Describing Finite Codimensional Polynomial Subalgebras Using Partial Derivatives

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Abstract

In this text we continue the work of describing subalgebras of $\mathbb{K}[x]$ of finite codimension that was started in [2]. In the referenced paper, the authors present how univariate subalgebras can be described by conditions based on evaluations in certain scalars, and proceed to develop a large theoretical framework to understand the nature of such conditions. The purpose of this thesis is to generalize as many of their results as possible to the multivariate setting $\mathbb{K}[\boldsymbol{x}]$. We include generalized definitions of the type, spectrum, clusters, $\boldsymbol{\alpha}$-derivations, and $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtractions. We also state and prove generalizations of most of the theorems relating to the spectrum, clusters, $\boldsymbol{\alpha}$-derivation spaces, as well as the Main Theorem. We also give a couple of new results pertaining to how $\boldsymbol{\alpha}$-derivation spaces behave when we apply subalgebra conditions on clusters not containing $\boldsymbol{\alpha}$.

Contents

1 Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $\boldsymbol{x} = \{x_1, x_2, \dots x_n\}$. Throughout this text we shall be concerned with polynomial subalgebras $A \subset \mathbb{K}[\boldsymbol{x}]$ of finite codimension. We will require that the subalgebras contain the scalar field $\mathbb{K} \subset A$. Usually such subalgebras would be described in terms of a basis. An example in the univariate case is the subalgebra $A_1 \subset \mathbb{K}[x]$ that is generated by the polynomials x^3, x^4 and x^5. Then A_1 consists of all polynomials where the first, and second degree terms are all 0. In [2], an entire theory is developed around the concept of describing such subalgebras in a new way, by the use of conditions. For example, the same subalgebra A_1 can be written using conditions as follows,

$$A_1 = \{f \in \mathbb{K}[\boldsymbol{x}] \mid f'(0) = f''(0) = 0\}.$$

Another, per-haps less obvious example, let $A_2 = \langle x^3 - x, x^2 \rangle$. This algebra can be written using conditions as,

$$A_2 = \{f \in \mathbb{K}[\boldsymbol{x}] \mid f(1) = f(-1)\}.$$

The theory which is developed in [2] is limited to the univariate case. The purpose of this thesis is to generalize some of their results to the multivariate case. We will still only treat the case of finite codimensional subalgebras though. A multivariate example is given by the algebra $A_3 \subset \mathbb{K}[x_1, x_2]$ which can be described by a generating set as

$$\begin{aligned} A_3 = \langle & x_2, x_1^2 - 2x_1, x_1^3 - 3x_1, \\ & x_2^2 x_1^2, x_2^2 x_1, x_2 x_1^3 - 2x_2 x_1^2 + x_2 x_1, \\ & x_2 x_1^4 - 3x_2 x_1^2 + 2x_2 x_1, x_2^3 x_1^3 \rangle, \end{aligned}$$

but also by conditions as

$$A_3 = \{f \in \mathbb{K}[\boldsymbol{x}] : f'_{x_1}(1,0) = f''_{x_1,x_2}(1,0) = 0\}.$$

It may be difficult to see that the descriptions by conditions are equivalent to the descriptions by generators right now, but it will hopefully be easier once we have introduced the necessary theory. Unfortunately, the requirement that a subalgebra has finite codimension is quite demanding when it comes to producing generators for- and interpreting multivariate examples. Therefore, there will be quite few examples in the text. The problem grows larger pretty quickly as we increase either the number of indeterminates or the codimension of the subalgebras we consider. Above we saw an example for a subalgebra of codimension 2 in $\mathbb{K}[x_1, x_2]$. To show what we mean, we'll include a computer generated example of a subalgebra of codimension 3 in $\mathbb{K}[x_1, x_2, x_3]$. Let $A_4 \subset \mathbb{K}[x_1, x_2, x_3]$ be defined by conditions as

$$\begin{aligned} A_4 = \{f \in \mathbb{K}[\boldsymbol{x}] : & f'_{x_3}(1, 0, -1) = 0, f(3, 2, 5) = f(1, -3, 2), \\ & f'_{x_1}(3, 2, 5) - 3f'_{x_2}(1, -3, 2) = 0\}. \end{aligned}$$

Then we can described A_4 via a generating set as

$$
\begin{aligned}
A_4 = \langle & x_1^2 + \frac{4}{11}x_2 - \frac{54}{11}x_1, \\
& x_2x_1 - x_2 - 2x_1, \\
& x_3x_1 - x_3 - 5x_1, \\
& x_1^3 + \frac{28}{11}x_2 - \frac{213}{11}x_1, \\
& x_3^2 - 3x_3x_1 + \frac{9}{4}x_1^2 + 5x_3 - \frac{18}{11}x_2 - \frac{75}{22}x_1, \\
& x_3x_2 - \frac{5}{2}x_3x_1 - \frac{3}{2}x_2x_1 + \frac{15}{4}x_1^2 + \frac{5}{2}x_3 + \frac{7}{22}x_2 - \frac{50}{11}x_1, \\
& x_3x_1^2 - \frac{3}{2}x_1^3 - 4x_3x_1 + 6x_1^2 + 3x_3 + \frac{2}{11}x_2 - \frac{109}{22}x_1, \\
& x_3x_2x_1 - \frac{3}{2}x_2x_1^2 - \frac{9}{2}x_3x_1 + \frac{27}{4}x_1^2 + \frac{9}{2}x_3 + \frac{7}{22}x_2 - \frac{83}{11}x_1, \\
& x_3^2x_1 - \frac{3}{2}x_3x_1^2 - \frac{13}{2}x_3x_1 + \frac{39}{4}x_1^2 + 10x_3 - \frac{18}{11}x_2 - \frac{120}{11}x_1, \\
& x_3^3 - \frac{9}{2}x_3^2x_1 + \frac{27}{4}x_3x_1^2 - \frac{27}{8}x_1^3 - \frac{75}{4}x_3 + \frac{54}{11}x_2 + \frac{1395}{88}x_1, \\
& x_2^2 - 5x_2x_1 + \frac{25}{4}x_1^2 + 11x_2 - \frac{55}{2}x_1, \\
& x_2x_1^2 - \frac{5}{2}x_1^3 - 4x_2x_1 + 10x_1^2 + x_2 - \frac{5}{2}x_1, \\
& x_2^2x_1 - \frac{5}{2}x_2x_1^2 - \frac{9}{2}x_2x_1 + \frac{45}{4}x_1^2 + 13x_2 - \frac{65}{2}x_1, \\
& x_3^2x_2 - \frac{5}{2}x_3^2x_1 - 3x_3x_2x_1 + \frac{15}{2}x_3x_1^2 + \frac{9}{4}x_2x_1^2 \\
& - \frac{45}{8}x_1^3 + 5x_3x_2 - \frac{25}{2}x_3x_1 - \frac{15}{2}x_2x_1 + \frac{75}{4}x_1^2 + \frac{25}{4}x_2 - \frac{125}{8}x_1, \\
& x_3x_2^2 - 5x_3x_2x_1 - \frac{3}{2}x_2^2x_1 + \frac{25}{4}x_3x_1^2 + \frac{15}{2}x_2x_1^2 - \frac{75}{8}x_1^3 + \frac{5}{2}x_3x_2 \\
& - \frac{25}{4}x_3x_1 - \frac{15}{4}x_2x_1 + \frac{75}{8}x_1^2 - \frac{1}{4}x_2 + \frac{5}{8}x_1, \\
& x_2^3 - \frac{15}{2}x_2^2x_1 + \frac{75}{4}x_2x_1^2 - \frac{125}{8}x_1^3 - \frac{363}{4}x_2 + \frac{1815}{8}x_1\rangle.
\end{aligned}
$$

The algorithm used is based on Theorem 25 which doesn't necessarily produce a minimal generating set. It does however produce a minimal SAGBI basis, a concept which will be explained shortly. A discussion on why we need so many generators can be found in Section 3.3.

