LU-TP 22-37
June 2022

Quantum Gauge Fields in Cosmology: Cosmic Inflation and Gravitational Waves

Emma Hermansson

Department of Astronomy and Theoretical Physics, Lund University

Bachelor thesis supervised by Roman Pasechnik

Abstract

In this thesis, the consequences of introducing Yang-Mills fields into cosmology are investigated. This is done by the effective action approach, in which the coupling constant is allowed to depend on the quantum fields. The equation of state and scale factor are then found for the flat universe filled with either chromoelectric or chromomagnetic fields. From this, we draw conclusions about the history and fate of the universe. Then, the effect of such cosmological Yang-Mills fields on the amplitude and amplification of primordial gravitational waves are discussed.

Populärvetenskaplig beskrivning

Universum tros ha en diameter på kring 93 miljarder ljusår. Det är därför svårt att tänka sig att hela universum en gång i tiden fick plats i en enda punkt för 13,8 miljarder år sedan. Sen kom den stora smällen! Det som följde var troligen en kort period av väldigt snabb expansion. På samma sätt som ett jetplan som accelererar förbi ljudvallen resulterar i en ljudbang, borde den snabba expansionen av universum ha lämnat krusningar i rumtiden i form av tidiga gravitationsvågor. Genom att observera dessa vågor hade vi fått stöd för våra teorier om vad som faktiskt hände tiden efter den stora smällen.

Idag kan det vara svårt att tänka sig att universum expanderar alls. Om man dock observerar ljuset från stjärnor omkring oss upptäcker man att de verkar vara rödare än de borde vara, på samma sätt som när ljudet från en ambulans får lägre frekvens när den åker ifrån en. På så sätt vet vi att universum expanderar. Med bara observationen av stjärnors skifte mot det röda kan det vara lätt att tro att jorden är i mitten av universum, och att alla himlakropparna rör sig ifrån jorden. Det som egentligen händer är att alla avstånd blir större, så som punkter på en ballong rör sig ifrån varandra när man blåser upp den.

Det återstår en mängd frågor om universum. Genom att formulera modeller om universums framtid och historia kan vi börja leta efter experimentella bevis som stödjer våra teorier. Detta leder till en bättre förståelse av universum, och förhoppningsvis kommer vi kunna svara på de uråldriga frågorna om universums början och slut.

Contents

1 Introduction

The aim of cosmology is to explain the structure and dynamics of the universe. This is done by formulating models that are consistent with observations. There have been many attempts to reconcile particle physics with modern cosmology. Particularly in models of the early universe, the interplay between the two fields of research becomes relevant, since at those early days the energy density was very high and so particle interactions dominated the cosmological evolution. As a solution to the flatness problem, that is the unexplained fact that the measured curvature of the universe is very small, a period of exponential expansion shortly after the Big Bang was introduced [1]. In many formulations of the inflation model, this expansion is driven by a scalar field, the inflaton field. Recently however, vector fields have become an attractive candidate for driving inflation, see for instance [2, 3, 4]. Vector fields have a natural place in for instance particle physics and fluid dynamics, and by finding a model for inflation with quantum Yang-Mills (YM) fields we take a step towards unifying quantum field theory with cosmology. As a result of quantum YM fields, we get the spatially homogeneous vacuum state, i.e. the lowest energy state with no real particles. The quantum fluctuations around this state will prove to have important consequences on cosmology.

There are many ways to derive information about the dynamics of the early universe, such as cosmic microwave background and large scale structure formation. With the first detection of gravitational waves (GWs) by LIGO in 2015 [5], however, we have entered a new era of multi-messenger astronomy, in which we can rely on multiple sources of information from the same events. GWs are ripples in the fabric of space-time. They occur as a result of catastrophic events in astrophysics, such as the merger of massive objects. The inflation period should have left an imprint in space-time in the form of stochastic primordial gravitational waves [6]. These waves are still propagating today as background radiation. As such, the primordial GWs provide a window into the early days of the universe. Detecting these waves is therefore essential to answer some fundamental questions about the origin of the universe, and would serve as a test of our models of the evolution of the early universe. The primordial GW background signal should be very weak, and so detection will be difficult. It is shown in [7], however, that interactions between primordial black holes should be within the range of detection of the proposed detector LISA [8]. As we will see shortly, there are conditions for the amplification of primordial GWs that should in principle facilitate their detection. Primordial GWs are the next big target for experimental cosmology, and there are major efforts going into the development of the next generation of detectors.

This thesis aims to explore some of the basic cosmological implications of the Yang-Mills field theory. In section 2, this is done by first defining the action, energy-momentum tensor and equation of state for classical field theory. The results are then applied to quantum field theory by introducing the renormalisation group (RG) running of the coupling constant. This is then generalised to the expanding universe with the use of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. In section 3, we look at two simple cosmological models based on quantum YM field theory. We find the equation of state and derive the scale factor for both models. The consequences of the YM fields on the time evolution of the universe are discussed. Lastly, we explore the amplitude and amplification of primordial GWs. Final remarks and conclusions are given in section 4.

2 Theory background

The Yang-Mills field theory (YMFT) is the very foundation of the Standard Model (SM) of particle physics, based upon a non-Abelian symmetry group, such as SU(N). A YMFT is a locally gauge invariant theory where the gauge symmetry generators do not commute. In other words, the Lagrangian is invariant under local gauge transformations, that is transformations which depend on the space-time point [9]. According to Noether's theorem, any continuous symmetry in a system described by an invariant Lagrangian leads to a conserved current [10]. One such symmetry is translations w.r.t. space-time, where the corresponding Noether's current is the energy-momentum tensor, $T^{\mu\nu}$. The energy-momentum tensor is essential for models of cosmology, as it describes the energy density and pressure of a system of fields, and is by definition a conserved quantity, i.e. $\partial_\mu T^{\mu\nu} = 0$. The effective (or quantum) YMFT builds upon the results of the corresponding classical YMFT, and so in the following sections, we will define the Lagrangian, equations of motion, and energy-momentum tensor in the case of classical YMFT, followed by the formulation of the effective YMFT. In what follows, we will use natural units, with $c = \hbar = 1$.

2.1 Classical Yang-Mills field theory

We start by defining the classical YM Lagrangian for the SU(N) symmetry group, where N is the number of colours. In the simplest case of two colours, $N = 2$, and we have for pure gauge fields without sources [10]

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_{YM}\varepsilon^{abc}A^b_\mu A^c_\nu, \tag{2.1}$$

where $A^a_\mu(x)$ is the gauge field in the adjoint representation, with $a, b, c = 1, ..., N^2 - 1$, g_{YM} is the coupling strength, and ε^{abc} is the Levi-Civita symbol. This Lagrangian has well-known applications in the theory of weak interactions in the SM framework. Compared to Maxwell electrodynamics, there is an extra term in the field strength tensor, which gives rise to the gauge field self-interactions [10].

The equations of motion of a field, known as the Euler-Lagrange (EL) equations, follow from the variational principle, which states that the variation of the action S w.r.t. all the fields in the system is required to be vanishing [10], i.e.

$$\delta S = \delta \int dx^4 \mathcal{L} = 0. \tag{2.2}$$

The EL equations describe the space-time dynamics of the fields. For the SU(2) Lagrangian above, the EL equations become

$$\partial_\mu F^{\mu\nu}_a + g_{YM}\epsilon^{abc}A^b_\nu F^{\mu\nu}_c = 0. \tag{2.3}$$

Further, the energy-momentum tensor describes the energy density and pressure of the system. For a system described by the Lagrangian density $\mathcal{L}$, the classical energy-momentum tensor is defined as [11]

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\rho)}\partial^\nu A^a_\rho - \eta^{\mu\nu}\mathcal{L}, \tag{2.4}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. This can be generalised to curved space-time with the FLRW metric $g^{\mu\nu}$ through $\eta^{\mu\nu} \to g^{\mu\nu}$. For an isotropic and homogeneous universe, $T^{00} = \rho$, and $T_{ij} = \frac{1}{3}\delta_{ij}p$, where ρ is the energy density, p is the effective pressure of the universe. For an ideal fluid p and ρ are functions only of time and not of space. The equation of state describes the relationship between the pressure and the energy density and is commonly parameterised as [6]

$$p = \omega \rho, \tag{2.5}$$

where ω is the equation of state parameter. If the energy of the universe is due to non-relativistic matter (dust), $\omega = 0$, and the pressure disappears. For relativistic matter, such as radiation, the equation of state parameter is $\omega = \frac{1}{3}$. This equation of state parameter is relevant for the time shortly after the Big Bang, when the universe was hot and expansion was dominated by ultra-relativistic plasma. Finally, there is the vacuum equation of state, where $\omega = -1$, giving $p = -\rho$ [12]. This is close to the case of the universe today. The vacuum energy density is also known as the cosmological constant [12]. The cosmological constant is much smaller than it should be if one considers contributions of strong, electroweak, and gravitational interactions [6]. This is known as the cosmological constant problem.

