# Hall's Marriage Theorem and Related Aspects

Rasmus Eklund

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**Popular scientific abstract**

Hall's Marriage Theorem was named after the English mathematician Philip Hall who proved it in 1935. Although as it turns out, a theorem proven by Karl Menger in 1927 is equivalent to the one proved by Hall [3]. It answers the Marriage Problem: "If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry the boys in such a way that each girl marries a boy she knows?"[8, p.112] It was formulated in terms of set theory rather than the nowadays more common formulations in terms of combinatorics or graph theory, the latter of which will be the formulation used in most of this thesis.

**Abstract**

The main focus of this thesis is to study Hall's Marriage Theorem, which was named after, and proven by the English mathematician Philip Hall in 1935 [6]. The theorem can be stated in terms of different mathematical fields and there are several equivalent theorems proven by other mathematicians [3]. This thesis is mainly going to focus on the graph-theoretic formulation. When stated in terms of graph theory, Hall's Marriage Theorem gives necessary and sufficient conditions for the existence of a special type of edge sets called perfect matchings. The second half of the thesis is dedicated to the applications of the theorem, as well as some of its connections to group theory.

# Contents

# 1 Introduction

When dealing with particular problems, it can be very useful to contextualise them in terms of graphs. For instance, two populations interacting with each other according to a relation (being friends for example) existing between some pairs of individuals in different populations can be represented by a graph with vertices representing the individuals and edges representing the relation. In this case there would be edges connecting vertices representing individuals in different populations, but no edges connecting vertices representing individuals in the same population (we only define the relation between individuals in *different* populations). A graph of this kind is called *bipartite*, since it partitions the vertices into two distinct sets. We study these graphs because they are common and possess many interesting properties.

Hall's Marriage Theorem states that for every bipartite graph, if every collection of vertices in the same partition is no larger than the set of neighbours of that collection, it is possible to find a set of edges, a *matching*, such that each vertex in the smaller partite set is incident to exactly one edge in the matching. If the matching is incident to every vertex in the graph, it is what is called a *perfect matching*, and as we shall see, the existence and applications of these can be very useful.

Before we can prove the theorem, we need to have the preliminary knowledge required for the proofs. This is why the second chapter is dedicated to laying the groundwork for the rest of the thesis by introducing useful concepts from graph theory, including bipartite graphs, paths and matchings. The chapter is concluded by proving two powerful theorems to be used in later chapters. The first of the two theorems, Berge's Theorem, gives a necessary and sufficient condition for the existence of perfect matchings while the second theorem, König's Theorem, connects perfect matchings to other objects called *minimum coverings*.

The reason we choose to prove these theorems is not only that they are important within graph theory, but also so that we can use this directly to prove Hall's Marriage Theorem in chapter three.

The third chapter can be divided into two parts. We start off by proving Hall's Marriage Theorem in three different ways: First by using Berge's Theorem, then by using König's Theorem, and finally without using any previous theorems. The rest of the chapter is used to exemplify how Hall's Marriage Theorem can be applied to solve problems. Some of these problems include the Worker Assignment Problem, Latin Squares and the proof of an important theorem called the König-Egeváry Theorem, which connects minimum coverings to $(0, 1)$-matrices. At this point we will have used König's Theorem to prove both the König-Egeváry Theorem and Hall's Marriage Theorem. As it turns out, these are three of several more theorems that are in fact equivalent, as shown by Robert D. Borgersen in *Equivalence of seven major theorems in combinatorics* [3].

The fourth and final chapter is dedicated to the connection between group theory and what has been discussed in the first three chapters. We start off by showing that for subgroups of finite groups, their cosets can be represented by any of its elements and any collection that represents all left cosets also represents all right cosets. The rest of the chapter is dedicated to analyzing the findings of the paper *Coset Intersection Graphs for Groups*[4] by Jack Button, Maurice Chiodo and Mariano Zeron-Medina. The coset intersection graph is a graph representing a left and a right coset, not necessarily of the same subgroup, as well as the intersection between each of them. This turns out to be a bipartite graph, and a maximum matching is thus a *transversal* of the smaller cosets.

## 2 Preliminary graph theory

In order to state and prove Hall's Marriage Theorem we first need to introduce the prerequisite parts of graph theory. The definitions given are inspired by definitions given in [8, Chap. 2, 3 and 13] and [1, Chap. 1 and 5].

**Definition 2.1.** An edge connected to a vertex is said to be *incident* to that vertex, and vice versa. Two edges are *adjacent* if they are incident to a common vertex and two vertices are adjacent if they are incident to a common edge.

**Definition 2.2.** A *neighbour* of a vertex $v$ is any vertex that is adjacent to $v$, and the set of neighbours of $v$ is denoted by $N(v)$.

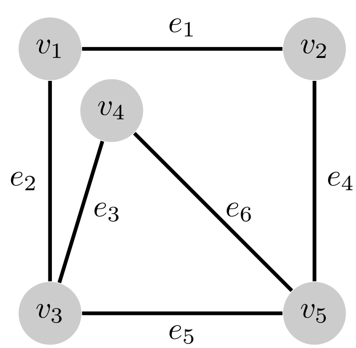


Figure 1:
•$v_5$ is incident to $e_4$, $e_5$ and $e_6$.
•$e_4$ and $e_5$ are adjacent.
•$N(v_4) = \{v_3, v_5\}$.

**Definition 2.3.** The *degree* of a vertex is the number of edges incident to that vertex.

**Definition 2.4.** If every vertex in a graph is a neighbour to all other vertices of the graph, the graph is called *complete*.

**Definition 2.5.** A graph $G$ is called *$k$-regular* if every vertex is of degree $k$. If $G$ is $k$-regular for some $k > 0$, then $G$ is called *regular*.

All $n$-regular graphs, such as 3-regular graphs, 1-regular graphs and 125-regular graphs are examples of *regular* graphs.

Note that any complete graph with $n$ vertices is $(n-1)$-regular.

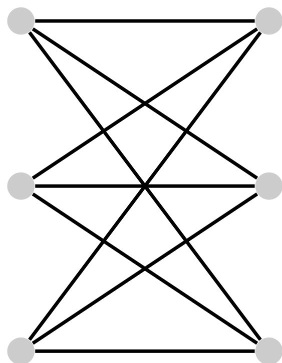

Figure 2: A 3-regular graph that is not complete.

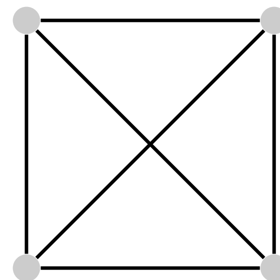

Figure 3: A complete graph (3-regular).

We shall mainly work with a special kind of graphs called bipartite graphs. As their name suggests, they can be partitioned into two distinct vertex sets and have some very useful properties as a result.

**Definition 2.6.** Let $G$ be a graph. Then $G$ is called a *bipartite graph* if it can be partitioned into two sets $X$ and $Y$ such that any edge of $G$ is incident to one vertex in $X$ and one vertex in $Y$. The partition $(X, Y)$ is called a *bipartition* of $G$.

Note that this definition does not require every vertex in a bipartite graph to be connected to an edge. In fact, any amount of such vertices can be added or removed from any bipartite graph, and the resulting graph would still be bipartite.

