NORMAL FAMILIES AND PICARD'S GREAT THEOREM

ENA JAHIC

Bachelor's thesis
2022:K5

LUND UNIVERSITY

Faculty of Science
Centre for Mathematical Sciences
Mathematics

CENTRUM SCIENTIARUM MATHEMATICARUM

Abstract

We establish the theory of normal families of meromorphic functions taking values in the extended complex plane, which can be regarded as a metric space equipped with the spherical metric. Using the notion of spherical derivatives, we state and prove Marty's theorem on normal families of meromorphic functions. As a result, we deduce the classical Montel's theorem on relatively compact families of analytic functions. With this knowledge, we obtain the fundamental normality test - Montel's three value theorem and prove it using the lemma of Zalcman. Applying these results, the proof of the celebrated Picard's great theorem easily follows.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given.

Acknowledgments

I am grateful to my supervisor Yacin Ameur for introducing me to this interesting topic and for his invaluable guidance and support throughout the writing of this thesis.

I would also like to thank my family and friends for their support throughout my studies in Lund.

Popular Scientific Summary

At the heart of the complex function theory lies the notion of a normal family of meromorphic functions. The subject has permeated through Picard's theorems, the Riemann mapping theorem, and many modern results such as the Bloch principle. This thesis deals with the concept of convergence of sequences of meromorphic functions taking values in the extended complex plane. A family of meromorphic functions is said to be normal if each sequence in the family converges uniformly in the chordal metric on each compact subset of the domain. This work culminates in proving different normality criteria for families of meromorphic functions, such as Marty's theorem (a criterion that uses the notion of spherical derivatives); Montel's three-value theorem (fundamental normality test); and the great Picard's theorem.

This work is dedicated to my parents and brothers

Contents

Chapter 1

Introduction

1.1 Historical Background

The beginning of a modern complex analysis dates back to 1907 when *P. Montel* introduced concepts of compactness into complex analysis [8]. In 1912, Montel presented the term *normal families* of analytic functions in his thesis, describing that the locally uniformly bounded family of analytic functions is normal [7]. Relying on the *Arzelá-Ascoli theorem* that is valid for any compact metric space, *Montel's theorem* became one of the key ingredients of the proof of the *Riemann mapping theorem* [2]. Here arises a question, what happens when we consider functions taking values at the point at infinity? The answer to this question was given by *F. Marty* in his dissertation in 1931 where he considered a family of meromorphic functions on the extended complex plane $\hat{\mathbb{C}}$ [6]. *Marty's theorem* became one of the commonly used criteria for determining the normality of such families. It can also be used for deriving some classical results in complex analysis such as *Picard's theorems* on omitted values.

1.2 General Notation

By a domain we mean an open and connected subset of the complex plane.

$\mathbb{R}$	Real line
$\mathbb{R}^n$	Real coordinate space of dimension n

$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$	Euclidean norm of $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$
$\mathbb{C} = \{x + iy\ ;\ x, y \in \mathbb{R}\}$	Complex numbers
$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$	Extended complex plane
$\mathbb{D} = \{z \in \mathbb{C}\ ;\ \|z\| < 1\}$	Unit disc
$\partial\mathbb{D} = \{z \in \mathbb{C}\ ;\ \|z\| = 1\}$	Circle
$D_\delta(z_0) = \{z \in \mathbb{C}\ ;\ \|z - z_0\| < \delta\}$ for $z_0 \in \mathbb{C}, \delta > 0$	Disc with center at z_0 and radius $\delta > 0$
$\mathbb{A}_{r,R} = \{z\ ;\ r < \|z\| < R\}$ for $0 \leq r < R \leq \infty$	Annulus
$B_\delta(x_0) = \{x\ ;\ \\|x - x_0\\| < \delta\}$	$n-$dimensional open ball with center x and radius $\delta > 0$
K	Compact Hausdorff space
$C(K)$	Continuous complex-valued functions on K
$\inf, \sup$	Infimum, Supremum
$\\|f\\|_K = \sup_{z \in K} \|f(z)\|$	Supremum-norm
$\sigma(z, w), z, w \in \hat{\mathbb{C}}$	Chordal metric
$f^\sharp$	Spherical derivative

Chapter 2

Compactness of Families of Continuous Functions

In this chapter, we collect some preliminaries from real analysis which are used throughout the thesis. As a general source on these matters, we refer to Gamelin's book [3].

The main result for our purposes is the Arzelá-Ascoli theorem, which characterizes the space $C(K, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued continuous functions on a compact set K with the metric of uniform convergence.

Preliminaries

We assume that the reader is familiar with the notion of a metric space $\mathcal{X} = (\mathcal{X}, d)$. In particular we assume some basic familiarity with the notion of a compact subset of $\mathcal{X}$, which can be defined either using sequential compactness (each sequence has a convergent subsequence) or by open coverings.

It is convenient to recall a few further basic definitions and basic results that we take for granted (cf. [3]):

Definition 2.1. A sequence $\{f_n\}_{n=1}^{\infty}$ in $(\mathcal{X}, d)$ is said to be a Cauchy sequence if

$$\lim_{n,m\to\infty} d(f_n, f_m) = 0,$$

that is, if for each $\varepsilon > 0$ there exists an N such that $d(f_n, f_m) < \varepsilon$ for all $n, m > N$.

Definition 2.2. $\mathcal{X}$ is said to be complete if every Cauchy sequence in $\mathcal{X}$ converges.

Theorem 2.3. (Heine-Borel Theorem). Let K be a subset of $\mathbb{C}$. Then K is compact if and only if at least one of the following equivalent conditions hold:

(a) K is closed and bounded.

(b) Each sequence $\{z_n\}_1^\infty \subset K$ has a convergent subsequence whose limit is also in K.

(c) Each open cover has a finite subcover.

Definition 2.4. Given a compact subset K of $\mathbb{C}$ we denote by $C(K, \mathbb{R}^n)$ the space of all continuous functions $f\colon K \to \mathbb{R}^n$ equipped with the sup-norm metric:

$$d(f, g) = \sup_{z \in K} \|f(z) - g(z)\|. \tag{2.1}$$

Definition 2.5. A sequence of functions $\{f_n(z)\}$ on K is said to *converge uniformly* to $f(z)$ if and only if $d(f_n, f) \to 0$ as $n \to \infty$.

Theorem 2.6. *If K is compact then each continuous function f in $C(K, \mathbb{R}^n)$ is uniformly continuous on K.*

We now have the main tools that we need. As an illustration we now prove the basic fact that the metric space $C(K, \mathbb{R}^n)$ is complete.

Theorem 2.7. *$C(K, \mathbb{R}^n)$ with the metric* (2.1) *is a complete metric space.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $C(K, \mathbb{R}^n)$. Then for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $d(f_n, f_m) < \varepsilon$ for all $n, m > N$. Thus for all $z \in K, \{f_n(z)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, that is, every Cauchy sequence in $\mathbb{C}$ converges, we can define $f\colon K \to \mathbb{C}$ as the pointwise limit $f(z) = \lim_{n\to\infty} f_n(z)$.

We claim that in fact f_n converges uniformly to f. Indeed if $m > n > N$ and $z \in K$ then, since $d(f_m, f_n) < \varepsilon$,

$$\|f(z)-f_n(z)\| \leq \|f(z)-f_m(z)\|+\|f_m(z)-f_n(z)\| < \|f(z)-f_m(z)\|+\varepsilon.$$

Here $m > n$ is completely arbitrary, so we may send $m \to \infty$. Using that $f_m(z) \to f(z)$ as $m \to \infty$ by the pointwise convergence we find $\|f_n(z) - f(z)\| \leq \varepsilon$. Since $z \in K$ is arbitrary we have shown $d(f_n, f) \leq \varepsilon$ when $n > N$. That is, $f_n \to f$ uniformly.

Next we want to show that f is continuous on K. To do this fix $n_0 > N$. Since f_{n_0} is uniformly continuous, by Theorem 2.6 above, there exists $\delta > 0$ such that for all $z, w \in K$

$$|z - w| < \delta \implies \|f_{n_0}(z) - f_{n_0}(w)\| < \varepsilon.$$

Hence for all $z, w \in K$, if $|z - w| < \delta$, it follows that (if n_0 is large enough)

$$\|f(z)-f(w)\| \leq \|f(z)-f_{n_0}(z)\|+\|f_{n_0}(z)-f_{n_0}(w)\|+\|f_{n_0}(w)-f(w)\| < 3\varepsilon.$$

This shows that $f \in C(K, \mathbb{R}^n)$ which completes the proof.