2.2 Effective Yang-Mills field theory

In effective YM field theory we take into account the quantum fluctuations of the fields, such as the spontaneous creation and annihilation of virtual particle-antiparticle pairs. This is a consequence of Heisenberg's uncertainty principle. The quantum fluctuations are accounted for by allowing the gauge coupling constant to depend on the quantum fields [10]. For the coming calculations, the fields are rescaled as $A^\mu_a \rightarrow \mathcal{A}^\mu_a = g_{YM} A^\mu_a$. We now let the gauge coupling g_{YM} depend on the quantum fields, such that $g_{YM} \rightarrow \overline{g} = g(\mathcal{J})$, where $\mathcal{J} = -\overline{g}^2 F^{\mu\nu}_a F^a_{\mu\nu} = -\mathcal{F}^{\mu\nu}_a \mathcal{F}^a_{\mu\nu}$ is the renormalisation group parameter. With the rescaling and the effective coupling, the effective Lagrangian of the pure YM theory (without fermions) becomes

$$\mathcal{L}_{eff} = \frac{\mathcal{J}}{4\overline{g}^2}, \tag{2.6}$$

where $\overline{g}$ satisfies the renormalisation group equation [13]

$$2\mathcal{J}\frac{\mathrm{d}\overline{g}^2}{\mathrm{d}\mathcal{J}} = \overline{g}^2 \beta(\overline{g}^2). \tag{2.7}$$

Here, β is the beta function that can be found as a perturbation theory series in the quantum field theory framework. Using the exact beta function takes into account all quantum fluctuations. As is shown in [13], the SU(2) gauge theory beta function corresponding to the lowest order fluctuation, the one-loop beta function, encompasses most of the behaviour of the exact beta function found in the framework of functional RG [14] and so this approximation will be adopted here. In the one-loop approximation, the beta function is [13]

$$\beta_1(\bar{g}^2) = -2b\bar{g}^2, \tag{2.8}$$

with $b = \dfrac{11N}{96\pi^2}$ for the SU(N) case. Solving for the inverse gauge coupling $\dfrac{1}{\bar{g}^2(\mathcal{J})}$ gives then

$$\frac{1}{\bar{g}^2} = \frac{g^2(\mu^4)}{bg^2(\mu^4)\ln\frac{|\mathcal{J}|}{\mu^4} + 1}, \tag{2.9}$$

where μ is the RG evolution scale, and so the effective Lagrangian in the one-loop approximation is

$$\mathcal{L}_{eff} = \frac{b}{4}\mathcal{J}\ln\frac{|\mathcal{J}|}{\Lambda^4}, \quad \Lambda^4 = \mu^4\exp\left[-\frac{1}{bg^2(\mu^4)}\right], \tag{2.10}$$

with Λ being the energy scale of the theory. With the effective YM Lagrangian given above, we can calculate the effective energy-momentum tensor, and hence find the energy density ρ and the pressure p as functionals of the solutions of the effective EL equations of motions of the quantum gauge field A_μ. The effective energy-momentum tensor is found as [15, 16]

$$T_{\mu\nu} = \left(F_{\mu\lambda}F^{\nu\lambda} - \eta_{\mu\nu}\frac{1}{4}F_{\lambda\rho}F^{\lambda\rho}\right)\frac{\partial\mathcal{L}}{\partial\mathcal{J}} - \eta_{\mu\nu}\left(\mathcal{L} - \mathcal{J}\frac{\partial\mathcal{L}}{\partial\mathcal{J}}\right). \tag{2.11}$$

With this, the equation of state (2.5) will depend on the quantum fields. The consequences of this on cosmology will be explored in section 3.

2.3 Friedmann universe and FLRW metric

The equations of sections 2.1 and 2.2 are given in flat Minkowski space, with metric $\eta^{\mu\nu}$. In order to generalise to the expanding universe, we use the FLRW metric, which is time dependent and takes into account the dynamics of space-time. The cosmological principle states that on a large enough scale, the universe is homogeneous and isotropic, meaning that the universe has the same properties at each point, and is uniform in all directions

[17]. Based on these assumptions, the Minkowski metric $\eta_{\mu\nu}$ is replaced with the FLRW metric $g_{\mu\nu}$. The line element of space-time is now given by [6]

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right), \tag{2.12}$$

where $k = 1$, $k = 0$, and $k = -1$ describe the spherical, flat, and hyperbolic universes respectively [6]. Here, the scale factor $a(t)$ is introduced, describing the distances between two points at physical time t compared to some earlier time t_0 [6]. The rate of change of the scale factor is a description of the rate of expansion of the universe. In conformal time η, $dt = a(\eta)d\eta$, and with $k = 0$, the FLRW metric takes the form [6]

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}. \tag{2.13}$$

The use of the FLRW metric leads to a rescaling of $\mathcal{J}$ through $\mathcal{J} \to \frac{\mathcal{J}}{\sqrt{-g}}$, where g is the determinant of $g_{\mu\nu}$. The equations of motion in curved space-time are then modified as

$$\frac{1}{\sqrt{-g}}\partial_\mu \frac{\sqrt{-g}}{g^2(\mathcal{J})}\left(1-\frac{\beta}{2}\right)\underset{a}{\mathcal{F}}^{\mu\nu} - \frac{1}{g^2(\mathcal{J})}\left(1-\frac{\beta}{2}\right)\epsilon^{abc}\mathcal{A}^b_\mu \underset{c}{\mathcal{F}}^{\mu\nu} = 0. \tag{2.14}$$

The relation between energy distribution in the universe and the space-time curvature is described by the Einstein field equations [17]

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \varkappa T_{\mu\nu}, \tag{2.15}$$

where $\varkappa = 8\pi G$, with G being the gravitational constant, and $G_{\mu\nu}$ is the Einstein tensor, with non-zero components $G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$ and $G_{ij} = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}$ [6]. The cosmological constant Λ represents the vacuum energy density and is absorbed into the energy-momentum tensor in the following equations, and so the 00 and ij components of the Einstein field equations are [6]

$$H^2 + \frac{k}{a^2} = \frac{\varkappa}{3}\rho, \tag{2.16}$$

$$2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} = -\varkappa p. \tag{2.17}$$

These are the first and second Friedmann equations, describing the evolution of the universe, where $H(t) = \frac{\dot{a}(t)}{a(t)}$ is the Hubble parameter. Combining the two Friedmann equations gives

$$\frac{\ddot{a}}{a} = -\frac{\varkappa}{6}(\rho + 3p), \tag{2.18}$$

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.19}$$

Equation (2.18) determines whether the expansion of the universe accelerates or decelerates, and equation (2.19) is the continuity equation, ensuring energy conservation in the comoving volume of the universe.