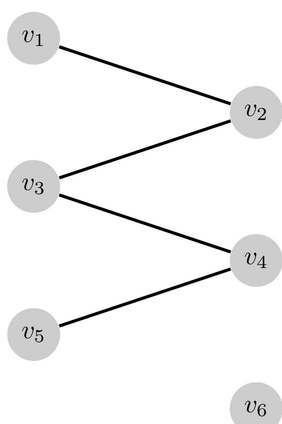


Figure 4: A bipartite graph.

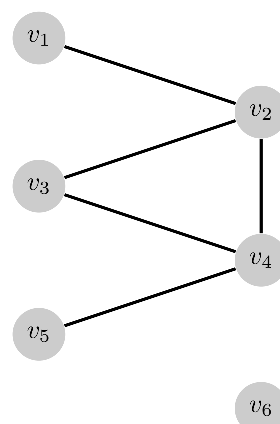


Figure 5: A non-bipartite graph.

In Figure 4, the partition $(\{v_1, v_3, v_5\}, \{v_2, v_4, v_6\})$ is a bipartition since any edge has an end in both sets. There is no such partition of the graph in Figure 5. If there were, no two vertices in the same partite set could be adjacent, so $v_4$ could not be in the same vertex set as either $v_2$ or $v_3$, which means $v_2$ and $v_3$ must be in the same vertex set. However, since they are adjacent, this partitioning does not satisfy a bipartite graph. Indeed, as we shall see later, the graph Figure 5 cannot possibly be a bipartite graph since it contains *an odd cycle*.

The definition of bipartite graphs does not allow them to be complete graphs unless each edge connects the same two vertices, but it is still useful to have a concept of what a bipartite graph would look like if every vertex is of highest possible degree given the constraints.

**Definition 2.7.** A *complete bipartite graph* $K_{s,t}$ is a bipartite graph with bipartition $(X, Y)$, such that $|X| = s$ and $|Y| = t$, where every vertex in $X$ is connected by an edge to each vertex in $Y$ and vice versa.

The 3-regular graph in Figure 2 is a complete bipartite graph. The complete graph shown in Figure 3 is not a complete bipartite graph since it is not a bipartite graph.

We would now like to prove the claim made earlier that bipartite graphs do not have odd cycles, and moreover, that graphs with no odd cycles are bipartite. To do this we need to introduce the concept of paths and distances.

**Definition 2.8.** A *walk* is a sequence alternating between vertices and edges such that neighbouring members of the sequence are incident. The start of a walk is a vertex called the *origin* and the end of a walk is a vertex called the *terminus*. If the origin and terminus are the same, the walk is said to be *closed*.

**Definition 2.9.** A *path* is a walk that has no repeated edges or vertices, except possibly the terminus and origin, and a closed path is called a *cycle*.

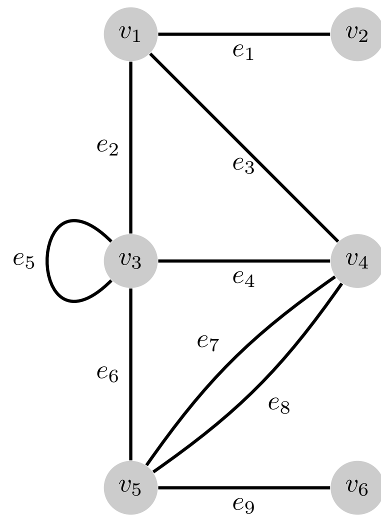


Figure 6: A walk: $v_4e_7v_5e_7v_4e_4v_3$. A closed walk: $v_4e_7v_5e_7v_4e_4v_3e_4v_4$. A path: $v_2e_1v_1e_3v_4e_8v_5e_9v_6$. A cycle: $v_1e_2v_3e_4v_4e_3v_1$.

You can always opt to exclude the vertices in the notation without causing any confusion, and if a walk only passes through vertices with single edges, you could opt to exclude the edges instead. For example: The path $v_6e_9v_5e_6v_3$ in the graph in Figure 6 can simply be written as $v_6v_5v_3$.

**Definition 2.10.** The *length* of a path $P$ is the number of edges that $P$ goes through.

**Definition 2.11.** Let $P$ be a path between $v_1$ and $v_2$. If there is no path between $v_1$ and $v_2$ of *shorter* length, then $P$ is called a *shortest path*.

**Definition 2.12.** The distance $d(v_1, v_2)$ between two vertices $v_1$ and $v_2$ is the length of any shortest path between $v_1$ and $v_2$.

The paths $P = v_5e_6v_3e_2v_1$ and $Q = v_1e_2v_4e_7v_5$ are both shortest paths in the graph illustrated in Figure 6 between $v_5$ and $v_1$. Both $P$ and $Q$ have length 2, and $d(v_1, v_5)$=2.

We now have all the tools we need to introduce the first theorem.

**Theorem 2.13.** *[1, Theorem 1.2] A graph is bipartite if and only if it contains no odd cycles.*

*Proof.* Let $G$ be a bipartite graph with bipartition $(X, Y)$ and $C = v_0v_1 \cdots v_kv_0$ a cycle in $G$. Without loss of generality, let $v_0 \in X$. Since $G$ is bipartite, we have $v_1 \in Y$, and in general $v_{2i} \in X$, and $v_{2i+1} \in Y$. However, we have that $v_k \in Y$ by our assumption, so $k = 2i + 1$, which implies that $C$ is even.

To prove the converse, it suffices to consider an arbitrary connected component of $G$ since if each component of $G$ is bipartite, it is clear that $G$ itself is bipartite. Let $G$ be a graph with no odd cycles. Choose any vertex $v \in G$ and partition $G$ into $(X, Y)$ such that

$$d(e, v) \text{ is even} \quad \Longleftrightarrow \quad e \in X \qquad \text{and} \qquad d(e, v) \text{ is odd} \quad \Longleftrightarrow \quad e \in Y.$$

Let $u$ and $w$ be vertices in the same partite set $X$ or $Y$ and consider a shortest path $P$ from $v$ to $u$, as well as another shortest path $Q$ from $v$ to $w$. Denote the last common vertex of $Q$ and $P$ by $u_1$. Since $P$ and $Q$ are shortest paths, the section from $v$ to $u_1$ must be of the same length for both $P$ and $Q$. Additionally since both $u$ and $w$ are in the same partite set, the section of $P$ from $u_1$ to $u$ and the section in $Q$ from $u_1$ to $w$ must be of the same parity. It follows that the sections of $Q$ and $P$ from $u_1$ to $w$ and $u$, respectively, form a path from $u$ to $w$ of even length. Hence, if $u$ and $w$ were adjacent there would be odd cycle in $G$. However, the graph $G$ contains no odd cycles, so no two vertices in the same partite set are connected by an edge. $\square$

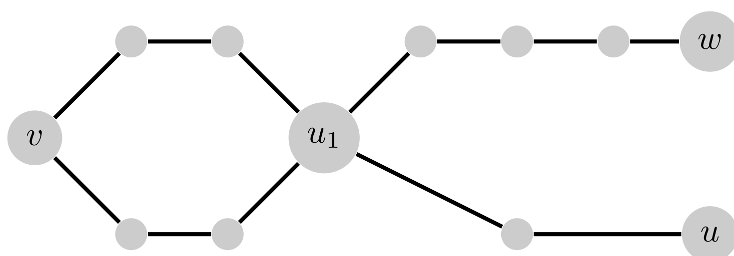


Figure 7: An illustration of the argument from the proof.

**Definition 2.14.** A *matching* $M$ in a graph $G$ is a subset of the edge set $E$ of $G$ such that no two edges in $M$ are adjacent and such that every edge in $M$ connects two distinct vertices.

