite trees, the rewrite system is clearly terminating. Further, given a tree if we can reduce it in two different ways, the carets involved in the different reductions will always be disjoint. This means that reducing in whichever order gives us the same tree diagram, and our system is locally confluent. Hence, we can use Newman's diamond lemma which will imply uniqueness. □

We call a tree *all-right* if all its leaves except for the left-most descend directly from the right edge of the tree. A positive element is a reduced tree diagram whose range tree is an all-right tree.

Definition 3.10. Define the elements $T_n \in F$ as the elements with the following breakpoints (using the notation discussed in Remark 3.2):

$$\left[\left(1-\frac{1}{2^n}, 1-\frac{1}{2^n}\right), \left(1-\frac{3}{2^{n+2}}, 1-\frac{1}{2^{n+1}}\right), \left(1-\frac{1}{2^{n+1}}, 1-\frac{1}{2^{n+2}}\right)\right]$$

A general T_n is given in Figure 9. Here the n leaves are just leaves coming from the right branch of the tree.

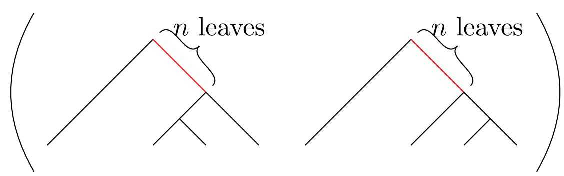

Figure 9: General example for T_n.

Lemma 3.11. [9, pg 23-24] *For all $k < n$, we have that*

$$T_k^{-1} T_n T_k = T_{n+1}.$$

Proof. This can be done using multiplication of trees for the cases $n = k+1$ and $n \geq k+2$. We give the case where $n \geq k+2$ in Figure 10. □

Definition 3.12. Given a tree with $n+1$ leaves ordered from 0 to n, let a_i be the maximal number of left edges ascending from leaf i that does not reach the right side of the tree. Then a_i is called the ith leaf exponent. An example for this is given in Figure 11

Theorem 3.13. *Any positive element $f \in F$ with tree diagram (R, T_n) can be written as*

$$T_0^{a_0} T_1^{a_1} \cdots T_n^{a_n}.$$

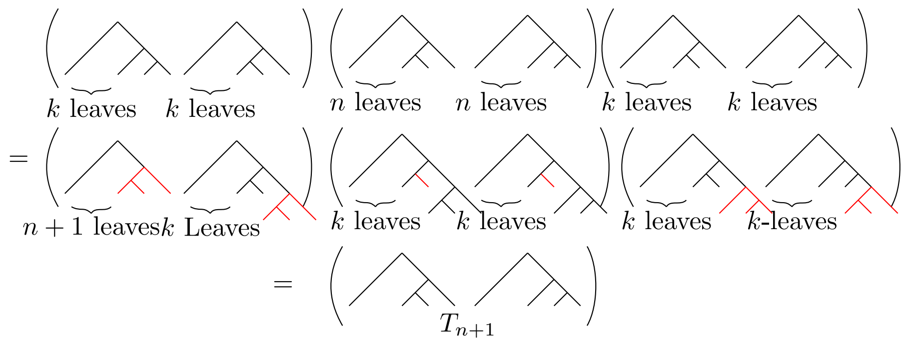

Figure 10: The case when $n \geq k+2$.

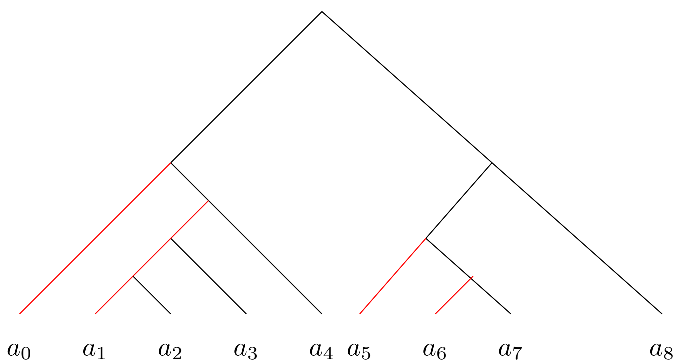

Figure 11: Tree with exponents $a_0 = 1$, $a_1 = 3$, $a_5 = 1$, $a_6 = 1$ and the other leaf exponents are 0.

Proof. The proof is by induction on $a = \sum_{i=0}^{n} a_i$. The statement is trivially true for $a = 0$ as R would have to be T_n as well. Hence, suppose the statement is true for $k-1$ and f is an element such that $k = \sum_{i=0}^{n} a_i$. Let m be the smallest number such that $a_m \neq 0$. We then multiply on the left by T_m^{-1} which gives the following tree diagram which has the same leaf exponents for all leaves except for the leaf exponent a_m which becomes $a_m - 1$. Hence, it follows from the induction hypothesis that this has form $T_m^{a_m-1} T_{m+1}^{a_{m+1}} \cdots T_n^{a_n}$. Hence, we get that the statement is true for f. $\square$

The above theorem was given by Burillo [9, Theorem 2.2.4], and the proof follows that given by Cannon, Floyd and Parry [11, Theorem 2.5].

Given an arbitrary element $f = (R, S)$ with $n+1$ leaves, we can write it as the product of two trees $(R, T_n)(T_n, S)$. Hence, we can write any element f as follows:

$$f = T_0^{a_0} T_1^{a_1} \cdots T^{a_n} T_n^{-b_n} \cdots T_0^{-b_0}$$

From this we can see that the elements $\{T_0, T_1, \ldots\}$ generate F. Unfortunately this form is not unique, but we can make it unique by the addition of one extra condition.

Theorem 3.14. [9, Theorem 2.3.3] *Every element of F admits a form*

$$f = T_0^{a_0} T_1^{a_1} \cdots T_n^{a_n} T_n^{-b_n} \cdots T_0^{-b_0},$$

where for all i, if a_i and b_i are simultaneously non-zero then a_{i+1} or b_{i+1} (or both) are non-zero as well. We call this the normal form of the normal form of f. Further, this form is unique and is the shortest expression in generators of F.

Proof. Suppose there exist elements of F which have multiple normal forms. Then there exists an element f with a pair of normal forms which has minimal total length. Let the two normal forms be

$$T_0^{a_0} T_1^{a_1} \cdots T_n^{a_n} T_n^{-b_n} \cdots T_0^{-b_0}$$

and

$$T_0^{c_0} T_1^{c_1} \cdots T_n^{c_n} T_n^{-d_n} \cdots T_0^{-d_0}$$

and let k be the smallest number such that T_k has non-zero exponent in at least one of the two normal forms. Then we have that the only generator in the above normal forms that affects the gradient on the right of $(1 - \frac{1}{2^k}, 1 - \frac{1}{2^k})$ is T_k, and the gradient there is given by $2^{a_k - b_k} = 2^{c_k - d_k}$. This implies that $a_k - b_k = c_k - d_k$. If $a_k - b_k = c_k - d_k > 0$, we would have that $a_k, c_k > 0$. This would mean we could cancel f_k in the two normal forms to get new normal forms with shorter length. This would contradict the minimality. Similarly, if $a_k - b_k = c_k - d_k < 0$ would contradict minimality of lengths as well. Hence, we get that $a_k - b_k = c_k - d_k = 0$.

Without loss of generality we can assume $a_k = b_k \neq 0$ and $c_k = d_k = 0$. Hence, we get that $f = T_k z T_k^{-1} = w$ where z, w are normal forms and w contains only generators with index greater than k in it. Hence, we can write $z = T_k^{-1} w T_k$. We can then use Lemma 3.11 to get $z = \bar{w}$ where $\bar{w}$ is just the normal form w with all T_i replaced by T_{i+1}. Hence, we arrive at a new pair of normal forms with shorter total length. Again we obtain a contradiction and there must be a unique normal form. This must be the shortest as existence of a shorter one would contradict the uniqueness. ☐

This proof is not strictly necessary for us to prove this theorem. It would be sufficient to note that this extra algebraic condition is the same as saying that the tree diagram obtained is reduced, by constructing the tree with the correct leaf exponents. However, the proof itself is interesting as it shows how it is useful to be able to switch freely between algebraic, combinatorial and analytic aspects of the group F.

3.2 Different presentations of F

In this section we will give two different presentations of the group F. The first will be an infinite presentation which is intuitive to understand but has an infinite presentation. Further, the tree diagrams T_0 and T_1 are sufficient to generate F, and this clearly shows that F is finitely generated. What is more surprising is the fact that F is finitely presented as well. In this section we will give a finite presentation for the group F.

Theorem 3.15. [9, pp. 23–29] *The group G given by the infinite presentation*

$$\langle x_0, x_1, x_2, \ldots \mid x_k^{-1} x_n x_k = x_{n+1}, \quad \forall k < n \rangle$$

is isomorphic to F.

Proof. We define a homomorphism from G to F by the map $x_i \mapsto T_i$. This is well defined by Lemma 3.11. As the elements T_i generate F this map is surjective.