2.4 Generation of gravitational waves

Gravitational waves (GWs) are propagating tensor perturbations of the metric $g_{\mu\nu}$, such that the metric becomes $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ [18]. The perturbation has amplitude $h(\eta)$, as a function of conformal time. In the inflationary model of the universe, a brief period of exponential expansion is introduced. This solves the flatness problem, which states that the current universe is too flat to have expanded since the Big Bang in the same rate as now. After the inflationary period, the expansion rate has reduced. The period of inflation should however have left an imprint in the fabric of the universe in the form of primordial GW [13]. These waves could also originate from other catastrophic events, such as the merger of primordial black holes. The primordial GWs could therefore serve as a probe into the early days of the universe. GWs interact weakly with matter, and so are in general very difficult to detect. However, this also means that the GWs produced in the inflationary period should be largely preserved today. In a Friedmann universe, the ratio of the initial amplitude A at initial conformal time η_0 and final amplitude B of the primordial GWs, as derived by Grishchuk [19], is

$$\left|\frac{B}{A}\right|^2 = 1 + \frac{1}{2}\frac{\alpha^2}{(n\eta_0)^2}, \quad \alpha = \frac{1-3\omega}{1+3\omega}, \tag{2.20}$$

where n is the wave number. The ratio differs from unity by an additional term, which depends on the equation of state parameter ω. We call the additional term the coefficient

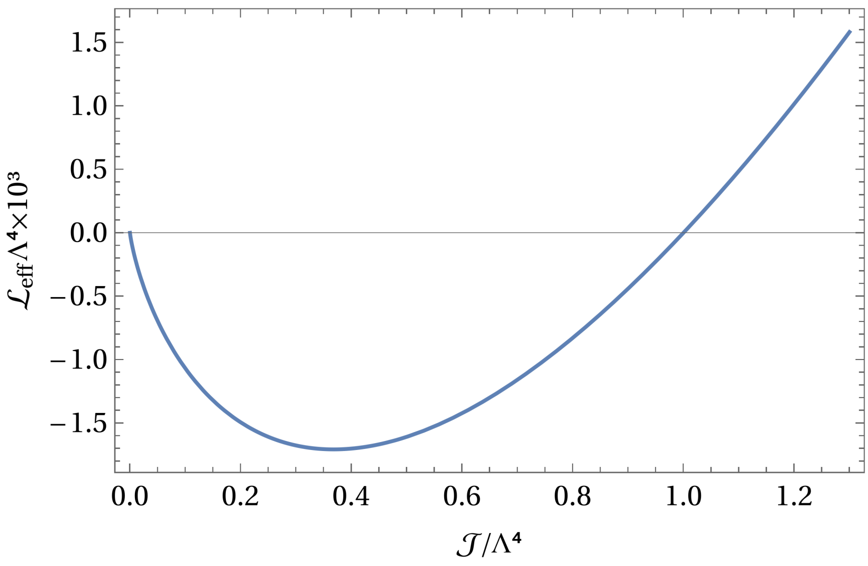

Figure 1: The behaviour of the effective Lagrangian, with the minimum at the vacuum expectation value of $\mathcal{J}$.

of amplification K, and it is defined as [19]

$$K = \frac{1}{2}\frac{\alpha^2}{(n\eta_0)^2}, \tag{2.21}$$

and so waves produced at the period of inflation, when η_0 would be small, are amplified. In the radiation dominated universe, where $\omega = \frac{1}{3}$, $\alpha = 0$, there is no amplification. The amplitude of the GW is then amplified compared to the initial amplitude as we move away from the radiation universe. Such a GW background is very weak, but should be present at almost all frequencies [17].

3 Cosmological implications

The behaviour of the effective Lagrangian (2.10) is shown in figure 1, as a function of $\mathcal{J}$. The effective Lagrangian has a minimum at the vacuum expectation value $\mathcal{J}_{vac}$ outside the classical minimum $\mathcal{J} = 0$ as a result of renormalisation and hence the vacuum polarisation. With the substitutions $\mathcal{J} \to -\frac{1}{4}\mathcal{J}$ and $\mu^4 \to \mathrm{e}\mu^4$, the effective Lagrangian in [2] is recovered, which will prove to simplify expressions further on. The Lagrangian becomes

$$\mathcal{L}_{eff} = -\frac{1}{2g^2(\mu^4)}\mathcal{J} - b\mathcal{J}\left[\ln\frac{|\mathcal{J}|}{\mu^4} - 1\right], \tag{3.1}$$

With equation 2.11 and the effective Lagrangian, the effective energy-momentum tensor becomes

$$T_{\mu\nu} = \left(F_{\mu\lambda}F^{\nu\lambda} - \eta_{\mu\nu}\frac{1}{4}F_{\lambda\rho}F^{\lambda\rho}\right)\frac{b}{4}(\ln\frac{|\mathcal{J}|}{\lambda^4} + 1) + \eta_{\mu\nu}\frac{b}{4}\mathcal{J}. \tag{3.2}$$

This expression will be used in section 3.1 to find the pressure, energy density and equation of state of the universe.

3.1 Equation of state

Assuming the field is homogeneous, we have for the electric and magnetic field E and B respectively, $E_x = E_y = E_z = E$ and $B_x = B_y = B_z = B$, where E_x is the x-component of the electric field and so on, and so $\mathcal{J} = \frac{B^2 - E^2}{2}$ [16]. To simplify further calculations, we let the gauge field condensate, i.e. let the corresponding gauge bosons form a Bose-Einstein condensate. We first study the chromomagnetic (CM) condensate, following the procedure in [2]. This is done by requiring the electric field to vanish, so that $\mathcal{J}$ only depends on the magnetic field, i.e. $\mathcal{J} = \frac{B^2}{2}$, and so $\delta_{ij}F^{i\lambda}F_{j\lambda} = -2B^2$, and $F^{0\lambda}F_{0\lambda} = 0$. Using equation (3.2) to find the effective energy-momentum tensor in the CM condensate, we have with the effective Lagrangian (3.1) the following expressions for the energy density and pressure

$$\rho_{CM} = T_{00}^{CM} = \frac{1}{2g^2(\mu^4)}\mathcal{J} + b\mathcal{J}\left(\ln\frac{\mathcal{J}}{\mu^4} - 1\right), \tag{3.3}$$

$$p_{CM} = \frac{1}{3}\delta_{ij}T_{ij}^{CM} = \frac{1}{6g^2(\mu^4)}\mathcal{J} + \frac{b}{3}\mathcal{J}\left(\ln\frac{\mathcal{J}}{\mu^4} + 3\right). \tag{3.4}$$

Absorbing the first term in each expression respectively into the scale $\mu^4 \to \Lambda^4 = \mu^4 \exp\left[-\frac{1}{2bg^2(\mu^4)}\right]$ gives

$$\rho_{CM} = b\mathcal{J}\left(\ln\frac{\mathcal{J}}{\Lambda^4} - 1\right), \tag{3.5}$$

$$p_{CM} = \frac{b}{3}\mathcal{J}\left(\ln\frac{\mathcal{J}}{\Lambda^4} + 3\right). \tag{3.6}$$

This leads to the equation of state parameter of the form

$$\omega_{CM} = \frac{p}{\rho} = \frac{1}{3}\left(\frac{\ln\frac{\mathcal{J}}{\Lambda^4}+3}{\ln\frac{\mathcal{J}}{\Lambda^4}-1}\right). \tag{3.7}$$

The equation of state parameter for the CM condensate is singular at $\ln\dfrac{\mathcal{J}}{\Lambda^4} = 1$, so its behaviour is shown before and after the singularity in figure 2.

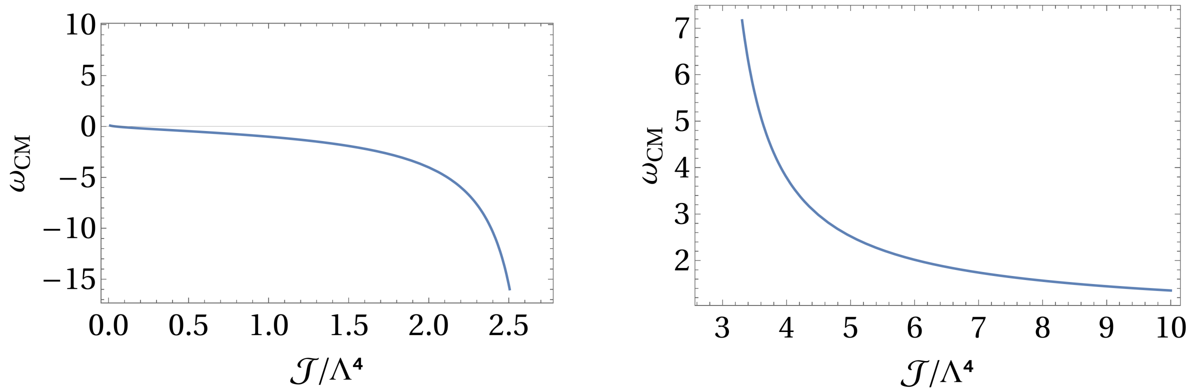

Figure 2: The equation of state parameter for the CM condensate.

