The Potential of Diverse Technologies in Growth

Abstract: A product-variety growth model is presented, featuring regions with diverse basic input prices, intermediate good technologies with randomly distributed usage intensities for those inputs, and technology improvement with usage eventually leading to the spread of these technologies. Exponential long-term growth is derived in the specified model. The probability of new technologies being immediately profitable in at least one region is found to be an important growth factor, and regions' contributions to increasing that probability depend both on having generally favorable conditions for making profits, and on having an input cost environment that differs from other regions, increasing the diversity of such environments.

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1 Introduction

In growth models with active research, the profit potential of new technologies is doubtlessly important, but is generally treated as being identical for all potential technologies. The model presented here extends this notion of profit potential to allow for heterogeneity in the degree to which technologies need various basic inputs as well as the environment of input prices in different regions. This sets up a distinction between the variation in arrival of technologies from how long it takes to invent a new technology and from whether its features render it useful or not to the regional economies in the model.

Furthermore, the early implementations of new technological ideas are rarely particularly efficient, needing to be refined with experience and experimentation before realizing their true potential. Once they have been developed further, they may be useful in more contexts than they initially were, creating a rational, profit-driven reason for technologies to spread.

By extending growth models to include these features, interactions between these features as well as their interactions with traditional growth model features can be modeled, allowing conclusions to be drawn about new mechanisms born from these interactions as well as about their effects on long-run growth. The main result of extending a product-variety model in this way is that the key quality of a new technology's features is that it is viable in at least one region, as that is what enables the quality improvements that in the long run make it viable in all regions. The probability of having this initial quality becomes a factor in the long-run growth rate, and is increased by regions being generally profitable to operate in, as well as these regions having diverse combinations of input prices, letting them each do their part in capturing new technologies with different input intensity niches.

The rest of this thesis is structured as follows: Section two reviews relevant literature. Section three sets up the model, including most assumptions (some technical assumptions are stated where they become relevant). Section four describes the mechanisms of the model, derives the long-run growth rate, and then discusses growth properties in the model. Section five concludes.

2 Relevant Literature

The main inspiration for the novel mechanics in the model presented here is the historical analysis of the industrial revolution by Allen (2009). He argues that most other explanations for the revolution either might have reversed causality, such as agricultural developments, or only present necessary but not sufficient factors for triggering it, such as social, institutional, and scientific developments. The book proposes that the supply of technology wasn't unique to Britain, in the sense that many countries had the capacity to invent the key technologies that would drive the revolution, but that the demand for these technologies was, in that it was the only place where early, inefficient versions of these labor-saving but capital- and coal-intensive technologies were cost-efficient, and thus profitable to refine from idea to invention. To support this, he verifies that wages were relatively high and that capital (notably including coal) was relatively cheap in Britain at the time (and especially so in certain regions). Finally, he shows that the timings of these technologies spreading to other countries generally corresponded to when they had become cost-efficient there too, driven by efficiency improvements in the places they had been used up to that point.

This has parallels with the induced innovation hypothesis, often summarized as: "A change in the relative prices of the factors of production is itself a spur to innovation, and to innovation of a particular kind – directed to economizing the use of a factor which has become relatively expensive." (Hicks, 1932, quoted in Ley, Stucki & Woerter, 2016, p.44) This theory has been subjected to various empirical tests, especially with regards to environmental and agricultural technological change. As an example of the former, Ley, Stucki, and Woerter (2016) investigate the effect of energy prices on green inventions (rather, patents for such inventions). They find a significant effect for both the number of green patents and the fraction of green inventions out of all inventions, where higher energy prices incentivize more development of green technologies. Furthermore, the subcategories of green inventions that relate the most to energy efficiency are also the ones where this effect is the strongest. An example for the latter is Thirtle, Schimmelpfennig, and Townsend (2002), who investigate how the actual technologies used in agriculture were influenced by factor prices in the long run. They use an error correction model to separate factor substitution (short-term) and technological change (long-term) in the time-series relationship between factor usage and their prices, finding that technological development does respond to factor prices, adjusting to use the relatively expensive factors less intensely. Many theoretical implementations of the induced innovation paradigm (see Irmen, 2020, for instance) are set up similarly to Thirtle, Schimmelpfennig, and Townsend (2002) in that they model technologies (often single or representative ones) which adapt over time to input prices over time following from directed research and development. While something analogous to such effects could be included here by letting researchers influence the distribution of new technology features, the model presented here only features a pseudo-evolutionary selection mechanism to the effect of induced innovation, where technologies whose features align well with the cost environment simply are more viable regardless of how new technologies' input intensity features are distributed.

On the growth-theoretical side, the paradigm the model presented here inherits the most from are product-variety models, a type of model that originated with Romer (1990). Here, a range of intermediate goods are used in the production of final goods, and the output of final goods can be increased by increasing the number of varieties of the intermediate goods available for such use. Motivated by patents that allow capture of the profits reaped from the intermediate goods invented, active research is conducted into creating new types of such goods, driving endogenous growth.

Beyond models of completely novel technologies, there are also models where growth is driven by increases in the productivity of existing technologies (or new, more efficient technologies that end up replacing old ones), from which the model presented here inherits some features. One branch of such models are the ones based on learning by doing (Arrow, 1962) or similar mechanisms, where firms maximize profits but also produce knowledge (or some other productivity-enhancing public good) as an unintentional byproduct of some of their operations. The most notable type of these are AK models (Frankel, 1962, for instance), in which some factor of production (usually capital) exhibits constant returns in its benefit to society as a whole, enabling economic growth, while suffering from diminishing returns for individual firms; however, do note that the AK approach allows for a wider range of interpretations of growth-causing "indirect effects" than just learning by doing.

Another branch of growth models based on increasing efficiency are "Schumpeterian" models, such as Aghion and Howitt's (1992). Much like in product-variety models, technological progress is driven by active research for the purposes of profit, leading to many overlapping conclusions about how to improve growth rates, but the internal workings are very different.

By conducting research, agents have a chance of discovering a more efficient technology for a specific good, letting them obtain the new monopoly rights for that good.

3 Model Setup

Two versions of the model are presented in this section and the next. The first is a rather general one, for which completely unambiguous results unfortunately cannot be derived; instead, it has two main purposes: To investigate which types of violated assumptions the results for the second model would and wouldn't be robust to. And to provide a framework in which some additional channels not present in the second version can be discussed. The second specification, meanwhile, leads to a closed-form long-run growth rate.

In this model, the world consists of a set of regions G, which only has two elements in the simple model. In each region g, the final goods sector and intermediate goods sector are explicitly modeled. The final goods sector uses the region's labor L_g, which is assumed to be constant over time, and some aggregate of the intermediate goods, also from the region. The intermediate goods sector uses basic inputs from the set V, which are also only procured within the region, to produce intermediate goods for the region. These inputs can be interpreted in a variety of ways, such as raw materials, rented capital, or hired human capital. The worldwide research sector, also explicitly modeled, uses final goods (sourced from any region) to attempt to produce new technologies for the intermediate goods sectors.

Time t is continuous, and within each phase s all variables follow smooth curves. The world shifts to the next phase whenever a technology is introduced to (or exits) a region, at which points discontinuities may arise at various orders of time differentiation for variables.

Consumption is not explicitly modeled here, but an interest rate is assumed to follow from consumers' intertemporal preferences, aligning the economy's growth rate with their most desired consumption path.

3.1 Final Goods

The perfectly competitive final goods sector uses labor and intermediate goods to produce and sell final goods at price one. Their production function is twice continuously differentiable, increasing in all inputs (positive first partial derivative), but with diminishing returns for any input on its own (negative second partial derivative) while having constant returns for all inputs together. An intermediate good's first partial derivative should tend to zero as usage of that good tends to positive infinity, and tend to positive infinity as its usage tends to zero. Since the number of intermediate goods inputs will change, it should be consistent with a fundamental production rule that takes all possible intermediate goods, discovered and undiscovered, used and not used, into account, where inactive inputs are used at quantity zero. For instance, the following constant-elasticity-of-substitution (CES) type of production function

$$Y_{g,t,s} = L_g^{1-\beta}\left(\sum_{i\in M_{g,s}} x_{i,g,t,s}^{\eta}\right)^{\frac{\beta}{\eta}} \tag{3.1.1}$$

fulfils these conditions for parameter values including $\beta \in (0,1)$, $\eta \in (0,1)$, which are substitutes cases between intermediate goods, but not for non-positive values of η, which are Cobb-Douglas and complements cases (these fail the last condition, as there will be no output as long as at least one input is zero). $M_{g,s}$ is the set of intermediate goods firms active in this region during this phase, and $x_{i,g,t,s}$ is the quantity used of firm i's good.