A vertex is said to be *matched*, *covered* or *saturated* by a matching if it is incident to an edge in the matching.

If $M$ is a matching such that it is not a proper subset of any other matching in the same graph, then $M$ is called a *maximal matching*.

If $M$ is a matching such that there is no larger matching in the same graph, then $M$ is called a *maximum matching*.

A matching is called a *perfect matching* in $G$ if every vertex in $G$ is saturated by that matching. If a matching saturates every vertex in a vertex set $S$, we call it an *$S$-perfect matching*.

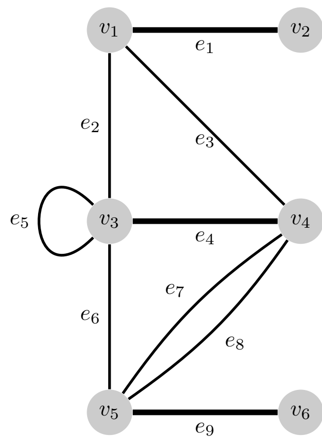


Figure 8: A perfect matching in $G$.

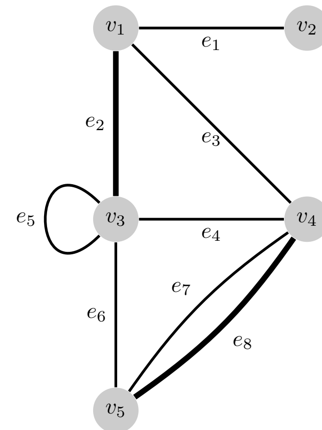


Figure 9: A maximal matching in $G \setminus \{e_9, v_6\}$.

**Definition 2.15.** Let $M$ be a matching in a graph $G$. An *alternating path* is a path with an unmatched vertex as origin and with edges alternating between being in $M$ and not being in $M$.

**Definition 2.16.** An alternating path that has an unmatched vertex as its terminus is called an *augmenting path*.

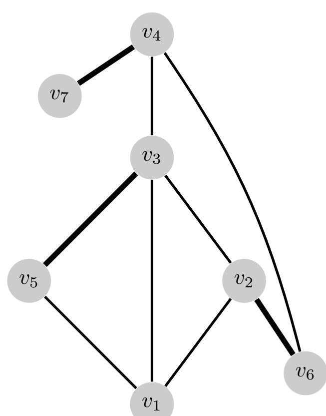


Figure 10: A graph $G$ with highlighted edges of a matching.

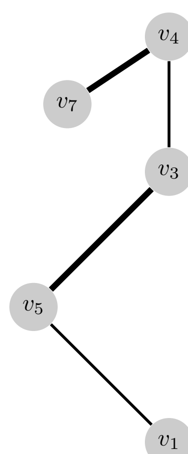


Figure 11: An $M$-alternating path in $G$.

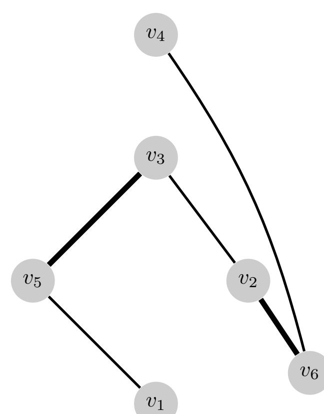


Figure 12: An $M$-augmenting path in $G$.

**Theorem 2.17** (Berge's Theorem)**.** *[5, Theorem 3.1] [1, Theorem 5.1] A matching $M$ in a graph $G$ is a maximum matching if and only if $G$ contains no $M$-augmenting path.*

*Proof.* Let $M$ be a matching in $G$, and let $G$ contain an $M$-augmenting path $P = v_0v_1v_2 \cdots v_n$. Additionally, let $v_iv_j$ denote the edge between $v_i$ and $v_j$. The matching $M' = (M \setminus \{v_1v_2, v_3v_4, \ldots, v_{n-2}v_{n-1}\}) \cup \{v_0v_1, v_2v_3, \ldots, v_{n-1}v_n\}$ contains one more edge than $M$, hence $M$ is not a maximum matching.

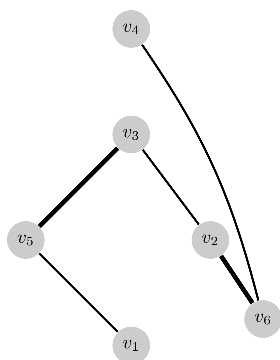


Figure 13: An $M$-augmenting path in the graph from Figure 10.

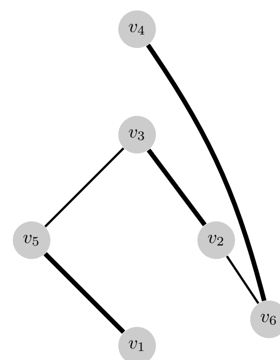


Figure 14: A larger matching in that part of the graph.

Conversely, suppose that $M$ is *not* a maximum matching. Then there is a larger matching $M'$. Consider the graph $H \subseteq G$ containing the edges that are in either $M$ or $M'$, but not in both, as well as their incident vertices. Then every vertex in $H$ has at least one and at most two incident edges; up to one from each matching $M$ and $M'$. It follows that every component of $H$ is a path (either cyclic or not) with edges alternating between $M$ and $M'$. Since $M'$ has more edges than $M$, at least one component of $H$ must be an alternating path with more edges in $M'$ than $M$, hence that path is an $M$-augmenting path. $\square$

Once familiar with matchings, it is natural to consider the corresponding concept in terms of vertices. We call these objects coverings.

**Definition 2.18.** Let $G$ be a graph with vertex set $V$. A *covering* $K$ of $G$ is a subset of $V$ such that every edge in $G$ is has at least one end in $K$.

A covering $K$ is said to be a *minimum covering* of $G$ if there is no other covering of $G$ that contains fewer vertices than $K$.

In every graph $G$, the entire vertex set is a covering of $G$. In Figure 8, the set $\{v_1, v_3, v_5\}$ is a minimum covering.

Note that any covering $K$ will necessarily contain distinct vertices incident to each edge in any matching $M$. Thus, we get that $|K| \geq |M|$ always holds. In fact, since this is true for all matchings and coverings, it is also true if $K$ is a minimum covering and $M$ is a maximum matching. It follows that equality never holds if the covering is not minimum or the matching not maximum. We state this as a lemma:

**Lemma 2.19.** *[1, Lemma 5.3] Let $K$ be a covering and let $M$ be a matching in a graph $G$. If $|K| = |M|$, then $K$ is a minimum covering in $G$ and $M$ is a maximum matching.*

**Theorem 2.20** (König's Theorem)**.** *[1, Theorem 5.3] The maximum matchings and minimum coverings in bipartite graphs are of the same size.*

*Proof.* Let $G$ be a bipartite graph with bipartition $(X, Y)$, and let $M$ be a maximum matching in $G$. Also, denote by $U$ the set of unmatched vertices in $X$, and let $Z$ be the set of all vertices connected to vertices in $U$ by $M$-alternating paths. Now consider the intersections $Z_X = Z \cap X$ and $Z_Y = Z \cap Y$.

Since $M$ is a maximum matching, the set $N(U)$ must consist of only $M$-saturated vertices. Additionally, each vertex in $N(Z_X \backslash U)$ must be $M$-saturated, since otherwise there would be an $M$-augmenting path in $G$, contradicting Berge's Theorem. Thus, we have that $N(Z_X) \subseteq Z_Y$. Clearly, $Z_Y \subseteq N(Z_X)$, so we get $N(Z_X) = Z_Y$.