□

Families of Continuous Functions and Arzelá-Ascoli's Theorem

In the following we fix a compact subset K of $\mathbb{C}$, and let $\mathcal{F} \subset C(K, \mathbb{R}^n)$ be a family of continuous functions on K. We recall the following basic definitions.

Definition 2.8. $\mathcal{F}$ is said to be *equicontinuous* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|z - w| < \delta \implies \|f(z) - f(w)\| < \varepsilon$ for all $f \in \mathcal{F}$ simultaneously.

Definition 2.9. $\mathcal{F}$ is said to be *uniformly bounded* if there is a constant $M > 0$ such that $|f(z)| \leq M$ for all $f \in \mathcal{F}$ and all $z \in K$.

Definition 2.10. $\mathcal{F}$ is said to be *relatively compact* if each sequence of functions $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ has a uniformly convergent subsequence, converging to some function $f \in C(K, \mathbb{R}^n)$. (We do not require that the limit function f be in the class $\mathcal{F}$.)

We are now ready to state and prove the main result of this section.

Theorem 2.11 (Arzelá-Ascoli Theorem). *Let $\mathcal{F} \subset C(K, \mathbb{R}^n)$ uniformly bounded. Then the following are equivalent.*

(a) $\mathcal{F}$ is equicontinuous.

(b) Each sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly on K, that is, $\mathcal{F}$ is relatively compact.

To prepare for the proof, we recall the following definitions from set-theory.

Definition 2.12. A set X is said to be *countable* if there is an enumeration $\{x_n\}_{n=1}^{\infty}$ of its elements, that is, if we can write $X = \{x_n\ ;\ n \in \mathbb{Z}_+\}$.

Lemma 2.13. *The Cartesian product of two countable sets is countable.*

Proof. Let X and Y be countable sets. Let $X = \{x_j\}_{j=1}^{\infty}$ and $Y = \{y_k\}_{k=1}^{\infty}$ be enumerations of X and Y respectively. To prove that there is an enumeration of all pairs (x_j, y_k), we list the pairs (x_j, y_k) with $j + k = 2$, that is, (x_1, y_1), then the pairs with $j + k = 3$, that is, $(x_1, y_2), (x_2, y_1)$, then $j + k = 4$, $(x_1, y_3), (x_2, y_2), (x_3, y_1)$, etc. Eventually, each pair will be on the list. □

It follows that the set of rational numbers $\mathbb{Q} = \{\frac{a}{b}\ ;\ a \in \mathbb{Z}, b \in \mathbb{Z}_+\}$ is countable, since it may be viewed as a subset of the set $\mathbb{Z} \times \mathbb{Z}_+$. (And a subset of a countable set is countable.) Using countability of $\mathbb{Q}$ and applying Lemma 2.13 $(n-1)$ times, it can be seen that $\mathbb{Q}^n$ is a countable set.

Definition 2.14. A subset X of a metric space $(\mathcal{X}, d)$ is said to be *dense* if for each $x \in \mathcal{X}$ and each $\varepsilon > 0$, there is $y \in X$ such that $d(x, y) < \varepsilon$.

Definition 2.15. A metric space $(\mathcal{X}, d)$ is said to be *separable* if there is a countable and dense subset.

Thus $\mathbb{R}^n$ is separable, since $\mathbb{Q}^n$ is a countable and dense subset. In particular, the complex plane $\mathbb{C}$ is separable, since $\mathbb{C}$ and $\mathbb{R}^2$ coincide as metric spaces.

A subset K of a metric space $(\mathcal{X}, d)$ is a metric space itself if we equip it with metric d of $\mathcal{X}$. The metric space (K, d) is called a *subspace* of $(\mathcal{X}, d)$. A subspace of a separable metric space is again separable. In our case $K \subset \mathbb{C}$ so K is separable since $\mathbb{C}$ is. Thus we can be assured that there exists a countable and dense subset $\{w_j\}_{j=1}^{\infty}$ of K.

Now we are equipped with the necessary material to prove the Theorem 2.11.

Proof of Theorem 2.11. The proof that (b) implies (a) is proof by contradiction. Assume that $\mathcal{F}$ is relatively compact. It is left to prove that $\mathcal{F}$ is equicontinuous. If we assume the opposite, then there would exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exist points $z, w \in K$ and $f \in \mathcal{F}$ so that

$$|z - w| < \delta \text{ and } \|f(z) - f(w)\| \geq 2\varepsilon.$$

Pick $\delta = \frac{1}{n}$. Then for each $n \in \mathbb{Z}^+$, we can find points $z_n, w_n \in K$ and $f_n \in \mathcal{F}$ so that

$$|z_n - w_n| < \frac{1}{n} \text{ and } \|f_n(z_n) - f_n(w_n)\| \geq 2\varepsilon.$$

To obtain a contradiction, we assume that $f_n \to f$ uniformly on K, where $f \in C(K, \mathbb{R}^n)$. Since f is uniformly continuous there is $\delta > 0$ such that for all $z, w \in K$ with $|z - w| < \delta$ we have $\|f(z) - f(w)\| < \frac{\varepsilon}{3}$.

By the uniform convergence we can pick N large enough that $d(f_n, f) < \frac{\varepsilon}{3}$ whenever $n \geq N$. Choosing N somewhat larger we can assume that $\frac{1}{N} < \delta$. Then for $n \geq N$,

$$\begin{aligned}\|f_n(z_n) - f_n(w_n)\| &\leq \|f_n(z_n) - f(z_n)\| \\ &\quad + \|f(z_n) - f(w_n)\| + \|f(w_n) - f_n(w_n)\| < \varepsilon.\end{aligned}$$

We have reached a contradiction since $\|f_n(z_n) - f_n(w_n)\| \geq 2\varepsilon$ for all n. The contradiction shows that the f_n's can't converge uniformly. In a similar way, we see that no subsequence of the f_n's can converge uniformly. Hence if each sequence f_n in $\mathcal{F}$ has a uniformly convergent subsequence, then $\mathcal{F}$ must be equicontinuous.

We now prove that (a) implies (b). Let $\{f_n\}$ be an arbitrary sequence in $\mathcal{F}$. Fix $\{w_j\}_{j=1}^{\infty}$ an arbitrary countable and dense subset of K. By the assumption, $\mathcal{F}$ is uniformly bounded, so the sequence of complex numbers $\{f_n(w_1)\}_{n=1}^{\infty}$ is bounded. By the *Bolzano-Weierstrass theorem*, we can find a convergent subsequence $\{f_{n_k}(w_1)\}$, which we rename as $\{f_{n,1}(w_1)\}$. The sequence $\{f_{n,1}(w_2)\}$ is bounded, and we choose $\{f_{n,2}(w_2)\}$ a convergent subsequence. Repeating the process, form a diagonal subsequence $\{g_n\} = \{f_{n,n}\}$ of the sequence $\{f_n\}$. It holds that $\{g_n(w_j)\}$ converges as $n \to \infty$ for each j.

Claim: $\{g_n\}$ converges uniformly on K, that is, $\{g_n\}$ is a *uniform Cauchy* sequence. To prove the claim, using the equicontinuity of $\mathcal{F}$, it is possible to fix $\varepsilon > 0$ and pick $\delta > 0$ such that

$$|z - w| < \delta \implies \|g(z) - g(w)\| < \frac{\varepsilon}{3} \tag{2.2}$$

for each n, and each $g \in \mathcal{F}$.

Secondly, consider discs $D_\delta(w_j)$, for $j = 1, 2, \ldots$ forming an open cover of K. Since K is compact, by the *Heine-Borel theorem* and renumbering the indices, we can pick a finite subcover $D_\delta(w_1), \ldots, D_\delta(w_N)$. Furthermore, fix n_0 such that

$$m, n \geq n_0 \implies \|g_n(w_j) - g_m(w_j)\| < \frac{\varepsilon}{3} \tag{2.3}$$

for $j = 1, 2, \ldots, N$.

Take arbitrary $z \in K$. Then $z \in D_\delta(w_j)$ for some $j \leq N$. Hence if $m, n \geq n_0$, it follows that

$$\|g_n(z)-g_m(z)\| \leq \|g_n(z)-g_n(w_j)\|+\|g_n(w_j)-g_m(w_j)\|+\|g_m(w_j)-g_m(z)\| < \varepsilon,$$

by (2.2) and (2.3).