Next, the equation of state in the chromoelectric (CE) condensate is explored. In the CE condensate, $\mathcal{J} = -\dfrac{E^2}{2}$, $F^{0\lambda}F_{0\lambda} = -2E^2$, and $\delta_{ij}F^{i\lambda}F_{j\lambda} = 0$. Once again, we find the energy-momentum tensor in the CE condensate. Following the procedure in [14], this gives the corresponding energy density and pressure of the CE condensate

$$\rho_{CE} = T_{00}^{CE} = \frac{b}{4}\mathcal{J}\left(\ln\frac{\mathcal{J}}{\Lambda^4}+2\right), \tag{3.8}$$

$$p_{CE} = \frac{1}{3}\delta_{ij}T_{ij}^{CM} = \frac{b}{12}\mathcal{J}\left(\ln\frac{\mathcal{J}}{\Lambda^4}-2\right). \tag{3.9}$$

The ω parameter in the CE condensate is then

$$\omega_{CE} = \frac{1}{3}\left(\frac{\ln\frac{\mathcal{J}}{\Lambda^4}-2}{\ln\frac{\mathcal{J}}{\Lambda^4}+2}\right). \tag{3.10}$$

The equation of state parameter that follow from the two condensates respectively both carry a logarithmic dependence on the fields. For the CM condensate, it has positive and

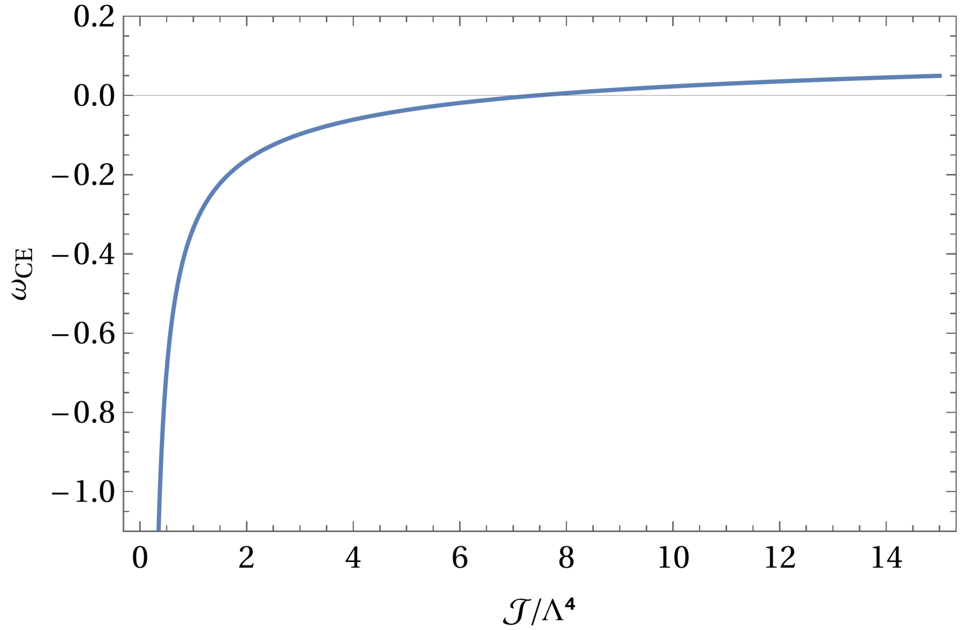

Figure 3: The equation of state parameter for the CE condensate.

negative branches, with a singularity at which its behaviour changes. The ω parameter for the CE condensate is monotonous. It is defined in the range $\mathcal{J} \in (\mathrm{e}^{-2}\Lambda^4, \infty)$, and does not exhibit singular behaviour in the whole range. ω_{CE} is shown in figure 3.

3.2 Scale factor for the flat universe

3.2.1 Scale factor for the chromomagnetic condensate

The universe as we observe it is to a good approximation very flat [6], and so the following calculations will be performed in the flat universe, with $k = 0$ in the Friedmann equations. In order to find the dimensionless scale factor $\tilde{a}(t)$, defined below, the continuity equation (2.19) is used. Following the procedure in [2], the continuity equation for the CM condensate, using the expression for the pressure (3.6), as well as the time derivative of the energy density (3.5), becomes

$$\dot{\mathcal{J}} + 4\mathcal{J}H = 0, \tag{3.11}$$

where $\dot{\mathcal{J}}$ refers to the time derivative of $\mathcal{J}$. This can be solved as $\mathcal{J}a^4 = \Lambda^4 a_0^4$, where $\Lambda^4 a_0^4$ is constant, and a_0 is the initial value of the scale factor. $\mathcal{J}$ then depends on the scale factor as

$$\mathcal{J} = \frac{\Lambda^4 a_0^4}{a^4}, \tag{3.12}$$

where Λ for the CM condensate is defined above. Using this expression for $\mathcal{J}$ in equation (3.6) and (3.5) gives the energy density and pressure in terms of the scale factor

$$\rho = \frac{b a_0^4}{a^4}\left(\ln\frac{a_0^4}{a^4} - 1\right), \quad p = \frac{b a_0^4}{3a^4}\left(\ln\frac{a_0^4}{a^4} + 1\right). \tag{3.13}$$

We now introduce the dimensionless scale factor $\tilde{a}(\tau) = \dfrac{a(\tau)}{a_0}$, as a function of dimensionless time $L\tau = ct$, where $\dfrac{1}{L^2} = \dfrac{1}{3} b\varkappa\Lambda^4$, such that $\dfrac{da}{cdt} = \dfrac{d\tilde{a}}{Ld\tau}$. The first Friedmann equation (2.16) then becomes

$$\frac{d\tilde{a}}{d\tau} = \pm\sqrt{\frac{1}{\tilde{a}^2}\left(\ln\frac{1}{\tilde{a}^4} - 1\right)}. \tag{3.14}$$

In order to solve for the scale factor, we look at the expression under the square root, and define the potential for the CM condensate

$$U_{CM}(\tilde{a}) = \left(\frac{d\tilde{a}}{d\tau}\right)^2 = \frac{1}{\tilde{a}^2}\left(\ln\frac{1}{\tilde{a}^4} - 1\right). \tag{3.15}$$

The potential is shown in figure 4. It turns to zero at one point, when $\tilde{a} = \tilde{\mu}_{CM} = \mathrm{e}^{-1/4}$, where e is the base of the natural logarithm. The scale factor should be real and non-negative, and so we restrict our discussion to values of the scale factor to $\tilde{a}(\tau) \in (0, \tilde{\mu}_{CM}]$. As in [2], we use the substitution $\tilde{a}^4 = \tilde{\mu}_{CM}^4 \mathrm{e}^{-g^2}$, and the expression becomes

$$\int_{-\infty}^{g} dg' \mathrm{e}^{-g'^2/2} = \frac{2\tau}{\tilde{\mu}_{CM}^2}, \tag{3.16}$$

where $\tau \in [0, \sqrt{\frac{\pi}{2\mathrm{e}}}]$. This can be solved using the inverse error function $\mathrm{erf}^{-1}(x)$. Solving for g and converting back to $\tilde{a}$ gives the final expression

$$\tilde{a}(\tau) = \exp\left[-\frac{1}{4} - \frac{1}{2}(\mathrm{erf}^{-1})^2\left(\sqrt{\frac{\pi}{8\mathrm{e}}}\tau - 1\right)\right]. \tag{3.17}$$

The energy density of the universe is related to the scale factor above through equation (3.5). The expression for the scale factor (3.17) is then used to find the time evolution of