Now let $K = (X \setminus Z_X) \cup Z_Y$. Every edge of $G$ must have at least one end in $K$, since otherwise there would be an edge with one end in $Z_X$ and the other end in $Y \setminus Z_Y$, which is impossible since $N(Z_X) = Z_Y$. Hence, $K$ is a covering of $G$.

Clearly $|K| = |M|$, so $K$ must be a minimum covering by Lemma 2.19, and the theorem follows. $\square$

# 3 Hall's Marriage Theorem

*Hall's Condition.* A bipartite graph $G$ with bipartition $(X, Y)$ is said to satisfy *Hall's Condition if*

$$|N(S)| \geq |S| \quad \forall S \subseteq X.$$

**Theorem 3.1** (*Hall's Marriage Theorem*)**.** *A bipartite graph $G$ with bipartition $(X, Y)$ contains an $X$-perfect matching if and only if it satisfies Hall's Condition.*

Hall's Condition is clearly necessary, since if $G$ has an $X$-perfect matching $M$, any subset $S \subseteq X$ would be saturated by the subset $M_S \subseteq M$, where $M_S$ is the set of edges in $M$ incident to any vertex in $S$, hence $|S| = |M_S| \leq |N(S)|$.

The real challenge lies in proving the sufficiency of the condition, and there are several ways one could solve it. In this chapter, we shall provide proofs using three different techniques.

## 3.1 Proof 1 - Alternating Paths [1, Theorem 5.2]

Suppose that $G$ satisfies Hall's Condition, but that there is no $X$-perfect matching in $G$. Consider a maximum matching $M$ and an $M$-unsaturated vertex $v \in X$. Let $Z$ be the set of all vertices connected to $v$ by $M$-alternating paths. Now consider the intersections $Z_X = Z \cap X$ and $Z_Y = Z \cap Y$. From the proof of König's Theorem, we know that $N(Z_X) = Z_Y$. Since every vertex in $Z_X \setminus \{v\}$ is $M$-saturated according to Berge's Theorem, the vertices in $Z_X \setminus \{v\}$ must be exactly matched to the vertices in $Z_Y$. This yields

$$|Z_Y| = |Z_X \setminus \{v\}| = |Z_X| - 1.$$

Also,

$$N(Z_X) = Z_Y \quad \Rightarrow \quad |Z_Y| = |N(Z_X)| = |Z_X| - 1 < |Z_X|.$$

This contradicts the assumption that Hall's Condition is met, hence there must exist an X-perfect matching in $G$. $\square$

This proof is strikingly similar to the proof of König's Theorem, and for good reason: König's Theorem is in fact one of the theorems equivalent to Hall's Marriage Theorem [3]. With this in mind, we shall use König's Theorem to prove Hall's Marriage Theorem by providing our own solution to a problem in [2].

### 3.2 Proof 2 - König implies Hall [2, Problem 16.2.6]

In order to deduce Hall's Marriage Theorem, we shall divide the proof into two parts:

(i) We show that every minimum covering of a bipartite graph $G(X, Y)$ is of the form $N(S) \cup (X \setminus S)$ for some subset $S$ of $X$.

(ii) We deduce Hall's Marriage Theorem by applying König's Theorem.

**(i)** Let $K$ be a minimum covering in $G$ that is not of the form $N(S) \cup (X \setminus S)$ for any subset $S \subseteq X$.

This means that $K$ must be of the form

$$K = [N(S) \cup (X \setminus S)] \setminus A \quad \text{or} \quad K = [N(S) \cup (X \setminus S)] \cup B \quad \forall S \subseteq X$$

where

$$A \subseteq N(S) \cup (X \setminus S) \quad \text{and} \quad B \subseteq X \setminus [N(S) \cup (X \setminus S)], \quad \text{such that } A \neq \emptyset \neq B.$$

Since each edge in $G$ has an end in $N(S) \cup (X \setminus S)$, no minimum covering can be of the form $K = [N(S) \cup (X \setminus S)] \cup B$. Consider the case where $K$ is of the form $K = [N(S) \cup (X \setminus S)] \setminus A$. Removing any vertex $u$ in $N(S)$ from $K$ would cause each edge between $u$ and $S$ to be $K$-unsaturated, hence $A \subseteq X \setminus S$. On the other hand, the only way to remove a set of vertices $A$ in $X \setminus S$ from $K$ such that $K$ still saturates every edge in $G$ is if $N(A) \subseteq N(S)$ holds.

It follows that

$$K = N(S) \cup [(X \setminus S)] \setminus A = N(S) \cup [(X \setminus S) \setminus A]$$
$$= N(S \cup A) \cup [X \setminus (S \cup A)].$$

Substituting $S' = S \cup A$ shows that $K$ is of the form $K = N(S') \cup (X \setminus S')$.

**(ii)** Let $M$ be a maximum matching and let $K$ be a minimum covering in $G$. By **(i)** we know that there is a subset $S$ of $X$ such that $K = N(S) \cup (X \setminus S)$. By König's Theorem, we know that $|K| = |M| = |N(S)| + |X| - |S|$.

If Hall's Condition is met, we get

$$|N(S)| \geq |S| \quad \Rightarrow \quad |N(S)| - |S| \geq 0 \quad \Rightarrow \quad |M| \geq |X|.$$

It follows that $M$ is an $X$-perfect matching, and Hall's Condition is sufficient. $\square$

There are, of course, ways of proving Hall's Marriage Theorem that do not have anything to do with König's Theorem, we shall provide one such proof next.

### 3.3 Proof 3 - Strong induction [8, Theorem 25.1]

Assume that Hall's Condition is sufficient for $|X| < m$. If $|X| = 1$ the condition is clearly sufficient. In order to prove that it is also sufficient for $|X| = m$ we consider two cases:

**Case 1.** Assume that every proper subset of $X$ has strictly more neighbours than vertices. In this case, consider the matching $M_1$ that matches $v \in X$ to one of its neighbours $u$. Now consider the set $X \setminus \{v\} = Z$. Since $Z$ is a proper subset of $X$ it has at least one neighbour more than vertices by our assumption. This gives:

$$|N(Z)| > |Z| \quad \iff \quad |N(Z)| \geq |Z| + 1 \quad \Rightarrow \quad |N(Z) \setminus \{u\}| \geq |Z| < m.$$

Hence, by our induction hypothesis there is a $Z$-perfect matching $M_2$ that matches vertices in $Z$ to vertices in $N(Z) \setminus \{u\}$. The matching $M = M_1 \cup M_2$ is an $X$-perfect matching in this case.

**Case 2.** If case 1 is not true, there must exist some proper subset of $X$ that has exactly as many neighbours as vertices, that is for some proper subset $S$ of $X$, we have $|N(S)| = |S|$, where $|S| = k$, for $1 \leq k < m$. By our induction hypothesis there is an $S$-perfect matching $M_1$ between $S$ and $N(S)$. Then $|X \setminus S| = m - k$. Additionally, for any subset $T \subseteq X \setminus S$ we have $|T| = h \leq m - k$. It follows that $T$ must have at least $h$ neighbours **not in** $N(S)$, since if not we would have $|T \cup S| = h + k > |N(T \cup S)|$, contradicting Hall's Condition since $T \cup S \subseteq X$. Hence, there is an $X \setminus S$-perfect matching $M_2$ to the vertex set $Y \setminus N(S)$. The matching $M = M_1 \cup M_2$ is an $X$-perfect matching in $G$, so the proof is completed by induction. □

A lot of the time it is hard to see if Hall's Condition holds, so it is useful to have a stronger condition that still implies the existence of an $X$-perfect matching, if the condition is sufficiently easy to check.