This proves that $d(g_n, g_m) < \varepsilon$ when $m, n \geq n_0$, that is, the sequence $\{g_n\}$ is uniformly Cauchy. Thus, $\{g_n\}$ is uniformly convergent sequence to some continuous function $g \in C(K, \mathbb{R}^n)$. □

Chapter 3

Stereographic Projection; The Spherical and Chordal Metrics on $\hat{\mathbb{C}}$

A meromorphic function can naturally be regarded as taking values in the extended plane $\hat{\mathbb{C}}$. When working with such functions, it is desirable to turn $\hat{\mathbb{C}}$ into a metric space.

In this chapter, we will describe two equivalent metrics on $\hat{\mathbb{C}}$ called the spherical metric and the chordal metric. For our future purposes, we could work with either of those metrics, but the chordal metric is more convenient, i.e. leads to shorter proofs. However, because of its geometrical appeal, we have chosen to include a discussion of the spherical metric as well.

Consider the sphere

$$S = \left\{ x \in \mathbb{R}^3 \ ; \ x_1^2 + x_2^2 + \left(x_3 - \frac{1}{2} \right)^2 = \frac{1}{4} \right\}$$

in $\mathbb{R}^3$ also called the *Riemann sphere*.

Let $N = (0, 0, 1)$ be the *north pole*, and $P = (x_1, x_2, x_3) \neq N$ any point on the sphere. The straight line segment $\overline{\text{NP}}$ intersects the complex x_1x_2-plane at some point $z \in \mathbb{C}$ via one-to-one correspondence $P \leftrightarrow z$ between the points of $S - \{N\}$ and $\mathbb{C}$. This mapping of the sphere S onto the plane $\mathbb{C}$ (and vice-versa) is *conformal*, and is known under then name *stereographic projection*. We extend this correspondence to one between the sphere S and the *extended complex plane* $\hat{\mathbb{C}}$ by identifying N with ∞.

Let $P = (x_1, x_2, x_3), Q = (x_1', x_2', x_3') \in S$ and their corresponding stereographic projections $z, w \in \mathbb{C}$. The Euclidean distance between P and Q is given by

$$\|P - Q\| = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}.$$

Definition 3.1. The *chordal distance* $\sigma(z, w)$ between two points $z, w \in \hat{\mathbb{C}}$ is the length of the straight line segment joining the points $P, Q \in S$ whose stereographic projections are $z, w \in \mathbb{C}$ respectively. It is given by

$$\sigma(z, w) = \begin{cases} \frac{|w-z|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, & z, w \in \mathbb{C} \\ \frac{1}{\sqrt{1+|z|^2}}, & z \in \mathbb{C}, w = \infty \\ 0, & z = w = \infty. \end{cases} \tag{3.1}$$

We will soon verify that σ is a metric on $\hat{\mathbb{C}}$.

Consider the following map: $Q \colon \hat{\mathbb{C}} \to \mathbb{R}^3$ given by

$$Q(z) = \begin{cases} \left(\frac{\operatorname{Re} z}{|z|^2+1}, \frac{\operatorname{Im} z}{|z|^2+1}, \frac{|z|^2}{|z|^2+1} \right), & z \in \mathbb{C} \\ (0, 0, 1), & z = \infty. \end{cases} \tag{3.2}$$

Proposition 3.2. (a)

$$\|Q(z) - (0, 0, 1/2)\|^2 = \frac{1}{4}. \tag{3.3}$$

(b) Q is a one-to-one map of $\hat{\mathbb{C}}$ onto S with (for $P = (x_1, x_2, x_3) \neq N$)

$$Q^{-1}(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}. \tag{3.4}$$

(c)

$$\|Q(z) - Q(w)\| = \sigma(z, w). \tag{3.5}$$

(d) $\sigma(z, w)$ satisfies the triangle inequality, that is, for $z_1, z_2, z_3 \in \mathbb{C}$,

$$\sigma(z_1, z_3) \leq \sigma(z_1, z_2) + \sigma(z_2, z_3).$$

(e) Define two maps π_θ and $\tilde{\pi}$ from the sphere S to itself by

$$\pi_\theta(x_1, x_2, x_3) = (x_1 \cos\theta - x_2 \sin\theta, x_2 \cos\theta + x_1 \sin\theta, x_3),$$

and

$$\tilde{\pi}(x_1, x_2, x_3) = (x_1, -x_2, 1 - x_3).$$

Geometrically, π_θ means rotation and $\tilde{\pi}$ means a kind of inversion. Then

$$Q(e^{i\theta}z) = \pi_\theta(Q(z))$$
$$Q(1/z) = \tilde{\pi}(Q(z)).$$

(f) The chordal metric σ is invariant under the inversion $z \mapsto \frac{1}{z}$, for all $z, w \in \hat{\mathbb{C}}$, it follows that

$$\sigma(z, w) = \sigma(z^{-1}, w^{-1}). \tag{3.6}$$

(g) The chordal metric is equivalent to the Euclidean metric on any fixed compact subset of $\mathbb{C}$,

$$|z|, |w| \leq R \implies \frac{|z - w|}{1 + R^2} \leq \sigma(z, w) \leq |z - w|. \tag{3.7}$$

(h) The chordal distance from z to w is equivalent to the Euclidean distance between the inverse points $\frac{1}{z}$ and $\frac{1}{w}$ on any compact subset of $\mathbb{C} \setminus \{0\}$,

$$|z|, |w| \geq R^{-1} \implies \frac{|z^{-1} - w^{-1}|}{1 + R^2} \leq \sigma(z, w) \leq |z^{-1} - w^{-1}|. \tag{3.8}$$

Proof. (a) The sphere centered at $\left(0, 0, \frac{1}{2}\right)$ of radius $\frac{1}{2}$ is given by

$$x_1^2 + x_2^2 + \left(x_3 - \frac{1}{2}\right)^2 = \frac{1}{4} \iff x_1^2 + x_2^2 + x_3^2 = x_3.$$

Any point on the sphere satisfies this equation and so (3.3) is

$$\frac{|z|^2}{(|z|^2 + 1)^2} + \frac{|z|^4}{(|z|^2 + 1)^2} = \frac{|z|^2}{|z|^2 + 1},$$

which completes the proof.

(b) If (3.4) holds, then $Q \circ Q^{-1}(x) = x$. Clearly $Q^{-1} \circ Q(z) = z$ for every $z \in \hat{\mathbb{C}}$, and thus Q is a bijection.

(c) Pick any two points $z = x+iy$, $w = u+iv \in \mathbb{C}$. To compute their distance, take the three-dimensional Euclidean distance between $Q(z)$ and $Q(w)$. From

$$\|Q(z)\|^2+\|Q(w)\|^2 = \frac{|z|^2}{1+|z|^2}+\frac{|w|^2}{1+|w|^2} = \frac{|z|^2+2|z|^2|w|^2+|w|^2}{(1+|z|^2)\,(1+|w|^2)},$$

and

$$Q(z)\cdot Q(w) = \frac{\operatorname{Re}(\bar{z}w)+|z|^2|w|^2}{(|z|^2+1)\,(|w|^2+1)},$$

it follows that

$$\|Q(z)-Q(w)\|^2 = \frac{|w-z|^2}{(1+|z|^2)(1+|w|^2)}.$$

which is (3.5).

(d) follows from (3.5). If z_1, z_2, z_3 are distinct, the inequality is strict.

(e) Write $z = x+iy$ and $(x_1, x_2, x_3) = Q(z) = \frac{1}{1+|z|^2}(x, y, |z|^2)$.

Since $e^{i\theta} = \cos\theta + i\sin\theta$ we have

$$\begin{aligned} Q(e^{i\theta}z) &= \frac{1}{1+|z|^2}\left(\operatorname{Re}(e^{i\theta}z), \operatorname{Im}(e^{i\theta}z), |z|^2\right) \\ &= \frac{1}{1+|z|^2}\left(\cos\theta\cdot x - \sin\theta\cdot y, \cos\theta\cdot y + \sin\theta\cdot x, |z|^2\right) \\ &= (\cos\theta\cdot x_1 - \sin\theta\cdot x_2, \cos\theta\cdot x_2 + \sin\theta\cdot x_1, x_3) \\ &= \pi_\theta(x_1, x_2, x_3) = \pi_\theta(Q(z)). \end{aligned}$$

A similar computation shows $\tilde{\pi}(Q(z)) = Q(1/z)$.

(f) $\sigma(z^{-1}, w^{-1}) = \frac{|1/w-1/z|}{\sqrt{1+1/|z|^2}\sqrt{1+1/|w|^2}} = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$.