For the specific version, assume a separable special case of this production function where $\eta = \beta$, yielding:

$$Y_{g,t,s} = L_g^{1-\beta}\sum_{i\in M_{g,s}} x_{i,g,t}^{\beta} \tag{3.1.2}$$

This specific production setup is similar to the one in Romer (1990), but the sum here won't be replaced by an integral. Note that the phase index s can be dropped in this case, because intermediate goods firms won't need to concern themselves with other firms' choices.

As this sector is perfectly competitive, intermediate goods can be sold for their marginal product, that is, their first partial derivative of the final goods production function. For the simple example version of the model, this is:

$$\frac{\partial Y_{g,t,s}}{\partial x_{i,g,t}} = \beta L_g^{1-\beta} x_{i,g,t}^{\beta-1} \tag{3.1.3}$$

3.2 Intermediate Goods

Intermediate goods firms produce according to a complements type of CES function:

$$x_{i,g,t,s} = A_{i,t}\left(\sum_{j\in V} \alpha_{i,j} v_{i,g,j,t}^{\varepsilon}\right)^{\frac{1}{\varepsilon}}$$

For which the substitutability parameter ε must be negative, and the input importance parameters $\alpha_{i,j}$ must be non-negative and sum to one, $\sum_{j\in V}\alpha_{i,j} = 1$. $A_{i,t}$ is the efficiency of their technology, V is the set of inputs, and $v_{i,g,j,t}$ is how much firm i uses of input j in region g at time t. For each quantity, they solve a cost minimization problem:

$$\min_{v}\sum_{j\in V} p_{g,j} v_{i,g,j,t}$$

Which results in the per-unit variable cost:

$$\lambda = A_{i,t}^{-1}\left(\sum_{j\in V}\alpha_{i,j}^{-\frac{1}{\varepsilon-1}} p_{g,j}^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}$$

For the general model, define:

$$c_{i,g} = \left(\sum_{j\in V}\alpha_{i,j}^{-\frac{1}{\varepsilon-1}} p_{g,j}^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}$$

Then, profits for firm i in region g at time t are:

$$\Pi_{i,g,t,s} = \frac{\partial Y_{g,t,s}}{\partial x_{i,g,t,s}} x_{i,g,t,s} - \frac{c_{i,g}}{A_{i,t}} x_{i,g,t,s} - C \qquad (3.2.1)$$

Where C is a constant fixed cost term. Firms compete monopolistically and only maximize profits short-term, not taking intertemporal effects into account.

In the example specification of the model, intermediate goods producers' technologies have the basic inputs as perfect complements:[1]

$$x_{i,g,t} = A_{i,t}\min_{j\in V}\left(\alpha_{i,j} v_{i,g,j,t}\right)$$

In this version, $c_{i,g}$ is simply defined as:

$$c_{i,g} = \sum_{j\in V}\alpha_{i,j} p_{g,j}$$

Furthermore, for the simple case, it is assumed there are only two inputs, in which case there is effectively only one variable for how important each input is, as $\alpha_{i,2} = 1 - \alpha_{i,1}$. Following from this and equation 3.1.3, intermediate goods profits will be:

$$\Pi_{i,g,t} = \beta L_g^{1-\beta} x_{i,g,t}^{\beta} - \frac{c_{i,g}}{A_{i,t}} x_{i,g,t} - C \qquad (3.2.2)$$

[1] Note that $\alpha_{i,j}$ has a technically different definition here, even if the substantive implication is still how important that good is for their production.

Firm technologies improve at the rate (in all versions of the model):

$$\dot{A}_{i,t,s} = \nu \sum_{g \in G_{i,s}} x_{i,t,g,s}^{\sigma} \tag{3.2.3}$$

Where ν is an efficiency parameter for technology improvement[2] and σ is a returns-to-scale parameter, which is sufficiently low enough for $A_{i,t}$ to grow less-than-exponentially. $G_{i,s}$ is the set of regions technology i has active firms in. Note that the technology level is not specific to region, so all regions contribute, and all regions benefit from firm activity. This could be seen as implementing a learning-by-doing principle as in Arrow (1962), but this setup is ultimately very different from the one in that paper.

3.3 Research

New technologies arrive randomly at the Poisson rate $\theta(R_{t,s})$ (that is, the time between each invention follows an exponential distribution with this function as its parameter; this applies a principle much like in Aghion and Howitt (1992), although for the invention of new intermediate good varieties instead of more efficient versions of such varieties), where $R_{t,s}$ is the final goods used in research (where the t index can be dropped for the simplified version), given by the relationship:

$$\theta(R_{t,s}) = \gamma |M_s|^{1-\psi} R_{t,s}^{\psi} \tag{3.3.1}$$

Where γ is a research efficiency parameter and ψ is a returns-to-scale parameter. M_s is defined as $M_s = \bigcup_{g \in G} M_{g,s}$, and is a public good for the research sector. The inclusion of a factor for some measure of the current stock of available knowledge is also present in Romer (1990); furthermore, such a measure was also found to have a significant positive effect on the level of invention in Ley, Stucki, and Woerter (2016). If one wants to see the invention chance function as a constant-returns-to-scale function in private goods, one can suppose there is some fixed-supply factor (some form of researcher labor or researcher human capital, for instance) raised to $1 - \psi$ included as part of γ.

Upon discovery of a new technology, the discoverer receives a perpetual monopoly on that technology, which they license to firms to produce in regions where it would be profitable to operate (firms are modeled as one per technology and per region). New technologies'

[2] ν is the Greek letter nu, not the v used for input quantities. Note that the latter has already had its last appearance in this thesis, and, for extra clarity, that the former does not have a subscript.

features (that is, how much they depend on each input) are random, following some distribution; for the simple specification, $\alpha_{i,1}$ is assumed to follow a uniform distribution between zero and one. When a technology is invented, its efficiency starts at the level $\tilde{A}_{t,s}$; in the specific example version of the model, this is constant across time and phases. If a technology ever isn't produced – that is, no firm wants to introduce it to any region immediately upon discovery, or the last firm producing its good exits the last remaining region – it is lost, never to return.

Agents in the sector maximize their expected profits:

$$\Pi_{R,t} = \gamma |M_s|^{1-\psi} R_{t,s}^{\psi} E(PV_t) - R_{t,s} \tag{3.3.2}$$

$E(PV_t)$ refers to the expected present value of a new technology invented at time t, based on future profits of firms using such a technology (calculated heuristically in the simple model).

4 Model Analysis

4.1 Intermediate Goods Market

There is no completely general definition of monopolistic competition that can readily be applied to any possible final goods production function (at least not to my knowledge), but for the purposes of this analysis it implies that firms see demand from the final goods sector in some simplified way $f(x_{i,g,t,s})$ where they take some aggregate condition for granted, even though their choices actually have a small effect on it. The naïve inverse demand function f retains all important properties of the actual relationship $\partial Y_{g,t,s}/\partial x_{i,g,t,s}$: Once continuously differentiable, infinity near zero quantity, zero at infinite quantity, and strictly decreasing. For example, in the CES case,

$$\frac{\partial Y_{g,t,s}}{\partial x_{i,g,t,s}} = \beta L_g^{1-\beta} \left(\sum_{j \in M_{g,s}} x_{j,g,t,s}^{\eta} \right)^{\frac{\beta}{\eta}} x_{i,g,t,s}^{\eta-1} \tag{4.1.1}$$

monopolistic competition typically entails firms seeing the third factor as a constant, which indeed produces a perception that fulfils the aforementioned description.