If we take the image of any word in G we get an element in F which we can reduce to a unique normal form by using Lemma 3.11. We can then use the same corresponding relations in G to transform that the initial word we used to the word corresponding to the unique normal form. This uniqueness implies injectivity and this map is an isomorphism. □

In order to show that the group F is finitely presented we follow the proofs given by Burillo [9, Theorem 3.13] and Floyd, Cannon and Parry [11, Theorem 3.1].

Theorem 3.16. *The groups given by the following presentations are isomorphic:*

$$F_1 = \langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle$$
$$F_2 = \langle x_0, x_1, x_2, \ldots \mid x_k^{-1} x_n x_k = x_{n+1} \quad \forall k < n \rangle$$

Proof. The idea of this proof is to define two surjective homomorphisms from the generators of F_1 to F_2 and vice-versa. In order to ensure that these are well defined we will show that the defining relations are in the respective kernels of the homomorphisms.

First we define a homomorphism from F_1 to F_2 that such that $a \mapsto x_0$ and $b \mapsto x_1$. Note that as $x_{n+1} = x_0^{-n} x_1 x_0^n$, we have that $a^{-n}ba^n \mapsto x_{n+1}$, for all $n \geq 2$ which in turn implies that this map is surjective.

Next we show that this map satisfies the relations of F_1:

$$[ab^{-1}, a^{-1}ba] \mapsto [x_0 x_1^{-1}, x_2] = x_0 x_1^{-1} x_2 x_1 x_0^{-1} x_2^{-1} = x_0 x_3 x_0^{-1} x_2^{-1} = 1$$
$$[ab^{-1}, a^{-2}ba^2] \mapsto [x_0 x_1^{-1}, x_3] = x_0 x_1^{-1} x_3 x_1 x_0^{-1} x_3^{-1} = x_0 x_4 x_0^{-1} x_3^{-1} = 1$$

The final equalities follow from the relations on F_2. Hence, we have a well defined surjective homomorphism from F_1 to F_2.

Next we show that there is a surjective homomorphism from F_2 to F_1. Naturally we define this such that $x_0 \mapsto a$, $x_1 \mapsto b$. This is sufficient to extend this map to the rest of the group, and it follows that $x_{n+1} \mapsto a^{-n}ba^n$.

We now need to show that this map satisfies the relations in order for this map to be well defined. In order to make the computations clear we define

$y_0 = a$ and $y_n = a^{-(n-1)}ba^{(n-1)}$. This means that what we need to show is that $y_k^{-1}y_ny_k = y_{n+1}$ for all $k < n$. Note that this clear for all n when $k = 0$.

In order to do this we first show that $[a^{-1}b, y_m] = 1 \implies y_k^{-1}y_ny_k = y_{n+1}$ for all $m = n - k + 2$. We have

$$\begin{aligned}
y_ny_k &= y_0^{-(n-1)}y_1y_0^{(n-1)}y_0^{-(k-1)}y_1y_0^{k-1} \\
&= y_0^{-(k-2)}\big(y_0^{-(n-k+1)}y_1y_0^{(n-k+1)}\big)y_0^{-1}y_1y_0^{k-1} \\
&= y_0y_0^{-(k-1)}\big(y_ma^{-1}b\big)y_0^{k-1} = y_0y_0^{-(k-1)}\big(a^{-1}by_m\big)y_0^{k-1} \\
&= \big(y_0^{-(k-1)}y_1y_0^{k-1}\big)y_0^{-(k-1)}y_my_0^{(k-1)} \\
&= y_k\big(y_0^{-n}y_1y_0^{n}\big) = y_ky_{n+1}.
\end{aligned}$$

Hence, if we show $[a^{-1}b, y_m] = 1$ for all $m \geq 3$ we would be done.

When $m = 3$,

$$\begin{aligned}
[ab^{-1}, a^{-1}ba] = 1 &\implies ab^{-1}a^{-1}baba^{-1}a^{-1}b^{-1}a = 1 \\
&\implies b^{-1}aa^{-2}ba^2a^{-1}ba^{-2}b^{-1}a^2 = 1 \\
&\implies [b^{-1}a, y_3] = 1 \\
&\implies [a^{-1}b, y_3] = 1.
\end{aligned} \tag{1}$$

When $m = 4$,

$$\begin{aligned}
[ab^{-1}, a^{-2}ba^2] = 1 &\implies ab^{-1}a^{-2}ba^2ba^{-1}a^{-2}b^{-1}a^2 = 1 \\
&\implies b^{-1}aa^{-3}ba^3a^{-1}ba^{-3}b^{-1}a^3 = 1 \\
&\implies [b^{-1}a, y_4] = 1 \\
&\implies [a^{-1}b, y_4] = 1.
\end{aligned} \tag{2}$$

As the statement is true for $m = 3$, we have that $y_{n-1}^{-1}y_ny_{n-1} = y_{n+1}$ for all n. We can now use this along with an induction argument to show that $[a^{-1}b, y_m] = 1$ for all $m \geq 3$. Indeed,

$$\begin{aligned}
[a^{-1}b, y_n] &= a^{-1}by_{(n-2)}^{-1}y_{(n-1)}y_{(n-2)}b^{-1}ay_{(n-2)}^{-1}y_{(n-1)}^{-1}y_{(n-2)} \\
&= a^{-1}by_{(n-2)}^{-1}y_{(n-1)}\big(y_{(n-2)}b^{-1}ay_{(n-2)}^{-1}\big)y_{(n-1)}^{-1}y_{(n-2)} \\
&= a^{-1}by_{(n-2)}^{-1}\big(y_{(n-1)}b^{-1}ay_{(n-1)}^{-1}\big)y_{(n-2)} \\
&= a^{-1}b\big(y_{(n-2)}^{-1}b^{-1}ay_{(n-2)}\big) \\
&= a^{-1}bb^{-1}a = 1.
\end{aligned}$$

Here we use $[a^{-1}b, y_{n-1}] = 1$ and $[a^{-1}b, y_{n-2}] = 1$ in order to prove $[a^{-1}b, y_n] = 1$ which is why we prove $[a^{-1}b, y_m] = 1$ when $m = 3$ and $m = 4$ separately.

Hence, it follows that F_1 and F_2 are isomorphic. □

This finite presentation for F that we obtain is useful to us, as it allows us to talk about the Cayley graph of the group F in a very reasonable manner. In this direction there is quite a remarkable result due to Fordham [14] which gives us a way of finding the exact length of an element in F, by counting the number of different types of caret pairs in the reduced tree diagram for that element.

Further, there is a topological generalisation of finite generation and presentation of groups, and it was shown by Brown and Geoghegan [8] that Thompson's group F was the first torsion free FP_∞ group (finitely generated groups are FP_1 and finitely presented groups are FP_2, and a group is FP_∞ if it is FP_n for all n). However, this result is beyond the author's current knowledge, and we will not delve into it further in this thesis.

3.3 Further properties of F

In this section we will look at how F acts on the dyadic rationals denoted by $\mathbb{Z}[\frac{1}{2}]$. In order to do this first we will discuss the idea of transitive actions.

First recall that we say that a group action $G \times X \to X$ is *transitive* if for all $x, y \in X$ there is a $g \in G$ such that $(g, x) \mapsto y$. Given such an action there is a natural extension of this action to an action on the Cartesian product X^n where the action acts pointwise. We consider the action of G on the subset of X^n which consists of n-tuples of distinct elements of X. If this action is transitive we say that G is *n-transitive* on X. We say that G is *highly transitive* if X is infinite and G is n-transitive on X for all $n \in \mathbb{N}$.

Similar to the above we consider the action of G on the subsets of X with n elements. We denote this collection of subsets as $X^{\{n\}}$. Then this action is called *n-homogeneous* if G acts transitively on $X^{\{n\}}$. We say that it is *highly homogeneous* if X is infinite and G is n-homogeneous on X for all $n \in \mathbb{N}$. By these definitions it is clear that n-transitive implies n-homogeneous.

By the definition of F it is clear that the elements of F act on the set $\mathbb{Z}[\frac{1}{2}] \cap [0, 1]$. Further, we can see that the orbit of the elements 0 and 1 are the singleton sets $\{0\}$ and $\{1\}$ respectively. Hence, it is clear that this action is not transitive nor homogeneous. As orbits of an action are disjoint, we can restrict this action to an action on $\mathbb{Z}[\frac{1}{2}] \cap (0, 1)$. As elements of F are always increasing it is clear that this action is not n-transitive for all $n \geq 2$. Using the next lemma we will show that this action is n-homogeneous for all n.

Lemma 3.17. [11, Lemma 4.2] *If $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ and $0 = y_0 < y_1 < \cdots < y_{n-1} < y_n = 1$ are partitions of $[0, 1]$ consisting of dyadic rationals, then there exists $f \in F$ such that $f(x_i) = y_i$ for all $i \in \{0, 1, \ldots, n\}$.*

Proof. Let $m \in \mathbb{Z}$ be such that $2^m x_i$ and $2^m y_i$ are integers for all i. Let $R = S$ be the tree where each leaf corresponds to an interval of $\frac{1}{2^m}$. Note that x_i and y_i appear as endpoints and start points of intervals except for the points 0 and 1. We can use this to extend R and S as follows to construct the required f.