1.1 Some Conventions

Before we move along, it will be useful to give some conventions upfront. We will speak of the degree of a multivariate polynomial as the tuple of exponents of each indeterminate in the leading term. When nothing else is indicated, we shall use lexicographical term ordering where $x_i < x_j$ if $i > j$, but any admissible order works (but results in a different definition of deg of course). An example,

$$\deg(4x_1^5x_2^2x_3^7 + x_1^2x_3^9) = (5, 2, 7).$$

In the case when A is a set of polynomials, we will write $\deg(A) = \{\deg(f) : f \in A\}$ for the set of degrees of polynomials in A.

Instead of writing $\mathbb{K}[x_1, x_2, \dots x_n]$ we shall simply write $\mathbb{K}[\boldsymbol{x}]$. The same holds in the case of monomials. Sometimes, instead of writing $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ we will write $\boldsymbol{x}^{\boldsymbol{a}}$ where $\boldsymbol{a} = (a_1, a_2, \dots a_n)$. Moreover, the variable n will always be used to denote the number of free variables $\boldsymbol{x} = \{x_1, x_2, \dots, x_n\}$ in our algebra.

Given a set G and a multiplicative operation on the elements of G, we will write G_{mon} to denote the set of all finite products of elements of G.

We will sometimes write the range of integers $\{x \in \mathbb{N} : 1 \leq x \leq n\}$ as $[1..n]$

When $f \in \mathbb{K}[\boldsymbol{x}]$ is a polynomial will write $\text{lm}(f)$ to denote the leading monomial of f. When $F \subset \mathbb{K}[\boldsymbol{x}]$ is a set of polynomials, we will write $\text{Lm}(F) = \{\text{lm}(f) : f \in F\}$ to denote the set of leading monomials. Similar terminology is used for leading coefficients and terms as $\text{lc}, \text{Lc}, \text{lt}, \text{Lt}$.

We will sometimes refer to a least or "smallest" polynomial which satisfies a property P. As it stands, this is informal terminology, but it can be made precise so we explain what we mean here. Given a term order $<$ on $\mathbb{K}[\boldsymbol{x}]$, we can introduce a partial order $\prec$ on $\mathbb{K}[\boldsymbol{x}]$ where $f_1 \prec f_2$ if $\text{lm}(f_1) < \text{lm}(f_2)$. When we say that f is a smallest polynomial which satisfies P among some set of polynomials A, we mean that $g \in A$ such that $g \prec f$ implies that g does not satisfy P.

2 Background

Really, the only non-standard prerequisite knowledge required to understand this thesis is that of SAGBI bases. Thus we give a short introduction here. We will also use a few well known results relating to linear functionals which are included afterwards.

2.1 SAGBI Basis of a Subalgebra

For the reader familiar with the Groebner Theory, a SAGBI basis is similar to a Groebner basis for a subalgebra. In fact, the acronym stands for Subalgebra Analogue to Groebner Bases for Ideals. For those who have not seen Groebner bases before, not to worry, the definition will be explained.

Definition 1. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra and $<$ a term order on A. Then a set $G \subseteq A$ is a SAGBI basis when $\mathrm{Lm}(G)_{\mathrm{mon}} = \mathrm{Lm}(A)$.

Every subalgebra A admits a SAGBI basis since A itself is a SAGBI basis. Unfortunately, there are some multivariate subalgebras which don't admit finite SAGBI bases. However, we will see pretty soon that any subalgebra of finite codimension in $\mathbb{K}[\boldsymbol{x}]$ does admit a finite SAGBI basis.

One of the main purposes of SAGBI bases is that they allow for a subalgebra membership algorithm when they are finite. We give a brief illustration.

The algorithm is based on something called subduction, which is performed in a sequence of steps. Let $f \in \mathbb{K}[\boldsymbol{x}]$. If $\mathrm{lm}(f) \in \mathrm{Lm}(G)_{\mathrm{mon}}$, then there exists a scalar $a_0 \in \mathbb{K}$ and product of elements in G, call it $h_0 = \prod_{g_i \in G} g_i^{b_i}$, such that $a_0 h_0$ has the same leading term as f. Then $\mathrm{lm}(f - a_0 h_0) < \mathrm{lm}(f)$. We now construct $f_0 = f - a_0 h_0$. If $\mathrm{lm}(f_0) \in \mathrm{Lm}(G)_{\mathrm{mon}}$, we can repeat the process above to obtain an even smaller polynomial, $f_1 = f_0 - a_1 h_1$. We can keep on repeating until we reach a point where either $f_n = 0$, or $\mathrm{lm}(f_n) \notin \mathrm{Lm}(G)_{\mathrm{mon}}$. We say that f subduces to f_n over G. If f subduces to 0, then we have obtained a decomposition

$$f = \sum_{i=0}^{n-1} a_i h_i$$

of f into a linear combination in the elements of G_{mon}. From this decomposition it's clear polynomials which subduce to 0 over G lie in A. Similarly, if $f \in A$, then $\mathrm{lm}(f) \in \mathrm{Lm}(A)$, but $\mathrm{Lm}(A) = \mathrm{Lm}(G)_{\mathrm{mon}}$ by definition, so we can subduce f one step further. Since $a_i h_i \in A$, it follows that any f_i during the subduction of f lies in A. Inductively we see that f subduces to 0.

2.2 Some Facts About Linear Functionals

We will call any linear function for a vector space V to it's scalar field $\mathbb{K}$ as a linear functional. Here are some results from linear algebra which will be useful later.

Lemma 2. Let V be a vector space over the field $\mathbb{K}$ and let $f : V \to \mathbb{K}$ be a linear functional. Then $\ker(f)$ is either trivial or has codimension 1.

Proof. This is nothing but a specialization of the Rank-Nullity Theorem. □

Lemma 3. Let f, g be two non-trivial linear functionals from V to $\mathbb{K}$. If $\ker(f) = \ker(g)$, then $f = cg$ for some $c \in \mathbb{K}$.

Proof. Let $v_0 \in V$ such that $f(v_0) = 1$. As above, we may write any $v \in V$ as $v = f(v)v_0 + u$ for some $u \in \ker(f)$. Looking at g we get $g(v) = g(v_0)f(v) + g(u) = g(v_0)f(v)$ as the functionals share the same kernel. Hence for any $v \in V$, $g(v) = g(v_0)f(v)$ and the statement of the lemma holds with $c = g(v_0)$. □

Lemma 4. Let V be a vector space over the field $\mathbb{K}$ and let $f_i : V \to \mathbb{K}$ for $i \in [1..m]$ be a finite set of linear functionals. If we denote

$$W = \bigcap_{f_i} \ker\ f_i,$$

and $f, g : V \to \mathbb{K}$ are linear functionals over V such that $f\big|_W = g\big|_W$. Then

$$f = g + \sum_{f_i} c_i f_i$$

for some set of scalars $c_i \in \mathbb{K}$.

Proof. We denote

$$W_k = \bigcap_{i=1}^{k} \ker\ f_i$$

and proceed by using induction on k in reverse to show that we can write $f\big|_{W_k} = g\big|_{W_k} + \sum_{i=k+1}^{m} c_i f_i$. The $k = m$ case is immediate since $f\big|_{W_m} = g\big|_{W_m}$.

Now let $k < m$ and assume that

$$f\big|_{W_{k+1}} = g\big|_{W_{k+1}} + \sum_{i=k+2}^{m} c_i f_i$$

Then

$$h_k = \left(f - g - \sum_{i=k+2}^{m} c_i f_i \right) \Bigg|_{W_k}$$

is zero on W_{k+1} whence $h_k = c_{k+1} f_{k+1}$ for some $c_{k+1} \in \mathbb{K}$ and our statement follows after rearrangement. □

3 The Type of a Multivariate Polynomial Algebra

3.1 Multivariate Numerical Semigroups

The type of an algebra is meant to capture the structure of the set of degrees of the algebra. In the multivariate case, we need to define the notion of a multivariate numerical semigroup to do so.

Definition 5. Let $S \subseteq \mathbb{N}^n$ be a semigroup such that $\mathbb{N}^n \setminus S$ is finite and $\mathbf{0} \in S$. Then we say that S is a multivariate numerical semigroup.

We will show later that $A \subset \mathbb{K}[\boldsymbol{x}]$ is a subalgebra of finite codimension if and only if $\deg(A)$ is a numerical semigroup.

We know that univariate numerical semigroups are finitely generated, and they admit unique minimal generating sets. A natural question is whether the same holds true in the multivariate case.