Then, naïve optimization of the profit function (equation 3.2.1) solves:

$$\frac{c_{i,g}}{A_{i,t}} = f(x_{i,g,t,s}) + f'(x_{i,g,t,s}) x_{i,g,t,s}$$

At specific points in time, $A_{i,t}$ is fixed, which makes the left-hand side a positive constant for short-term optimization. When $x_{i,g,t,s} = 0$, the first term of the right-hand side dominates, and that side is positive infinity. Meanwhile, as quantity tends toward infinity, the first term approaches zero, implying that the second term, which is negative as f is strictly decreasing, eventually dominates, resulting in a negative right-hand side. Thus, equality is achieved for some positive quantity, which is the solution to the profit maximization problem.[3] This solution is the best response as long as the firm would enjoy positive profits, otherwise the best response is to not operate in the region under consideration. The condition on which it is profitable be active in a region is the viability condition (sometimes also called the profitability condition), which is that $c_{i,g}/A_{i,t}$ is less than some critical value (which is not generally constant over time).

Slightly more specifically, the right-hand side is assumed to be strictly decreasing in quantity,[4] implying that smaller marginal costs (lower left-hand side, higher technology level for a specific $c_{i,g}$) lead to higher optimal quantities (as lowering the right-hand side to restore equality requires raising quantity), all other firms' actions equal. But if a firm actually changes its quantity because of changed marginal costs, all else will not be equal, as even though the effect may be small, the aggregate level of intermediate goods does move with individual firms' actions. If the equilibrium response is to move in the same direction as the aggregate, the aggregate obviously also ultimately moves in that direction, but responses here must be appropriately small, else there could be an explosive feedback loop from increased quantities. If the response by other firms is to move their quantities in the opposite direction, these responses must also be small enough for the aggregate to still move in the same direction as the first (whose quantity change is dampened), as their original prompt would otherwise be

[3] Strictly speaking, it may be possible to construct an f such that there are multiple extreme points. But since profits are increasing near zero quantity, decreasing at very large quantities, and continuous (excluding actually zero quantity, where the firm avoids the fixed cost), there must be at least one point where it shifts from increasing to decreasing – a local maximum. At least one of these must be a global maximum for positive quantities, with that then being the solution. Note that the equality can only hold for one value if the right-hand side is strictly decreasing, in which case the extreme point is unique, and is thus also the global maximum.

[4] I seriously doubt that it's actually possible to construct a reasonable f which doesn't have a strictly decreasing right-hand side, because of the final goods production function having constant returns to scale overall.

more than nullified, which would make their actions contradictory. In both cases, as the aggregate follows the first firm, so does output.

The set of general assumptions used here need not be the only way to achieve an intermediate goods market that behaves nicely, in the sense that it determines a unique equilibrium (that is, if there are multiple, there is also a selection mechanism for picking one), wherein lowered marginal costs imply increased quantity for the firm and more output for the region. The purpose here is not to investigate the behavior of extremely general oligopoly setups, it is to create a reasonable environment in which existing technologies can improve and new technologies can be introduced.

In the closed-form specification, maximizing profits (equation 3.2.2) leads to:

$$x_{i,g,t} = L_g \beta^{\frac{2}{1-\beta}} \left(\frac{A_{i,t}}{c_{i,g}}\right)^{\frac{1}{1-\beta}} \tag{4.1.2}$$

Then, evaluating output (equation 3.1.2) and profits (equation 3.2.2) using this expression respectively yield:

$$Y_{g,t,s} = L_g \beta^{\frac{2\beta}{1-\beta}} \sum_{i \in M_{g,s}} \left(\frac{A_{i,t}}{c_{i,g}}\right)^{\frac{\beta}{1-\beta}} \tag{4.1.3}$$

$$\Pi_{i,g,t} = L_g \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right) \left(\frac{A_{i,t}}{c_{i,g}}\right)^{\frac{\beta}{1-\beta}} - C \tag{4.1.4}$$

And solving the latter for the viability condition (expressed for $A_{i,t}/c_{i,g}$ instead of $c_{i,g}/A_{i,t}$, reversing the inequality) in this version of the model results in:

$$\frac{A_{i,t}}{c_{i,g}} > \left(\frac{C}{L_g}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{-\frac{1-\beta}{\beta}} \tag{4.1.5}$$

4.2 Change over Time within Phases

All active technologies have firms producing positive quantities in at least one region, else they are, by definition, lost. Thus, the within-phase rate of change of efficiency (equation 3.2.3) is also positive for all used technologies. Then, there is a push toward increasing quantities for all firms, before equilibrium effects. With these equilibrium effects, however, some of them may actually have declining quantities (similarly, there is no safeguard that ensures that all firms experience increasing profits), but not all, nor to such a degree that the aggregate level ends up decreasing, as that would once again lead to a contradiction where the original effect is more than nullified. As each of these productivity improvements contribute positively to the output of all regions they operate in, through increased aggregate intermediate goods, economic progress occurs between phase shifts, although at a slower-than-exponential pace, inheriting efficiency levels assumed lack of exponential growth.

With a specific setup, it is possible to go further by setting up a differential equation for the efficiency rate of change, but care must be taken as this can change from phase to phase. For the simple specification, expanding equation 3.2.3 using equation 4.1.2:

$$\dot{A}_{i,t,s} = \nu\beta^{\frac{2\sigma}{1-\beta}} A_{i,t}^{\frac{\sigma}{1-\beta}} \sum_{g\in G_{i,s}} L_g^{\sigma} c_{i,g}^{-\frac{\sigma}{1-\beta}} \tag{4.2.1}$$

The condition for $A_{i,t}$'s growth to be slower than exponential is that the exponent on $A_{i,t}$ is less than one, which is equivalent to $\sigma + \beta < 1$. The general solution to this differential equation is:

$$A_{i,t,s} = \left((t + B_s)\nu\beta^{\frac{2\sigma}{1-\beta}} \frac{1-\beta-\sigma}{1-\beta} \sum_{g\in G_{i,s}} L_g^{\sigma} c_{i,g}^{-\frac{\sigma}{1-\beta}} \right)^{\frac{1-\beta}{1-\beta-\sigma}} \tag{4.2.2}$$

Where B_s is a constant of integration that aligns this function with actual values. In this specification, each technology can have one or two phase-dependent instances of this function. Letting $\tilde{t}$ be the time since that technology was invented, solving for the starting level of technological efficiency $\tilde{A}$ specifies the first version as:

$$A_{i,\tilde{t}} = \left(\tilde{t}\nu\beta^{\frac{2\sigma}{1-\beta}} \frac{1-\beta-\sigma}{1-\beta} \sum_{g\in G_{i,s}} L_g^{\sigma} c_{i,g}^{-\frac{\sigma}{1-\beta}} + \tilde{A}^{\frac{1-\beta-\sigma}{1-\beta}} \right)^{\frac{1-\beta}{1-\beta-\sigma}} \tag{4.2.3}$$

If the technology is initially profitable in both regions, this is the only instance of this function, lasting forever.

If it was only viable in one region at first, then $G_{i,s}$ eventually changes, as the technology will be efficient enough to be viable in the other region as well after enough time. This level of technology is (following from equation 4.1.5):

$$\hat{A}_i = \left(\frac{C}{L_{\hat{g}}}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{-\frac{1-\beta}{\beta}} c_{i,\hat{g}}$$

Where $\hat{g}$ is the region it was not initially viable in. Let $\tilde{g}$ be the region it started in. Then, solving equation 4.2.3 for $\hat{A}_i = A_{i,\hat{t}}$, where $\hat{t}$ is the point in time where the technology becomes profitable in the other region, yields:

$$\hat{t} = \nu^{-1} L_{\tilde{g}}^{-\sigma} c_{i,\tilde{g}}^{\frac{\sigma}{1-\beta}} \beta^{-\frac{2\sigma}{1-\beta}} \frac{1-\beta}{1-\beta-\sigma} \left(\left(\left(\frac{C}{L_{\hat{g}}} \right)^{\frac{1-\beta-\sigma}{\beta}} c_{i,\hat{g}}^{\frac{1-\beta-\sigma}{1-\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}} \right) \right)^{-\frac{1-\beta-\sigma}{\beta}} - \tilde{A}^{\frac{1-\beta-\sigma}{1-\beta}} \right)$$

At the time where $\tilde{t} = \hat{t}$, this technology spreads to the other region, triggering a phase shift, which is also the only phase shift that affects this technology's efficiency's rate of change, which then follows (from solving 4.2.2 again for a new constant of integration):

$$A_{i,\tilde{t}} = \left((\tilde{t} - \hat{t}) \nu \beta^{\frac{2\sigma}{1-\beta}} \frac{1-\beta-\sigma}{1-\beta} \sum_{g \in G_{i,s}} L_g^{\sigma} c_{i,g}^{-\frac{\sigma}{1-\beta}} + \hat{A}^{\frac{1-\beta-\sigma}{1-\beta}} \right)^{\frac{1-\beta}{1-\beta-\sigma}} \qquad (4.2.3')$$