(g) If $|z| \le R$, then $1 \le \sqrt{1+|z|^2} \le \sqrt{1+R^2}$ which means $\frac{1}{\sqrt{1+R^2}} \le \frac{1}{\sqrt{1+|z|^2}} \le 1$. So if $|z|, |w| \le R$, then $\frac{|z-w|}{1+R^2} \le \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \le |z-w|$.

(h) (3.8) follows from (3.7) and (3.6).

□

Theorem 3.3. *The arc-distance $\theta(z, w)$, that is, the shortest length of the arc of the great circle in the sphere connecting z with w is explicitly given by*

$$\theta(z, w) = \arctan \left| \frac{w - z}{1 + z\bar{w}} \right| . \tag{3.9}$$

Proof. Let S be a sphere of radius $\frac{1}{2}$ centered at a point $p_0 \in \mathbb{R}^3$, that is,
$S = \{x \in \mathbb{R}^3 \ ; \ \|x - p_0\| = \frac{1}{2}\}$, where $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Pick two points $x, y \in S$ and let $\tilde{\theta}(x, y)$ be the angle between them (as seen from the center p_0). We can choose $\tilde{\theta}$ between 0 and π.

Let $\tilde{x} = x - p_0$ and $\tilde{y} = y - p_0$. Then $\tilde{x}$ and $\tilde{y}$ are on the sphere $\tilde{S} = \{x \in \mathbb{R}^3 \ ; \ \|x\| = \frac{1}{2}\}$ centered at the origin. The angle between $\tilde{x}$ and $\tilde{y}$ is the same as the angle $\tilde{\theta}(x, y)$ between x and y. Thus we compute

$$\begin{aligned} \|x - y\|^2 = \|\tilde{x} - \tilde{y}\|^2 = (\tilde{x} - \tilde{y}) \cdot (\tilde{x} - \tilde{y}) = \|\tilde{x}\|^2 - 2\|\tilde{x}\|\|\tilde{y}\| \cos \tilde{\theta} + \|\tilde{y}\|^2 \\ = \frac{1}{4} - \frac{1}{2} \cos \tilde{\theta} + \frac{1}{4} = \frac{1}{2}(1 - \cos \tilde{\theta}(x, y)). \end{aligned}$$

Next for two points x, y on the sphere S, we let $\theta(x, y)$ be the arclength-distance between x and y on the sphere. We claim that $\theta(x, y) = \frac{1}{2}\tilde{\theta}(x, y)$.

To see this we can translate and rotate the sphere S so that it is centered at the origin and $x = \left(\frac{1}{2}, 0, 0\right), y = (y_1, y_2, 0)$, for some y_1, y_2. Since $\tilde{\theta}$ is the angle between x and y, we must have $y = \frac{1}{2}(\cos \tilde{\theta}, \sin \tilde{\theta}, 0)$. Thus a curve on the sphere S which connects x with y is

$$\gamma(t) = \frac{1}{2}(\cos t, \sin t, 0), \quad (0 \leq t \leq \tilde{\theta}).$$

A geometric consideration shows that $\gamma(t)$ must be the shortest path between x and y on the sphere. Thus

$$\theta(x, y) = \int_0^{\tilde{\theta}} \|\gamma'(t)\| dt.$$

But $\gamma'(t) = \frac{1}{2}(-\sin t, \cos t, 0)$ which has $\|\gamma'(t)\| = \frac{1}{2}$ so indeed

$$\theta(x, y) = \frac{1}{2}\tilde{\theta}(x, y).$$

It now follows that

$$\|x-y\|^2 = \frac{1}{2}(1-\cos(2\theta(x,y))). \tag{3.10}$$

Now let S be the sphere centered at $p_0 = (\frac{1}{2}, 0, 0)$ and set $x = Q(z), y = Q(w)$ and use the formula (3.5). In the following we change notation and write $\theta(z,w)$ for $\theta(Q(z), Q(w))$.

Using (3.10) and the double angle formula we can write

$$\|Q(z) - Q(w)\|^2 = \sin^2(\theta(z,w))$$

and also

$$\|Q(z) - Q(w)\|^2 = 1 - \cos^2(\theta(z,w)).$$

It follows that

$$\tan(\theta(z,w)) = \frac{\sin\theta}{\cos\theta} = \frac{\sigma(z,w)}{\sqrt{1-\sigma(z,w)^2}}.$$

A computation shows that (in case neither of the points is ∞)

$$1 - \sigma(z,w)^2 = \frac{|1+z\bar{w}|^2}{(1+|z|^2)(1+|w|^2)},$$

so using (3.1) we find that

$$\tan(\theta(z,w)) = \left|\frac{w-z}{1+z\bar{w}}\right|$$

as desired. (The case when one of z and w are ∞ is omitted.)

Note that

$$\theta(z,w) = \arctan\left|\frac{w-z}{1+z\bar{w}}\right|$$

is always between 0 and $\pi/2$, and the maximal distance is attained when $z = -1/\bar{w}$. In this case we say that the points z and w are *antipodal* on the Riemann sphere. □

It is useful to have the infinitesimal form of the spherical metric.

Theorem 3.4. *Let $\theta(z,w)$ be the spherical metric defined explicitly as in* (3.9)*. The infinitesimal form of this metric is given by*

$$\frac{|dz|}{(1+|z|^2)}.$$

Proof. Let h be a small complex number. We have that

$$\theta(z, z+h) = \arctan\left(\frac{|h|}{|1+|z|^2+z\bar{h}|}\right).$$

Let $f(t) = 1/(1+|z|^2+t)$, where $t = z\bar{h}$. Using Taylor series expansions of arctan about $t = 0$, it follows that

$$\arctan|hf(t)| = |f(t)h| + \mathcal{O}(|hf(t)|^3) \quad \text{as} \quad hf(t) \to 0$$

and so the right-hand side clearly equals to

$$f(0)|h| + \mathcal{O}(|h|^2) \quad \text{as} \quad h \to 0,$$

that is,

$$\theta(z, z+h) = \frac{|h|}{(1+|z|^2)} + \mathcal{O}(|h|^2).$$

If we replace h by dz and think of dz as being a very small complex number, we obtain the approximation

$$\theta(z, z+dz) = \frac{|dz|}{(1+|z|^2)}(1+\mathcal{O}(dz)).$$

□

If $\gamma(t), a \leq t \leq b$ is a curve on the sphere S, we define its spherical length to be

$$L(\gamma) = \int_\gamma \frac{|dz|}{(1+|z|^2)}.$$

The spherical distance between two points z, w on the Riemann sphere S is defined to be the infimum taken over all paths on S joining z and w, that is,

$$\theta(z, w) = \inf\{L(\gamma)\}.$$

And this coincides precisely with the $\theta(z, w)$ computed in the Theorem 3.3.

$\theta(z, w)$ represents the Euclidean length of the shortest arc of any great circle on the sphere joining z and w. It defines a metric on the sphere known as the *spehrical metric.* Since $2\theta/\pi \leq 2\sin(\theta/2) \leq \theta$, we have that

$$\frac{2}{\pi}\theta(z, w) \leq \sigma(z, w) \leq \theta(z, w).$$

Hence, θ and σ are *uniformly equivalent* metrics.

Having defined the spherical metric on the Riemann sphere, in the next section, we study spherical derivative and normal convergence of a sequence of meromorphic functions in this metric.

Chapter 4

Normal Families

We shall now introduce the notion of a normal family. This is a very broad and useful concept, which characterizes compact sets of analytic or meromorphic functions. As all meromorphic functions are continuous when regarded as functions into the Riemann sphere $\hat{\mathbb{C}}$, the Arzelá-Ascoli theorem can be applied to families of meromorphic functions. However, since meromorphic functions are typically defined on domains (which are open), it is not suitable to use a simple sup-norm metric. We have the following definitions.

In the following we fix an open subset $\Omega \subset \mathbb{C}$.

Definition 4.1. A sequence of compact subsets $\{K_l\}_{l=1}^{\infty}$ is called a compact exhaustion of Ω if

(a) K_l is contained in the interior of K_{l+1}, that is, $K_l \subset K_{l+1}^{\text{int}}$ for each l;

(b) $\bigcup_{l=1}^{\infty} K_l = \Omega$.

We remark that compact exhaustions always exist. For example we may take $K_l = \{z \in \Omega\,;\, \text{dist}(z, \partial\Omega) \geq 1/l,\ |z| \leq l\}$.

In the following we fix an arbitrary exhaustion $K_1, K_2, \ldots$ of Ω.

Definition 4.2. We denote by $\mathcal{A}(\Omega)$ is the space of analytic functions $\Omega \to \mathbb{C}$ with metric

$$d(f,g) = \sum_{k=1}^{\infty} \min\{2^{-k}, \|f-g\|_{K_k}\}$$

where $\|h\|_K = \sup_{z \in K} |h(z)|$.