First we show that a unique minimal generating set exist.

Theorem 6. Let S be a multivariate semigroup. Then S has a unique minimal generating set $Gn = Gn(S) = S \setminus (S^* + S^*)$ where $S^* = S \setminus \{0\}$.

Proof. We first show that Gn is a generating set. Let $<$ be a lexicographical ordering on the elements in S. We prove that all $c \in S$ lie in $\langle Gn \rangle$ via induction over $<$. Let m be the minimal element in S with respect to $<$. Then $m \in Gn$ for if there exist $a, b \in S^*$ such that $a + b = m$, then $a, b < m$. Now let c be an element in S and assume that all elements $< c$ lie in $\langle Gn \rangle$. We are done if $c \in Gn$ so assume that there exist $a, b \in S^*$ such that $a + b = c$. Then $a, b < c$ so $a, b \in \langle Gn \rangle$ whence $c \in \langle Gn \rangle$ and we are done.

Now we show that Gn is minimal. Let G' be a generating set of S. Given any $g \in Gn \subset \langle G' \rangle$, we can write

$$g = \sum_{g_i \in G'} x_i g_i,$$

for some natural numbers x_i. But as $Gn \cap (S^* + S^*) = \emptyset$, we have $\sum x_i = 1$, and we can write $g = g_i$ for some index i. It follows that $g \in G'$ and $Gn \subset G'$.

□

And now we show finiteness.

Theorem 7. The minimal generating set $Gn(S)$ of a multivariate semigroup is finite.

Proof. For this proof, let $<$ denote the partial order on $\mathbb{N}^n$ where $(a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots b_n)$ if and only if $a_i \leq b_i$ for all i and $a_i \neq b_i$ for at least one i.

Let $M \subset Gn(S)$ be the set of minimal elements in $Gn(S)$ with respect to $<$. Then M is finite by Dickson's Lemma.

Aiming towards a contradiction, assume that $Gn(S)$ is infinite. For each $m_i \in M$, let $S_i = \{s \in Gn(S) : m_i < s\}$. We now claim that some S_i must be infinite. To see this, note that $Gn(S) \setminus M$ is infinite since M is finite, and given any non-minimal element $s \in Gn(S) \setminus M$, we know that some $m_i \in M$ will satisfy $m_i < s$. It follows that $Gn(S) = M \cup \bigcup_{i=1}^{n} S_i$, whence at least one S_i must be infinite since $Gn(S)$ is.

Let S_i be one of the infinite sets guaranteed above. Then the elements $s' = s - m_i, s \in S_i$ makes an infinite set. Even more none of the s' lies in S since

$s \in S \setminus S^* + S^*$ and $m \in S$. This is a contradiction to the requirement that $\mathbb{N}^n \setminus S$ is finite. □

In the sequel, we shall simply say numerical semigroup when we are discussing multivariate numerical semigroups.

3.2 The Definition of $T(A)$ and Finiteness of SAGBI Bases for Finite Codimensional Subalgebras

We are now ready to give the definition of the type (with respect to some term ordering) of a multivariate subalgebra of finite codimension.

Definition 8. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. We define the *type* of A, written $T(A)$ as $T(A) = Gn(\deg(A))$ the minimal generating set of $\deg(A)$.

One of the main purposes of the definition is the following theorem.

Theorem 9. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. Then any subset $G \subset A$ such that $\deg(G) \supseteq T(A)$ is a SAGBI basis.

Proof. Let $f \in A$. Then $\deg(f) \in \langle T(A) \rangle = \langle \deg(G) \rangle$ and it follows that $\mathrm{lm}(f) \in \mathrm{Lm}(G)_{\mathrm{mon}}$ whence G is a SAGBI basis. □

Note that there exist a polynomial $g \in A$ with degree d for each $d \in T(A)$, and we've shown that $T(A)$ is finite. Thus we get the following corollary.

Corollary 10. Any subalgebra $A \subset \mathbb{K}[\boldsymbol{x}]$ of finite codimension admits a finite SAGBI basis.

3.3 Some Notes on Finite Codimension

As we increase the number of indeterminates, we will see that the requirement of finite codimension becomes more and more significant. In this section we will discuss the requirement and some consequences. First we formally prove the following intuitive fact.

Lemma 11. Let $A \subset \mathbb{K}[x]$ be a subalgebra. Then A has finite codimension if and only if $\deg(A)$ is a numerical semigroup with respect to any term order.

Proof. First of all, it's clear that $\deg(A)$ is a semigroup in $\mathbb{N}^n$ that contains $\mathbf{0}$.

If there is some term order that results in $C = \mathbb{N}^n \setminus \deg(A)$ being infinite, then for each $d \in C$ we can find a corresponding monomial not in A. Note that any linear combination of such monomials won't lie in A since the degree of such a linear combination would lie in C. Thus they span an infinite dimensional subspace of $\mathbb{K}[\boldsymbol{x}] \setminus A$ and codim A is infinite.

If instead A has infinite codimension, let $K \subset \mathbb{K}[x] \setminus A$ be a infinite linearly independent set. We can reduce the elements in K so that no two polynomials have the same degree. It follows that $C = \deg(K)$ is an infinite set that lies in $C \subset \mathbb{N}^n \setminus \deg(A)$ so $\deg(A)$ is not a numerical semigroup. $\square$

One interesting fact that follows from codim $A =$ finite is that for any monomial $\boldsymbol{x}^{\boldsymbol{a}}$, and indeterminate x_i, there must be some $k \in \mathbb{N}$ such that $x_i^k \boldsymbol{x}^{\boldsymbol{a}} \in \mathrm{Lm}(A)$. Otherwise we would have an infinite set of elements in $\mathbb{N}^n \setminus T(A)$ which is contradictory. There are probably many other similar arguments which rely on the finiteness of $\mathbb{N}^n \setminus T(A)$.

This results explains why it is so laborious to correctly produce a set of generators which generate a subalgebra of finite codimension - and it becomes much harder the more indeterminates that are involved.

4 Subalgebra Conditions

This section will develop definitions of different kinds of subalgebra conditions. We say that a linear functional $L : A \to \mathbb{K}$ is a subalgebra condition if $\ker L$ is a subalgebra of A. We really only use two types of subalgebra conditions, evaluation subtractions and $\boldsymbol{\alpha}$-derivations. We define them bellow.

4.1 Evaluation Subtractions

Definition 12. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{K}^n, \boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $c \in \mathbb{K}$, $c \neq 0$. We define an evaluation subtraction to be a function $E : \mathbb{K}[\boldsymbol{x}] \to \mathbb{K}$ of the kind $E(f) = c(f(\boldsymbol{\alpha}) - f(\boldsymbol{\beta}))$. When we want to emphasize the scalars used, we call E an $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction. The scalar c will be irrelevant for all of our discussions and can be assumed to equal 1 throughout the remainder of this text.

Theorem 13. Let $E : A \to \mathbb{K}$ be an evaluation subtraction given as $E(f) = f(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})$. Then ker E is a subalgebra of A.

Proof. Let $f, g \in \ker E$ and $c \in \mathbb{K}$. It's clear that E is linear whence $f + g, cf \in \ker E$. Moreover, we have

$$\begin{aligned} E(fg) &= f(\boldsymbol{\alpha})g(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})g(\boldsymbol{\beta}) \\ &= f(\boldsymbol{\alpha})g(\boldsymbol{\alpha}) - f(\boldsymbol{\alpha})g(\boldsymbol{\alpha}) \\ &= 0, \end{aligned}$$

so $fg \in \ker E$. Finally E vanishes on $\mathbb{K}$ and ker E is an algebra. $\square$

4.2 $\boldsymbol{\alpha}$-Derivations

Definition 14. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra and $\boldsymbol{\alpha} \in \mathbb{K}^n$. Then a linear functional $D : A \to \mathbb{K}$ is said to be a $\boldsymbol{\alpha}$-derivation if for all $f, g \in A$, we have

$$D(fg) = f(\boldsymbol{\alpha})D(g) + D(f)g(\boldsymbol{\alpha}).$$

The previous condition will be referred to as the derivation condition.

The definition above is implicit in the sense that it doesn't tell how to construct $\boldsymbol{\alpha}$-derivations. This is a non-trivial issue that will get more and more resolved in the coming sections. For now, we can give a couple of examples.