Consider the leaves of I_1 and I_2 of R where x_i is the left endpoint and x_{i+1} is the right end point, and the leaves J_1 and J_2 of S where y_i is the left endpoint and y_{i+1} is the right end point. Without loss of generality suppose that the number of leaves appearing in R between the leaves I_1 and I_2 is greater than the number of leaves between J_1 and J_2. Then we can add carets to leaves between J_1 and J_2 until there are the same number of leaves between I_1 and I_2 and J_1 and J_2. We can map these leaves linearly in an order preserving manner to get the function f that we desire. $\square$

Corollary 3.18. *The group F acts highly homogeneously on $\mathbb{Z}[\frac{1}{2}] \cap (0,1)$.*

Proof. The previous lemma gives that F acts n-homogeneously on $\mathbb{Z}[\frac{1}{2}] \cap (0,1)$ for all n. $\square$

Define $F[a,b]$ to be the subgroup of F whose elements have support in $[a,b]$. Our next goal is to show that $F[a,b]$ is isomorphic to F if a and b are dyadic rationals in $[0,1]$. In order to do this we first prove it in the special case when $a-b$ is a power of 2.

Lemma 3.19. [11, Lemma 4.4] *Let a,b be dyadic rational numbers such that $a-b$ is a power of 2. Then $F \cong F[a,b]$.*

Proof. We define $\phi : [a,b] \to [0,1]$ as the map $x \mapsto \frac{1}{b-a}x - \frac{a}{b-a}$. Then $\phi^{-1}(x) = (b-a)x + a$. We use this to define an isomorphism from F to $F[a,b]$ that maps $f \mapsto \phi f \phi^{-1}$.

We see that $\phi f \phi^{-1}$ is in $F[a,b]$ as it maps dyadic rationals to dyadic rationals and the derivative is $f'(\phi(x))$ whenever the derivative exists which is a power of 2. This can be easily seen to be a group isomorphism. $\square$

From this we see that F is isomorphic to $F[0,\frac{1}{2}], F[\frac{1}{2},\frac{3}{4}]$ and $F[\frac{1}{2},1]$. We will use these three subgroups of F to show that $F \cong F[a,b]$ for arbitrary dyadic rationals a and b.

Theorem 3.20. [9, Theorem 3.1.3] *Let a and b dyadic rationals in $[0,1]$. Then $F \cong F[a,b]$.*

Proof. We split the proof into three different cases. First we consider the case where $a \neq 0$ and $b \neq 1$. Then we can find an element $\alpha \in F$ such that $\alpha(\frac{1}{4}) = a$ and $\alpha(\frac{3}{4}) = b$. Now we define an isomorphism from $F[\frac{1}{4},\frac{3}{4}]$ to $F[a,b]$ which maps f to$\alpha^{-1} f \alpha$.

Similarly, if $a = 0$ we can use of $F[0,\frac{1}{2}]$ and if $b = 1$ use $F[\frac{1}{2},1]$. In this way we get that $F[a,b] \cong F$. $\square$

As elements with disjoint supports commute we can also use this to show that F has finite products of itself as a subgroup of itself. Further it also contains $\bigoplus_{i=1}^{\infty} F$ in itself. As any element has at most finitely many breakpoints it cannot contain an infinite product of itself. Similarly, we can construct free abelian groups of arbitrary finite order contained in F using Lemma 2.14. Hence, F also contains $\bigoplus_{i=1}^{\infty} \mathbb{Z}$.

Our next goal is to show that F has exponential growth.

Lemma 3.21. [2, Proposition 1.5.9] *The submonoid of F which is generated by x_0^{-1} and x_1 is free.*

Proof. Take any word x generated by x_0^{-1} and x_1.

$$x = x_1^{a_1} x_0^{-1} x_1^{a_2} \cdots x_0^{-1} x_1^{a_n}$$

where $a_i \geq 0$ for all i.

Then we can move all the x_0^{-1} to the right, by using the fact that $x_0^{-1} x_i = x_{i+1} x_0^{-1}$. Then we get

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} x_0^{-(n-1)}.$$

This normal form is in the unique form, so different words in x_0^{-1} and x_1 correspond to different words in F. Hence, the submonoid is free. □

Corollary 3.22. *The group F has exponential growth.*

Proof. This follows from the fact that F has a free submonoid generated by two generators and Lemma 2.26. □

3.4 Properties of F'

In this section we will study this group as well as some related groups. We will have two main goals in this section. The first is to show that F' is simple and, the second is to show that every quotient group of F is abelian. In order to prove these we will begin by proving a few interesting properties of F.

Theorem 3.23. [9, Theorem 4.1] *Let $[F, F]$ be the commutator subgroup of F. Then,*

$$F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Further the commutator subgroup contains elements of F which have support contained in a proper subset of $[0, 1]$.

Proof. We define a map $\phi : F \to \mathbb{Z} \oplus \mathbb{Z}$ such that for $f \in F$,

$$\phi(f) = (m, n)$$

where the right derivative of f at 0 is 2^m and the left derivative at 1 is 2^n. This is a homomorphism and is surjective as $\phi(a) = (1, -1)$ and $\phi(b) = (0, -1)$.

Next, we proceed to show that $[F, F] = \mathrm{Ker}(\phi)$. Clearly $[F, F] \subseteq \mathrm{Ker}(\phi)$ as

$$\phi(xyx^{-1}y^{-1}) = \phi(x) + \phi(y) - \phi(x) - \phi(y) = 0.$$

In order to show $\mathrm{Ker}(\phi) \subseteq [F, F]$ we consider $f \in \mathrm{Ker}(\phi)$. We can write $f = a^{i_1} b^{j_1} a^{i_2} \cdots a^{i_k} b^{j_k}$ where only i_1 and j_k are allowed to be 0.

Then we have that,

$$\begin{aligned} \phi(f) = {} & \phi(a)^{(i_1 + \cdots + i_k)} + \phi(b)^{(j_1 + \cdots + j_k)} \\ & = (i_1 + \cdots + i_k)(1, -1) + (j_1 + \cdots + j_k)(0, -1) = 0. \end{aligned}$$

This implies that $i_1 + \cdots + i_k = j_1 + \cdots + j_k = 0$. We can now use induction on k to prove that $f \in [F, F]$.

Clearly this is true for $k = 1, 2$. Suppose it is true for $k = n-1$. Then it follows that,

$$\begin{aligned} a^{i_1}b^{j_1}\cdots a^{i_n}b^{j_n} =& a^{i_1}b^{j_1}\cdots a^{i_{n-1}+i_n}b^{j_{n-1}+j_n} \\ & \times (b^{-(j_{n-1}+j_n)}a^{-i_n}b^{j_{n-1}+j_n}a^{i_n})(a^{-i_n}b^{-j_n}a^{i_n}b^{j_n}). \end{aligned}$$

From the induction hypothesis it follows that $a^{i_1}b^{j_1}\cdots a^{i_{n-1}+i_n}b^{j_{n-1}+j_n} \in [F, F]$ and hence $f \in [F, F]$. From the first isomorphism theorem we are done.

The fact that the support of any element in $[F, F]$ is contained strictly in [0,1] can be seen clearly by the fact that any element must be the identity for some neighborhood of 0 and 1. □

Theorem 3.24. *The group F has trivial center.*

Proof. Let p be in $[0, 1]$. Then p can be written in the two following ways:

$$p = \sum_{i=1}^{\infty} \frac{a_i}{2^i} = 1 - \sum_{i=1}^{\infty} \frac{b_i}{2^i},$$

where $a_i, b_i \in \{0, 1\}$ and $A_n = \sum_{i=1}^{n} \frac{a_i}{2^i} < 1 - \sum_{i=1}^{n} \frac{b_i}{2^i} = B_n$.
Define the family of functions $f_n \in F$ as follows:

$$f_n(x) = \begin{cases} 2x, & 0 \le x \le \frac{A_n}{4}, \\ x + \frac{A_n}{4}, & \frac{A_n}{4} \le x \le \frac{A_n}{2}, \\ \frac{x}{2} + \frac{A_n}{2}, & \frac{A_n}{2} \le x \le A_n, \\ x, & A_n \le x \le B_n, \\ 2x - B_n, & B_n \le x \le B_n + \frac{1-B_n}{4}, \\ x + \frac{1-B_n}{4} & B_n + \frac{1-B_n}{4} \le x \le B_n + \frac{1-B_n}{2}, \\ \frac{x}{2} + \frac{1}{2} & B_n + \frac{1-B_n}{2} \le x \le 1. \end{cases}$$

Note that these functions have support in $[0, 1] \backslash [A_n, B_n]$ with $f_n(p) = p$.