Definition 4.3. If $d(f_n, f) \to 0$ we say that f_n converges *locally uniformly* on Ω to f. Equivalently, $d(f_n, f) \to 0$ means that f_n converges to f uniformly on any given compact subset of Ω.

Definition 4.4. A family $\mathcal{F} \subset \mathcal{A}(\Omega)$ is called *relatively compact* if each sequence $\{f_n\}$ in $\mathcal{F}$ has a locally uniformly convergent subsequence.

Definition 4.5. We denote by $\mathcal{M}(\Omega)$ the space of meromorphic functions on $\Omega \to \hat{\mathbb{C}}$ (adjoined by the constant function ∞) with metric

$$\rho(f, g) = \sum_{k=1}^{\infty} \min\{2^{-k}, \sigma_{K_k}(f, g)\} \tag{4.1}$$

where $\sigma_K(f, g) = \sup_{z \in K} \sigma(f(z), g(z))$.

We say that $f_n \to f$ *normally* on Ω if $\rho(f_n, f) \to 0$. It's easy to see that this happens if and only if $\sigma_{K_m}(f_n, f) \to 0$ as $n \to \infty$ for each fixed $m = 1, 2, \ldots$

Definition 4.6. A family $\mathcal{F} \subset \mathcal{M}(\Omega)$ is called *normal* if each sequence $\{f_n\}$ in $\mathcal{F}$ has a uniformly convergent subsequence with respect to the chordal metric on compact subsets of Ω.

4.1 Completeness of the Spaces $\mathcal{M}(\Omega)$ and $\mathcal{A}(\Omega)$

We now prove that the above spaces are complete. The proof uses Morera's and Hurwitz's theorems from complex analysis, plus some elementary facts about uniform convergence of continuous functions from Chapter 2.

We first prove that $\mathcal{M}(\Omega)$ is complete in the metric (4.1).

Theorem 4.7. *(a) If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}(\Omega)$ and $f_n \to f$ normally, then $f \in \mathcal{M}(\Omega)$.*

(b) $(\mathcal{M}(\Omega), \rho)$ is a complete metric space.

Proof. Recall that $\|Q(f_n(z)) - Q(f(z))\| = \sigma(f_n(z), f(z))$. So for each m,

$$\|Q \circ f_n - Q \circ f\|_{K_m} = \sigma_{K_m}(f_n, f) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $Q \circ f_n \to Q \circ f$ uniformly on each compact subset of Ω. As each $Q \circ f_n \colon \Omega \to S$ is continuous, it follows that $Q \circ f$ is continuous as well.

(a) Now pick $z_0 \in \Omega$. Two cases emerge.
Case (i): Assume $f(z_0) \neq \infty$, i.e. $Q \circ f(z_0) \neq N = (0,0,1)$. By continuity there are then $\varepsilon > 0$ and $\delta > 0$ such that

$$\|Q \circ f(z) - N\| > 3\varepsilon \quad \text{when} \quad |z - z_0| \leq \delta. \tag{4.2}$$

If $\delta > 0$ is small enough, then $Q \circ f_n \to Q \circ f$ uniformly on the disc $|z - z_0| \leq \delta$ so we can find n_0 such that

$$\|Q \circ f_n - Q \circ f\|_{\overline{D_\delta(z_0)}} < \varepsilon, \quad n \geq n_0. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\|Q \circ f_n(z) - N\| > 2\varepsilon \quad \text{when} \quad n \geq n_0, z \in \overline{D_\delta(z_0)}.$$

Applying stereographic projection Q^{-1}, we see that there must be some large radius C such that

$$|f_n(z)| \leq C, \quad |f(z)| \leq C \quad \text{when} \quad n \geq n_0, z \in \overline{D_\delta(z_0)}. \tag{4.4}$$

By (4.4) and (3.7), we thus have

$$\begin{aligned} \|f_n(z) - f(z)\| &\leq (1 + C^2)\sigma(f_n(z), f(z)), \quad \Big(n \geq n_0, z \in \overline{D_\delta(z_0)}\Big) \\ &\leq (1 + C^2)\sigma_{\overline{D_\delta(z_0)}}(f_n, f) \to 0. \end{aligned}$$

Thus $f_n \to f$ uniformly on $\overline{D_\delta(z_0)}$.
It now follows from Morera's theorem that f is analytic in $D_\delta(z_0)$.
Indeed, if $R \subset D_\delta(z_0)$ is a closed rectangle

$$\int_{\partial R} f(z)dz = \lim_{n\to\infty} \int_{\partial R} f_n(z)dz = 0.$$

Thus f is analytic on $D_\delta(z_0)$ by Morera.
Case (ii): If $f(z_0) = \infty$ we use (3.6) and conclude as above that $1/f$ is analytic in a neighbourhood $D_\delta(z_0)$.
If $1/f \equiv 0$, we have $f \equiv \infty$. Otherwise the zero is isolated and we can write $1/f(z) = (z - z_0)^m h(z), \quad h(z_0) \neq 0$.

(b) Suppose $\{f_n\}$ is a Cauchy sequence with respect to the metric ρ, i.e. $\sigma_{K_m}(f_i, f_j) \to 0$ as $i, j \to \infty$ for each fixed m. In other words,

$$\|Q \circ f_i - Q \circ f_j\|_{K_m} \to 0 \quad \text{as} \quad i, j \to \infty.$$

We know that the space $C(K_m, \mathbb{R}^3)$ is complete, so $Q \circ f_i$ must converge uniformly on K_m to some continuous function F_m.

Clearly, $m_1 < m_2 \implies F_{m_1} = F_{m_2}$ on K_{m_1} so the F_m's piece together to a continuous function $F\colon \Omega \to \mathbb{R}^3$ and $Q \circ f_n \to F$ uniformly on compact subsets. Since each $Q \circ f_n$ has values in the sphere S we see easily that F takes values in S as well.

Putting $f = Q^{-1} \circ F$ we have that $Q \circ f_n \to Q \circ f$ locally uniformly, i.e. $\rho(f_n, f) \to 0$. By (a) we obtain $f \in \mathcal{M}(\Omega)$.

□

The following lemma is a variant of the Hurwitz's theorem.

Lemma 4.8. *Suppose $\{f_n\} \subset \mathcal{A}(\Omega)$ and $f \in \mathcal{M}(\Omega)$ and $\rho(f_n, f) \to 0$. If each f_n is nonvanishing, then either f never vanishes or $f \equiv 0$.*

Proof. Suppose there is some point $z_0 \in \Omega$ with $f(z_0) = 0$. We will prove that $f \equiv 0$.

Indeed, as in the proof above, we can find a disc $D_\delta(z_0)$ in which f is analytic and $f_n \to f$ uniformly on $\overline{D_\delta(z_0)}$.

If $f \not\equiv 0$, then the zero at z_0 is isolated and we can write $f(z) = (z - z_0)^m h(z)$ where $h(z_0) \neq 0$ and $m \geq 1$. Choosing $\delta > 0$ somewhat smaller, we can assume that $|h(z)| \geq |h(z_0)|/2$ in $\overline{D_\delta(z_0)}$.

Then $1/f_n \to 1/f$ uniformly on the circle $|z - z_0| = \delta/2$ and likewise $f_n' \to f'$ uniformly on that circle. So

$$\frac{f_n'}{f_n} \to \frac{f'}{f} \quad \text{uniformly on} \quad |z - z_0| = \frac{\delta}{2}.$$

But by the argument principle,

$$\frac{1}{2\pi i} \int_{|z-z_0|=\frac{\delta}{2}} \frac{f_n'}{f_n} dz = 0 \quad \text{and} \quad \frac{1}{2\pi i} \int_{|z-z_0|=\frac{\delta}{2}} \frac{f'}{f} dz = m$$

so we get the contradiction $m = 0$. □

Theorem 4.9. *Suppose* $\{f_n\}_{n=1}^{\infty} \in \mathcal{A}(\Omega)$ *and* $f \in \mathcal{M}(\Omega)$ *and* $\rho(f_n, f) \to 0$ *as* $n \to \infty$. *Then either* $f \in \mathcal{A}(\Omega)$ *or* $f \equiv \infty$. *Moreover, if* $f \in \mathcal{A}(\Omega)$ *then* $d(f_n, f) \to 0$ *as* $n \to \infty$.