Let $A = \mathbb{K}[\boldsymbol{x}]$. Then $D : A \to \mathbb{K}$ given by

$$D(f) = \sum_{i=1}^{n} a_i f'_{x_i}(\boldsymbol{\alpha})$$

is an $\boldsymbol{\alpha}$-derivation over A for all possible choices of scalars a_i. We state an important fact before we give our next example.

Theorem 15. Let $D : A \to \mathbb{K}$ be an $\boldsymbol{\alpha}$-derivation. Then ker D is a subalgebra of A.

Proof. Let $f, g \in \ker D$ and $c \in \mathbb{K}$. We know that D is is linear by definition whence $f + g, cf \in \ker D$. Moreover, we have

$$\begin{aligned} D(fg) &= f(\boldsymbol{\alpha})D(g) - D(f)g(\boldsymbol{\alpha}) \\ &= 0, \end{aligned}$$

so $fg \in \ker D$. Finally, we need $D(1) = D(1^2) = 2D(1)$ so D vanishes on $\mathbb{K}$ and ker D is an algebra. □

If we now construct the algebra $A' = \mathbb{K}[\boldsymbol{x}] \cap \ker f \mapsto f'_{x_1}(\boldsymbol{\alpha})$, then

$$D'(f) = b_2 f''_{x_1x_1}(\boldsymbol{\alpha}) + b_3 f'''_{x_1x_1x_1}(\boldsymbol{\alpha}) + \sum_{i=2}^{n} a_i f'_{x_i}(\boldsymbol{\alpha})$$

is an $\boldsymbol{\alpha}$-derivation over A' but not over A. In a similar fashion,

$$f \mapsto b_1 f^{(4)}_{x_1x_1x_1x_1}(\boldsymbol{\alpha}) + \sum_{i=2}^{n} a_i f'_{x_i}(\boldsymbol{\alpha})$$

is an $\boldsymbol{\alpha}$-derivation over $\mathbb{K}[\boldsymbol{x}] \cap \ker f \mapsto f'_{x_1}(\boldsymbol{\alpha}) \cap \ker f \mapsto f''_{x_1x_1}(\boldsymbol{\alpha})$, but not over A or A'. In some sense, the more $\boldsymbol{\alpha}$-derivations that we've kerneled by to obtain an algebra, the more $\boldsymbol{\alpha}$-derivations we can expect to exist over the algebra. The intuition is that for example if $D : \mathbb{K}[\boldsymbol{x}] \to K$ is given by $D(f) = f''_{x_1x_1}(\boldsymbol{\alpha})$, then

$$D(fg) = f''_{x_1x_1}(fg) = f''_{x_1x_1}(\boldsymbol{\alpha})g(\boldsymbol{\alpha}) + 2f'_{x_1}(\boldsymbol{\alpha})g'_{x_1}(\boldsymbol{\alpha}) + f(\boldsymbol{\alpha})g''_{x_1x_1}(\boldsymbol{\alpha})$$

and if D is supposed to be an $\boldsymbol{\alpha}$-derivation over some algebra A, then we need for $f'_{x_1}(\boldsymbol{\alpha})g'_{x_1}(\boldsymbol{\alpha}) = 0$ for all $f, g \in A$, which is exactly what we get when $f'_{x_1}(\boldsymbol{\alpha}) = 0$ for all $f \in A$. In a similar way, for $f \mapsto f^{(n)}(\boldsymbol{\alpha})$ to be an $\boldsymbol{\alpha}$-derivation over some algebra, we need all the middle terms which result from expanding $(fg)^{(n)}(\boldsymbol{\alpha})$

via the Leibniz rule to vanish in the algebra.

We will see later in the Main Theorem of $\boldsymbol{\alpha}$-Derivations (Theorem 43), that all $\boldsymbol{\alpha}$-derivations can be expressed this way as linear combinations of derivative evaluations. There is also an alternate and equally important linear algebraic interpretation given in Theorem 20.

We will take great interest in the $\boldsymbol{\alpha}$-derivation space.

Definition 16. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra and $\boldsymbol{\alpha} \in \mathbb{K}^n$. The set of $\boldsymbol{\alpha}$-derivations over A form a vector space which we will call the $\boldsymbol{\alpha}$-derivation space, denoted $\mathcal{D}_{\boldsymbol{\alpha}}(A)$.

4.3 Unification of α-Derivations and Evaluation Subtractions

There is a perspective which unifies the definition of evaluation subtractions with that of $\boldsymbol{\alpha}$-derivations. We give it in the following lemma.

Lemma 17. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension where there exist some $f \in A$ such that $f(\boldsymbol{\alpha}) \neq f(\boldsymbol{\beta})$. Let E be a non-trivial linear functional $A \to \mathbb{K}$. Then E is an $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction if and only if

$$E(fg) = f(\boldsymbol{\alpha})E(g) + g(\boldsymbol{\beta})E(f)$$

for all $f, g \in \mathbb{K}[\boldsymbol{x}]$.

Proof. First let E be an $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction. Then

$$\begin{aligned} E(fg) &= c(f(\boldsymbol{\alpha})g(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})g(\boldsymbol{\beta})) \\ &= c(f(\boldsymbol{\alpha})g(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})g(\boldsymbol{\alpha})) + c(f(\boldsymbol{\beta})g(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})g(\boldsymbol{\beta})) \\ &= g(\boldsymbol{\alpha})E(f) + f(\boldsymbol{\beta})E(g). \end{aligned}$$

For the other direction, if E is a linear function $A \to \mathbb{K}$ which satisfies the condition $E(fg) = f(\boldsymbol{\alpha})E(g) + g(\boldsymbol{\beta})E(f)$ for all f, g in A. Pick f, g such that $E(f) \neq 0, E(g) = 0$. Then since f and g commute we need

$$\begin{aligned} 0 &= E(fg) - E(gf) \\ &= f(\boldsymbol{\alpha})E(g) + g(\boldsymbol{\beta})E(f) - g(\boldsymbol{\alpha})E(f) - f(\boldsymbol{\beta})E(g) \\ &= E(g)(f(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})) - E(f)(g(\boldsymbol{\alpha}) - g(\boldsymbol{\beta})) \\ &= -E(f)(g(\boldsymbol{\alpha}) - g(\boldsymbol{\beta})), \end{aligned}$$

and it follows that $g(\boldsymbol{\alpha}) - g(\boldsymbol{\beta}) = 0$. If we let $E' : A \to \mathbb{K}$ be defined as $E'(f) = f(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})$, then we've shown that $\ker E' \supseteq \ker E$. It follows from Lemmas 2 and 3 that $E = cE'$ and we are done. ☐

So, we could unify $\boldsymbol{\alpha}$-derivations and evaluation subtractions by talking of the set of linear functionals $L : A \to \mathbb{K}$ which satisfy $L(fg) = f(\boldsymbol{\alpha})L(g) + g(\boldsymbol{\beta})L(f)$ where possibly $\boldsymbol{\alpha} = \boldsymbol{\beta}$. In practice, we won't use this too much. It turns out that evaluation subtractions and $\boldsymbol{\alpha}$-derivations differ in many significant ways, and it will almost always benefit us to consider them as different kinds of functions. The perspective above is still interesting to maintain in the back of ones head though, and it will be used to shorten proofs where applicable.

4.4 The Connection Between $\mathcal{D}_{\boldsymbol{\alpha}}(A)$ and $M_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A)$

In the univariate case, the $\boldsymbol{\alpha}$-derivation space of an algebra corresponds to a particular dual space based on A. We will see that the same correspondence is true in the multivariate case as well. First we need some definitions.

Definition 18. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra and $\boldsymbol{\alpha} \in \mathbb{K}^n$. Then we denote the $\boldsymbol{\alpha}$-vanishing subspace of A by $M_{\boldsymbol{\alpha}}(A) = \{f \in A : f(\boldsymbol{\alpha}) = 0\}$.

From this definition we immediately see that $A = M_{\boldsymbol{\alpha}}(A) \oplus \mathbb{K}$.

We will also need to name a few functions.