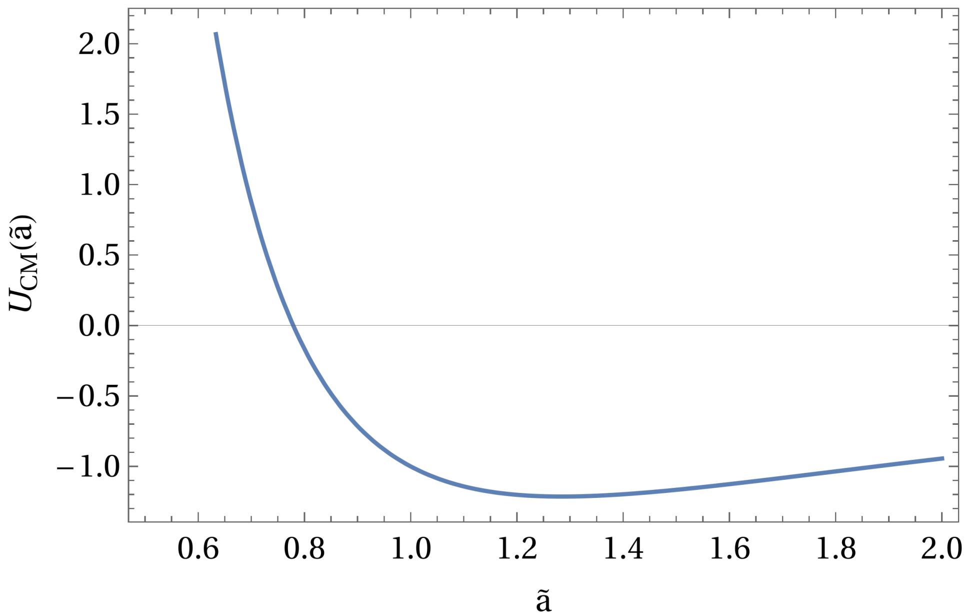

Figure 4: The potential for the CM condensate.

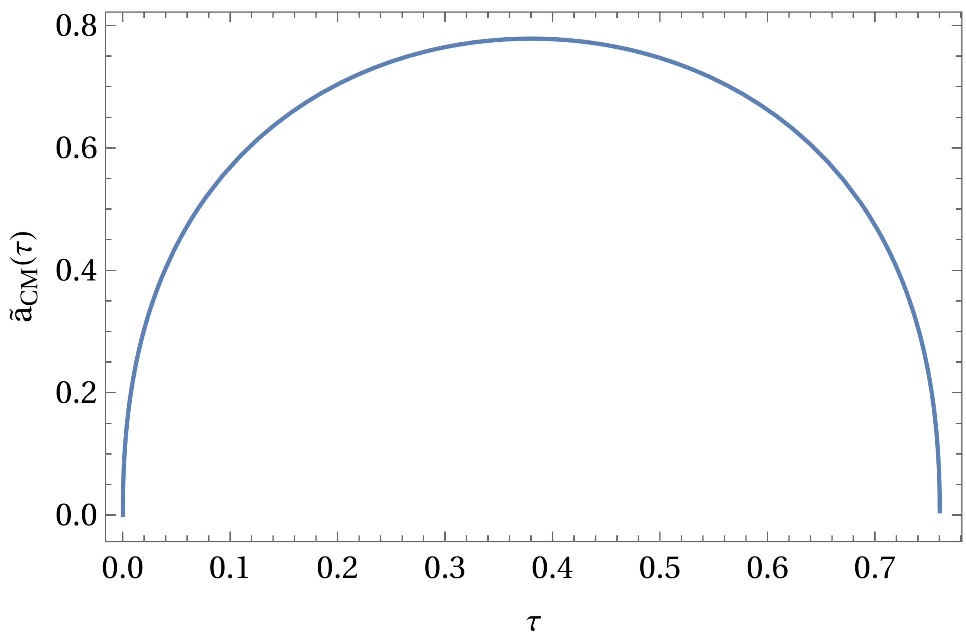

Figure 5: The scale factor of the flat universe filled with the CM condensate.

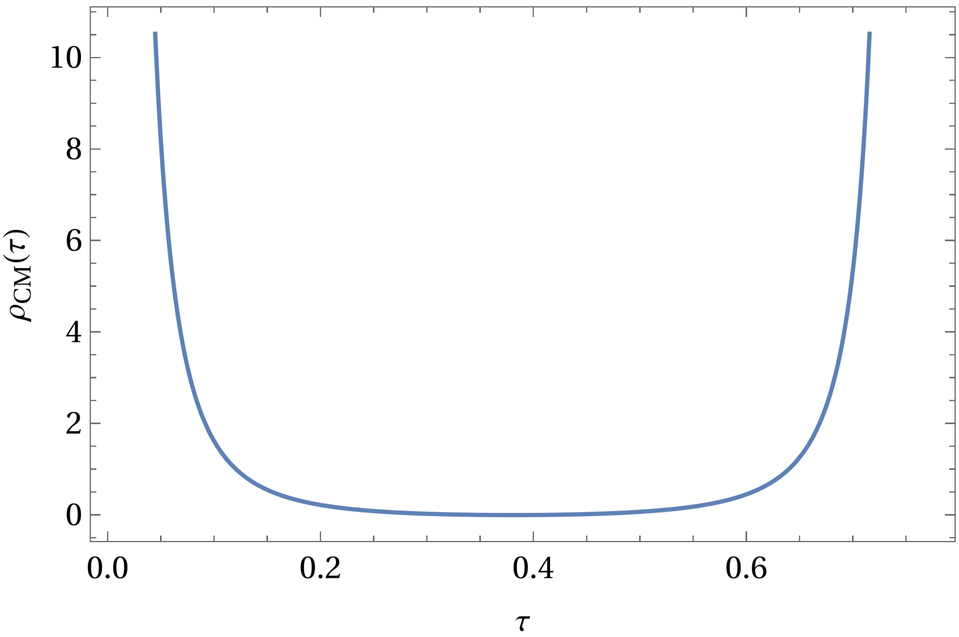

Figure 6: The qualitative behaviour of the energy density of the flat universe filled with the CE condensate, with fixed Λ.

the energy density in the universe. The energy density for the CM condensate is then

$$\rho_{CM} = (\mathrm{erf}^{-1})^2 \left(\sqrt{\frac{\pi}{8\mathrm{e}}}\tau - 1\right) \exp\left[2(\mathrm{erf}^{-1})^2 \left(\sqrt{\frac{\pi}{8\mathrm{e}}}\tau - 1\right) + 1\right] b\Lambda^4. \tag{3.18}$$

The scale factor here is periodic, with period $\tau = \sqrt{\frac{\pi}{2\mathrm{e}}}$ [2]. The energy density has the same period. This means that in this model, the universe expands until the scale factor reaches its maximum value $\tilde{a}_{max} = \tilde{\mu}_{CM}$, at half the period $\tau = \sqrt{\frac{\pi}{8\mathrm{e}}}$. The universe then starts to contract, since the second time derivative of the scale factor is less than 0 when the energy density is 0, as is predicted by equation (2.18). This means that the expansion velocity of the expanding universe is less than the escape velocity. With some appropriate mechanism, the processes could then restart, and the universe would oscillate between the minimum and maximum value of the scale factor indefinitely [20]. Converting to physical time, with $ct = L\tau$ and $L = 1.25 \cdot 10^{25} \left(\frac{\mathrm{eV}}{\Lambda}\right)^2$ cm, where Λ is of the order of a few eV [2], the half period of the scale factor is of order $t \sim 10^6$ years, so the universe has time to reach a significant size before recollapsing.

The universe modelled here is cyclic, and is a version of the Big Bounce. The Big Bounce is a model of the universe in which the Big Bang follows the collapse of the previous universe. The universe collapses until the energy density is so large that the expansion is restarted. This is the bounce point [21]. In some cyclic models of the universe, where $\tilde{a}_{max}$ increases with each cycle, for sufficiently old universes, there could be solutions to the flatness problem even without inflation, see for instance [22]. In the model here, however, $\tilde{a}_{max}$ is constant over the cycles, and there is no apparent solution to the flatness problem.