Let f be any element in the center of F. Then we have that $f(p) = f(f_n(p)) = f_n(f(p))$ for all n. Then $f(p) \in [A_n, B_n]$ for all n. This implies that $f(p) = p$ and as we can do this for an arbitrary $p \in [0, 1]$ it follows that f is the identity map. Therefore, the center of F is trivial. □

It is interesting to note that Burillo gives a purely algebraic proof for Theorem 3.24 [9, Proposition 3.3.5].

Lemma 3.25. [9, Proposition 3.3.2] *Let F'' be the double commutator of F. Then $F' = F''$.*

Proof. This argument relies on the fact that the support of elements in F' are proper subsets of $(0, 1)$ (Theorem 3.23). Let $f \in F'$ and $[a, b]$ be its support. Let c, d dyadics such that $0 < c < a$, and $b < d < 1$. Then $f \in F[a, b] \subset F[c, d] \subset F'$. However $F[a, b] \subset F'[c, d] \subset F''$ as $[a, b] \subset [c, d]$. Hence, we have that $f \in F''$. From this we get that $F'' = F'$. □

Theorem 3.26. [9, Theorem 3.3.1] *The commutator subgroup F' is simple.*

Proof. We will show that F' satisfies the Higman condition which will imply that F'' is simple. As we have shown that $F' = F''$ this will imply that F' is simple.

Let $\alpha, \beta \in F'$ and let $S = \operatorname{supp}(\alpha) \cup \operatorname{supp}(\beta)$. As $\alpha, \beta \in F'$ we have that $S \subseteq [\epsilon, 1-\epsilon]$ for some $\epsilon > 0$. Let γ be a non-trivial element of F'. Then there is an interval I such that $\gamma(I) \cap I = \emptyset$. Then the highly homogeneous properties of F (Corollary 3.18) allows us to construct an element $\rho \in F'$ that takes $[\epsilon, 1-\epsilon]$ into I. Hence F' satisfies the Higman condition and the theorem follows. $\square$

Theorem 3.27. [9, Theorem 3.3.6] *Every proper quotient group of F is abelian.*

Proof. Let F/N be a proper quotient of F and x a non-trivial element of N. As F has a trivial center there is an element $y \in F$ such that $yxy^{-1}x^{-1} \neq 1$. As $yxy^{-1} \in N$ and $x \in N$ we have that $[y, x] \in N \cap F'$. Hence, we have that $N \cap F'$ is a non-trivial normal subgroup of F'. As F' is simple we have that $N \cap F' = F'$ which implies that $F' \subset N$ and F/N is abelian. $\square$

4 Thompson's group T

In the group F we took two binary rooted trees with the same number of leaves and mapped them to each other in such away that the map preserved the index of the leaf when ordered from left to right. In Thompson's group T we do the same thing, but we allow cyclic permutations of the leaves in order to obtain a larger group which contains F as a subgroup of it.

4.1 Describing T

To begin with we define the circle $S^1 = [0, 1]/\sim$ where $\sim$ denotes the identification of the points 0 and 1.

Definition 4.1. We define Thompson's group T to be the subgroup of homeomorphisms of S^1 with the following properties:

- All elements are piecewise linear.
- When differentiable, the derivatives are a power of 2.
- Dyadic rationals are mapped to dyadic rationals and there are only finitely many break points.

As with the group F it can be shown that this is indeed a group when defined as above. We can also view elements of T as tree diagrams and unique reduced tree diagrams, the only difference being that we now have to explicitly define which leaf of the range tree the first leaf of the domain tree maps to. We denote this leaf with a bold vertex. We can number the leaves in the range tree cyclically and map the leaves such that the map preserves the ordering. When multiplying it is also important to preserve this order as seen in Figure 13.

Example 4.2. As an example for an element in T which is not in F we give the following.

$$c(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & 0 \le x \le \frac{1}{2} \\ x + \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - \frac{3}{2}, & \frac{3}{4} \le x \le 1 \end{cases}$$

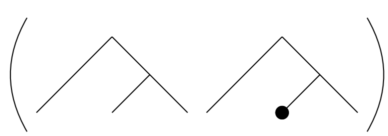

Figure 12: Tree diagram for $c(x)$.

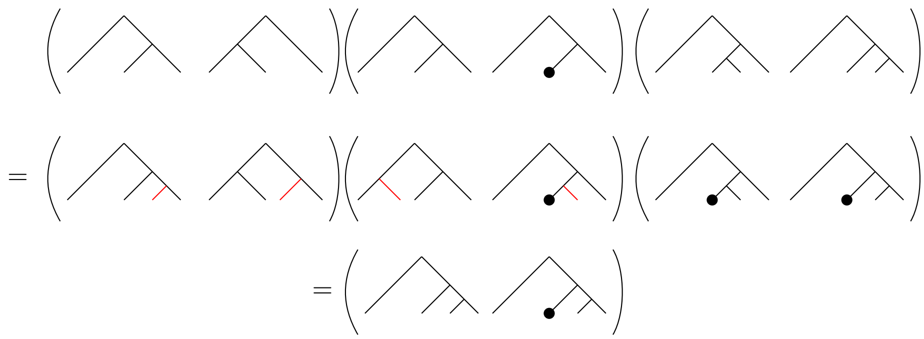

Figure 13: Example for multiplication: $a^{-1}cb$.

4.2 A presentation for T and its simplicity

There are two main goals for this section. We will give a finite presentation for T and show that T is simple. We will mostly follow the discussion of this done by Cannon, Floyd and Parry [11, Chapter 5].
To do this we first define the group T_1 as follows:

$$T_1 = \langle a, b, c \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2], c^{-1}b(a^{-1}cb), ((a^{-1}cb)(a^{-1}ba))^{-1}b(a^{-2}cb^2), (ca)^{-1}(a^{-1}cb)^2, c^3\rangle$$

While this is a presentation of T, in order to show this we will first define a surjective group homomorphism from T_1 to T. After which we will show that T_1 is a simple group. From this it will follow that $T \cong T_1$, and hence T is also simple.

Lemma 4.3. [11, Lemma 5.2] *The group T is generated by the elements a, b, c and satisfy the following relations:*

(i) $[ab^{-1}, a^{-1}ba] = 1$

(ii) $[ab^{-1}, a^{-1}ba] = 1$

(iii) $c = b(a^{-1}cb)$

(iv) $(a^{-1}cb)(a^{-1}ba) = b(a^{-2}cb^2)$

(v) $ca = (a^{-1}cb)^2$

(vi) $c^3 = 1$

Proof. Let H be the subgroup of T generated by a, b and c. Then H contains F as F is generated by a and b. Let f be an arbitrary element in T. If $f([0]) = [0]$ then f is an element of F and hence H. So suppose that $f([0]) = [x]$ such that $[x] \neq [0]$. Then there is $h \in F$ such that $h(x) = 3/4$. Then the map $g = fhc$ is such that $g([0]) = c(h(f([0]))) = c(h([x])) = c([\frac{3}{4}]) = [0]$. This gives that g is in F which in turn gives that $h = f^{-1}gc^{-1} \in H$. From this we see that $T = H$ and so T is generated by a, b and c.

Next we check if the relations hold. The first two are clear as they follow from the finite presentation of F. We can use tree diagrams to show the rest, and we give one such computation in Figure 14. In this we use the previously computed tree diagram for $a^{-1}cb$ in Figure 13. $\square$

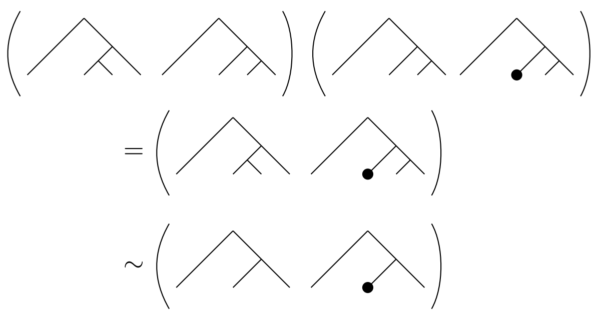

Figure 14: $c = b(a^{-1}cb)$.

As a consequence of Lemma 4.3 the following holds.

Corollary 4.4. [11, Lemma 5.3] *There is a surjective homomorphism from T_1 to T.*

We now proceed to show that T_1 is simple. In order to do this we first define the following elements in T_1:

$$X_0 = a \text{ and } X_n = a^{-(n-1)}ba^{(n-1)} \text{ for all } n \geq 1.$$

It follows from our discussion of F that $X_nX_k = X_kX_{n+1}$ for all $k < n$. We also define $C_n = a^{-(n-1)}cb^{(n-1)}$.

Our next goal is to get a similar normal form to that of the group F. In order to do this we need to study how elements commute with each other, and we derive a few computational rules for this.

Lemma 4.5. [11, Lemma 5.5] *For all $k \in \{1, 2, \ldots n\}$, we have the following:*

(i) $C_n = X_nC_{n+1}$,

(ii) $C_nX_k = X_{k-1}C_{n+1}$,

(iii) $C_nX_0 = C_{n+1}^2$.