Proof. Set $g_n = 1/f_n, g = 1/f$. Then $g_n \in \mathcal{M}(\Omega)$ are nonvanishing, and $\rho(g_n, g) = \rho(f_n, f) \to 0$. By the Lemma 4.8, either $g \equiv 0$ or g never vanishes.

Assume g never vanishes, so that $f \in \mathcal{A}(\Omega)$. We must shows that $d(f_n, f) \to 0$. But if $f \in \mathcal{A}(\Omega)$, then $|f|$ attains its maximum value on each K_m, i.e. there is some constant C_m such that $|f(z)| \leq C_m$ for all $z \in K_m$.

Hence there is $\delta_m > 0$ such that $\sigma(f(z), \infty) \geq \delta_m$ for all $z \in K_m$. Now fix n_0 such that $\sigma_{K_m}(f_n, f) \leq \delta_m/2$ when $n \geq n_0$.

For $n \geq n_0$ we have that

$$\sigma(f_n(z), \infty) \geq \delta_m/2 \quad \text{for all} \quad z \in K_m.$$

It follows that there is another constant $C' = C'(n_0, m)$ such that $|f_n(z)| \leq C'$ for all $n \geq n_0, z \in K_m$, and also $|f(z)| \leq C'$ for all $n \geq n_0, z \in K_m$.

By (3.7) we have

$$\sup_{z \in K_m} \|f_n(z) - f(z)\| \leq \left(1 + (C')^2\right) \sigma_{K_m}(f_n, f) \to 0 \quad \text{as} \quad n \to \infty,$$

i.e. $f_n \to f$ uniformly on K_m. Since m is arbitrary, we conclude $d(f_n, f) \to 0$ as $n \to \infty$. □

4.2 Spherical Derivatives and Marty's Theorem

In this section, we state and prove the fundamental Marty's theorem, which gives a convenient criterion for normality of a subfamily $\mathcal{F}$ of $\mathcal{M}(\Omega)$. The criterion uses the notion of spherical derivatives. As a consequence we will deduce the classical Montel's theorem on relatively compact subfamilies of $\mathcal{A}(\Omega)$.

Let us note first that assuming that the family $\mathcal{F}$ of meromorphic functions is normal does not imply that their derivatives constitute a normal family. Consider the following example.

Example 4.10. Let $f_n = n(z^2 - n)$ be a sequence of functions in the whole plane. Since $f_n \to \infty$ uniformly on each compact set, $\mathcal{F}$ is normal. The derivatives $f_n' = 2nz$ tend to ∞ for $z \neq 0$ and to 0 for $z = 0$. Therefore, these derivatives do not form a normal family.

Now that we are working with the family $\mathcal{F} \subset \mathcal{M}(\Omega)$, functions in $\mathcal{F}$ take values on the Riemann sphere, that is, in the extended complex plane $\hat{\mathbb{C}}$ equipped with the spherical metric. We may raise the question about a derivative of such functions.

Definition 4.11. The *spherical derivative* of a function in $\mathcal{M}(\Omega)$ is defined by

$$f^\sharp(z) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{\sigma(f(z), f(z_0))}{|z - z_0|}. \tag{4.5}$$

Proposition 4.12. Let $f(z) \in \mathcal{M}(\Omega)$.

(a) If $f(z_0) \in \mathbb{C}$, then

$$f^\sharp(z_0) = \frac{|f'(z_0)|}{1 + |f(z_0)|^2}. \tag{4.6}$$

(b) (Chain rule)

$$(g \circ f)^\sharp(z) = g^\sharp(f(z))|f'(z)|.$$

(c) The spherical derivative is invariant under the inversion,

$$(1/f)^\sharp = f^\sharp. \tag{4.7}$$

(d) If $f \in \mathcal{M}(\Omega)$ and γ is a curve in Ω, then

$$\sigma(f(\gamma(0)), f(\gamma(1))) \leq \int_0^1 f^\sharp(\gamma(s))|d\gamma(s)|. \tag{4.8}$$

Proof. (a) Since $\frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$ as $z \to z_0$, it follows that

$$\frac{\sigma(f(z), f(z_0))}{|z - z_0|} = \frac{1}{\sqrt{1 + |f(z)|^2}\sqrt{1 + |f(z_0)|^2}} \frac{|f(z) - f(z_0)|}{|z - z_0|} \to \frac{|f'(z_0)|}{1 + |f(z_0)|^2}$$

as $z \to z_0$.

(b) $(g \circ f)^\sharp(z) = |(g \circ f)'(z)|/(1 + |g(f(z))|^2) = |g'(f(z))f'(z)|/(1 + |g(f(z))|^2) = g^\sharp(f(z))|f'(z)|$.

(c) The spherical metric is invariant under the inversion $z \mapsto \frac{1}{z}$, so the result immediately follows from (3.6) and (4.5).

(d) Assume γ piecewise C^1 and $f \circ \gamma(s)$ finite image curve for all s. Let $j = 1, \dots, n$ and $j/n \to s$ as $n \to \infty$. Then uniformly in s,

$$\frac{\sigma\left(f\left(\gamma\left(\frac{j}{n}\right)\right), f\left(\gamma\left(\frac{j-1}{n}\right)\right)\right)}{|\gamma\left(\frac{j}{n}\right) - \gamma\left(\frac{j-1}{n}\right)|} \to f^\sharp(\gamma(s)).$$

Using the the triangle inequality,

$$\sigma(f(\gamma(0)), f(\gamma(1))) \leq \sum_{j=1}^{n} f^\sharp\left(\gamma\left(\frac{j}{n}\right)\right)\left|\gamma\left(\frac{j}{n}\right) - \gamma\left(\frac{j-1}{n}\right)\right|\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Passing the limit as $n \to \infty$, we obtain (4.8) which holds for all rectifiable curves.

Since σ and $f^\sharp$ are invariant under the inversion, we perform the same computations for the finite $1/f(\gamma(s))$ for all s.

□

Lemma 4.13. *If a sequence $\{f_n\}_{n=1}^{\infty} \in \mathcal{M}(\Omega)$ converges normally to f, then the spherical derivatives $f_n^\sharp$ converge locally uniformly to $f^\sharp$.*

Proof. We must show for each compact subset $K \subset \Omega$ that

$$\|f_n^\sharp - f^\sharp\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.9}$$

Fix a small $\delta > 0$. By Heine-Borel Theorem 2.3, we can find finitely many points $z_1, \dots, z_w \in K$ such that $K \subset \bigcup_{j=1}^{N} D_{2\delta}(z_j)$.

Choosing $\delta > 0$ smaller we can also assume that $D_{4\delta}(z_j) \subset \Omega$. This shows that it suffices to show (4.9) when K is a small closed disc, say $K = \overline{D_{2\delta}(z_0)}$. Similar to earlier proofs, two cases emerge.

Case (i): If $f(z_0) \neq \infty$, then by the proof of the Theorem 4.7 (a), we can find $\delta > 0$ and n_0 such that all f_n with $n \geq n_0$ are analytic in $D_{3\delta}(z_0)$ and $f_n \to f$ uniformly on $\overline{D_{2\delta}(z_0)}$ as $n \to \infty$.

So f is analytic in $D_{2\delta}(z_0)$ by Morera and continuous up to the boundary.

If $z \in \overline{D_\delta(z_0)}$, then

$$f_n'(z) - f'(z) = \frac{1}{2\pi i} \int_{|w-z|=\delta} \frac{f_n(w) - f(w)}{(w-z)^2} dw$$

so

$$|f_n'(z) - f'(z)| \leq \frac{1}{\delta}\|f_n - f\|_{\overline{D_{2\delta}(z_0)}} \to 0 \quad \text{as} \quad n \to \infty.$$

We have shown $f_n' \to f'$ uniformly on $K = \overline{D_\delta(z_0)}$.

It follows that $|f_n| \to |f|$ and $|f_n'| \to |f'|$ uniformly on K, so $f_n^\sharp(z) = \frac{|f_n'(z)|}{|f_n(z)|^2+1} \to \frac{|f'(z)|}{|f(z)|^2+1} = f^\sharp(z)$ uniformly on K.

Case (ii): If $f(z_0) = \infty$ we apply Case (i) to the function $g_n = 1/f_n$ and $g = 1/f$, and recall that $g_n^\sharp = f_n^\sharp$ and $g^\sharp = f^\sharp$.

□

Theorem 4.14 (Marty's Theorem)**.** *Let $\mathcal{F} \subset \mathcal{M}(\Omega)$. Then $\mathcal{F}$ is a normal family on Ω if and only if the spherical derivatives $\{f^\sharp(z) \colon f \in \mathcal{F}\}$ are uniformly bounded on each compact subset of Ω.*

Proof. As always, let S be the sphere $\|x - \left(0, 0, \frac{1}{2}\right)\| = \frac{1}{2}$ in $\mathbb{R}^3$.