3.2.2 Scale factor for the chromoelectric condensate

Next, we study the implications of the CE condensate on the evolution of the universe. The same procedure as above is repeated for the CE condensate. By limiting the values that $\mathcal{J}$ can take to a region close to Λ^4, the continuity equation (2.19) simplifies to

$$\dot{\mathcal{J}} + \frac{4}{3} H \mathcal{J} = 0, \tag{3.19}$$

and so the $\mathcal{J}$ can be expressed in terms of the scale factor

$$\mathcal{J} = \frac{\Lambda^4 a_0^{4/3}}{a^{4/3}}, \tag{3.20}$$

where $\Lambda^4 a_0^{4/3}$ is constant, and Λ for the CE condensate is defined above. With equation (3.8), the energy density in terms of the dimensionless scale factor reads then

$$\rho_{CE} = \frac{b}{4} \tilde{a}^{2/3} \left(\ln \frac{1}{\tilde{a}^{2/3}} + 2 \right) \Lambda^4. \tag{3.21}$$

The time derivative of the scale factor (3.14) becomes

$$\frac{d\tilde{a}(\tau)}{d\tau} = \pm \sqrt{\tilde{a}^{2/3} \left(\ln \frac{1}{\tilde{a}^{4/3}} + 2 \right)}. \tag{3.22}$$

As with the CM condensate, we now define the potential for the CE condensate

$$U_{CE}(\tau) = \tilde{a}^{2/3} \left(\ln \frac{1}{\tilde{a}^{4/3}} + 2 \right), \tag{3.23}$$

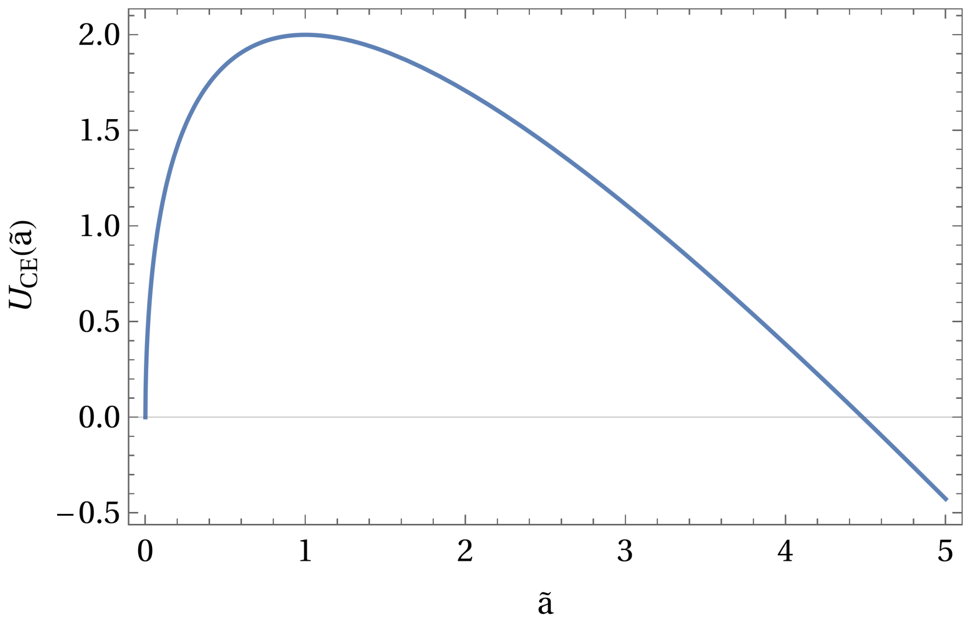

Figure 7: The potential of the CE condensate with in the flat universe. The potential is 0 at $\tilde{a} = e^{3/2}$.

where $U_{CE} = 0$, at $\tilde{a} = \mu = e^{3/2}$. The potential for the CE solution is shown in figure 7. For real and physical values of $\tilde{a}(\tau)$, this discussion is limited to values such that $\ln \dfrac{1}{\tilde{a}^{4/3}} \geq -2$, and so the range for the scale factor is $\tilde{a} \in (0, \mathrm{e}^{3/2}]$, such that the potential is non-negative. Using the expression for the potential, the scale factor is found numerically using the differential equation solver provided by Wolfram Mathematica. The time dependence of the scale factor is shown in figure 8.

In the flat universe filled with the CE condensate, the scale factor has close-to-linear behaviour throughout its range defined above, meaning that the rate of expansion of this universe is close to a constant. Assuming then that such a close-to-linear behaviour carries on indefinitely, this means that there is unbounded expansion. As opposed to the CM condensate universe, this universe never collapses. With the scale factor, the energy density is found numerically and is shown in figure 9. Considering the behaviour of the scale factor and the energy density, the latter drops off quickly as the scale factor increases. It seems that the expansion slows down slightly as the energy density reaches 0, but the energy density is not enough to completely halt the expansion.

In the $k = 0$ case, we were not able to find a period of exponential growth. This means that in these models, there is no inflation, and so we cannot solve the flatness problem

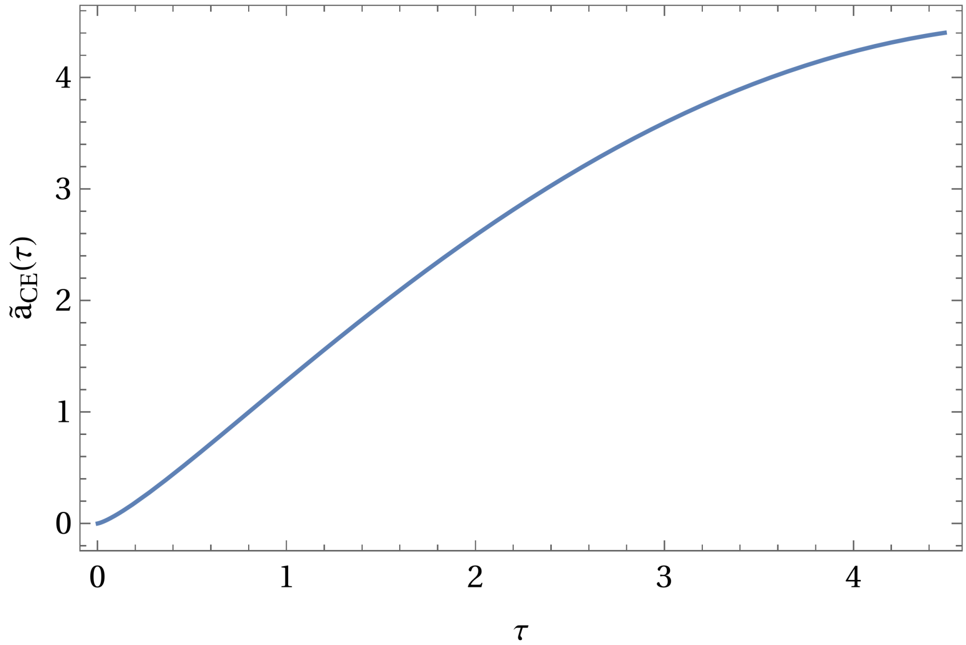

Figure 8: The time evolution of the scale factor in the flat universe filled with the CE condensate.

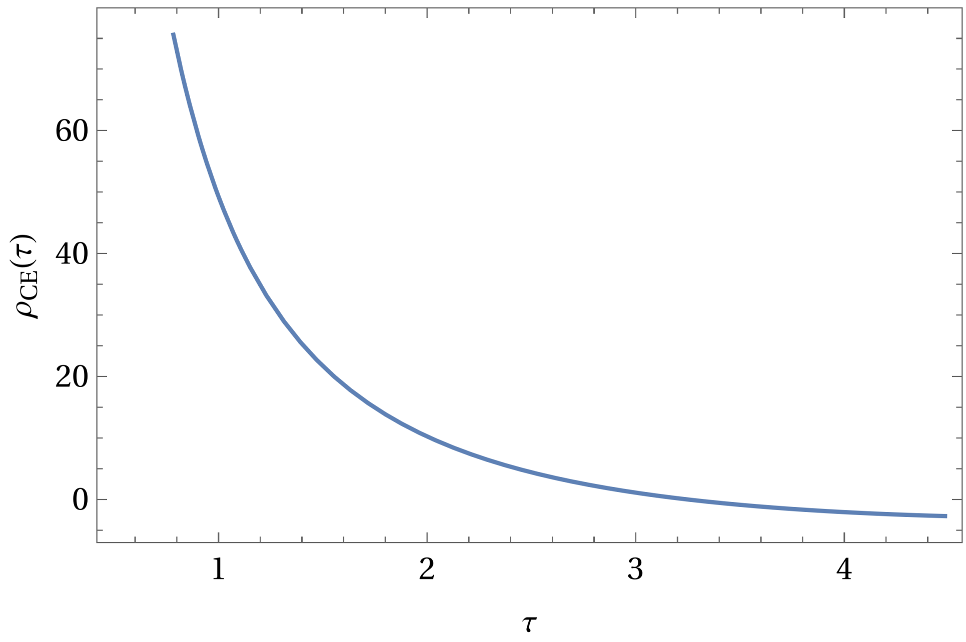

Figure 9: The qualitative behaviour of the energy density in the flat universe filled with the CE condensate, with fixed Λ.

this way. There could however be other solutions, some of which are discussed in [23], and there are recent attempts to find inflation using scalar-vector-tensor theories [24, 25]. This could be a candidate for inflation even in the $k = 0$ case. It is also possible that some kind of combination of the condensates would result in a period of inflation. In [2], inflation is found in the $k = -1$ case.