Definition 19. Let $\boldsymbol{\alpha} \in \mathbb{K}$ then we define $z_{\boldsymbol{\alpha}} : \mathbb{K}[\boldsymbol{x}] \to M_{\boldsymbol{\alpha}}(\mathbb{K}[\boldsymbol{x}])$ as the linear function

$$z_{\boldsymbol{\alpha}}(f) = f - f(\boldsymbol{\alpha}).$$

We will take great interest in the quotient space $M_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A)$, and introduce a function related to it. Given a subalgebra $A \subset \mathbb{K}[\boldsymbol{x}]$ of finite codimension, let $\phi_{\boldsymbol{\alpha}} : M_{\boldsymbol{\alpha}}(A) \to M_{\boldsymbol{\alpha}}/M^2_{\boldsymbol{\alpha}}(A)$ denote the linear function $\phi_{\boldsymbol{\alpha}}(f) = f + M^2_{\boldsymbol{\alpha}}(A)$.

Finally, the composition of the functions above will be denoted as $\varphi_{\boldsymbol{\alpha}} = \phi_{\boldsymbol{\alpha}} \circ z_{\boldsymbol{\alpha}}$.

In [2] it is shown that the following holds when $A \subset \mathbb{K}[x]$ is a univariate subalgebra of finite codimension,

$$\dim \mathcal{D}_{\boldsymbol{\alpha}}(A) = \dim M_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A),$$

and that the $\boldsymbol{\alpha}$-derivations are precisely the compositions $L \circ \varphi_{\boldsymbol{\alpha}}$ for linear functionals $L : M_{\boldsymbol{\alpha}}/M^2_{\boldsymbol{\alpha}} \to \mathbb{K}$. We will show that this holds in the multivariate case too.

Let D be an $\boldsymbol{\alpha}$-derivation. The first observation to make is that $D(\mathbb{K}) = 0$ so $D \circ z_{\boldsymbol{\alpha}} = D$. The second observation we need is that $\boldsymbol{\alpha}$-derivations vanish on $M^2_{\boldsymbol{\alpha}}(A)$. Indeed, given $f, g \in M_{\boldsymbol{\alpha}}(A)$ we have $D(fg) = f(\boldsymbol{\alpha})D(g) + g(\boldsymbol{\alpha})D(f) = 0$. It follows that $D(f) = D(g)$ whenever $f - g \in M^2_{\boldsymbol{\alpha}}(A)$ and there exist a well-defined linear functional $D' : M_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A) \to \mathbb{K}$ such that $D'(z_{\boldsymbol{\alpha}}(f) + M^2_{\boldsymbol{\alpha}}(A)) = D(z_{\boldsymbol{\alpha}}(f)) = D(f)$. In other words, we have $D = D' \circ \varphi_{\boldsymbol{\alpha}}$ where D' is a linear functional $D' : M_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A) \to \mathbb{K}$, and we've verified that all $\boldsymbol{\alpha}$-derivations can be written in the desired way. It remains to verify that any composition $L \circ \varphi_{\boldsymbol{\alpha}}$ is an $\boldsymbol{\alpha}$-derivation.

To see this fact, let L be a linear functional $M_{\boldsymbol{\alpha}}(A)/M_{\boldsymbol{\alpha}}^2(A)$ and $f, g \in A$. Then we can write

$$\begin{aligned} f &= f_1 + f_2 + a, \\ g &= g_1 + g_2 + b \end{aligned}$$

where $f_1, g_1 \in M_{\boldsymbol{\alpha}}(A), f_2, g_2 \in M_{\boldsymbol{\alpha}}^2(A), a, b \in \mathbb{K}$. Our construction implies that $f(\boldsymbol{\alpha}) = a, g(\boldsymbol{\alpha}) = b$ and $\varphi_{\boldsymbol{\alpha}}(f) = f_1 + M_{\boldsymbol{\alpha}}^2(A), \varphi_{\boldsymbol{\alpha}}(g) = g_1 + M_{\boldsymbol{\alpha}}^2(A)$. It follows that

$$\begin{aligned} L \circ \varphi_{\boldsymbol{\alpha}}(fg) &= L \circ \phi_{\boldsymbol{\alpha}} \circ z_{\boldsymbol{\alpha}}\left((f_1 + f_2 + a)(g_1 + g_2 + b)\right) \\ &= L \circ \phi_{\boldsymbol{\alpha}} \circ z_{\boldsymbol{\alpha}}\left(f_1g_1 + f_1g_2 + bf_1 + f_2g_1 + f_2g_2 + bf_2 + ag_1 + ag_2 + ab\right) \\ &= L \circ \phi_{\boldsymbol{\alpha}}\left(f_1g_1 + f_1g_2 + bf_1 + f_2g_1 + f_2g_2 + bf_2 + ag_1 + ag_2\right) \\ &= L\left(bf_1 + ag_1 + M_{\boldsymbol{\alpha}}^2(A)\right) \\ &= bL\left(f_1 + M_{\boldsymbol{\alpha}}^2(A)\right) + aL\left(g_1 + M_{\boldsymbol{\alpha}}^2(A)\right) \\ &= g(\boldsymbol{\alpha})L \circ \varphi_{\boldsymbol{\alpha}}(f) + f(\boldsymbol{\alpha})L \circ \varphi_{\boldsymbol{\alpha}}(g), \end{aligned}$$

and $L \circ \varphi_{\boldsymbol{\alpha}}$ is an $\boldsymbol{\alpha}$-derivation.

We summarize our result as a Theorem.

Theorem 20. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension and $\boldsymbol{\alpha} \in \mathbb{K}^n$. Then

$$(M_{\boldsymbol{\alpha}}(A)/M_{\boldsymbol{\alpha}}^2(A))^* \xrightarrow[\sim]{\circ\phi_{\alpha}} \mathcal{D}_{\boldsymbol{\alpha}}(A)$$

where $(M_{\boldsymbol{\alpha}}(A)/M_{\boldsymbol{\alpha}}^2(A))^*$ denotes the dual of $M_{\boldsymbol{\alpha}}(A)/M_{\boldsymbol{\alpha}}^2(A)$.

This is one of the best tools we have to understand $\mathcal{D}_{\boldsymbol{\alpha}}(A)$. Unfortunately, it is still very difficult to understand $M_{\boldsymbol{\alpha}}^2(A)$, which will be one of our biggest obstacles in making sense of the $\boldsymbol{\alpha}$-derivation space of A.

We also have the following connection to SAGBI bases.

Theorem 21. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension and G a SAGBI basis for A inside $M_{\boldsymbol{\alpha}}(A)$. Then

$$\operatorname{span}\left\{g + M_{\boldsymbol{\alpha}}^2(A) : g \in G\right\} = M_{\boldsymbol{\alpha}}(A)/M_{\boldsymbol{\alpha}}^2(A)$$

Proof. Let $f \in M_{\boldsymbol{\alpha}}(A)$. Then we can subduce f by G and write f as a polynomial in G_{mon}. Any non-linear terms of the polynomial will lie in $M_{\boldsymbol{\alpha}}^2(A)$ and we won't require a constant term since $G \subset M_{\boldsymbol{\alpha}}(A)$ and $f \in M_{\boldsymbol{\alpha}}(A)$. It follows that $f \in \operatorname{span}\left\{g + M_{\boldsymbol{\alpha}}^2(A) : g \in G\right\}$ and we are done. □

Note that the previous theorem and proof would remain valid even if G is just a normal algebra basis. The previous theorem along with the fact that any finite codimensional subalgebra admits a finite SAGBI basis yields the following corollary.

Corollary 22. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. Then $\dim \mathcal{D}_{\boldsymbol{\alpha}}(A)$ is finite for all $\boldsymbol{\alpha} \in \mathbb{K}$ and if G is a generating set for A, then we have the bound

$$\dim \mathcal{D}_{\boldsymbol{\alpha}}(A) \leq |G|.$$

Moreover, we claim that codim $M^2_{\boldsymbol{\alpha}}(A)$ is finite as well. If it were not, then $\dim M_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A)$ would be infinite since dim $M_{\boldsymbol{\alpha}}(A)$ is infinite, which contradicts the previous corollary. Thus we get the following corollary as well.

Corollary 23. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. Then codim $M^2_{\boldsymbol{\alpha}}(A)$ is finite for all $\boldsymbol{\alpha} \in \mathbb{K}$.