Assume the spherical derivatives are bounded near some point $z_0 \in \Omega$, that is, $f^\sharp(z) \leq C$ for all $z \in \overline{D_\delta(z_0)}$ and for all $f \in \mathcal{F}$. Then for all $w \in \overline{D_\delta(z_0)}$, and all $f \in \mathcal{F}$, by (4.8) follows

$$\sigma(f(w), f(z)) = \|(Q \circ f)(w) - (Q \circ f)(z)\| \leq \int_\gamma f^\sharp(z)|dz| \leq C|z - w|,$$

where γ is the shortest path between z and w. Hence, the family $\mathcal{G} = \{Q \circ f \colon \Omega \to S \ ; \ f \in \mathcal{F}\}$ is locally uniformly equicontinuous. By the Arzelá-Ascoli Theorem 2.11, each sequence $g_j = Q \circ f_j$ of functions in $\mathcal{G}$ has a further subsequence (relabeled as g_j for convenience) converging locally uniformly on Ω to a continuous function $g \colon \Omega \to S$.

Now define a function $f \colon \Omega \to \hat{\mathbb{C}}$ by $f(z) = Q^{-1}(g(z))$. f is continuous, since both Q^{-1} and g are so.
Note that for each z (by Proposition 3.2)

$$\|g_n(z) - g(z)\| = \|Q \circ f_n(z) - Q \circ f(z)\| = \sigma(f_n(z), f(z)).$$

For each compact set K_j in the exhaustion we thus have

$$\|g_n - g\|_{K_j} = \sigma_{K_j}(f_n, f). \tag{4.10}$$

Since $g_n \to g$ locally uniformly, we know that $d(g_n, g) \to 0$ where $d(g_n, g) = \sum_{j=1}^{\infty} \min\{2^{-j}, \|g_n - g\|_{K_j}\}$.
By (4.10) this is equivalent to $\rho(f_n, f) \to 0$, i.e. $f_n \to f$ normally.

We have shown that each sequence f_j in $\mathcal{F}$ has a further subsequence converging normally to a limit $f \in \mathcal{M}(\Omega)$.
In other words, $\mathcal{F}$ is a normal family.

Now assume that $f^\sharp$ are not uniformly bounded on compacts of Ω. Then there exist $f_n \in \mathcal{F}$ and some compact subset K such that

$$\lim_{n\to\infty} \sup_{z\in K} f_n^\sharp = \infty.$$

By the Lemma 4.13, f_n cannot have a normally convergent subsequence implying that $\mathcal{F}$ is not normal. □

Let us finally deduce the classical Montel's theorem, which historically preceded Marty's theorem.

Theorem 4.15 (Montel's Theorem)**.** *Suppose that $\mathcal{F}$ is a locally uniformly bounded family of analytic functions defined on an open set $\Omega \subset \mathbb{C}$. Then $\mathcal{F}$ is relatively compact.*

Proof. By the assumption, $\mathcal{F}$ is locally uniformly bounded on Ω. Pick $z_0 \in \Omega$. Then there is some $\delta > 0$ such that $\overline{D_\delta(z_0)} \subset \Omega$, and there exists some constant $C = C(z_0)$ such that $|f(z)| \leq C$ for all $f \in \mathcal{F}$ and all $z \in \overline{D_\delta(z_0)}$.

If $z \in \overline{D_{\delta/2}(z_0)}$, then

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|w-z|=\frac{\delta}{2}} \frac{|f(w)|}{|w-z|^2}|dw| < \frac{4C}{\delta} \quad \text{for all } f \in \mathcal{F} \text{ and all } z \in \overline{D_{\delta/2}(z_0)}.$$

Hence, the family $\mathcal{F}' = \{f' \ ; \ f \in \mathcal{F}\}$ is locally uniformly bounded.

Since $f^\sharp(z) = \frac{|f'(z)|}{|f(z)|^2+1}$, it follows that

$$|f^\sharp(z)| < \frac{4C}{\delta} \quad \text{for all } f \in \mathcal{F} \text{ and all } z \in \overline{D_{\delta/2}(z_0)}.$$

We conclude that the family $\mathcal{F}^\sharp = \{f^\sharp \ ; \ f \in \mathcal{F}\}$ is locally uniformly bounded, so by Marty's Theorem 4.14, $\mathcal{F}$ is a normal family.

Since $\mathcal{F} \subset \mathcal{A}(\Omega)$, it follows from Theorem 4.9 that each sequence in $\mathcal{F}$ has a locally convergent subsequence, i.e., $\mathcal{F}$ is relatively compact in $\mathcal{A}(\Omega)$. □

Chapter 5

Montel's Three-Value Theorem and Picard's Great Theorem

In this chapter, we will see that the theory of normal families can be used to prove the famous Picard's theorem, which says that in an arbitrarily small deleted neighborhood of an essential singularity, an analytic function assumes all complex values with one possible exception. (Note that $f(z) = e^{1/z}$ has an essential singularity at $z = 0$ and omits the value 0.)

So, what does normal families have to do with this? As we shall see, Picard's theorem follows quite easily from a theorem of Montel, which says that a family $\mathcal{F} \subset \mathcal{M}(\Omega)$ of functions omitting three values (say $0, 1$ and ∞) is necessarily a normal family. The proof of this Montel's Three-Value Theorem is a main goal in this chapter.

We now turn to some details. We shall follow an approach discovered by Larry Zalcman [8], based on a clever lemma on non-normal families. The following discussion is based on several textbooks, for example [2, 4, 7].

5.1 Zalcman's Lemma

One of the most important results in the theory of normal families is the *Fundamental Normality Test*, also called *Montel's Three-Value Theorem*, introduced by *Paul Montel* in 1912 [6]. The proof of this theorem almost immediately follows from *Zalcman's Lemma*, which gives characterization of non-normality.

Proposition 5.1 (Zalcman's Lemma)**.** Let $\mathcal{F}$ be a family of meromorphic functions on a region $\Omega \subset \mathbb{C}$, which is not normal. Then there exist $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, points $z_n \in \Omega$, $z_n \to z_\infty \in \Omega$, and nonnegative numbers $\rho_n \to 0$ so that if

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta)$$

then for some entire meromorphic function g_∞, we have

(a) $g_n \to g_\infty$ normally on each $\mathbb{D}_R$ with $R < \infty$.

(b) $g_\infty^\sharp(0) = 1$.

(c) $g_\infty^\sharp(\zeta) \leq 1$ for all $\zeta \in \mathbb{C}$.

Proof. Assume $\mathcal{F}$ not normal. By Marty's Theorem there exists a compact $K \subset \Omega$ such that

$$\sup_{f \in \mathcal{F}, z \in K} \|f^\sharp(z)\| = \infty.$$

There exists a sequence $(w_n) \in K$, $w_n \to w_\infty \in K$ and $f_n \in \mathcal{F}$ so that

$$f_n^\sharp(w_n) \geq n. \tag{5.1}$$

For simplicity take $w_\infty = 0$ and $\overline{\mathbb{D}} \subset \Omega$.

Define

$$M_n = \sup_{|z| \leq 1} (1 - |z|^2) f_n^\sharp(z).$$

Then by (5.1), $M_n \geq (1 - |w_n|^2)n \to \infty$ since $w_n \to 0$. Since $(1 - |z|^2) f_n^\sharp(z)$ is continuous and vanishing on $\partial\mathbb{D}$, it attains its maximum at some point $z_n \in \mathbb{D}$ so $M_n = (1 - |z_n|^2) f_n^\sharp(z_n)$ and $z_n \to z_\infty \in \mathbb{D} \subset \Omega$.

Define

$$\rho_n = [f_n^\sharp(z_n)]^{-1} = (1 - |z_n|^2) M_n^{-1} \leq M_n^{-1} \to 0 \text{ as } n \to \infty, \tag{5.2}$$

so that

$$(1 - |z_n|)^{-1} \rho_n = M_n^{-1}(1 + |z_n|) \leq 2M_n^{-1}. \tag{5.3}$$

For $R > 0$ fixed and n large enough with $|\zeta| \leq R < R_n = \frac{1}{2} M_n$, we have $|z_n + \rho_n \zeta| < 1$.