**Corollary 3.2.** *[1, Corollary 5.2] Every regular bipartite graph has a perfect matching.*

*Proof.* Let $G$ be a $k$-regular bipartite graph with bipartition $(X, Y)$, where $S \subseteq X$ and let $E_1$ and $E_2$ denote the set of edges incident to vertices in $S$ and $N(S)$, respectively. Then

$$E_1 \subseteq E_2 \quad \Rightarrow \quad |E_1| \leq |E_2|. \tag{3.1}$$

Also, since every vertex in $G$, has degree $k$ it follows that

$$|E_1| = k|S|, \quad \text{and similarly} \quad |E_2| = k|N(S)|. \tag{3.2}$$

The equations (3.1) and (3.2) together imply $|S| \leq |N(S)|$, hence there is an $X$-perfect matching by Hall's Marriage Theorem. Then any $X$-perfect matching is a perfect matching because $k|X| = |E_G| = k|Y|$, which implies that $|X| = |Y|$. $\square$

## 3.4 Applications of Hall's Marriage Theorem

### 3.4.1 Transversals

**Definition 3.3.** Given a family of sets $S = \{S_1, \ldots, S_n\}$, a *transversal* $T$ of $S$ is a set of $n$ distinct elements, such that $T \cap S_i = 1$, for $i \in \{1, \ldots, n\}$.

**Example 3.4.** Let $S = \{S_1, S_2, S_3\}$, where

$$S_1 = \{1, 2, 3\}, \quad S_2 = \{3, 4, 5\}, \quad S_3 = \{5, 6, 7\}.$$

Then $T_1 = \{1, 4, 6\}$ is a transversal of $S$ and $T_2 = \{2, 4, 7\}$ is a transversal of $S$, but $T_3 = \{3, 7\}$ is not a transversal of $S$, nor is $T_4 = \{1, 4, 5\}$.

Transversals are naturally connected to graph theory and Hall's Marriage Theorem. In fact, the theorem can be restated in terms of set theory and transversals. We do this ourselves in the following way:

**Theorem 3.5.** *[8, Theorem 26.1] Let $E$ be a non-empty finite set and let $S = \{S_1, \ldots, S_n\}$ be a family of non-empty subsets of $E$. There exists a transversal $T$ of $S$ if and only if, for any collection of $k$ sets $S_i$ (where $i \in \{1, \ldots, n\}$) their union contains at least $k$ elements.*

*Proof.* Consider a graph $G$ with vertex sets $(X, Y)$, such that each set $S_i$ (where $i \in \{1, \ldots, n\}$) is represented by a vertex in $X$, and each element in the set $E$ is represented by a vertex in $Y$. Draw an edge from a vertex $x$ in $X$ to a vertex $y$ in $Y$ if the set represented by $x$ contains the element represented by $y$. Then $G$ is a bipartite graph and, by Hall's Marriage Theorem, has an $X$-perfect matching $M$ if and only if it satisfies Hall's Condition. By virtue of how $G$ was constructed, $G$ satisfies Hall's Condition if and only if for each collection of $k$ subsets $\{S_i\}$, their union contains at least $k$ elements.

Vertices in $Y$ that are incident to an edge in an $X$-perfect matching $M$ form a transversal of $S$. Additionally, if $T$ is a transversal of $S$, the edges that connect the vertices representing $\{S_i\}$ and the vertices representing their respective contributions to $T$ form an $X$-perfect matching. Hence, a transversal of $S$ exists if and only if an $X$-perfect matching in $G$ exists. $\square$

**Example 3.6.** Consider the previous example of the family $S = \{S_1, S_2, S_3\}$ and the sets $T_1, T_2, T_3, T_4$. Each one of the sets $S_1$, $S_2$, and $S_3$ contain at least one element that is not shared by either of the other sets. Hence, for any collection of them, their union will contain at least as many elements as there are sets in the union. Thus, Theorem 3.5 states that there exists a transversal of $S$, which we have already shown is true.

### 3.4.2 Worker Assignment Problem [1, Chapter 5.4]

If a company has a group of several workers, each of whom is qualified for one or more jobs out of many, when and how can the company assign every worker to a job they are qualified to?

We can construct a bipartite graph $(X, Y)$ such that $X = \{x_1, x_2, \ldots, x_n\}$ represents the workers and $Y = \{y_1, y_2, \ldots, y_n\}$ represents the jobs. We draw an edge between a worker and a job if the worker is qualified to do that job. The problem now boils down to finding an $X$-perfect matching, and Hall's Marriage Theorem tells us that there is one if and only if *any* group of workers are qualified for at least as many distinct jobs as they are workers.

We could also state the problem in terms of transversals:
Consider, again, the set $X = \{x_1, x_2, \ldots, x_n\}$ where every element represents a worker. This time, let $x_i$ be a set containing elements of $Y = \{y_1, y_2, \ldots, y_n\}$ where every element represents a job. Let $y_i \in x_j$ if the worker represented by $x_j$ is qualified for the job represented by $y_i$.

Now instead of trying to find a matching of a bipartite graph, we are trying to find a transversal of $X$.

Hall's Marriage Theorem tells us under what conditions there are matchings or transversals to be found, but does not tell us *how* to find them. For this purpose, there are algorithms such as Edmonds' Blossom Algorithm [7] that can find them.

### 3.4.3 Latin Squares

An $m \times n$ Latin rectangle is an $m \times n$ matrix whose rows have integer entries ranging from 1 to $n$, such that there are no repeated integers in any row or column.

An $n \times n$ Latin rectangle is called a Latin square. For example:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & 1 & 3 & 2 \\ 2 & 4 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

On the left, we see a $2 \times 3$ Latin rectangle, and on the right we see a $4 \times 4$ Latin square.

With the help of Hall's Marriage Theorem (stated in terms of transversals), we can easily prove that any Latin rectangle can be extended to a Latin square.

**Theorem 3.7.** *[8, Theorem 27.1] Any $m \times n$ $(m < n)$ Latin rectangle $M$ can be extended to a Latin square by adding new rows.*

*Proof.* Consider the family $F = \{N_1, N_2, \dots, N_n\}$ of subsets of $I = \{1, \dots, n\}$ where $N_i$ is the set of missing elements in column $i \in I$ of $M$. Each $N_i$ contains exactly $n - m$ elements, and if $n = m$, the Latin rectangle $M$ is already a Latin square. Theorem 3.5 states that if every collection of $k$ sets $N_i$ contains at least $k$ distinct elements, there exists a transversal $T$ of $F$. This is clearly the case, since each union of any $k$ sets $N_i$ contains exactly $k(n - m)$ elements, including repetition. If there were fewer than $k$ distinct elements in the collection, some element would have to included twice in the same set.

We can extend $M$ with a row of elements from $T$ by extending the $i$th column with the element in $T \cap N_i$. Doing so extends the $m \times n$ Latin rectangle $M$ to a new $(m + 1) \times n$ Latin rectangle.

This process can be repeated until $M$ is extended to a $n \times n$ Latin square. □

**Example 3.8.** Using the method in the proof on the $2 \times 3$ Latin rectangle in the example above, $N_1, N_2, N_3$ would be the sets $\{1\}$,$\{3\}$, and $\{2\}$, respectively. Thus $T = \{1, 2, 3\}$ is a trivial transversal of the set $N = \{N_1, N_2, N_3\}$, and the resulting Latin rectangle would be:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

### 3.4.4 Application to matrices

We shall now demonstrate the strength of Hall's Marriage Theorem and König's Theorem by providing our own proofs of the König-Egeváry Theorem and Theorem 4.1. Both of these theorems do not seem to have anything to do with graph theory, yet can be proven by using it.