Proof. For *(i)*,

$$\begin{aligned} C_n &= a^{-(n-1)}cb^{(n-1)} = a^{-(n-1)}(b(a^{-1}cb))b^{(n-1)} \\ &= (a^{-(n-1)}ba^{(n-1)})(a^{-n}cb^n) = X_nC_{n+1}. \end{aligned}$$

For *(ii)*, if $k = 1$, it follows from how we define C_n. If $k = 2$ and $n = 2$, it follows from fourth relation in the group presentation.

If $k = 2$ and $n > 2$, we use induction on n to get

$$C_n X_2 = X_{n-1}^{-1} C_{n-1} X_2 = X_{n-1}^{-1} X_1 C_n = X_1 X_n^{-1} C_n = X_1 C_{n+1}.$$

If $k \geq 3$ we use induction on k,

$$\begin{aligned} C_n X_k &= a^{-1} C_{n-1} b X_k = a^{-1} C_{n-1} X_{k-1} b = a^{-1} X_{k-2} C_n b \\ &= X_{k-1} a^{-1} C_n b = X_{k-1} C_{n+1}. \end{aligned}$$

For *(iii)*, If $n = 1$ it is exactly fifth relation in the presentation. If $n > 1$, we use induction to get

$$\begin{aligned} C_n X_0 &= X_0^{-1} C_{n-1} X_1 X_0 = X_0^{-1} C_{n-1} X_0 X_2 \\ &= X_0^{-1} C_n^2 X_2 = X_0^{-1} X_0 C_{n+1}^2 = C_{n+1}^2. \end{aligned}$$ □

Lemma 4.6. *For all $n, m, s \in \mathbb{N}$ such that $m \leq n+1$ and $r, s \leq n$ we have the following;*

(i)

$$C_n^m X_r = \begin{cases} X_{r-m} C_{n+1}^m, & r \geq m \\ C_{n+1}^{m+1}, & r = m-1 \\ X_{r+(n+2-m)} C_{n+1}^{m+1}, & r < m-1, \end{cases}$$

(ii)

$$X_s^{-1} C_n^m = \begin{cases} C_{n+1}^{m+1} X_{(s+m)-(n+2)}^{-1}, & s \geq (n+2)-m \\ C_{n+1}^m, & s = n+1-m \\ C_{n+1}^m X_{s+m}^{-1}, & s \leq n-m, \end{cases}$$

(iii) $C_n^m = X_{(n+1)-m} C_{n+1}^m$,

(iv) $C_n^m = C_{n+1}^{m+1} X_{m-1}^{-1}$,

(v) $C_n^{n+2} = 1$.

Proof. We refer to the proof given by Cannon, Floyd and Parry [11, Lemma 5.6]. □

Similarly to the group F we define a positive element to be the product of non-negative powers of X_i, and negative elements to be those which are inverse to positive elements. In the previous two lemmas what we have shown are the results required to move positive elements to the left of elements of form C_n^m and negative elements to the right of C_n^m. This combined with the theory developed in discussing F we can see how we can get normal forms of T_1.

Lemma 4.7. *Let i, j, k and l be positive integers such that $i < j+2$ and $k < l+2$. Then there are positive elements p and q, and non-negative integers n and m such that $m < n+2$ such that $C_j^i C_l^k = p C_n^m q^{-1}$.*

Proof. Let $n \geq \max(j,l)$. Then Lemma 4.6 gives us positive elements p and q such that $C_j^i = pC_n^i$ and $C_l^k = C_n^{i+r}q^{-1}$. Hence $C_j^i C_l^k = pC_n^{2i+r}q^{-1}$. □

We can use Lemma 4.7 along with all the computational rules we have proven for the group T_1 in order to prove that every element in T_1 can be written in the form $pC_n^m q^{-1}$. However, it is important to note that this form is not unique. This can be seen clearly from rules *(iii)* and *(iv)* in Lemma 4.6.

Theorem 4.8. *If $g \in T_1$, then $g = pC_n^m q^{-1}$ where for some positive elements p, q where $m < n+2$.*

Proof. We give the main idea of the proof and refer to the proof given by Cannon, Floyd and Parry [11, Theorem 5.7] for all the details.

We consider the set of elements in T_1 of the for $pC_n^m q^{-1}$, where p and q are positive elements and $m < n+2$. We can then use our computational rules in T_1 to show that this set is closed under multiplication. Then it follows that this is a subgroup of T_1. As this subgroup also contains a, b and c it follows that this is the group T_1, and every element $g \in T_1$ can be written in the form $pC_n^m q^{-1}$. □

Theorem 4.9. [11, Theorem 5.8] *The group T_1 is simple.*

Proof. Suppose that N is a non-trivial normal subgroup of N, and let $\Theta : T_1 \to T_1/N$ be its quotient homomorphism. Then there is a non-trivial element g which maps to 1 by Θ.

Let $g = pC_n^m q^{-1}$ where $m < n+2$. We then have that $\Theta(p^{-1}q) = \Theta(C_n^m)$ which in turn implies that $\Theta((p^{-1}q)^{n+2}) = 1$.

Let $\alpha : F \to T_1/N$ be the restriction of Θ on the group generated by a and b in T_1. Then this is a well-defined homomorphism.

If $p^{-1}q \neq 1$ we have that $(p^{-1}q)^{n+2} \neq 1$ as F is torsion-free. This means that $\alpha(F)$ is isomorphic to a proper quotient group of F. As every proper quotient group of F is abelian by Theorem 3.27 it follows that $\Theta(ab) = \Theta(ba)$.

If $p^{-1}q = 1$, we have $\Theta(C_{n+1}^{m+1} X_{m-1}^{-1}) = \Theta(C_n^m) = \Theta(p^{-1}q) = 1$ and $C_n^m \neq 1$. This implies that $\Theta(X_{m-1}^{n+2}) = 1$, as before this will imply that $\Theta(ab) = \Theta(ba)$. Then from the fourth relation of the group T_1 we have that,

$$\Theta(a^{-1}cb)\Theta(a^{-1}ba) = \Theta(ba^{-2}cb^2) \implies \Theta(a^{-1}c) = \Theta(ba^{-2}c) \implies \Theta(a) = \Theta(b).$$

Now from the third relation we get that,

$$\Theta(c) = \Theta(ba^{-1}cb) = \Theta(c)\Theta(b) \implies \Theta(b) = 1,$$

which also means that $\Theta(a) = 1$.

From the fifth relation we get

$$\Theta(ca) = \Theta((a^{-1}cb)^2) \implies \Theta(c) = 1.$$

Hence, we get that $N = T_1$, and it follows that group T_1 is simple. □

Corollary 4.10. [11, Corrollary 5.9] *The group T_1 is isomorphic to Thompson's group T.*

Proof. As there is a surjective homomorphism from T_1 to T we have that T is isomorphic to a quotient of T_1. As T_1 is simple and T is non-trivial, we must have that T_1 is isomorphic to T. □

The problem of uniqueness of the form $pC_n^m q^{-1}$ is an interesting one to work with. An obvious way to remedy this would be to use our tree diagrams in order to form what we call our normal form which would be unique. As our tree diagrams in T consist of a domain tree and range tree with the leaf mapping to the first leaf of the domain tree being marked. We can then reduce it and get a unique form as follows.

First suppose that our reduced tree diagram (R, S) has $n+2$ leaves. We can then use our all-right trees T_{n-1} and our elements C_n to write

$$(R, S) = (R, T_{n-1})C_n^m(T_{n-1}, S).$$

This would then give us a form $pC_n^m q^{-1}$ which fits in well with our intuition.

However, it is not so obvious as to how we could characterise this unique normal form as we do in Thompson's group F using Theorem 3.14. Such an algebraic characterisation, as well as some other interesting aspects of T were given by Burillo, Cleary, Stein and Taback [10]. Unfortunately, due to a lack of time this will not be discussed further in this thesis.

5 Piecewise linear homeomorphisms of the real line

In the previous sections we have seen how Thompson's group F can be viewed as either a group of homeomorphisms of the unit interval or as a group of maps between finite binary rooted trees. In order to motivate the following section we look at yet another way of viewing this group.

Theorem 5.1. [11, Theorem 1.4.1] *Thompson's group F is isomorphic to the subgroup of homeomorphisms f of $\mathbb{R}$ with the following properties:*

- *They are piecewise linear and orientation preserving.*
- *They have finitely many breakpoints which are all dyadic rationals.*
- *There is a number $M \in \mathbb{N}$ and $k, l \in \mathbb{Z}$ such that $f(x) = x + k$ for all $x \leq -M$ and $f(x) = x + l$ for all $x \geq M$.*

Proof. Consider the piecewise linear function $\phi : \mathbb{R} \to [0, 1]$ defined on its integer breakpoints as,

$$\phi(k) = \begin{cases} 1 - \frac{1}{2^{k+1}}, & k \in \mathbb{Z}, k \geq 0 \\ 2^{k-1}, & k \in \mathbb{Z}, k < 0 \end{cases}$$

and is linear in between them.