On the set where $|z_n + \rho_n \zeta| < 1$, define

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta).$$

Note that g_n is defined on $\mathbb{D}$. Then by (5.2)

$$|g_n^\sharp(0)| = \rho_n f_n^\sharp(z_n) = 1 \text{ for each } n, \tag{5.4}$$

and

$$
\begin{aligned}
|g_n^\sharp(\zeta)| &= \rho_n f_n^\sharp(z_n + \rho_n\zeta) \\
&\le \rho_n M_n (1 - |z_n + \rho_n\zeta|^2)^{-1} \\
&= \frac{1 - |z_n|^2}{1 - |z_n + \rho_n\zeta|^2} && (5.5) \\
&\le \frac{1 + |z_n|}{1 + |z_n| + \rho_n|\zeta|}\frac{1 - |z_n|}{1 - |z_n| - \rho_n|\zeta|} && (5.6) \\
&\le \frac{1}{1 - 2|\zeta|M_n^{-1}}. && (5.7)
\end{aligned}
$$

In the above, we obtain (5.6) from (5.5) using that $|z_n + \rho_n\zeta|^2 \le (|z_n| + \rho_n|\zeta|)^2$ and $(1-a^2) = (1-a)(1+a)$. Then from (5.6) we obtain (5.7) since the first term is bounded by 1 and by (5.3) we have that $\rho_n|\zeta|(1 - |z_n|)^{-1} \le 2|\zeta|M_n^{-1}$.

Hence for each fixed R,

$$\lim_{n\to\infty} \sup_{|\zeta|\le R} g_n^\sharp(\zeta) = 1. \tag{5.8}$$

By Marty's Theorem, $\{g_n\}$ is normal on $\mathbb{C}$ and there exists g_∞ meromorphic on $\mathbb{C}$ and a subsequence, also denoted as g_n, such that $g_n \to g_\infty$ uniformly on each compact subset of $\mathbb{C}$. Furthermore, (b) and (c) hold by (5.4) and (5.8).

Since

$$g_\infty^\sharp(0) = \lim_{n\to\infty} g_n^\sharp(0) = 1$$

by Lemma 4.13, g_∞ is a nonconstant entire meromorphic function. □

5.2 Montel's Three Value Theorem and Picard's Theorem

We now state and prove Montel's Three-Value Theorem and Picard's Great Theorem.

In order to simplify, we shall only prove these theorems for functions defined on simply connected subsets of the plane. This assumption is convenient, for if the domain Ω is simply connected, and if $f : \Omega \to \mathbb{C}$ omits the value zero, then we can define an analytic branch of $\log f$ on Ω, and hence also analytic branches of the power functions

$f(z)^\alpha = e^{\alpha \log f(z)}$. (Indeed, simple connectivity of Ω is equivalent to the possibility of forming such analytic branches of the logarithm, see for example Chapter 8 of Gamelin [2].)

Definition 5.2. Assume that function f is meromorphic on some punctured disk $0 < |z - z_0| < r$. A value $w_0 \in \hat{\mathbb{C}}$ is said to be an *omitted value* of f at z_0 if there exists $\delta > 0$ such that $f(z) \neq w_0$ for $0 < |z - z_0| < \delta$. So w_0 is *not* an omitted value of f at z_0 if and only if there exists a sequence $z_n \to z_0, z_n \neq z_0$, so that $f(z_n) = w_0$.

Theorem 5.3 (Montel's Three-Value Theorem)**.** *A family $\mathcal{F}$ of meromorphic functions on a simply connected domain Ω which omits three distinct values α, β, γ in $\hat{\mathbb{C}}$ is normal.*

Proof. Let $\alpha, \beta, \gamma \in \hat{\mathbb{C}}$ be distinct omitted values. Composing each element of a family $\mathcal{F}$ with a fractional linear transformation, without a loss, we take $\alpha = 0, \beta = 1, \gamma = \infty$. Since such elements of $\mathcal{F}$ are analytic and nonvanishing functions on Ω, they have roots of all orders.

Fix k and define

$$\mathcal{F}_k = \{f^{1/2^k} : f \in \mathcal{F}\}$$

the family consisting of all 2^k-th roots of functions in $\mathcal{F}$. Clearly $\mathcal{F}_k$ is normal if and only if $\mathcal{F}$ is normal. The functions in $\mathcal{F}_k$ omit the values $0, 1, \infty$ and all 2^k-th roots of unity.

To obtain a contradiction, we assume that $\mathcal{F}$ is not normal. So for each $k, \mathcal{F}_k$ is not normal. Then by Zalcman's lemma, there exist entire functions $g_\infty^{(k)}$ such that $(g_\infty^{(k)})^\sharp(0) = 1, (g_\infty^{(k)})^\sharp(z) \leq 1$, for all k and all z, and $(g_\infty^{(k)})$ is a limit of restrictions of functions in $\mathcal{F}_k$. As functions in $\mathcal{F}_k$ omit the 2^k-th roots of unity so do their translations, dilations and restrictions, and by *Hurwitz's theorem* (see Simon [7]), so does any nonconstant normal limit. Hence, $g_\infty^{(k)}$ omits the values which are the 2^k-th roots of unity.

By Marty's theorem, family $\{g_\infty^{(k)}\}$ is normal so there exists a converging subsequence to an entire function g_∞ satisfying $g_\infty^\sharp(0) = 1$ so that g_∞ is nonconstant and $g_\infty^\sharp(z) \leq 1$ for all z. Again, by Hurwitz's theorem, g_∞ omits all 2^k-th roots of unity for all k. Since g_∞ is an open mapping, and roots of unity are dense in the circle, it omits the unit circle, that is, $|g_\infty(z)| < 1$ or $|g_\infty(z)^{-1}| < 1$ for all z. By *Liouville's theorem* (see Chapter 4 of Gamelin [2]), g_∞ is constant which contradicts that $g_\infty^\sharp(0) = 1$. Thus, $\mathcal{F}$ is normal. □

As a consequence of the above theorem, we obtain the following *Picard's theorem.*

Theorem 5.4 (Picard's Great Theorem)**.** *In any neighborhood of an isolated essential singularity, a function f takes every value with at most one exception infinitely often.*

Proof. Assume that the function f has an essential singularity at $z_0 = 0$ and that it omits two distinct complex values.

Let $g_n(z) = f(4^{-n}\rho z)$ be defined on the annulus $\mathbb{A}_{\frac{1}{2},2} = \{\frac{1}{2} < |z| < 2\}$, where ρ is chosen so that f is analytic in a punctured disc around the origin $\mathbb{D}_{\frac{1}{2}\rho}(0) \setminus \{0\}$. By Theorem 5.3, $\{g_n\}$ is a normal family. Thus, there exists a subsequence g_{n_k} converging uniformly to some function g on compact subsets of $\mathbb{A}_{\frac{1}{2},2}$, where g is either an analytic function or $\equiv \infty$.

If g is analytic, then f is uniformly bounded on a sequence of concentric circles of radius $4^{-n_k}\rho$ converging to the origin. By the maximum principle, on each annulus $\mathbb{A}_{4^{-n_{k-1}}\rho, 4^{-n_k}\rho}$ maximum appears on one of these circles. Hence f is bounded around 0. Therefore, by the *Riemann's theorem on removable singularities* (see Chapter 6 of Gamelin [2]), f has a removable singularity at 0 and not an essential singularity.

If $g \equiv \infty$, we apply the same argument to $1/f$ to conclude that $1/f$ extends analytically to 0. Thus, f has a pole at 0. □

Bibliography

[1] L. V. Ahlfors, *Complex Analysis. An Introduction to the Theory of Analytic Functions of One Complex Variable*, 3rd edition, McGraw-Hill, New York, 1979.

[2] T. W. Gamelin, *Complex Analysis*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2001.

[3] T. W. Gamelin and R. E. Greene, *Introduction to Topology*, 2nd edition, Dover Publications, New York, 1999.

[4] S. G. Krantz, *Geometric Function Theory: Explorations in Complex Analysis*, Birkhäuser, Boston, 2006.

[5] A. I. Markushevich, *Theory of Functions of a Complex Variable. Vol. I*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[6] J. L. Schiff, *Normal Families*, Universitext, Springer-Verlag, New York, 1993.

[7] B. Simon, *Basic Complex Analysis*, A Comprehensive Course in Analysis; Part 2A, American Mathematical Society, Providence, RI, 2015.

[8] L. Zalcman. *A Tale of Three Theorems*, The American Mathematical Monthly, vol. 123, no. 7, (2016), 643-656.

Bachelor’s Theses in Mathematical Sciences 2022:K5
ISSN 1654-6229

LUNFMA-4132-2022

Mathematics
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund, Sweden

http://www.maths.lth.se/