**Definition 3.9.** The *line* of a matrix is a row or column of the matrix.

**Theorem 3.10** (The König-Egeváry Theorem)**.** *[2, Problem 16.2.3] The minimum number of lines containing all the* $1$*s in a* $(0, 1)$*-matrix is equal to the maximum number of* $1$*s such that no two* $1$*s lie in the same line.*

*Proof.* Consider a graph $G$ with vertices representing the lines of a matrix. If the common entry of two distinct lines is a one, draw an edge between the vertices in $G$ representing those lines. It follows that any covering $K$ of $G$ represents a set of lines such that all nonzero elements are contained in those lines. Also, any matching $M$ in $G$ represents nonzero entries, and two distinct edges in $M$ cannot represent entries in the same line since edges in $M$ cannot be adjacent. Hence, the matching $M$ represents nonzero entries no two of which lie in the same line.

If $M$ and $K$ are maximum and minimum, respectively, it follows from König's Theorem that $|K| = |M|$, and the statement of the problem follows. □

**Example 3.11.** Consider the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

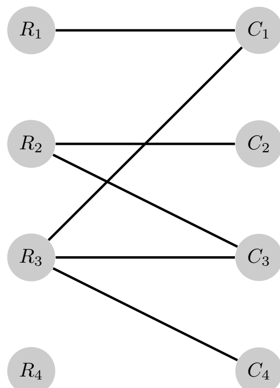


Figure 15: The graph constructed from [2, Problem 16.2.3]

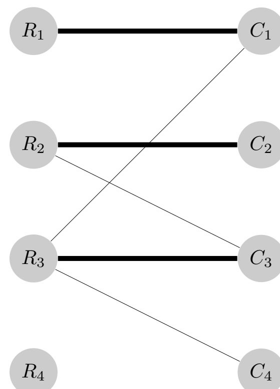


Figure 16: A maximum matching.

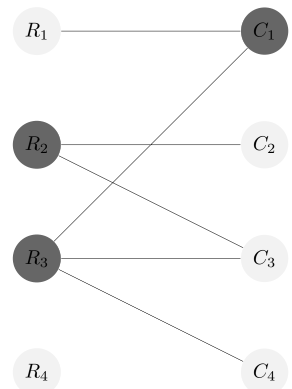


Figure 17: A minimum covering.

The maximum matching in Figure 16 corresponds to the choice of nonzero entries on the diagonal, and the minimum covering in Figure 17 corresponds to the choice of row 2, row 3 and column 1.

As we can see, this problem was immediately solved by applying König's Theorem; it would be fitting to call it a corollary. Sometimes the statement proven in this problem is called the König-Egeváry Theorem, but there is very little consistency between texts regarding the naming of these theorems. If fact, the theorem we call König's Theorem is referred to as both König's Theorem and The König-Egeváry Theorem by Bondy and Murty, albeit in different books [2] [1].

# 4 Connections to group theory

We shall start off this chapter by proving a theorem in group theory with the help of Hall's Marriage theorem. The proof of it is provided by us.

**Theorem 4.1.** *[2, Problem 16.2.20] Let $H$ be a finite group and let $K$ be a subgroup of $H$. Then there exist elements $h_1, h_2, \ldots, h_k \in H$ such that $h_1K, h_2K, \ldots, h_kK$ are the left cosets of $K$ and $Kh_1, Kh_2, \ldots, Kh_k$ are the right cosets of $K$.*

*Proof.* Consider a graph $G$ with vertices representing the left and right cosets of $K$. If a left and a right coset of $K$ are equal, they are still represented by distinct vertices. Additionally, draw an edge between two vertices if their intersection is non-empty, and let that edge represent said intersection.

Same-sided cosets are disjoint, of equal size, and their union contains exactly all the elements of their group, in this case $H$. It follows that $G$ is bipartite graph partitioned between the left and right cosets, and that any $k$ left cosets contain $k|K|$ elements, so must intersect at least $k$ right cosets, since otherwise all the elements of the left cosets would not fit in the right cosets.

Thus, Hall's Condition is fulfilled and there exists a perfect matching $M$ in $G$. Since the sets represented by $M$ are disjoint it is trivial to find a transversal $T = \{h_1, \ldots, h_k\}$ of them. Moreover,

$$h_1 \in hK \quad \Rightarrow \quad \exists a \in K : ha = h_1 \quad \Rightarrow \quad h_1K = (ha)K = h(aK) = hK.$$

The same argument goes for right cosets, hence we get that $h_1K, h_2K, \ldots, h_kK$ are the left cosets of $K$ and $Kh_1, Kh_2, \ldots, Kh_k$ are the right cosets of $K$. □

This theorem shall prove very useful and relevant in the next part of this section, which is dedicated to analyzing some of the findings in the paper named *Coset Intersection Graphs for Groups*, written by Jack Button, Maurice Chiodo and Mariano Zeron-Medina Laris in 2014 [4]. They expand upon common knowledge about cosets by studying the intersection of left cosets with right cosets, not necessarily cosets of the same subgroups. They then use this to prove a generalization of Hall's Marriage Theorem for transversals.

We will start by presenting the necessary definitions presented in the paper.

**Definition 4.2.** Let $H$ be a subgroup of a group $G$. A left-transversal of $H$ is a set of elements $\{l_i\} \subseteq G$ such that $\{l_i\}$, where $i \in I$ and $I$ is an index set, contains exactly one element from each left coset of $H$. A right-transversal is defined analogously, and we write $\{r_j\}$. A left-right transversal is a set that is both a left- and right-transversal of the same group.

According to the previous theorem, a coset can be represented by any element within itself, which means that we can represent the left cosets of $H$ as $\{l_iH\}$ and, conversely, the right cosets as $\{Hr_j\}$.

**Example 4.3.** Consider the symmetric group

$$S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$$

and one of its subgroups $|H| = \{e, (1,3)\}$.

The left cosets of $H$ are

$$\begin{aligned} eH &= H = \{e, (1,3)\}, \\ (1,2)H &= \{(1,2), (1,3,2)\} \text{ and} \\ (2,3)H &= \{(2,3), (1,2,3)\}. \end{aligned}$$

The right cosets of H are

$$\begin{aligned} He &= H = \{e, (1,3)\}, \\ H(1,2) &= \{(1,2), (1,2,3)\} \text{ and} \\ H(2,3) &= \{(2,3), (1,3,2)\}. \end{aligned}$$

We see that

$T_1 = \{e, (1,2), (1,2,3)\}$ is a left transversal of H,
$T_2 = \{(1,3), (1,2), (1,3,2)\}$ is a right transversal of H and
$T_3 = \{e, (1,2), (2,3)\}$ is a left-right transversal of H.

**Definition 4.4.** [4, Definition 2] Let $H$ and $K$ be subgroups of a group $G$. We define *the coset intersection graph* $\Gamma^G_{H,K}$ as a graph where every left coset $l_i H$ and every right coset $Kr_j$, is represented by a vertex. If a left coset of $H$ is the same as a right coset of $K$, they still correspond to two distinct vertices. If a left coset has a non-empty intersection with a right coset, they are joined by an edge representing the intersection.