Here, each firm's level of technology progresses according to a power function of a linear expression of time. This is weaker than exponential growth, which means the growth rate will decrease as time passes and firms' technologies become more advanced. This can most clearly be seen by dividing each side of equation 4.2.1 by $A_{i,t}$ to obtain its growth rate:

$$\frac{\dot{A}_{i,t,s}}{A_{i,t}} = \nu \beta^{\frac{2\sigma}{1-\beta}} A_{i,t}^{\frac{\sigma+\beta-1}{1-\beta}} \sum_{g \in G_{i,s}} L_g^{\sigma} c_{i,g}^{-\frac{\sigma}{1-\beta}}$$

The exponent for $A_{i,t}$ on the right-hand side is negative, revealing the diminishing growth. Within phases, the appearances of $A_{i,t}$ in the equilibrium expressions for quantity, profits, and output (equations 4.1.2 to 4.1.4) are all as bases in exponentiation. This means they are all also ultimately (linear functions of) power functions of (a linear function of) time, thus inheriting the diminishing-growth quality within phases. For long-term exponential growth, then, one must look toward shifts between phases.[5]

[5] If one instead were to assume that technology efficiency levels actually do grow exponentially, this would be the point where final goods outputs are shown to grow exponentially. For the closed-form specification, this would use the assumption $\sigma + \beta = 1$. The resulting model would be a learning-by-doing model, but a rather strange one, as the neglected benefits are actually internal to the firm, if external to the current moment in time.

4.3 Research Behavior

For perfectly rational profit maximization in research, the expected value of a successful invention must take movement of profit within phases, changes in profit between phases, and how these behaviors are influenced by the distribution of technology features into account, which must then also be discounted. The movement of profit within phases was discussed in the last subsection. The changes between phases depends both on the timing of phase shifts, which is determined by what happens in the research sector, and on the sizes and directions of jumps in variables, which does not directly depend on the research sector, but may do so indirectly if timing matters for the exact characteristics of the jumps.

The most direct effect of the feature distribution is on the distribution of regions where the technology is initially viable. For the simple version, where there are two regions and two inputs to intermediate production, a fair number of scenarios can already be described. For the general model, this very rapidly becomes more complicated as more regions and basic inputs become part of the model. Furthermore, a more elaborate intermediate goods production function is used, and the viability condition may vary over time; the latter of these implies that the starting level of technology $\tilde{A}_{t,s}$ may need to change over time to let new technologies retain a fair chance of viability as competition tightens (or prevent it from becoming too likely if conditions instead become more favorable over time). Despite these complications, it is still endogenously determined, and some of the consequences of that will be discussed in the growth subsection.

In the simpler specification, as the variation in the features of technologies can be described using only a single variable, $\alpha_{i,1}$, clear critical values can be derived for that variable, and technologies will only be immediately viable in the focal region if they're on the right side of it. By combining the definition of $c_{i,g}$ in this version with $A_{i,t} = \tilde{A}$ and equation 4.1.5, the critical value can be written:

$$\tilde{\alpha}_g = \frac{\tilde{A}\left(\frac{L_g}{C}\right)^{\frac{1-\beta}{\beta}}\left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} - p_{g,2}}{p_{g,1} - p_{g,2}} \tag{4.3.1}$$

Out of the two inputs, let the first one be picked such that it is the relatively expensive one in the first region (with no loss of generality). The critical value is an upper bound for regions where the first input is the expensive one, and a lower bound for regions where the first input is the cheap one. Many scenarios are possible, but two of them manage to capture most possible notable features.

The first scenario (A) is one where both regions have the same relatively expensive good. One region will have the higher critical value, which will be the critical value determining if the new technology is viable at all. Let that region be the first region. Then, the probability of a new technology not being viable anywhere is $P(\tilde{G} = \emptyset) = 1 - \tilde{\alpha}_1$ (where $\tilde{G}$ is the set of regions in which a new technology is profitable upon discovery), the probability of it being viable in one region (region one) is $P(\tilde{G} = \{1\}) = \tilde{\alpha}_1 - \tilde{\alpha}_2$, and the probability of being profitable in both regions is $P(\tilde{G} = \{1,2\}) = \tilde{\alpha}_2$.

For the second scenario (B), suppose they have different relatively expensive goods, and there is no overlap between their ranges ($\tilde{\alpha}_1 < \tilde{\alpha}_2$). In that case, the probability of not being viable anywhere is $P(\tilde{G} = \emptyset) = \tilde{\alpha}_2 - \tilde{\alpha}_1$, and the probability of being profitable in one region is $P(\tilde{G} = \{1\}) + P(\tilde{G} = \{2\}) = \tilde{\alpha}_1 + (1 - \tilde{\alpha}_2)$, while the probability of being viable in both is zero.

Beyond these two, there are a number of additional scenarios, but most of them aren't particularly interesting. If the general profitability in a region (the first term in the numerator of the critical value formula) is too low compared to the price of the relatively cheap input for new technologies to ever be viable in one of the regions, let that be the second region, and a scenario that is essentially a simplification of the last two is created. If general profitability is too low in both regions, one of the main mechanisms of the model is nullified, and long-term growth will be based on efficiency levels and diminishing over time – this type of scenario can serve as a growth trap, but given how the model is primarily designed for international growth, this explanation shouldn't be particularly applicable for the present day.[6] There are also a couple of scenarios where new technologies are guaranteed to be profitable at least somewhere, which, while a nice goal and not too implausible for two inputs, becomes a poor approximation for reality. And finally, there are scenarios where input prices are equal in at least one region, in which case the critical value formula fails for that region, and viability will depend entirely on its numerator being positive (yielding an always-viable or never-viable case).

[6] It can, however, potentially serve as a partial explanation for the world economy before the industrial revolution, but as shown and noted by Allen (2009), a more complete explanation should also involve the differing potentials of different basic input factors and the population growth responses to increased economic output.

The distribution of technology features does not only impact the probability of viability, but also just how profitable the technology will be if viable. Calculating the expected present value then involves two integrals. First one to accumulate discounted expected future profits into a present value for a given set of features. Then one which takes the expectation of these conditional present values across the distribution of features to obtain the unconditional expected present value. Note that this expected present value must be positive (excluding never-viable scenarios, where it is zero), as firms have been assumed to only operate when the immediate profit flow is positive.

The example implementation of the model manages to avoid most of the complications introduced by phase shifts and time. Except one, that being the shift when the focal technology spreads. Between the timing of that phase shift and the development of profit over time depending on its usage in both regions, the properly calculated expected value lacks a closed-form expression. However, by treating each region separately – that is, assuming that the research sector neglects the accelerated profit growth when active in both regions as well as profits in regions it isn't immediately profitable in – a closed-form expression can be derived.[7] This heuristic retains all fundamental features in the model, as profits are still perceived as higher if the technology is used in more regions, if profits are generally higher in the regions, and if the technology's efficiency increases faster. Therefore, note that the only result that wouldn't survive if the research sector calculated expected present value properly is that the final expression for long-run growth has a closed-form expression. The heuristic expression for $E(PV)$ (which is constant over time for the simple model) can be found at the end of appendix A (preceded by its derivation; placed there due to its bulky nature).

The expected present value for a successful invention enters into the research sector maximization problem on the revenue side. Their profits (equation 3.3.2) are maximized by:

$$R_{t,s} = \psi^{\frac{1}{1-\psi}} \gamma^{\frac{1}{1-\psi}} |M_s| E(PV_t)^{\frac{1}{1-\psi}} \tag{4.3.2}$$

Substituting this into the arrival rate of inventions (equation 3.3.1) results in:

$$\theta(R_{t,s}) = \psi^{\frac{\psi}{1-\psi}} \gamma^{\frac{1}{1-\psi}} |M_s| E(PV_t)^{\frac{\psi}{1-\psi}} \tag{4.3.3}$$

[7] This heuristic systematically undervalues the expected present value. This is not much of a problem for an analysis such as this one, as qualitative effects of parameters are the focus. However, if one wants to calculate quantitative predictions using this framework, one would either want to use a closer approximation or introduce some kind of increasing adjustment of the heuristic value.