Given any $f \in F$ we can conjugate it with ϕ to get a map $\phi f \phi^{-1} : \mathbb{R} \to \mathbb{R}$. Then $\phi f \phi^{-1}$ maps dyadics to dyadics. Further, it is piecewise linear and whenever the derivative exists the derivative is a power of 2. Also note that if $f \in F$ has derivative 2^a in some neighborhood of 0, and the interval $[\frac{1}{2^m}, \frac{1}{2^{m-1}}]$ is contained in this contained in that interval we have that $\phi f \phi^{-1}$ takes $[-(m+1), -m]$ to $[-(m-a+1), -(m-a)]$ linearly. Hence, we have that $\phi f \phi^{-1}(x) = x - a$ near $-\infty$. Similarly, if the derivative of f near 1 is 2^b, we have that $\phi f \phi^{-1}(x) = x - b$ near ∞. This implies that $\phi f \phi^{-1}$ has finitely many breakpoints. Hence, we are done. $\square$

Motivated by this we proceed to study the group of piecewise linear homeomorphisms of $\mathbb{R}$ following the paper due to Brin and Squier [7]. We define this group as follows.

Definition 5.2. We define the group $PL(\mathbb{R})$, consisting of piecewise linear homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ which have the following properties:

- The element f is not differentiable on a discrete set $B(f)$.
- It has constant derivative when it is differentiable.
- It is positively oriented.

If we only allow elements f where $B(f)$ is finite we get a new group $PLF(\mathbb{R})$.

5.1 A presentation for $PLF(\mathbb{R})$ and its subgroups

Our next goal is to give a presentation for $PLF(\mathbb{R})$. In order to do this we first define a family of elements of $PLF(\mathbb{R})$ (which will end up being a generating set for $PLF(\mathbb{R})$) and show how they interact with each other.

Definition 5.3. We define the elements of M_p, T_a and $X_{b,q}$ where $a, b \in \mathbb{R}$ and $p, q \in \mathbb{R}^+$ as follows;

(i) $M_p(t) = pt$,

(ii) $T_a(t) = t + a$,

(iii)
$$X_{b,q}(t) = \begin{cases} t & t \leq b, \\ b + q(t-b) & t \geq b. \end{cases}$$

Lemma 5.4. [7, Lemma 2.2] *Let $a, b \in \mathbb{R}$ and $p, q \in \mathbb{R}^+$, then we have the following;*

(i) $M_p M_q = M_{pq}$,

(ii) $T_a M_p = M_p T_{ap}$,

(iii) $T_a T_b = T_{a+b}$,

(iv) $X_{b,q} M_p = M_p X_{pb,q}$,

(v) $X_{b,q} T_a = T_a X_{a+b,q}$,

(vi) $X_{b,q} X_{b,p} = X_{b,pq}$,

(vii) $X_{b,q} X_{a,q} = X_{a,p} X_{a+p(b-a),q}$ *where* $a < b$.

Proof. Parts *(i)*, *(ii)* and *(iii)* are clear.

For *(iv)* we have

$$X_{b,q} M_p(t) = \begin{cases} pt & t \leq b \\ p(b + q(t-b)) & > b \end{cases}$$

and

$$M_p X_{pb,q}(t) = \begin{cases} pt & pt \leq pb \\ pb + q(pt - pb) & pt \geq pb \end{cases}$$

which show that $X_{b,q} M_p = M_p X_{pb,q}$. Parts *(v)*, *(vi)* and *(vii)* can be shown in the same manner. $\square$

Our next goal is to show that any element in $PLF(\mathbb{R})$ can be written in a unique normal form using the above relations. This normal form is constructed by using the fact the set of non-differentiable points is finite. We first use elements M_p and T_a to make the element into one which is the identity near $-\infty$. We then move along the real line from left to right getting rid of all non-differentiable points one by one using elements of the form $X_{b,q}$.

Lemma 5.5. [7, Theorem 2.3] *Let $g \in PLF(\mathbb{R})$ be such that g is the identity near $-\infty$. Then g can be written uniquely as,*

$$g = X_{b_1,q_1} \cdots X_{b_n,q_n},$$

where $b_1 < b_2 < \cdots < b_n$ and $q_i \neq 1$.

Proof. We proceed by induction on $|B(g)|$. If $B(g) = \emptyset$ then g is the identity. Suppose the statement is true for all $k \leq n-1$ and let g be an element such that $B(g) = n$. Let b_1 be the smallest number in $B(g)$, and $q_1 = g'(b_1^+)$. Then consider the function $g_1 = X_{b_1,q_1}^{-1} g$. As $|B(g_1)| < |B(g)|$ the existence of the above form follows by the induction hypothesis.

To show uniqueness we can again proceed by induction. The statement is clearly true for the identity map where $B(g)$ is empty. After which the choice of each b_i and q_i is forced. □

Theorem 5.6. [7, Theorem 2.3] *Any $f \in PLF(\mathbb{R})$ can be written uniquely as,*

$$f = M_p T_a X_{b_1,q_1} \cdots X_{b_n,q_n},$$

where $b_1 < b_2 < \cdots < b_n$ and $q_i \neq 1$.

Proof. We simplify the situation to a setting where we can use Lemma 5.5. Let p be the derivative of f near $-\infty$. Then we can consider the element $f_1 = M_p^{-1} f$ which has slope 1 near $-\infty$. Hence $f_1(t) = t + a$ near $-\infty$. Then let g be the element $T_a^{-1} M_p^{-1} f$. From this our theorem follows. Uniqueness follows as our choice of p and a are forced. We call this form the normal form of f. □

Note that this theorem also shows that the elements M_p, T_a and $X_{b,q}$ generate $PLF(\mathbb{R})$. In the next theorem we show that the relations proven in Lemma 5.4 are sufficient to give a presentation for $PLF(\mathbb{R})$.

Theorem 5.7. [7, Theorem 2.4] *The group $PLF(\mathbb{R})$ is isomorphic to the group G given by the presentation,*

$$\begin{aligned} G = \langle M_p, T_a, X_{b,q}, p, q \in \mathbb{R}^+ \mid\ & M_p M_q = M_{pq}, T_a M_p = M_p T_{pa}, T_a T_b = T_{a+b}, \\ & X_{b,q} T_a = T_a X_{a+b,q}, X_{b,q} X_{b,p} = X_{b,pq}, \\ & X_{b,q} X_{a,p} = X_{a,p} X_{a+p(b-a),q} \text{ where } a < b \rangle. \end{aligned}$$

Proof. We define a map $\theta : G \to PLF(\mathbb{R})$ which maps the generators of G to the functions in $PLF(\mathbb{R})$ with the same name. We can then extend this map to a well-defined group homomorphism as $PLF(\mathbb{R})$ satisfies the relations of this group, which we call θ as well. The map θ is surjective as M_p, T_a and $X_{b,q}$ generate $PLF(\mathbb{R})$. The map θ is injective as the relations of the abstract group of G are sufficient to get a form corresponding to the unique normal form in $PLF(\mathbb{R})$. Hence, we are done. □

Note that this proof works exactly the same when we consider the subgroup $PLF_+(\mathbb{R})$, where the generators are of type $X_{b,q}$, and the relations are the relevant relations of the above group G. Similar to this we also define the subgroup $PLF_\alpha(\mathbb{R})$ which consists of elements which are the identity for all $t \leq \alpha$. This is generated by elements of type $X_{b,q}$ where $b \geq \alpha$, and has a presentation with the obvious relations from the presentation for G.

Let K be a multiplicative subgroup of $\mathbb{R}^+$. Define $PLF^K(\mathbb{R})$ to be the subgroup of $PLF(\mathbb{R})$, where $f \in PLF^K(\mathbb{R})$ if $f'(t) \in K$ whenever the derivative exists. This group has a presentation of all generators and relations of G with the added restriction that $p, q \in K$.

Let A be an additive subgroup of $\mathbb{R}$ such that for all $p \in K$ and $a \in A$, $pa \in A$. Then define $PLF_A^K(\mathbb{R})$ to be the subgroup of $PLF^K(\mathbb{R})$, whose elements have breakpoints contained in A. This group will again have the same presentation of G with the added restriction that $p, q \in K$ and $a, b \in A$.

5.2 A family of subgroups of $PLF_A^K(\mathbb{R})$

In this section we will study a specific family of subgroups of $PLF_A^K(\mathbb{R})$, and show that they are finitely presented. Let $p \in \mathbb{Z}^+$ and K be the multiplicative subgroup of $\mathbb{R}^+$ consisting all integral powers of p. Let A be the additive subgroup of rational numbers whose denominators are integral powers of p.
Then define $G(p) = PLF_0(\mathbb{R}) \cap PLF_A^K(\mathbb{R})$. Notice that $G(p)$ has the following infinite presentation:

$$G(p) = \langle X_{b,p^j}, b \in A^+, j \in \mathbb{Z} \mid X_{b,p^i} X_{b,p^j} = X_{b,p^{i+j}},$$
$$X_{b,p^i} X_{a,p^j} = X_{a,p^j} X_{a+p^j(b-a),p^i} \textit{ where } a < b \rangle.$$

In this section we will show that $G(p)$ is finitely presented. In order to do this we first transform the above generators and relations to a more manageable infinite presentation.