Since same-sided cosets are disjoint, edges only join left cosets with right cosets, hence $\Gamma^G_{H,K}$ is a bipartite graph with bipartition $(\{l_i H\}, \{Kr_j\})$.

**Example 4.5.** Consider the group $G = \mathbb{Z}$ of integers under addition and the subgroups $H = 5\mathbb{Z}$ and $K = 7\mathbb{Z}$. Then we have $H = \{5k\}$ and $K = \{7k\}$, where $k \in \mathbb{Z}$.

In this case, the left cosets of $H$ are

$$\begin{aligned} &0 \oplus H = H, \\ &1 \oplus H = \{1 + 5k\}, \\ &2 \oplus H = \{2 + 5k\}, \\ &3 \oplus H = \{3 + 5k\} \text{ and} \\ &4 \oplus H = \{4 + 5k\}. \end{aligned}$$

Also, the right cosets of $K$ are

$$\begin{aligned}
K \oplus 0 &= K, \\
K \oplus 1 &= \{1 + 7k\}, \\
K \oplus 2 &= \{2 + 7k\}, \\
K \oplus 3 &= \{3 + 7k\}, \\
K \oplus 4 &= \{4 + 7k\}, \\
K \oplus 5 &= \{5 + 7k\} \text{ and} \\
K \oplus 6 &= \{6 + 7k\}.
\end{aligned}$$

The coset intersection graph $\Gamma^{\mathbb{Z}}_{5\mathbb{Z},7\mathbb{Z}}$ has five vertices representing the left cosets of $H$ and seven vertices representing the right cosets of $K$.

For two arbitrary cosets $m \oplus H$ and $K \oplus n$ we can calculate their intersection by solving the following system of congruences:

$$\begin{cases} x \equiv n \pmod 7 \\ x \equiv m \pmod 5). \end{cases}$$

We have

$$\begin{aligned}
x = 7b + n &\equiv m \pmod 5 \\
2b &\equiv m - n \pmod 5 \\
6b &\equiv 3(m - n) \pmod 5 \\
b &\equiv 3(m - n) \pmod 5.
\end{aligned}$$

This gives us

$$x = 7b+n = 7(5k+3(m-n))+n = 35k+21(m-n)+n \equiv 21(m-n)+n \pmod{35}.$$

Using this result, we can easily determine the intersection of any two cosets. For example: $1 \oplus H \cap K \oplus 2 = \{16 + 35k\}$.

As we can see, each intersection represents exactly one congruence class modulo 35. Clearly, each integer is contained in a coset of either side, so since there are $5 \cdot 7 = 35$ intersections, they must all contain distinct congruence classes modulo 35.

Hence, each edge in the coset intersection graph represents a distinct congruence class modulo 35.

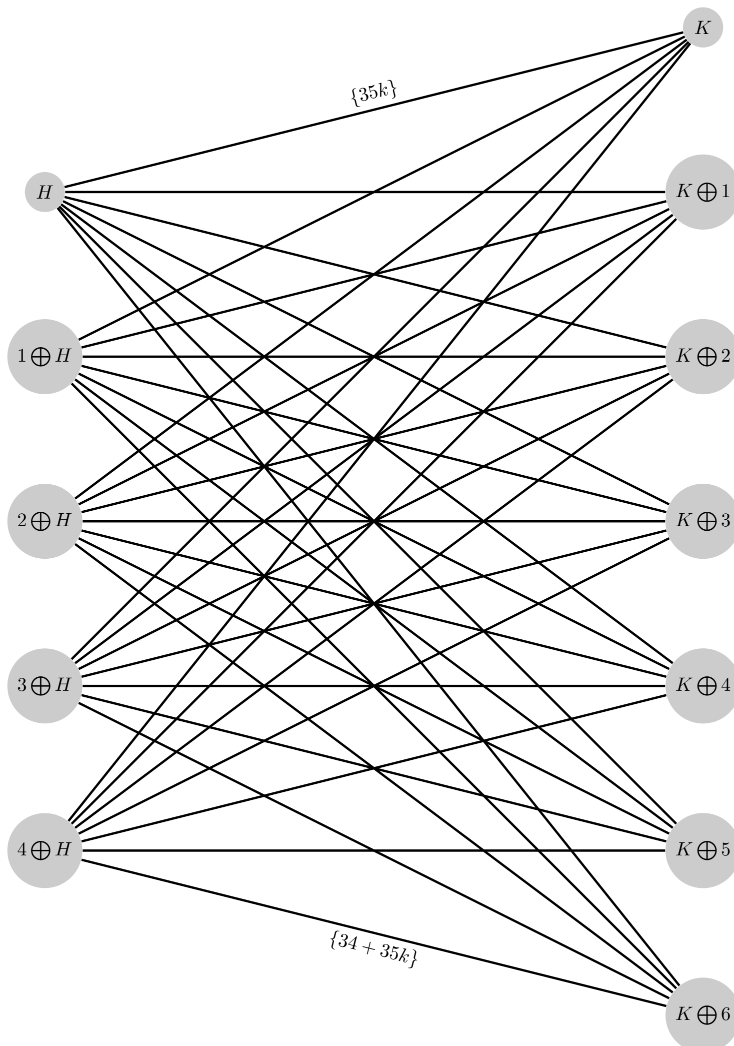


Figure 18: The coset intersection graph $\Gamma^{\mathbb{Z}}_{5\mathbb{Z},7\mathbb{Z}}$ from Example 4.5.

**Example 4.6.** Let $\Gamma^G_{H,K}$ be the coset intersection graph where $G = S_3$ and $H$ the same subgroup of $S_3$ as in the first example in this chapter, and $K$ is the subgroup $\{e, (1,2)\}$ of $S_3$.

The right cosets of $K$ are

$$\begin{aligned} Ke = K &= \{e, (1,2)\}, \\ K(1,3) &= \{(1,3), (1,3,2)\} \text{ and} \\ K(2,3) &= \{(2,3), (1,2,3)\}. \end{aligned}$$

The non-empty intersections $l_iH \cap Kr_j$ are

$$\begin{aligned} eH \cap Ke &= \{e\}, \\ eH \cap K(1,3) &= \{1,3\}, \\ (1,2)H \cap Ke &= \{(1,2)\}, \\ (1,2)H \cap K(1,3) &= \{1,3,2\} \text{ and} \\ (2,3)H \cap K(2,3) &= \{(2,3), (1,2,3)\}. \end{aligned}$$

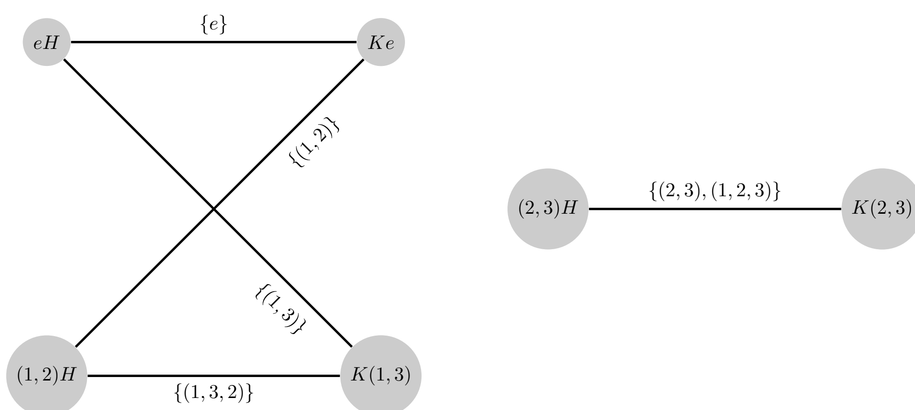


Figure 19: The coset intersection graph $\Gamma^G_{H,K}$ from Example 4.6.