4.5 Gorin's Result

We will now give one of the pillars upon which this theory rests. A result which implies that any finite codimensional subalgebra of $\mathbb{K}[\boldsymbol{x}]$ can be described as the kernel of a set of equality conditions and $\boldsymbol{\alpha}$-derivations (where in this sentence, $\boldsymbol{\alpha}$ is not a set scalar, but just part of the name, I.e $\boldsymbol{\alpha}$ can vary between the conditions). A more general version of the following result is given in [1].

Theorem 24. Let $A \subseteq \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. Then $A \subset B$ where codim $B =$ codim $A - 1$ and $A = \ker L \cap B$ where L is either an evaluation subtraction or $\boldsymbol{\alpha}$-derivation over B.

Proof. The formulation above differs slightly from that in [1]. In [1], we are given that L is either of the form $L(f) = \varphi(f) - \varphi(g)$ for some algebra homomorphism $\varphi : A \to \mathbb{K}$, or L is a linear function $A \to \mathbb{K}$ such that $L(fg) = \varphi(f)L(g) + \varphi(g)L(f)$. In [1], we also learn that any such homomorphism φ can be lifted to $B \to \mathbb{K}$, after which induction over the codimension on A yields that φ can be extended to a homomorphism $\mathbb{K}[\boldsymbol{x}] \to \mathbb{K}$, whence φ must be an evaluation, giving us the formulation above. ☐

It follows that any subalgebra $A \subset \mathbb{K}[\boldsymbol{x}]$ can be written as

$$A = \bigcap_{L \in \mathcal{L}} \ker L$$

where $\mathcal{L}$ is a set of $\boldsymbol{\alpha}$-derivations and evaluation subtractions.

4.6 Subalgebra Conditions and SAGBI bases

If we have a SAGBI basis G for a finite codimensional algebra $A \subset \mathbb{K}[\boldsymbol{x}]$, and a subalgebra condition L over A, then we can determine a SAGBI basis $\widehat{G}$ for $A \cap \ker L$ as described in the following theorem.

Theorem 25. Let $G = \{g_i : i \in [1..n]\}$ be a ordered SAGBI basis for A and $\widehat{A} = A \cap \ker(L)$ where L is an equality condition or $\boldsymbol{\alpha}$-derivation over A. Let

j be the smallest index such that $L(g_j) \neq 0$. Then a (not necessarily minimal) SAGBI basis for $\widehat{A}$ is given by

$$\widehat{G} = \left\{ g_i - \frac{L(g_i)}{L(g_j)} g_j : i \neq j \right\}$$
$$\cup$$
$$\left\{ g_i g_j - \frac{L(g_i g_j)}{L(g_j)} g_j : i \in [1..n] \right\}$$
$$\cup$$
$$\left\{ g_j^3 - \frac{L(g_j^3)}{L(g_j)} g_j \right\}.$$

Moreover, if G is inside $M_{\boldsymbol{\alpha}}$, then so is $\widehat{G}$.

Proof. First of, for any $f \in A$ we have $f - \frac{L(g)}{L(f)} g \in \widehat{A}$ so $\widehat{G} \subset \widehat{A}$.

Now we show that $\widehat{G}$ is a SAGBI basis. As L is a linear functional, we have $\mathrm{Lm}(A) \setminus \mathrm{Lm}(\widehat{A}) = \mathrm{lm}(g)$. So, let $f \in \widehat{A}$, then $\mathrm{lm}(f) \in \mathrm{Lm}(A) \setminus \mathrm{lm}(g) = \mathrm{Lm}(G)_{\mathrm{mon}} \setminus \{g\}$, and staring at the leading monomials in $\widehat{G}$, we see that any such monomial $\mathrm{lm}(f)$ can be found in $\mathrm{Lm}(\widehat{G})_{\mathrm{mon}}$. $\square$

This procedure can be used to generate examples and is how the examples of A_3 and A_4 from the introduction were produced. The SAGBI basis $\widehat{G}$ can also be pruned with subduction to a minimal SAGBI basis. This is fairly slow though, and there is probably a faster way to prune a $\widehat{G}$ via calculating $T(\widehat{A})$ from $T(A)$ and $\deg(g_j)$, but such an algorithm is outside of the scope of this thesis.

Note that the above theorem when combined with Theorems 20 and 21 gives an upper bound for how much a $\boldsymbol{\alpha}$-derivation space can grow when we apply a condition as,

$$\dim\left(\mathcal{D}_{\boldsymbol{\alpha}}(A \cap \ker \text{Ł})\right) \leq 2|G|,$$

when G is a SAGBI basis for a finite codimensional subalgebra $A \subset \mathbb{K}[\boldsymbol{x}]$. This will not be used a whole bunch, but is useful for intuition.

5 The Spectrum

5.1 Definition and General Results

We will extend the univariate definition of the spectrum and prove that many of the results we know from the univariate case still hold when we consider polynomials in several variables.

Definition 26. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. Then we define the spectrum of A, written $\mathrm{sp}(A) \subset \mathbb{K}^n$, as the set of points $\boldsymbol{\alpha}$ such that either $f'_{\boldsymbol{u}}(\boldsymbol{\alpha}) = 0$ for all $f \in A$ and some $\boldsymbol{u} \in \mathbb{K}^n \setminus \{\mathbf{0}\}$, or there exist some $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$ such that $f(\boldsymbol{\alpha}) = f(\boldsymbol{\beta})$ for all $f \in A$.

Here $f'_{\boldsymbol{u}}$ denotes the directional derivative

$$f'_{\boldsymbol{u}} = \sum_{u_i \in \boldsymbol{u}} u_i f'_{x_i}.$$

We will show that the spectrum is non-empty, but to do this, we first need to classify all derivations on $\mathbb{K}[\boldsymbol{x}]$.

A SAGBI basis for $\mathbb{K}[\boldsymbol{x}]$ inside $M_{\boldsymbol{\alpha}}$ is given by

$$G = \{x_i - \alpha_i : x_i \in \boldsymbol{x}\}.$$

Moreover, these are all linearly independent modulo M_α^2, hence $\dim M_\alpha / M_\alpha^2 = n$. But for each $i \in [1..n]$, we have that $D_i(f) = f'_{x_i}(\boldsymbol{\alpha})$ is an $\boldsymbol{\alpha}$-derivation. Linearity is known, and the derivation condition is the product rule of derivatives. Alternatively, one could verify that D_i is an $\boldsymbol{\alpha}$-derivation by verifying that it's a projection onto $x_i - \alpha_i$. To do this, note that G_{mon}, is an extension of G to a vector space basis of $M_{\boldsymbol{\alpha}}$, and that D_i vanishes on all elements of G_{mon} except $x_i - \alpha_i$ where it attains unit value.

As there are n of these and they're all linearly independent, these form a basis for $\mathcal{D}_{\boldsymbol{\alpha}}$. We summarize the result in a theorem.

Theorem 27. The $\boldsymbol{\alpha}$-derivation space $\mathcal{D}_{\boldsymbol{\alpha}}$ of $\mathbb{K}[\boldsymbol{x}]$ is spanned by the basis $D_i(f) = f'_{x_i}(\boldsymbol{\alpha})$ for $i \in [1..n]$.

Theorems 27 and 24 imply that if A is a non-trivial subalgebra of $\mathbb{K}[\boldsymbol{x}]$ of finite codimension then $\text{sp}(A) \neq \emptyset$, which we summarize in a theorem.

Theorem 28. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a non-trivial subalgebra. Then $\text{sp}(A) \neq \emptyset$.

One natural question is whether the spectrum contains any 'unexpected' elements. It does not, as we see in the following theorem.

Theorem 29. Let $B \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension and consider an algebra A of codimension 1 in B.

- If A is the kernel of some $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction E in B, then $\text{sp}(A) = \text{sp}(B) \cup \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$.
- If A is the kernel of some $\boldsymbol{\alpha}$-derivation D over B, then $\text{sp}(A) = \text{sp}(B) \cup \{\boldsymbol{\alpha}\}$.

Proof. One direction of inclusion is immediate. We will direct our attention to proving that no unexpected elements appear in the spectrum of the derived subalgebra.