The first thing we notice is that the first relation in this presentation allows us to write $X_{b,p^j} = X_{b,p}^j$, this will allow us to disregard the generators of the form X_{b,p^j} where $j > 1$ in future discussions. Then we would also like to show that we can write the second relation in an equivalent form involving only elements of the form $X_{b,p}$.

$$X_{b,p^k} X_{a,p^j} = X_{b,p}^k X_{a,p^j} = X_{b,p}^{k-1} X_{a,p^j} X_{a+p^j(b-a),p} = \cdots = X_{a,p^j} X_{a+p^j(b-a),p}^k.$$

From this it follows that this relation is independent of k. Showing independence of j is slightly more tricky.

We define a function $\beta_j(a, b) = a + p^j(b - a)$. Then we get that,

$$\beta_i(a, \beta_j(a, b)) = \beta_{i+j}(a, b).$$

Using this it follows that,

$$X_{b,p} X_{a,p^j} = X_{b,p} X_{a,p}^j = X_{a,p} X_{\beta_1(a,b),p} X_{a,p}^{j-1} = X_{(a,p)}^2 X_{\beta_1(a,\beta_1(a,p)),p} X_{a,p}^{j-2}.$$

$$= X^2_{(a,p)} X_{\beta_2(a,p),p} X^{j-2}_{a,p} = \cdots = X^j_{(a,p)} X_{\beta_j(a,p),p}$$

Hence it is independent of j. Then we can re-label $X_{a,p} = x_a$, and get the following presentation for $G(p)$:

$$\langle x_a, a \geq 0, a \in A \mid x_b x_a = x_a x_{a+p(b-a)} \text{ where } b > a \rangle.$$

We can further reduce the necessary generators to the case where $a, n \in \mathbb{N}$ as follows.

If x_b a generating element of $G(p)$ then $b \in A$ which means that $bp^k = a \in \mathbb{N}$ for some $k \in \mathbb{Z}$. Then we can use the relation $x_0^{-1} x_b x_0 = x_{pb}$ to write $x_b = x_0^k x_a x_0^{-1}$. We can also use this to reduce the relation $x_a x_b = x_a x_{a+p(b-a)}$ to the case where $a, b \in \mathbb{N}$. Hence, we get that

$$G(p) = \langle x_a, a \geq 0, a \in \mathbb{N} \mid x_b x_a = x_a x_{a+p(b-a)} \text{ where } b > a \rangle.$$

We can finally proceed to show that $G(p)$ is finitely presented.

Theorem 5.8. [7, Theorem 2.9] *The group $G(p)$ is finitely presented.*

Proof. Define $z = x_1^{-1} x_0$. Then for all $b \geq 1$,

$$z^{-1} x_b z = x_0^{-1} x_1 x_b x_1^{-1} x_0 = x_0^{-1} x_{p^{-1}(b-1)+1} x_0 = x_{b+(p-1)}.$$

Hence, the set $\{z, x_1, x_2 \ldots, x_{p-1}\}$ generate $G(p)$.

Next we show that the relation $x_0^{-1} x_b x_0 = x_{pb}$ holds if and only if $z^{-1} x_b z = x_{b+(p-1)}$ for all $b > 0$. We have:

$$\begin{aligned} x_0^{-1} x_b x_0 = x_{pb} &\iff x_1^{-1} x_b x_1 = z x_{pb} z^{-1} \\ &\iff z^{-1} x_{1+p(b-1)} z = x_{pb}. \end{aligned}$$

Let $1 + p(b-1) = b'$. Then we have that

$$x_0^{-1} x_b x_0 = x_{pb} \iff z^{-1} x_b z = x_{b+(p-1)}.$$

In this way we no longer need to consider x_0 in our set of generators and relations.

Our next goal is to show that for all $a, b \in \mathbb{Z}^+, n \in \mathbb{N}$ and for $a', b' \in \{1, 2, \ldots, p-1\}$ we have

$$x_b x_a = x_a x_{a+p(b-a)} \iff [z^{b'-a'+np} x_{a'}^{-1} z^{-n}, x_{b'}] = 1.$$

However, we have that $x_b x_a = x_a x_{a+p(b-a)} = x_a z^{-(b-a)} x_b z^{b-a}$, which is equivalent to saying that $[z^{b-a} x_a^{-1}, x_b] = 1$.

As $a, b \geq 1$ we have $q, q' \in \mathbb{N}$ and $r, r' \in \{1, 2, \ldots, p-1\}$ such that $a = r + q(p-1)$ and $b = r' + q'(p-1)$. Then we can write $x_a = z^{-q} x_r z^q$ and $x_b = z^{-q'} x_{r'} z^{q'}$. Using this we get

$$1 = [z^{b-a} x_a^{-1}, x_b] = [z^{b-a} z^{-q} x_r^{-1} z^q, z^{-q'} x_r z^{q'}]$$

$$\iff [z^{q'-q}z^{b-a}x_r^{-1}z^{q-q'}, x_{r'}] = 1 \iff [z^{(r'-r)+(q'-q)p}x_r^{-1}z^{q-q'}, x_{r'}] = 1.$$

Letting $q' - q = n$ and renaming r', r as b', a' we get that

$$[z^{b'-a'+np}x_{a'}^{-1}z^{-n}, x_{b'}] = 1$$

where either $n > 0$ or $n = 0$ and $b' > a'$.

Hence, we have simplified our problem to a presentation with finite generators and relations which are indexed by (b, a, n) where a, b are taken from a finite set and $n \in \mathbb{N}$. Next we define $A_{(b,a,n)} = z^{b-a+np}x_a^{-1}z^{-n}$, then our relations can be written as $[A_{(b,a,n)}, x_b] = 1$.

Now we show that $[A_{(b,a,n)}, x_b] = 1 \implies A_{b,a,n+p} = A_{b,b,p}A_{b,a,n+1}A_{b,b,1}^{-1}$:

$$\begin{aligned}
A_{b,a,n+p} &= z^{b-a+p+p^2}x_a^{-1}z^{-(n+p)} \\
&= z^{p^2}z^{b-a+np}x_a^{-1}z^{-n}z^{-p} \\
&= z^{p^2}A_{b,a,n}z^{-p} \\
&= z^{p^2}x_b^{-1}A_{b,a,n}x_bz^{-p} \\
&= (z^{p^2}x_b^{-1}z^{-p})(z^pA_{b,a,n}z^{-1})(zx_bz^{-p}) \\
&= A_{b,b,p}A_{b,a,n+1}A_{b,b,1}^{-1}.
\end{aligned}$$

As x_b commutes with $A_{b,b,p}, A_{b,a,n+1}$ and $A_{b,b,1}^{-1}$, we have that $[A_{(b,a,n)}, x_b] = 1$. Hence, we have that we only need to consider finitely many relations where $a, b \in \{1, \dots, p-1\}$ and $n \in \{1, \dots, p\}$, and we are done. $\square$

In the case that $p = 2$, the group that we just studied has finitely many dyadic breakpoints, slopes the power of 2 and has support contained in $[0, \infty)$. Trivially F is contained in this group, as we can see F as being the group with the above properties but with support in $[0, 1]$. Here it is interesting to note that $G(2)$ has infinite presentation structure,

$$G(2) = \langle x_0, x_1, \dots \mid x_i^{-1}x_jx_i = x_{2j-i}, \forall i < j\rangle.$$

Further it is an interesting question as to what properties of the subgroups of $PLF(\mathbb{R})$ we considered are needed for it to be finitely generated and presented.

5.3 $PLF(\mathbb{R})$ has no free subgroups

We say that a group G is metabelian if its commutator subgroup is abelian. In this section we will prove that $PLF(\mathbb{R})$ has no free subgroups generated by more than one element. In order to do this we will prove that any subgroup of $PLF(\mathbb{R})$ is either metabelian or contains a free abelian subgroup isomorphic to $\mathbb{Z}^2$. Then as a free subgroup of rank greater than 1 is neither metabelian nor does it contain abelian free subgroups it will follow that $PLF(\mathbb{R})$ does not contain any free subgroup of rank greater than 1.

In order to show this we begin our discussion with some facts about the commutator subgroup of $PLF(\mathbb{R})$.

Lemma 5.9. [9, Lemma 6.1.4] *Let $f, g \in PLF(\mathbb{R})$ then we have that following:*

(i) The element $[f, g]$ has slope 1 at $\pm\infty$.

(ii) If f, g have slope 1 at $\pm\infty$, we have that supp($[f, g]$) has compact closure.

(iii) If f, g have common fixed point t, then $[f, g]$ is the identity in a neighborhood of t.