There are two events which can change M_s. The first is that a new technology is invented, which adds an element as long as the technology is immediately viable at least somewhere. The second is when a technology's last active firm exits its region, removing that technology from the set, which is possible in some specifications with nonseperabilities; but from the alternative setups I have explored, this seems to be extremely rare. Thus, focusing on the first type of event, the expected rate of change for total number of active technologies is:

$$E(|\dot{M}_t|) = \left(1 - P(\tilde{G} = \emptyset)\right) \psi^{\frac{\psi}{1-\psi}} \gamma^{\frac{1}{1-\psi}} E(|M|) E(PV_t)^{\frac{\psi}{1-\psi}} \quad (4.3.4)$$

Recall that $E(PV_t)$ is from the perspective of the research sector while the other expectations here describe the expected movement of model variables in equilibrium. As $R_{t,s}$ and $\theta(R_{t,s})$ are proportional to $|M_s|$, they will grow at the same expected rate, which is:

$$E\left(\frac{\dot{R}_t}{R_t}\right) = E\left(\frac{\theta(\dot{R}_t)}{\theta(R_t)}\right) = E\left(\frac{|\dot{M}_t|}{|M_t|}\right) = \left(1 - P(\tilde{G} = \emptyset)\right) \psi^{\frac{\psi}{1-\psi}} \gamma^{\frac{1}{1-\psi}} E(PV_t)^{\frac{\psi}{1-\psi}} \quad (4.3.5)$$

The exponential growth in economic output in this model will follow from the exponential growth of $\theta(R_t)$. In the specific version of the model, this expression is constant as the expected present value of inventions does not change over time, leading to long-term exponential growth. The same is not guaranteed for all specifications, however, and if $E(PV_t)$ is diminishing over time, so will the growth rate be.

When the expression is constant, such as in the simple specification, equation 4.3.4 or 4.3.5 can easily be solved as differential equations, resulting in:

$$E(|M_t|) = e^{t\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}} |\widetilde{M}| \quad (4.3.6)$$

Where $\widetilde{M}$ is the set of technologies in use at $t = 0$.

4.4 Growth

Within phases, the economies are expanding, but at a less-than-exponential pace. For sustained long-term exponential economic growth, then, shifts in phase are necessary. These shifts, from new or spreading technologies (the latter of which are just delayed new technologies), cause jumps in the variables themselves or one of their time derivatives. In expectation, these appear as additive terms one time derivative deeper in the form of the arrival rate of that type of jump multiplied by the size of that jump. If the jumps (and their delays, if applicable) are constant in expectation, while their frequencies are growing exponentially, these terms grow exponentially. As all of these effects are integrated to the variable level, it will then consist of terms that grow exponentially and terms that grow less-than-exponentially. In the long run, the exponential terms dominate. Then, transforming output as a function of time into a growth rate leaves only the multiplier in the exponent remaining, showing that the final goods output $Y_{g,t}$ grows at the same as the arrival rate of inventions $\theta(R_t)$ in the long run, in expectation:

$$E\left(\frac{\dot{Y}_{g,t}}{Y_{g,t}}\right) = E\left(\frac{\theta(\dot{R_t})}{\theta(R_t)}\right) = \left(1 - P(\tilde{G} = \emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV_t)^{\frac{\psi}{1-\psi}} \tag{4.4.1}$$

This applies to all regions, although their output will be at different levels due to level effects. For a mathematical demonstration of this argument for the fully specified version of the model, see appendix B. To obtain the full closed-form expression, $\left(1 - P(\tilde{G} = \emptyset)\right)$ needs to be expanded depending on the relevant scenario (see the last subsection), and $E(PV_t)$ needs to be expanded using the expression in appendix A (also dependent on scenario); the latter of these is especially bulky, but at least both of these have very clear interpretations.

This argument of constant jumps but exponentially growing frequencies is essentially the same as that in Romer (1990), except adapted to randomly arriving inventions instead of continuous innovation. The argument holds as long as at least one of the jumps (including its delay, if applicable) is constant while its frequency of arrival grows exponentially; if either one of these change over time for all jumps, the regions grow at some non-exponential rate in the long run. This means the exponential-growth result is specific to the specific version of the model covered in detail here, and will not necessarily be part of all implementations of the framework; this result is especially sensitive to nonseperabilities, where jumps may fail to be constant, and frequencies may diminish over time. With that said, however, even less-than-exponential growth can be impressive for quite some time, and by the time it ceases to be impressive, the world may very well have run into the next impossible-to-model paradigm shift creating a new economic reality.

With final goods output and final goods usage in research both growing at the same rate for the world economy, the latter has to be some constant fraction of the former in the long run. The other use for final goods, consumption, must then take up the remaining fraction, and thus also grow at the same rate. This also extends to specific regions if new research expenditures source final goods from the regions in fixed ratios.

That the long-run growth rate in the closed-form specification includes the factor $\left(1 - P(\tilde{G} = \emptyset)\right)$, the probability of a new technology being initially profitable in at least one region, is the main new result of the model. While the exact expression for this probability changes with altered assumptions, the fundamental result still holds for more regions, more basic inputs, and more elaborate intermediate goods production functions. It should also be far more robust to nonseperabilities than the result that output grows exponentially is, as the required efficiency threshold must be rising faster than the technologies are improving for them to not be guaranteed to eventually spread to every region.

In the basic two-inputs two-regions case, each region contributes to this probability by covering a part of the line along which new technology features can vary. For more inputs, this space expands to more effective dimensions, with the total number being one less than the number of basic inputs, due to the constraint (that can also be relaxed) that the dependence degrees have to sum to one. Each region still captures part of this space, but the notion of the size and location of each covered region becomes richer. The more generally profitable a region is (the only general profitability feature unique to each region in the model as presented here is labor force, but this can be extended), the larger the area it covers is, and the more potential technologies it can save from being lost to unviability. Furthermore, the more a region's input cost environment differs from other regions, the farther away the center, so to speak, of its area will be from the others, decreasing the risk of their areas overlapping; in this sense, diversity in input costs also enhances long-term growth. Without nonseperabilities, the idea of increasing diversity pretty much only extends to having low prices for inputs that are expensive elsewhere, but with them, it may also somewhat apply to having higher prices for some (but not all) inputs, as technologies that use the more expensive inputs lightly may get a competitive advantage from being harmed far less by it than technologies that use the expensive inputs intensively. While the importance of general profitability features is found in most growth models driven by active research, the importance of diverse price conditions is not. This has the notable implication that any region can make meaningful contributions to the economic development of the world, by creating an environment where some niche ofnew technologies is able to thrive.

Additionally, some other, mostly unsurprising effects also feature in the model. Various technology-related parameters enhance growth if high, including efficiency in research, efficiency in technology improvement, and a high initial technology level for new technologies. General profitability and cost diversity also enter into the expected present value factor in the final growth rate, although the latter has almost no effect in this channel as it mostly just moves profits from one outcome to another, which would roughly be a net-zero effect when risk aversion isn't in play. A lower interest rate increases the value of future profits, incentivizing more research, but one must take care to avoid using different measures in comparisons involving changed interest rates, as the interest rate is supposed to implement consumers' intertemporal preferences given feasible growth paths. And finally, low fixed costs also increase profits, also increasing incentives to research more and thus bringing about more growth.

5 Conclusion

A closed-form model of exponential long-run growth has been derived, in which new technologies may or may not immediately be viable in various regions. As long as they're initially profitable at least somewhere, they can improve over time and eventually spread to all regions. This is the foundation for the key original result of this thesis, that the probability being viable in at least one region enters as a factor into the long-run growth rate. This probability, and growth in extension, is enhanced by having more profitable conditions in regions in general, but also by having diverse cost environments for inputs to intermediate goods production, letting the regions capture more of the feature space for new technologies together.

Furthermore, some implications of generalizing the framework have been discussed, including which results are and aren't robust to various types of changed assumptions. The importance of the probability of immediate viability and the role of cost environment diversity in determining that remain in face of most changes, with the latter concept becoming richer as more elaborations are introduced. The exponential nature of the growth path is not as robust, but with that said, this result should not be underestimated, as the key assumptions which lead to this result – an additively separable final goods production function and an innovation probability function whose growing factors' exponents sum to one – both have equally strong analogous assumptions in the original product-variety growth model (Romer, 1990).