3.3 Amplitude of primordial gravitational waves

As was demonstrated earlier, for the flat universe case, for both the CE and CM condensates, see section 3.2, there is no inflationary period. We shift our attention to the case of the open universe ($k = -1$) filled with the CM condensate. In this case, the first Friedmann equation (2.16) in conformal time becomes [2]

$$\tilde{a}'^2 = \frac{1}{\gamma^2 \tilde{a}^2}\left(\ln\frac{1}{\tilde{a}^4} - 1\right) + 1, \tag{3.24}$$

where γ^2 is a constant, and $\tilde{a}' = \dfrac{d\tilde{a}}{d\eta}$. With the potential $U_{-1}(\tilde{a}) = \dfrac{1}{\tilde{a}^2}\left(\ln\frac{1}{\tilde{a}^4} - 1\right) + \gamma^2$, the value of γ^2 determines the number of solutions to $U_{-1} = 0$. For $0 < \gamma^2 < \gamma_c^2$, where $\gamma_c^2 = 2\mathrm{e}^{-1/2}$, there are two solutions. For $\gamma^2 = \gamma_c^2$, there is one solution, and for $\gamma^2 > \gamma_c^2$, there are no solutions. In the coming calculations, we will look at the first case, specifically with $\tilde{a} \in [\mu_2, \infty]$, where μ_2 is a solution to $U_{-1} = 0$, such that the potential is always non-negative. Combining equation 2.18 and 3.24 then gives

$$\frac{\tilde{a}''}{\tilde{a}} - \frac{\tilde{a}'^2}{\tilde{a}^2} = -\frac{2}{\gamma^2\tilde{a}^2}\left(\ln\frac{1}{\tilde{a}^4} - 1\right). \tag{3.25}$$

We express the pressure p in the open universe as a function of the corresponding energy density ρ

$$p = \frac{1}{3}\rho + \frac{4}{3}b\mathcal{J}, \tag{3.26}$$

which differs from the radiation dominated universe by an additional term. Expressing $\mathcal{J}$ in terms of the scale factor, as $\mathcal{J} = \dfrac{\Lambda^4}{\tilde{a}^4}$, and so the additional term disappears as $a \to \infty$. To avoid the cosmological singularity, we explore the universe in the conformal time η such that $\eta > \eta_0$. Introducing $h(\eta) = \dfrac{\theta(\eta)}{a(\eta)}$, the perturbation has spatial components [2]

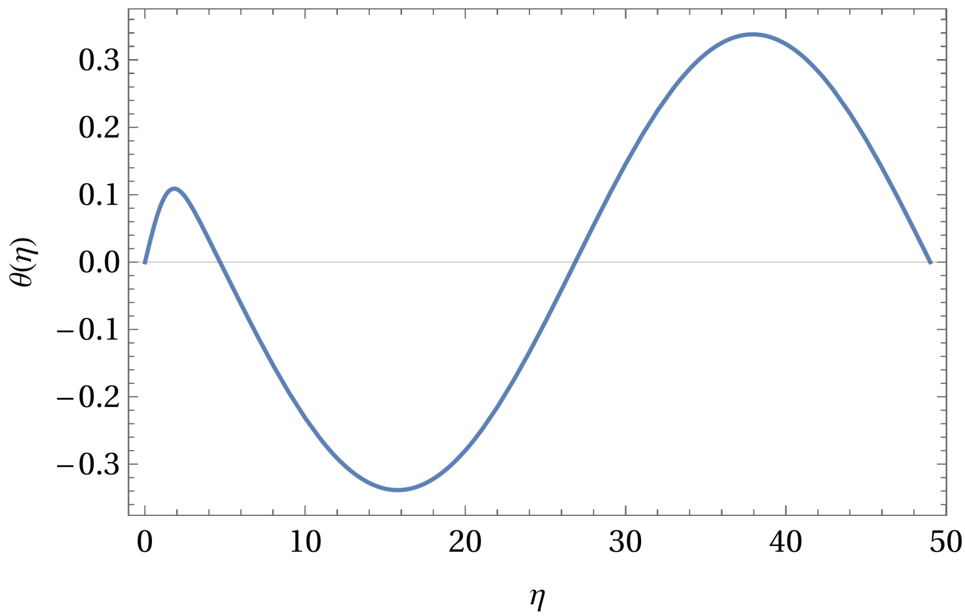

Figure 10: The amplitude of the perturbation of the FLRW metric as a function of conformal time with $\gamma^2 = 1.213$ and $\mu_2 = 1.32$.

$$h_i^j = h(\eta) Y_j^i e^{inx} = \frac{\theta(\eta)}{a(\eta)} Y_j^i e^{inx}, \tag{3.27}$$

where $\theta(\eta)$ is the amplitude of the metric perturbations at conformal time η, and Y_j^i is the eigenfunction of the Laplace operator. Each component of the perturbation obeys [26]

$$h'' + 2\frac{a'}{a}h' + n^2 h = 0. \tag{3.28}$$

Combining equation (3.27) and (3.28) with the expression for the scale factor found by adding equation (3.24) to (3.25) gives

$$\theta'' + \theta\left(n^2 + \frac{2}{\gamma^2 \tilde{a}^2} - 1\right) = 0. \tag{3.29}$$

We solve for the amplitude of the perturbation, which corresponds to the amplitudes of the primoridial GWs, numerically with Wolfram Mathematica using equation (3.29), with the initial conditions $\gamma^2 = 1.231$ and $\tilde{a}(0) = 1.32$, reproducing the results in [2]. The amplitude is shown in figure 10. The maximum of the wave amplitude starts at a lower value in the first period and then reaches a constant value as it propagates.

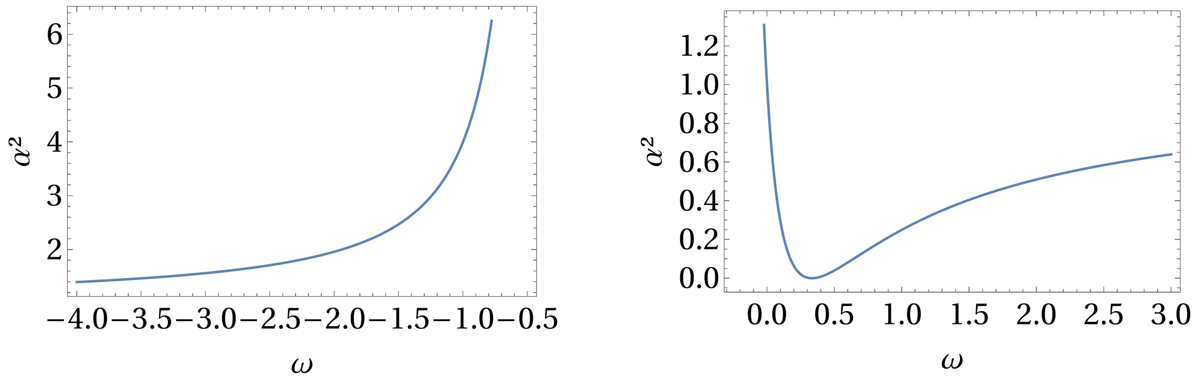

Figure 11: α^2 as a function of the equation of state parameter. To the left is α^2 leading up to the singularity at $\omega = -\frac{1}{3}$. To the right is α^2 after the singularity.

The increase of the amplitude θ in figure 10 is determined by the coefficient of amplification K (2.21) and depends on the equation of state parameter through $\alpha^2 = \left(\frac{1-3\omega}{1+3\omega}\right)^2$. The behaviour of α^2 is shown in figure 11. As expected, the amplification is 0 for $\omega = \frac{1}{3}$. For the vacuum and dust equations of state then, we will have an amplification. As the amplification coefficient also depends on the wave number, only certain frequencies will be amplified. Given that the amplification conditions are fulfilled, the amplitude at time η, $\theta(\eta)$, is amplified compared to the initial amplitude $\theta(\eta_0)$ by an additional term as specified by equation (2.21). The predictions of equation (2.21) collapse around the singularity $\omega = -\frac{1}{3}$.