Proof. *(i)* As elements have finitely many break points gradients commute near $\pm\infty$. Hence, we have that $[f, g]$ has slope 1 near $\pm\infty$.

(ii) If f, g have slope 1 near $\pm\infty$, then f and g can be written as $t + a$ and $t + b$ near $\pm\infty$. Hence, it will then follow that $[f, g]$ is the identity near $\pm\infty$.

(iii) If f, g have common fixed point t, then $[f, g]$ will also have fixed point t. Further, the slopes will commute at this point, and we will get that $[f, g]$ is the identity on some neighborhood of t. □

Further, we will also need the two purely dynamical results given by Lemma 5.10 and Lemma 5.11.

Lemma 5.10. [7, Lemma 3.4] *Let f be an orientation preserving homeomorphism of $\mathbb{R}$, and $c, d \in \mathbb{R}$ such that $[c, d] \subseteq$ supp(f). Then there exists an $n \in \mathbb{Z}$ such that $f^n(c) > d$.*

Proof. As $[c, d] \subseteq \operatorname{supp}(f)$, we have that $f(t) > t$ or $f(t) < t$ for all $t \in [c, d]$.

Suppose $f(t) > t$ for all $t \in [c, d]$. If $\lim_{n\to\infty} f^n(c) = \infty$, we are done. If $f^n(c)$ converges to some number c_1, we can see that c_1 will be a fixed point of f. This implies that $c_1 > d$, and we are done.

Similarly, if $f(t) < t$ for all $t \in [c, d]$, we can repeat the proof with f^{-1}, and we are done. □

Lemma 5.11. *Let f, g be orientation preserving homeomorphisms of $\mathbb{R}$, and $c, d \in \mathbb{R}$ such that $[c, d] \subseteq$ supp(f) $\cup$ supp(g). Then there is a word $w \in \langle f, g\rangle$ such that $w(c) > d$.*

Proof. The idea of this proof is to get a finite partition of $[c, d]$ in such away that each interval of the partition lies entirely in one of $\operatorname{supp}(f)$ or $\operatorname{supp}(g)$. Then we can repeatedly use Lemma 5.10 on each of the partitions to get a word w such that $w(c) > d$.

We begin by noticing that $\operatorname{supp}(f)$ can be written as the union of disjoint open intervals in $\mathbb{R}$. We can do the same with $\operatorname{supp}(g)$. Then the union of these open intervals form a cover for $[c, d]$. Compactness implies that we then have a finite subcover of $[c, d]$ given by the union of the elements of $\{(c_i, d_i)\}_{i=1}^n$. After which we can repeatedly travel along (c, d) getting partitioning points $c = t_0 < \cdots < t_k = d$ such that $[t_i, t_i + 1] \subseteq (c_k, d_k)$ for some k. Hence, our theorem follows. □

Note that it was essential for us to be able to get the finite cover of our interval. In the case that $f, g \in PLF(\mathbb{R})$, we have that $\text{supp}(f)$ can be written as a finite union of open intervals, and we would not need the compactness argument to get a finite cover of $[c, d]$.

Theorem 5.12. [9, Theorem 6.1.7] *Let G be a subgroup of $PLF'(\mathbb{R})$. Then either G is abelian or G contains a subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. Suppose G is not abelian. Our goal is to create two elements with disjoint supports in order to generate a group isomorphic to $\mathbb{Z}$.

Then there are elements $f, g \in G$ such that $[f, g] \neq 1$. As $f, g \in PLF(\mathbb{R})$ it follows that $\text{supp}(f) \cup \text{supp}(g)$ can be written as the disjoint union of finitely many open intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$. Lemma 5.9*(ii)* and *(iii)* imply that $\text{supp}([f, g])$ is contained in a compact subset of $\text{supp}(f) \cup \text{supp}(g)$.

Let W denote the set of non-trivial words in $\langle f, g\rangle$ which have support contained in a compact subset of $\text{supp}(f) \cup \text{supp}(g)$. As $[f, g] \in W$, we have that W is non-empty. Let $w \in W$ such that $\text{supp}(w)$ intersects the minimal number of components of $\text{supp}(f) \cup \text{supp}(g)$. Let (a_i, b_i) be a component such that $\text{supp}(w) \cap (a_i, b_i)$ is non-empty. Then we have that $\text{supp}(w) \cap (a_i, b_i) \subseteq [c, d] \subseteq (a_i, b_i)$. Then Lemma 5.11 gives us an element $u \in \langle f, g\rangle$ such that $u(c) > u(d)$. Then $\text{supp}(u^{-1}wu) \cap (a_i, b_i) \subseteq (u(c), u(d))$. Hence, we have that,

$$\text{supp}(w) \cap \text{supp}(u^{-1}wu) \cap (a_i, b_i) = \emptyset.$$

As $\text{supp}([u^{-1}wu, w]) \subseteq \text{supp}(w) \cap \text{supp}(u^{-1}wu)$, we have that $\text{supp}([u^{-1}wu, w])$ has non-empty intersection less components than w. Hence, minimality implies that $\text{supp}([u^{-1}wu, w]) = 1$ and they commute. Hence, they generate a free abelian group isomorphic to $\mathbb{Z}^2$. □

Corollary 5.13. *Let G be a subgroup of $PLF(\mathbb{R})$. Then G is either metabelian or contains a free abelian subgroup of rank 2.*

Proof. Apply Theorem 5.12 to G'. Then G' is either abelian or contains a free subgroup of rank 2. □

Corollary 5.14. *Any subgroup of $PLF(\mathbb{R})$ contains no free subgroups of rank greater than 1.*

Proof. It is well-known that the subgroup of a free group is free (Nielsen-Schreier theorem, see e.g., [18, Corollary 4.2.8]). As free groups of rank greater than 2 are not metabelian the theorem follows. □

We note that this implies that Thompson's group F does not contain any free subgroups, and this has interesting ramifications on the amenability of F. Unfortunately due to a lack of time we will not delve into this further.

Further, in any discussion of $PL(\mathbb{R})$ we must note the importance of the monograph due to Bieri and Strebel [5]. In this the authors collect and refine many results regarding the subgroups of $PL(\mathbb{R})$ and seems to be a natural place to go to for further reading on these groups.

6 Further avenues

In this section we briefly discuss a few related topics that would have been interesting to cover in this thesis.

The most natural place to start is Thompson's third group which we will call V. If Thompson's group T can be thought of as tree pair diagrams where we allow cyclic permutations of leaves, we can think of V as the group of tree diagrams where we allow arbitrary permutations of V. More concretely V can be defined similar to Definition 4.1 with the relaxation of the requirement of the function to be continuous. In V we only want the elements to be right continuous at its breakpoints. Thompson's group V turns out to be another finitely presented infinite simple group and the proof for simplicity given by Cannon, Floyd and Parry [11, Chapter 6] follows a similar structure to the proof of the fact that T is simple. Similar to this idea, a braided version of Thompson's groups were introduced independently by Brin [6] and Dehornoy [12] and would be an interesting area of study in the future.

Another interesting problem is the conjugacy problem in Thompson's groups F, T and V. A unified solution to this problem in all three groups was given by Belk and Matucci [4] and the solution shows quite a beautiful relationship between group theory and geometry.

In order to do this the authors introduce yet another way to view elements of F, T and V. Given an element f in tree diagram form we create a *strand diagram* embedded in the unit square as seen in Figure 15.

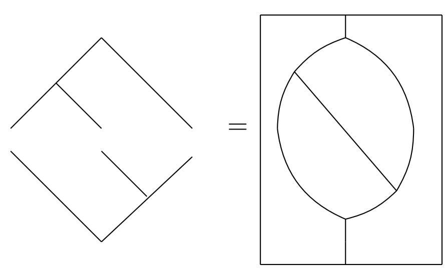

Figure 15: The element $a(x)$
as a strand diagram.

Then we can glue the two ends of the strand diagram in order to obtain an *annular strand diagram* (a strand diagram embedded in an annulus). The main result here is that two elements of F are conjugate if and only if they have the same reduced annular diagram. Similar to this result the groups T and V have solutions to the conjugacy problem where the added difficulty is that you can no longer embed elements of T and V in the unit square, and hence their closed strand diagrams will not be embedded in two-dimensional space (in the case of T it embeds in the torus and in the case of V it does not embed on the surface of any three-dimensional object). Strand diagrams also allow us to construct a very simple groupoid whose orbit group at any point will be Thompson's group F, and this would have also been an interesting idea to explore further.

Finally, the embedding of $\mathbb{Q}$ as an additive abelian into a finitely presented subgroup $\bar{T}$ of $PL(\mathbb{R})$ as seen in the paper due to Belk, Hyde and Matucci [3] would have been interesting to include in this thesis. The group $\bar{T}$ is the lift of Thompson's group T to the real line through the covering map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. The proof of this is purely dynamical with arguments similar to those seen in Section 5.3. This result gives a solution to a question in the Kourovka notebook [17, Problem 14.10(a)], and it is quite surprising that this embedding was not noticed before as the argument used in this paper is remarkably simple.

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