Assume there exist some $\boldsymbol{\gamma} \in \mathrm{sp}(A) \setminus \mathrm{sp}(B)$. By the definition of the spectrum we have $L(f) = 0$ for all $f \in A$ and $L(g) \neq 0$ for some $g \in B$ where either $L = E$ is an $\boldsymbol{\gamma}, \boldsymbol{\delta}$-evaluation subtraction or $L = D$ is the $\boldsymbol{\gamma}$-derivation given by $D(f) = f'_{x_i}(\boldsymbol{\gamma})$ for some $x_i \in \boldsymbol{x}$. In either case, we denote the linear functional by L and consider both cases simultaneously.

Combine the two cases of the theorem statement and denote the linear functional that was used to obtain A from B by L'. We have that $\ker L' \subset \ker L$, but as L is non-trivial over B, Lemma 2 yields $\dim \ker L' = \dim \ker L$, and $\ker L' = \ker L$. Applying lemma 3 yields $L' = cL$ from which the theorem statement follows. $\square$

Finally, we define a class of $\boldsymbol{\alpha}$-derivations which we call trivial.

Definition 30. An $\boldsymbol{\alpha}$-derivation over some finite codimensional subalgebra $A \subset \mathbb{K}[\boldsymbol{x}]$ is said to be trivial when $\boldsymbol{\alpha} \notin \mathrm{sp}(A)$.

We call them trivial as they all exhibit the form given in Theorem 27 and therefore $\mathcal{D}_{\boldsymbol{\alpha}}(A)$ is only interesting when $\boldsymbol{\alpha} \in \mathrm{sp}(A)$. This is non-trivial to show however, but we will obtain this result before the end of the next section.

5.2 Clusters

We now define a central equivalence relation on the spectrum.

Definition 31. Two elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{sp}(A)$ are said to be equivalent, written $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if $f(\boldsymbol{\alpha}) = f(\boldsymbol{\beta})$ for all $f \in A$. Moreover, we say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to the same cluster, where the clusters are the parts of $\mathrm{sp}(A)$ resulting from a partitioning by $\sim$.

We are going to need slightly more flexible notation for juggling derivation spaces of multiple subalgebras simultaneously. If $A' \subseteq A$ is a subalgebra, we write $\mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'}$ for the space of functions $\mathcal{D}_{\boldsymbol{\alpha}}(A)$ restricted to elements in A'.

Clusters are nice in that equivalent spectral elements $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ give rise to the same derivation spaces $\mathcal{D}_{\boldsymbol{\alpha}}(A) = \mathcal{D}_{\boldsymbol{\beta}}(A)$.

Lemma 32. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension where $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$. Then

$$\mathcal{D}_{\boldsymbol{\alpha}}(A) = \mathcal{D}_{\boldsymbol{\beta}}(A)$$

Proof. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be cofinite algebra where $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$. Since $f(\boldsymbol{\alpha}) = f(\boldsymbol{\beta})$ for all $f \in A$, if $D : A \to \mathbb{K}$ is a linear functional such that $D(fg) = f(\boldsymbol{\alpha})D(g) + D(f)g(\boldsymbol{\alpha})$ then $D(fg) = f(\boldsymbol{\beta})D(g) + D(f)g(\boldsymbol{\beta})$. $\square$

Moreover, these spaces are disjoint as seen in the following lemma.

Lemma 33. Let $A \subset \mathbb{K}[x]$ be a subalgebra of finite codimension where $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$. Also let D_1 be a non-zero $\boldsymbol{\alpha}$-derivation and D_2 be a $\boldsymbol{\beta}$-derivation. Then $D_1 \neq D_2$. Even more, denote $A' = A \cap \ker f \to f(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})$. Then $D_1\big|_{A'} \neq D_2\big|_{A'}$.

Proof. Aiming towards a contradiction, assume $D_1 = D_2$ is non-trivial. By hypothesis there exist some $f \in A$ such that $f(\boldsymbol{\alpha}) \neq f(\boldsymbol{\beta})$. Then we have

$$\begin{aligned} D_1 = D_2 &\Rightarrow D_1(f^2) = D_2(f^2) \\ &\Rightarrow 2f(\boldsymbol{\alpha})D_1(f) = 2f(\boldsymbol{\beta})D_2(f) \\ &\Rightarrow D_1(f) = D_2(f) = 0 \end{aligned}$$

Also, as $D_1 = D_2$ is non-trivial so there exist some $g \in A$ such that $D_1(g) = D_2(g) = 1$. Then

$$D_1(fg) = D_1(g)f(\boldsymbol{\alpha}) + D_1(f)g(\boldsymbol{\beta}) = f(\boldsymbol{\alpha}),$$

but by similar calculation we also get $D_1(fg) = D_2(fg) = f(\boldsymbol{\beta})$, a contradiction.

For the second statement, assume towards a contradiction that $D_1\big|_{A'} = D_2\big|_{A'}$. We consider a polynomial $f \in M_{\boldsymbol{\beta}}(A) \setminus A'$ and denote $f(\boldsymbol{\alpha}) = a$. Note that $f(\boldsymbol{\beta}) = 0$ as $f \in M_{\boldsymbol{\beta}}(A)$, and $a \neq 0$ as $f \notin A'$. Then let $h_k = f^k - a^{k-1}f$ for $k > 1$ and note that $h_k \in A'$. As D_1 and D_2 coincides here, we get

$$\begin{aligned} 0 &= D_1(h_k) - D_2(h_k) \\ &= D_1(f^k - a^{k-1}f) - D_2(f^k - a^{k-1}f) \\ &= ka^{k-1}D_1(f) - a^{k-1}D_1(f) - k0^{k-1}D_2(f) + a^{k-1}D_2(f) \\ &= a^{k-1}((k-1)D_1(f) + D_2(f)). \end{aligned}$$

As this needs to hold for all $k > 1$, we see that $D_1(f) = D_2(f) = 0$. But f was an arbitrary element in $M_{\boldsymbol{\beta}}(A) \setminus A'$, and as $D_1 = D_2$ on A' by assumption, we see that $D_1 = D_2$ on all of $M_{\boldsymbol{\beta}}(A)$, and in turn all of A since derivations vanish on scalars. This contradicts the first statement. □

The goal of this section is to investigate what happens to $\mathcal{D}_{\boldsymbol{\gamma}}$ when we kernel with $\boldsymbol{\alpha}$-derivations and $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtractions where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ lie outside the cluster of $\boldsymbol{\gamma}$. We will see that $\mathcal{D}_{\boldsymbol{\gamma}}$ remains unchanged and significant corrolaries will follow from this. After this we will also investigate what happens when we kernel by $\boldsymbol{\alpha}, \boldsymbol{\gamma}$-evaluation subtractions, I.e when we merge the cluster of $\boldsymbol{\gamma}$ with that of $\boldsymbol{\alpha}$. We will see that when we merge two clusters in such a way, the resulting derivation space is the direct sum of the $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$-derivation spaces of the original subalgebra.

Let's begin with some lemmas!

Lemma 34. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. If $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$ over A and E is a $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction over A, then E is not a $\boldsymbol{\gamma}$-derivation for any $\boldsymbol{\gamma}$.

Proof. By hypothesis, we can find $f \in A$ such that $f(\boldsymbol{\alpha}) = 0, f(\boldsymbol{\beta}) = -1$.

Assume towards a contradiction that E is a $\boldsymbol{\gamma}$-derivation. Then

$$\begin{aligned} 1 &= f(\boldsymbol{\beta})^2 \\ &= E(f^2) \\ &= 2f(\boldsymbol{\gamma})E(f) \\ &= -2f(\boldsymbol{\gamma}), \end{aligned}$$

but also

$$\begin{aligned} -1 &= f(\boldsymbol{\beta})^3 \\ &= E(f^3) \\ &= 3f(\boldsymbol{\gamma})^2E(f) \\ &= -3f(\boldsymbol{\gamma})^2, \end{aligned}$$

and we reach a contradiction since these two equations can't both simultaneously be true. □

Lemma 35. Let A be a subalgebra of $\mathbb{K}[\boldsymbol{x}]$ where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{K}^n$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are all pairwise mutually inequivalent elements. Then if E_1 is an $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction, and E_2 is an $\boldsymbol{\gamma}, \boldsymbol{\delta}$-evaluation subtraction, we have $E_1 \neq E_2$ over A.