This investigation into the relationship between economic growth and factor price conditions could be continued in a variety of ways. As a pure macroeconomic model, explicitly including consumption could bring more clarity into how the interest rate ties together consumption and production in equilibrium, as well as set the stage for welfare analyses. Similarly, modeling input markets could tie input prices to underlying degrees of scarcity. Numerical methods will likely be necessary for such analyses, however; numerical methods will likely also be needed for analyzing the various features only touched upon in the discussions of a general version of the model, such as more advanced intermediate goods production functions as well as competition between intermediate goods firms due to nonseperabilities in the final goods production function. Introducing features from trade theory could clarify the role of countries in the framework, which currently only considers the regional as well as international level, while implicitly assuming free spread of information and only allowing intermediate and input goods to be used in their local region. Apart from the degree of dependence on the basic inputs, all technologies are equal in this model, but as noted by Ley, Stucki, and Woerter (2016), among many others, usage of certain inputs is desirable, while usage of certain other inputs (fossil fuel, for instance) is not. The model already includes the effect that making preferred inputs relatively cheap will increase the bias of technology features toward those inputs, but it could still potentially be extended to include differing long-term potentials of different types of technologies as well as externalities from using various input goods. And finally, of course, empirical tests of the model would be valuable; while time series investigations face many difficulties, both economic output on the dependent side and input prices on the independent side are quite observable in theory, at least.

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Appendices

Appendix A: Perceived Expected Present Value of Inventions

Using the heuristic present value calculation method, the present value of profits (equation 4.1.4) in an active region for a new technology with given features is:

$$PV_{i,g} = \int_0^\infty e^{-r\tilde{t}} \left(L_g \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}} \right) \left(\frac{A_{i,\tilde{t}}}{c_{i,g}} \right)^{\frac{\beta}{1-\beta}} - C \right) d\tilde{t}$$

Where r is the discount parameter following from consumption preferences. Expanding this using the expression for $A_{i,\tilde{t}}$ (equation 4.2.3) yields:

$$PV_{i,g} = \int_0^\infty e^{-r\tilde{t}} \left(L_g \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}} \right) \left(\tilde{t} \nu \beta^{\frac{2\sigma}{1-\beta}} \frac{1-\beta-\sigma}{1-\beta} L_g^{\sigma} c_{i,g}^{-\frac{\sigma}{1-\beta}} + \tilde{A}^{\frac{1-\beta-\sigma}{1-\beta}} \right)^{\frac{\beta}{1-\beta-\sigma}} c_{i,g}^{-\frac{\beta}{1-\beta}} - C \right) d\tilde{t}$$

To proceed, $\beta/(1-\beta-\sigma)$ is assumed to equal a natural number. The method of expanding the expression and then integrating by parts works for any natural number, but the higher that number is, the more repeated iterations the method has to work through and the larger the expressions will be, without altering the qualitative features of the results; therefore, it is assumed to equal one, implying $\beta = 1-\beta-\sigma$. The per-country present value can then be expanded to:

$$PV_{i,g} = L_g \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}} \right) c_{i,g}^{-\frac{\beta}{1-\beta}} \left(\nu \beta^{\frac{2\sigma}{1-\beta}} \frac{\beta}{1-\beta} L_g^{\sigma} c_{i,g}^{-\frac{\sigma}{1-\beta}} \int_0^\infty e^{-r\tilde{t}} \tilde{t} d\tilde{t} + \tilde{A}^{\frac{\beta}{1-\beta}} \int_0^\infty e^{-r\tilde{t}} d\tilde{t} \right) - \frac{C}{r}$$

Which can then be expressed in closed form using integration by parts:

$$PV_{i,g} = L_g \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}} \right) r^{-1} \left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_g^{\sigma} c_{i,g}^{-1} r^{-1} + \tilde{A}^{\frac{\beta}{1-\beta}} c_{i,g}^{-\frac{\beta}{1-\beta}} \right) - \frac{C}{r} \qquad (A.1)$$

Then, for scenario A, the heuristic expected present value is:

$$E(PV) = P(\tilde{G} = \{1\}) E(PV_{i,1}) + P(\tilde{G} = \{1,2\}) \left(E(PV_{i,1}) + E(PV_{i,2}) \right)$$

Which becomes:

$$E(PV) = \int_0^{\tilde{\alpha}_1} PV_{i,1} d\alpha_{i,1} + \int_0^{\tilde{\alpha}_2} PV_{i,2} d\alpha_{i,1}$$

Expanding this using equation A.1 and the definition for $c_{i,g}$, and then splitting the integrals for each additive term results in:

$$E(PV) = L_1\left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)r^{-1}$$

$$* \left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_1^{\sigma} r^{-1} \int_0^{\tilde{\alpha}_1} \left(\alpha_{i,1}\left(p_{1,1} - p_{1,2}\right) + p_{1,2}\right)^{-1} d\alpha_{i,1}\right.$$

$$\left. + \tilde{A}^{\frac{\beta}{1-\beta}} \int_0^{\tilde{\alpha}_1} \left(\alpha_{i,1}\left(p_{1,1} - p_{1,2}\right) + p_{1,2}\right)^{-\frac{\beta}{1-\beta}} d\alpha_{i,1}\right) + L_2\left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)r^{-1}$$

$$* \left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_2^{\sigma} r^{-1} \int_0^{\tilde{\alpha}_2} \left(\alpha_{i,1}\left(p_{2,1} - p_{2,2}\right) + p_{2,2}\right)^{-1} d\alpha_{i,1}\right.$$

$$\left. + \tilde{A}^{\frac{\beta}{1-\beta}} \int_0^{\tilde{\alpha}_2} \left(\alpha_{i,1}\left(p_{2,1} - p_{2,2}\right) + p_{2,2}\right)^{-\frac{\beta}{1-\beta}} d\alpha_{i,1}\right) - (\tilde{\alpha}_1 + \tilde{\alpha}_2)\frac{C}{r}$$

Then, using integration by substitution:

$$E(PV) = \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)r^{-1}$$

$$* \left(L_1\left(p_{1,1} - p_{1,2}\right)^{-1}\left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_1^{\sigma} r^{-1}\left(\log\left(\tilde{\alpha}_1\left(p_{1,1} - p_{1,2}\right) + p_{1,2}\right)\right.\right.\right.$$

$$\left.\left. - \log\left(p_{1,2}\right)\right) + \tilde{A}^{\frac{\beta}{1-\beta}} \frac{1-\beta}{1-2\beta}\left(\left(\tilde{\alpha}_1\left(p_{1,1} - p_{1,2}\right) + p_{1,2}\right)^{\frac{1-2\beta}{1-\beta}} - p_{1,2}^{\frac{1-2\beta}{1-\beta}}\right)\right)$$

$$+ L_2\left(p_{2,1} - p_{2,2}\right)^{-1}\left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_2^{\sigma} r^{-1}\left(\log\left(\tilde{\alpha}_2\left(p_{2,1} - p_{2,2}\right) + p_{2,2}\right)\right.\right.$$

$$\left.\left.\left. - \log\left(p_{2,2}\right)\right) + \tilde{A}^{\frac{\beta}{1-\beta}} \frac{1-\beta}{1-2\beta}\left(\left(\tilde{\alpha}_2\left(p_{2,1} - p_{2,2}\right) + p_{2,2}\right)^{\frac{1-2\beta}{1-\beta}} - p_{2,2}^{\frac{1-2\beta}{1-\beta}}\right)\right)\right)$$

$$- (\tilde{\alpha}_1 + \tilde{\alpha}_2) C r^{-1}$$

Which, when expanded using the expression for $\tilde{\alpha}_g$ (equation 4.3.1), yields the final expression:

$$E(PV) = \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right) r^{-1}$$

$$* \left(\frac{1}{p_{1,1} - p_{1,2}} L_1 \left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_1^{\sigma} r^{-1} \left(\log\left(\tilde{A} \left(\frac{L_1}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} \right) \right.\right.\right.$$

$$\left.\left. - \log(p_{1,2}) \right) + \frac{1-\beta}{1-2\beta} \left(\tilde{A} \left(\frac{L_1}{C}\right)^{\frac{1-2\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-2\beta}{\beta}} - \tilde{A}^{\frac{\beta}{1-\beta}} p_{1,2}^{\frac{1-2\beta}{1-\beta}} \right) \right)$$

$$+ \frac{1}{p_{2,1} - p_{2,2}} L_2 \left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_2^{\sigma} r^{-1} \left(\log\left(\tilde{A} \left(\frac{L_2}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} \right) \right.\right.$$

$$\left.\left.\left. - \log(p_{2,2}) \right) + \frac{1-\beta}{1-2\beta} \left(\tilde{A} \left(\frac{L_2}{C}\right)^{\frac{1-2\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-2\beta}{\beta}} - \tilde{A}^{\frac{\beta}{1-\beta}} p_{2,2}^{\frac{1-2\beta}{1-\beta}} \right) \right) \right)$$

$$- \left(\frac{\tilde{A} \left(\frac{L_1}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} - p_{1,2}}{p_{1,1} - p_{1,2}} \right.$$

$$\left. + \frac{\tilde{A} \left(\frac{L_2}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} - p_{2,2}}{p_{2,1} - p_{2,2}} \right) C r^{-1}$$