4 Conclusion

In this thesis, we have explored some cosmological implications of YMFT. We have done this by studying two simple cosmological models of the spatially flat universe. The time evolution of the two universes is found to be very different. The universe filled with the CM condensate expands and contracts periodically, while in the case of the CE condensate, the universe seems to expand indefinitely. The results for the CM condensate were consistent with those in [2]. However, in these models, there is no period of exponential expansion, and so alternative solutions to the flatness problem would have to be found. Furthermore, an assumption of the universe filled with the condensate of one type only is a simplification, and a more accurate description of the universe would likely be a combination of the condensates localised in spatially separated domains, see e.g. [13].

In section 3, we examined some conditions for the amplification of primordial GWs. We found that the primordial GWs fulfilling certain conditions will experience an amplification. The GW amplitude and its amplification is of high relevance when developing future GW experiments.

As a complement to the results in this thesis, it would be interesting to find more general expressions for the scale factor and energy density in the CE case, and explore the consequences this has on the evolution of the universe. Furthermore, building on the results in [2], one could construct models in the CE condensate case for the $k = -1$ and $k = 1$ universes. Furthermore, another aspect that could be considered is how the condensates interact and perhaps coexist in the ground-state of the universe, and what implications a cosmological model taking into account both the condensates would have.

Acknowledgments

I would like to thank the supervisor of this project Roman Pasechnik for unlimited patience and wisdom. I would also like to thank Agnes Hermansson and Maja Skans for support in the writing process.

References

[1] A. Linde. “Inflationary Cosmology”. In: *Inflationary Cosmology*. Springer Berlin Heidelberg, pp. 1–54. DOI: 10.1007/978-3-540-74353-8_1. URL: https://doi.org/10.1007%2F978-3-540-74353-8_1.

[2] G. Savvidy. “Gauge field theory vacuum and cosmological inflation without scalar field”. In: *Annals Phys.* 436 (2022), p. 168681. DOI: 10.1016/j.aop.2021.168681. arXiv: 2109.02162 [hep-th].

[3] A. Maleknejad and M. M. Sheikh-Jabbari. “Non-Abelian gauge field inflation”. In: *Physical Review D* 84.4 (Aug. 2011). DOI: 10.1103/physrevd.84.043515. URL: https://doi.org/10.1103%2Fphysrevd.84.043515.

[4] R. Emami et al. “Stable solutions of inflation driven by vector fields”. In: *Journal of Cosmology and Astroparticle Physics* 2017.03 (Mar. 2017), pp. 058–058. DOI: 10.1088/1475-7516/2017/03/058. URL: https://doi.org/10.1088/1475-7516/2017/03/058.

[5] B. P. Abbott et al. “Observation of Gravitational Waves from a Binary Black Hole Merger”. In: *Physical Review Letters* 116.6 (Feb. 2016). DOI: 10.1103/physrevlett.116.061102. URL: https://doi.org/10.1103%2Fphysrevlett.116.061102.

[6] D. S. Gorbunov and V. A. Rubakov. *Introduction to the Theory of the Early Universe: Hot Big Bang Theory*. World Scientific Publishing, 2011.

[7] J. García-Bellido and S. Nesseris. “Gravitational wave energy emission and detection rates of Primordial Black Hole hyperbolic encounters”. In: *Phys. Dark Univ.* 21 (2018), pp. 61–69. DOI: 10.1016/j.dark.2018.06.001. arXiv: 1711.09702 [astro-ph.HE].

[8] C. Caprini et al. “Detecting gravitational waves from cosmological phase transitions with LISA: an update”. In: *Journal of Cosmology and Astroparticle Physics* 2020.03 (Mar. 2020), pp. 024–024. DOI: 10.1088/1475-7516/2020/03/024. URL: https://doi.org/10.1088%2F1475-7516%2F2020%2F03%2F024.

[9] G. Kane. *Modern Elementary Particle Physics*. Cambridge University Press, 2017.

[10] M. Maggiore. *A Modern Introduction to Quantum Field Theory*. Oxford University Press, 2005.

[11] M. R. Baker, N. Linnemann, and C. Smeenk. “Noether’s first theorem and the energy-momentum tensor ambiguity problem”. In: (2021). DOI: 10.48550/ARXIV.2107.10329. URL: https://arxiv.org/abs/2107.10329.

[12] S. Weinberg. *Cosmology*. McGraw-Hill, 2005.

[13] R. Pasechnik, G. Prokhorov, and O. Teryaev. “Mirror QCD and Cosmological Constant”. In: *Universe* 3.2 (2017), p. 43. DOI: 10.3390/universe3020043. arXiv: 1609.09249 [hep-ph].

[14] P. Donà et al. “Yang-Mills condensate as dark energy: A nonperturbative approach”. In: *Physical Review D* 93.4 (Feb. 2016). DOI: 10.1103/physrevd.93.043012. URL: https://doi.org/10.1103%2Fphysrevd.93.043012.

[15] J. S. Schwinger. “On gauge invariance and vacuum polarization”. In: *Phys. Rev.* 82 (1951). Ed. by K. A. Milton, pp. 664–679. DOI: 10.1103/PhysRev.82.664.

[16] G. Savvidy. “From Heisenberg–Euler Lagrangian to the discovery of Chromomagnetic Gluon Condensation”. In: *Eur. Phys. J. C* 80.2 (2020), p. 165. DOI: 10.1140/epjc/s10052-020-7711-6. arXiv: 1910.00654 [hep-th].

[17] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum. *Relativistic Cosmology*. Cambridge University Press, 2012.

[18] E. Lifshitz. “Republication of: On the gravitational stability of the expanding universe”. In: *J. Phys. (USSR)* 10.2 (1946), p. 116. DOI: 10.1007/s10714-016-2165-8.

[19] L. P. Grishchuk. “Amplification of gravitational waves in an istropic universe”. In: *Zh. Eksp. Teor. Fiz.* 67 (1974), pp. 825–838.

[20] G. Prokhorov and R. Pasechnik. “Light meson gas in the QCD vacuum and oscillating Universe”. In: *JCAP* 01 (2018), p. 017. DOI: 10.1088/1475-7516/2018/01/017. arXiv: 1711.08317 [hep-ph].

[21] Y.-K. Cheung, C. Li, and J. Vergados. “Big Bounce Genesis and Possible Experimental Tests: A Brief Review”. In: *Symmetry* 8.11 (Nov. 2016), p. 136. DOI: 10.3390/sym8110136. URL: https://doi.org/10.3390%2Fsym8110136.

[22] M. Novello and S. Bergliaffa. “Bouncing cosmologies”. In: *Physics Reports* 463.4 (July 2008), pp. 127–213. DOI: 10.1016/j.physrep.2008.04.006. URL: https://doi.org/10.1016%2Fj.physrep.2008.04.006.

[23] S. F. Bramberger et al. “Solving the flatness problem with an anisotropic instanton in Hořava-Lifshitz gravity”. In: *Physical Review D* 97.4 (Feb. 2018). DOI: 10.1103/physrevd.97.043512. URL: https://doi.org/10.1103%2Fphysrevd.97.043512.

[24] A. Oliveros and C. J. Rodríguez. “Inflation in a scalar–vector–tensor theory”. In: *General Relativity and Gravitation* 54.1 (Jan. 2022). DOI: 10.1007/s10714-022-02901-y. URL: https://doi.org/10.1007%2Fs10714-022-02901-y.

[25] L. Heisenberg, R. Kase, and S. Tsujikawa. “Cosmology in scalar-vector-tensor theories”. In: *Physical Review D* 98.2 (July 2018). DOI: 10.1103/physrevd.98.024038. URL: https://doi.org/10.1103%2Fphysrevd.98.024038.

[26] L. P. Grishchuk. “Graviton Creation in the Early Universe”. In: *Annals N. Y. Acad. Sci.* 302 (1977), p. 439. DOI: 10.1111/j.1749-6632.1977.tb37064.x.