Looking at Figure 19, you might observe that $\Gamma^G_{H,K}$ is divided into complete bipartite graphs. This is in fact always the case and something to be familiar with going forward:

**Theorem 4.7.** *[4, Theorem 3] Every component of the coset intersection graph is complete.*

Knowing this, it follows from Theorem 3.2 that there exists a left-right transversal of any subgroup $H$ of a finite group $G$. The authors of this paper, however, set out to prove a stronger statement without the use of Hall's Marriage Theorem.

**Theorem 4.8.** *[4, Theorem 4] Let $H$ and $K$ be finite subgroups of $G$. Then $\Gamma^G_{H,K}$ is a union of finite, disjoint, complete bipartite graphs $K_{s_i,t_i}$ such that $s_i/t_i = |K|/|H|$.*

*Proof.* Consider any connected component of $\Gamma^G_{H,K}$. We know that it must be finite since $H$ and $K$ are, and they cannot intersect infinitely many times. Thus, this component of $\Gamma^G_{H,K}$ is a complete bipartite graph with vertex sets of size $s$ and $t$, respectively. Since the union of all same-sided cosets cover $G$, all elements of any given coset must be included in an edge in $\Gamma^G_{H,K}$. It follows that the union of the $s$ left-sided cosets in the connected component must contain exactly the same elements as the union of the $t$ right-sided cosets. Additionally since any coset of $H$ has size $n = |H|$ and any coset of $K$ has size $m = |K|$, we get $s|H| = t|K|$, which mean that we have $s/t = n/m$ . □

With this theorem in mind, we can look back at the previous example and see that coset intersection graph depicted Figure 19 has two components, each of which has a one-to-one ratio between the vertices in the bipartition $(H, K)$, as the theorem would suggest since $|H| = |K|$ in this case. The component to the left in Figure 19 is a complete bipartite graph $K_{2,2}$, and the one to the right a complete bipartite graph $K_{1,1}$.

**Corollary 4.9.** *[4, Corollary 5] Let $H, K$ be finite subgroups of a group $G$ where $|H| = m$ and $|K| = n$, with $m \geq n$. Then there exists a left transversal $T \subseteq G$ for $H$ that can be extended to a right transversal for $K$. If $H = K$, then $T$ is a left-right transversal of $H$.*

*Proof.* Denote the components of $\Gamma^G_{H,K}$ by $K^i_{s_i,t_j}$. From the previous theorem we know that each such component is a complete bipartite graph with $s_i \leq t_i$. Choose a matching $T_i$ that saturates all $s_i$ vertices on the $H$-side of $K^i_{s_i,t_j}$. Then $T = \bigcup_{i \in I} T_i$ is a matching saturating the left cosets of $H$, and $T$ is therefore a left-transversal for $H$. Since every element in $T$ exist in exactly one right coset of $K$, it follows that $T$ can be extended to a right-transversal of $K$ by including any element from each of the right cosets of $K$ with no element in $T$. If $H = K$, this extension of $T$ is a left-right transversal of $H$. □

**Example 4.10.** Let $\Gamma^G_{H,K}$ be the coset intersection graph where $G = S_3$ and $K$ is the subgroup $\{e, (1, 3)\}$ of $S_3$, and this time let $H$ be the subgroup $\{e, (1, 2, 3), (1, 3, 2)\}$ of $S_3$.

The right cosets of $H$ are

$$\begin{aligned} He = H = \{e, (1, 2, 3), (1, 3, 2)\} \text{ and} \\ H(1, 3) = \{(1, 3), (1, 2), (2, 3)\}. \end{aligned}$$

The left cosets of $K$ are

$$\begin{aligned} eK &= K = \{e, (1, 3)\}, \\ (1, 2)K &= \{(1, 2), (1, 3, 2)\} \text{ and} \\ (2, 3)K &= \{(2, 3), (1, 2, 3)\}. \end{aligned}$$

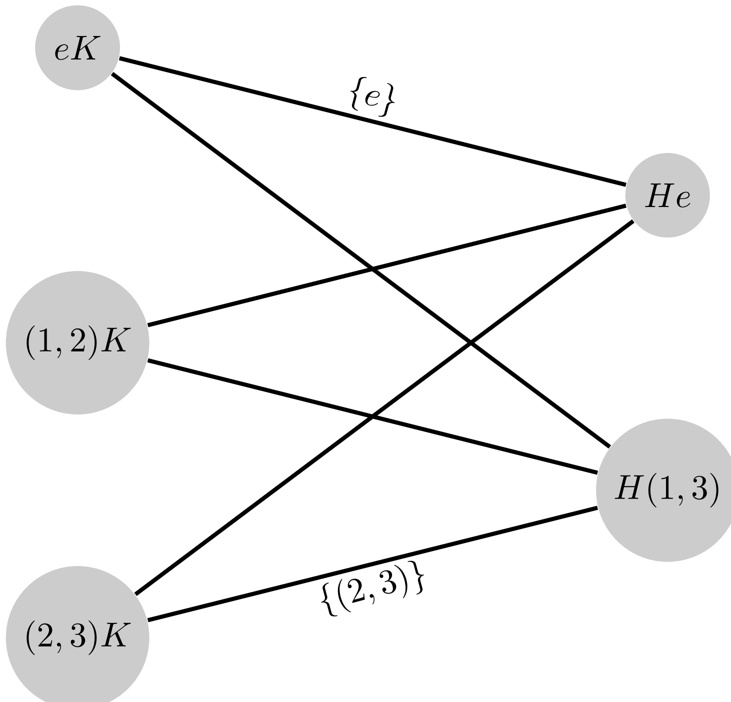


Figure 20: The coset intersection graph $\Gamma^G_{H,K}$.

Here we can choose the edges $\{e\}$ and $\{(2,3)\}$ to obtain a right-transversal $T_1 = \{e, (2,3)\}$ of $H$, and $T_1$ can be extended to a left-transversal of $K$ by adding any element from $(1,2)K$. For example, choosing the element $(1,2)$, we see that the set $T_2 = T_1 \cup \{(1,2)\} = \{e, (1,2), (2,3)\}$ is such a right-transversal.

By doing this, the authors have proven the existence of a left-right transversal of any finite subgroup $H$ of $G$ without using Hall's Marriage Theorem.

However, there are in fact stronger versions of Theorem 4.8 and Corollary 4.9. Looking at Example 4.5, we can see that the results from both of them seem to be true even though $\mathbb{Z}$ is infinite. This is in fact true, the authors of [4] proved that the statements hold in a more general case: The subgroups $H$ and $K$ do not need to be finite, they just need to have finite index.

**Theorem 4.11.** *[4, Theorem 7] Let $H$ and $K$ be subgroups of $G$ with finite index. Then $\Gamma^G_{H,K}$ is a union of finite, disjoint, complete bipartite graphs $K_{s_i,t_i}$ such that $s_i/t_i = |G : H|/|G : K|$.*

**Corollary 4.12.** *[4, Corollary 8] Let $H, K$ be subgroups of a group $G$ where $|G : H| = m$ and $|G : K| = n$, with $m \geq n$. Then there exists a left transversal $T \subseteq G$ for $H$ that can be extended to a right transversal for $K$. If $H = K$, then $T$ is a left-right transversal of $H$.*

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