For scenario B:

$$E(PV) = P(\tilde{G} = \{1\}) E(PV_{i,1}) + P(\tilde{G} = \{2\}) E(PV_{i,2})$$

$$E(PV) = \int_0^{\tilde{\alpha}_1} PV_1 \, d\alpha_{i,1} + \int_{\tilde{\alpha}_2}^{1} PV_2 \, d\alpha_{i,1}$$

Which is solved the same way, having the similar final expression:

$$E(PV) = \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right) r^{-1}$$

$$* \left(\frac{1}{p_{1,1} - p_{1,2}} L_1 \left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_1^{\sigma} r^{-1} \left(\log\left(\tilde{A} \left(\frac{L_1}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} \right) \right.\right.\right.$$

$$\left.\left. - \log(p_{1,2}) \right) + \frac{1-\beta}{1-2\beta} \left(\tilde{A} \left(\frac{L_1}{C}\right)^{\frac{1-2\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-2\beta}{\beta}} - \tilde{A}^{\frac{\beta}{1-\beta}} p_{1,2}^{\frac{1-2\beta}{1-\beta}} \right) \right)$$

$$+ \frac{1}{p_{2,2} - p_{2,1}} L_2 \left(\nu \frac{\beta^{\frac{1-\beta+2\sigma}{1-\beta}}}{1-\beta} L_2^{\sigma} r^{-1} \left(\log\left(\tilde{A} \left(\frac{L_2}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} \right) \right.\right.$$

$$\left.\left.\left. - \log(p_{2,1}) \right) + \frac{1-\beta}{1-2\beta} \left(\tilde{A} \left(\frac{L_2}{C}\right)^{\frac{1-2\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-2\beta}{\beta}} - \tilde{A}^{\frac{\beta}{1-\beta}} p_{2,1}^{\frac{1-2\beta}{1-\beta}} \right) \right) \right)$$

$$- \left(\frac{\tilde{A} \left(\frac{L_1}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} - p_{1,2}}{p_{1,1} - p_{1,2}} \right.$$

$$\left. + \frac{\tilde{A} \left(\frac{L_2}{C}\right)^{\frac{1-\beta}{\beta}} \left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}} - p_{2,1}}{p_{2,2} - p_{2,1}} \right) C r^{-1}$$

Appendix B: Detailed Derivation of Long-Run Growth Rate

In the closed-form model, from the perspective of a single region, three phase shifts are possible: A new technology is introduced to the region. An old technology is introduced to the region. And a technology already viable in the focal region is introduced to the other.

The first two of these have direct effects on the region's economic output (equation 4.1.3), and the steps they bring about are, for new or spreading technology i, respectively:

$$Y_{g,t,s+1} - Y_{g,t,s} = L_g \beta^{\frac{2\beta}{1-\beta}} \left(\frac{\tilde{A}}{c_{i,g}}\right)^{\frac{\beta}{1-\beta}}$$

$$Y_{g,t,s+1} - Y_{g,t,s} = L_g \beta^{\frac{2\beta}{1-\beta}} \left(\frac{\hat{A}_i}{c_{i,g}} \right)^{\frac{\beta}{1-\beta}} = C(\beta - \beta^2)^{-1}$$

The latter step is definitely constant, while the former step is random, but does have a constant expectation.

All of the shifts cause kinks in the trajectory of the region's output. But to see how large, output's rate of change within phases is first needed. This can be derived by differentiating equation 4.1.3 with respect to time, and then expanding and simplifying using 4.2.1 and 4.2.2 while using the $1 - \beta - \sigma = \beta$ simplification from appendix A, ultimately resulting in:

$$\dot{Y}_{g,t,s} = \nu L_g \frac{\beta^{\frac{1+\beta+2\sigma}{1-\beta}}}{1-\beta} \sum_{i \in M_{g,s}} c_{i,g}^{-\frac{\beta}{1-\beta}} \sum_{h \in G_{i,s}} L_h^\sigma c_{i,h}^{-\frac{\sigma}{1-\beta}}$$

Notably, when that exponent is assumed to be one as in appendix A, output grows linearly within phases. This rate of change changes by, for the first two types of phase shifts and for the last type (spreading to the other region h), respectively:

$$\dot{Y}_{g,t,s+1} - \dot{Y}_{g,t,s} = \nu L_g \frac{\beta^{\frac{1+\beta+2\sigma}{1-\beta}}}{1-\beta} c_{i,g}^{-\frac{\beta}{1-\beta}} \sum_{h \in G_{i,s+1}} L_h^\sigma c_{i,h}^{-\frac{\sigma}{1-\beta}}$$

$$\dot{Y}_{g,t,s+1} - \dot{Y}_{g,t,s} = \nu L_g \frac{\beta^{\frac{1+\beta+2\sigma}{1-\beta}}}{1-\beta} c_{i,g}^{-\frac{\beta}{1-\beta}} L_h^\sigma c_{i,h}^{-\frac{\sigma}{1-\beta}}$$

These changes are also constant in expectation.

The effect of the step changes on the first derivative of expected output sum to (using equations 4.3.3 and 4.3.6):

$$L_g \beta^{\frac{2\beta}{1-\beta}} \tilde{A}^{\frac{\beta}{1-\beta}} E\big(\theta(R_t)\big) P\big(g \in \tilde{G}\big) E\left(c_{i,g}^{-\frac{\beta}{1-\beta}} \right) + E\big(\theta(R_{t-\hat{t}})\big) P\big(g \notin \tilde{G}, \tilde{G} \neq \emptyset\big) C(\beta - \beta^2)^{-1}$$

$$L_g \beta^{\frac{2\beta}{1-\beta}} \tilde{A}^{\frac{\beta}{1-\beta}} E\big(\theta(R_t)\big) P\big(g \in \tilde{G}\big) E\left(c_{i,g}^{-\frac{\beta}{1-\beta}} \right)$$

$$+ \psi^{\frac{\psi}{1-\psi}} \gamma^{\frac{1}{1-\psi}} E(PV)^{\frac{\psi}{1-\psi}} E\left(e^{(t-\hat{t})(1-P(\tilde{G}=\emptyset))\psi^{\frac{\psi}{1-\psi}} \gamma^{\frac{1}{1-\psi}} E(PV)^{\frac{\psi}{1-\psi}}} \right) |\widetilde{M}|$$

$$* P\big(g \notin \tilde{G}, \tilde{G} \neq \emptyset\big) C(\beta - \beta^2)^{-1}$$

$$E\big(\theta(R_t)\big)\left(L_g\beta^{\frac{2\beta}{1-\beta}}\tilde{A}^{\frac{\beta}{1-\beta}}P\big(g\in\tilde{G}\big)E\left(c_{i,g}^{-\frac{\beta}{1-\beta}}\right)\right.$$

$$\left.+E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\right)P\big(g\notin\tilde{G},\tilde{G}\neq\emptyset\big)C(\beta-\beta^2)^{-1}\right)$$

Note that these expectations can be split because the fundamental timing of phase shifts depends only on events before the discovery of the focal technology, while the delay and expected jump size depend on the features of the focal technology, making them independent random variables (while jump sizes and delays are jointly distributed). Here, the expectation of the first type of shift can be solved in closed form (for the first region, either scenario A or B), while the latter can't:

$$E\big(\theta(R_t)\big)\left(L_g\beta^{\frac{2\beta}{1-\beta}}\tilde{A}^{\frac{\beta}{1-\beta}}\big(p_{g,1}-p_{g,2}\big)^{-1}\frac{1-\beta}{1-2\beta}\right.$$

$$*\left(\tilde{A}^{\frac{1-2\beta}{1-\beta}}\left(\frac{L_g}{C}\right)^{\frac{1-2\beta}{\beta}}\left(\beta^{\frac{1+\beta}{1-\beta}}-\beta^{\frac{2}{1-\beta}}\right)^{\frac{1-2\beta}{\beta}}-p_{g,2}^{\frac{1-2\beta}{1-\beta}}\right)$$

$$\left.+E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\right)P\big(g\notin\tilde{G},\tilde{G}\neq\emptyset\big)C(\beta-\beta^2)^{-1}\right)$$

The kink effects sum to, using similar steps:

$$\nu L_g\frac{\beta^{\frac{1+\beta+2\sigma}{1-\beta}}}{1-\beta}\left(E\big(\theta(R_t)\big)P\big(g\in\tilde{G}\big)L_g^{\sigma}E\big(c_{i,g}^{-1}\big)+E\big(\theta(R_t)\big)P\big(\tilde{G}=G\big)L_h^{\sigma}E\left(c_{i,g}^{-\frac{\beta}{1-\beta}}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right.$$

$$+P\big(g\notin\tilde{G},\tilde{G}\neq\emptyset\big)E\left(\theta(R_{t-\hat{t}})\left(L_g^{\sigma}c_{i,g}^{-1}+c_{i,g}^{-\frac{\beta}{1-\beta}}L_h^{\sigma}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

$$\left.+L_h^{\sigma}P\big(\tilde{G}=\{g\}\big)E\left(\theta(R_{t-\hat{t}})c_{i,g}^{-\frac{\beta}{1-\beta}}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

$$\nu L_g \frac{\beta^{\frac{1+\beta+2\sigma}{1-\beta}}}{1-\beta} E\big(\theta(R_t)\big)\left(L_g^{\sigma}\big(p_{g,1}-p_{g,2}\big)^{-1}\left(\log\left(\tilde{A}\left(\frac{L_g}{C}\right)^{\frac{1-\beta}{\beta}}\left(\beta^{\frac{1+\beta}{1-\beta}}-\beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}}\right)\right.\right.$$

$$\left.\left. -\log\big(p_{g,2}\big)\right) + L_h^{\sigma} P\big(\tilde{G}=G\big) E\left(c_{i,g}^{-\frac{\beta}{1-\beta}} c_{i,h}^{-\frac{\sigma}{1-\beta}}\right) + P\big(g\notin\tilde{G},\tilde{G}\neq\emptyset\big)$$

$$* E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\left(L_g^{\sigma}c_{i,g}^{-1}+c_{i,g}^{-\frac{\beta}{1-\beta}}L_h^{\sigma}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

$$\left. + L_h^{\sigma}P\big(\tilde{G}=\{g\}\big)E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}c_{i,g}^{-\frac{\beta}{1-\beta}}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

Adding all of them together (kink effects integrated with respect to time):

$$E\big(\dot{Y}_{g,t}\big) = E\big(\theta(R_t)\big)\left(L_g \beta^{\frac{2\beta}{1-\beta}}\tilde{A}^{\frac{\beta}{1-\beta}}\big(p_{g,1}-p_{g,2}\big)^{-1}\frac{1-\beta}{1-2\beta}\right.$$

$$* \left(\tilde{A}^{\frac{1-2\beta}{1-\beta}}\left(\frac{L_g}{C}\right)^{\frac{1-2\beta}{\beta}}\left(\beta^{\frac{1+\beta}{1-\beta}}-\beta^{\frac{2}{1-\beta}}\right)^{\frac{1-2\beta}{\beta}} - p_{g,2}^{\frac{1-2\beta}{1-\beta}}\right)$$

$$\left. + E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\right)P\big(g\notin\tilde{G},\tilde{G}\neq\emptyset\big)C\big(\beta-\beta^2\big)^{-1}\right)$$

$$+ \frac{E\big(\theta(R_t)\big)}{\Big(1-P\big(\tilde{G}=\emptyset\big)\Big)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}$$

$$* \nu L_g \frac{\beta^{\frac{1+\beta+2\sigma}{1-\beta}}}{1-\beta}\left(L_g^{\sigma}\big(p_{g,1}-p_{g,2}\big)^{-1}\left(\log\left(\tilde{A}\left(\frac{L_g}{C}\right)^{\frac{1-\beta}{\beta}}\left(\beta^{\frac{1+\beta}{1-\beta}}-\beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}}\right)\right.\right.$$

$$\left.\left. -\log\big(p_{g,2}\big)\right) + L_h^{\sigma} P\big(\tilde{G}=G\big) E\left(c_{i,g}^{-\frac{\beta}{1-\beta}} c_{i,h}^{-\frac{\sigma}{1-\beta}}\right) + P\big(g\notin\tilde{G},\tilde{G}\neq\emptyset\big)$$

$$* E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\left(L_g^{\sigma}c_{i,g}^{-1}+c_{i,g}^{-\frac{\beta}{1-\beta}}L_h^{\sigma}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

$$\left. + L_h^{\sigma}P\big(\tilde{G}=\{g\}\big)E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}c_{i,g}^{-\frac{\beta}{1-\beta}}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

Technically there should be a constant of integration here, and there should be two for the next expression, aligning these functions with actual values, but because they only grow linearly and quadratically they get overshadowed by the exponential terms in the long run. Then, by integrating the last expression with respect to time, output in the long run as a function of time is:

$$E(Y_{g,t}) = \frac{E(\theta(R_t))}{\left(1 - P(\tilde{G} = \emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\left(L_g \beta^{\frac{2\beta}{1-\beta}}\tilde{A}^{\frac{\beta}{1-\beta}}(p_{g,1} - p_{g,2})^{-1}\frac{1-\beta}{1-2\beta}\right.$$

$$* \left(\tilde{A}^{\frac{1-2\beta}{1-\beta}}\left(\frac{L_g}{C}\right)^{\frac{1-2\beta}{\beta}}\left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-2\beta}{\beta}} - p_{g,2}^{\frac{1-2\beta}{1-\beta}}\right)$$

$$+ E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\right)P(g \notin \tilde{G}, \tilde{G} \neq \emptyset)C(\beta - \beta^2)^{-1}$$

$$+ \nu L_g \frac{\beta^{\frac{1+\beta+2\sigma}{1-\beta}}}{1-\beta}\left(L_g^{\sigma}(p_{g,1} - p_{g,2})^{-1}\left(\log\left(\tilde{A}\left(\frac{L_g}{C}\right)^{\frac{1-\beta}{\beta}}\left(\beta^{\frac{1+\beta}{1-\beta}} - \beta^{\frac{2}{1-\beta}}\right)^{\frac{1-\beta}{\beta}}\right)\right.\right.$$

$$\left. - \log(p_{g,2})\right) + L_h^{\sigma}P(\tilde{G} = G)E\left(c_{i,g}^{-\frac{\beta}{1-\beta}}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right) + P(g \notin \tilde{G}, \tilde{G} \neq \emptyset)$$

$$* E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}\left(L_g^{\sigma}c_{i,g}^{-1} + c_{i,g}^{-\frac{\beta}{1-\beta}}L_h^{\sigma}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

$$\left. + L_h^{\sigma}P(\tilde{G} = \{g\})E\left(e^{-\hat{t}\left(1-P(\tilde{G}=\emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}}c_{i,g}^{-\frac{\beta}{1-\beta}}c_{i,h}^{-\frac{\sigma}{1-\beta}}\right)\right)$$

$$\left. * \left(\left(1 - P(\tilde{G} = \emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}\right)^{-1}\right)$$

Now, while neither of these expressions (notably including the level of output) are closed-form – their long-run ratio is, and that ratio is the long-run growth rate. Dividing the former by the latter verifies equation 4.4.1 for the closed-form specification:

$$E\left(\frac{\dot{Y}_{g,t}}{Y_{g,t}}\right) = \left(1 - P(\tilde{G} = \emptyset)\right)\psi^{\frac{\psi}{1-\psi}}\gamma^{\frac{1}{1-\psi}}E(PV)^{\frac{\psi}{1-\psi}}$$