Proof. Assume towards a contradiction that $E_1 = E_2$ over A. Let $f \in A$ such that $f(\boldsymbol{\alpha}) = 0 \neq f(\boldsymbol{\beta})$. Then we have

$$f(\boldsymbol{\alpha}) + f(\boldsymbol{\delta}) = f(\boldsymbol{\gamma}) + f(\boldsymbol{\beta})$$

as $E_1(f) = E_2(f)$, and

$$f^2(\boldsymbol{\alpha}) + f^2(\boldsymbol{\delta}) = f^2(\boldsymbol{\gamma}) + f^2(\boldsymbol{\beta})$$

as $E_1(f^2) = E_2(f^2)$. But then it follows that

$$\begin{aligned} (f(\boldsymbol{\alpha}) + f(\boldsymbol{\delta}))^2 - f^2(\boldsymbol{\alpha}) + f^2(\boldsymbol{\delta}) &= (f(\boldsymbol{\gamma}) + f(\boldsymbol{\beta}))^2 - f^2(\boldsymbol{\gamma}) + f^2(\boldsymbol{\beta}) \\ &\Rightarrow \\ 2f(\boldsymbol{\alpha})f(\boldsymbol{\delta}) &= 2f(\boldsymbol{\gamma})f(\boldsymbol{\beta}) \\ &\Rightarrow \\ f(\boldsymbol{\gamma}) &= 0, \end{aligned}$$

but this means that $M_{\boldsymbol{\alpha}}(A) = M_{\boldsymbol{\gamma}}(A)$ and in turn $\boldsymbol{\alpha} \sim \boldsymbol{\gamma}$ in A, contradicting our hypothesis. □

Lemma 36. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension, L a subalgebra condition over A, and $A' = A \cap \ker L$. If L is not an $\boldsymbol{\alpha}$-derivation, then

$$\dim \mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'} = \dim \mathcal{D}_{\boldsymbol{\alpha}}(A)$$

Proof. Let $D_1, D_2, \dots D_N \in \mathcal{D}_{\boldsymbol{\alpha}}(A)$ be a vector space basis for $\mathcal{D}_{\boldsymbol{\alpha}}(A)$. Assume towards a contradiction that the D_i admit a non-trivial linear dependency when restricted to A',

$$0 = \sum_{i=1}^{N} a_i D_i\big|_{A'}.$$

Then by Lemma 4, we have

$$\sum_{i=1}^{N} a_i D_i = L$$

but the expression on the left is a non-trivial $\boldsymbol{\alpha}$-derivation, and the expression on the right is either an evaluation subtraction or a $\boldsymbol{\beta}$-derivation for some $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$. This contradicts either Lemma 33 or 34. ☐

We can use the previous lemma along with Theorem 20 to prove a lemma which will be our main tool for this section.

Lemma 37. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension, and L a subalgebra condition over A which is not an $\boldsymbol{\alpha}$-derivation. Denote $A' = A \cap \ker L$. Then

$$\mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'} \subseteq \mathcal{D}_{\boldsymbol{\alpha}}(A'),$$

$$M^2_{\boldsymbol{\alpha}}(A') \subset M^2_{\boldsymbol{\alpha}}(A),$$

and

$$\dim\left(\mathcal{D}_{\boldsymbol{\alpha}}(A')/\mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'}\right) = \dim\left(M^2_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A')\right) - 1.$$

Proof. The first statement follows directly from the fact that any $\boldsymbol{\alpha}$-derivation over A is a $\boldsymbol{\alpha}$-derivation over any subalgebra of A. The second statement follows from the fact that $A' \subset A$, and $A = M_{\boldsymbol{\alpha}}(A) \oplus \mathbb{K}, A' = M_{\boldsymbol{\alpha}}(A') \oplus \mathbb{K}$, whence $M_{\boldsymbol{\alpha}}(A') \subset M_{\boldsymbol{\alpha}}(A)$.

For the third statement, Lemma 36 and Theorem 20 yields

$$\begin{aligned}
\dim\left(\mathcal{D}_{\boldsymbol{\alpha}}(A')/\mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'}\right) &= \dim(\mathcal{D}_{\boldsymbol{\alpha}}(A')) - \dim(\mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'}) \\
&= \dim(\mathcal{D}_{\boldsymbol{\alpha}}(A')) - \dim(\mathcal{D}_{\boldsymbol{\alpha}}(A)) \\
&= \dim\left((M_{\boldsymbol{\alpha}}(A')/M^2_{\boldsymbol{\alpha}}(A'))\right) - \dim\left((M_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A))\right) \\
&= \mathrm{codim}(M^2_{\boldsymbol{\alpha}}(A')) - \mathrm{codim}(M_{\boldsymbol{\alpha}}(A')) \\
&\quad - \mathrm{codim}(M^2_{\boldsymbol{\alpha}}(A)) + \mathrm{codim}(M_{\boldsymbol{\alpha}}(A)) \\
&= \mathrm{codim}(M^2_{\boldsymbol{\alpha}}(A')) - \mathrm{codim}(M^2_{\boldsymbol{\alpha}}(A)) - 1 \\
&= \dim(M^2_{\boldsymbol{\alpha}}(A)/M^2_{\boldsymbol{\alpha}}(A')) - 1,
\end{aligned}$$

where we used the fact that $\mathrm{codim} M_{\boldsymbol{\alpha}}(A) = \mathrm{codim} M_{\boldsymbol{\alpha}}(A') - 1$ follows from L being a linear functional. ☐

We can specify the previous lemma further. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension and consider two polynomials $f_1, f_2 \in M_{\boldsymbol{\alpha}}(A)$. Also, let g be a polynomial in $M_{\boldsymbol{\alpha}}(A) \setminus A' = M_{\boldsymbol{\alpha}}(A) \setminus M_{\boldsymbol{\alpha}}(A')$.Then $M_{\boldsymbol{\alpha}}(A) = M_{\boldsymbol{\alpha}}(A') + g\mathbb{K}$ since codim $M_{\boldsymbol{\alpha}}(A) =$ codim $M_{\boldsymbol{\alpha}}(A') + 1$, and we can write $f_i = h_i + a_i g$ for some $h_i \in M_{\boldsymbol{\alpha}}(A')$ and $a_i \in \mathbb{K}$. Moreover,

$$f_1 f_2 = h_1 h_2 + (a_2 h_1 + a_1 h_2 + a_1 a_2 g)g,$$

and any polynomial in $M^2_{\boldsymbol{\alpha}}(A)$ is a linear combination of such products. Thus any polynomial in $M^2_{\boldsymbol{\alpha}}(A)$ is congruent to a product fg for some $f \in M_{\boldsymbol{\alpha}}(A)$ when taken modulo $M^2_{\boldsymbol{\alpha}}(A')$. We summarize our result.

Lemma 38. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension, and L a subalgebra condition over A which is not an $\boldsymbol{\alpha}$-derivation. Let g be a polynomial in $M_{\boldsymbol{\alpha}}(A) \setminus A'$. Denote $A' = A \cap \ker L$. Then

$$\dim \left(\mathcal{D}_{\boldsymbol{\alpha}}(A') / \mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'} \right) = \dim \left(\left(g M_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A') \right) / M^2_{\boldsymbol{\alpha}}(A') \right) - 1.$$

Proof. From Lemma 37 we know that

$$\dim \left(\mathcal{D}_{\boldsymbol{\alpha}}(A') / \mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'} \right) = \dim \left(M^2_{\boldsymbol{\alpha}}(A) / M^2_{\boldsymbol{\alpha}}(A') \right) - 1,$$

so our goal is to show that

$$M^2_{\boldsymbol{\alpha}}(A) / M^2_{\boldsymbol{\alpha}}(A') = \left(g M_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A') \right) / M^2_{\boldsymbol{\alpha}}(A')$$

but this follows immediately from the fact that any $h_i \in M^2_{\boldsymbol{\alpha}}(A)$ is congruent to some $f_i g$ modulo $M^2_{\boldsymbol{\alpha}}(A')$ and that $g M_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A') \subseteq M^2_{\boldsymbol{\alpha}}(A)$. ☐

The following lemma will also prove useful.

Lemma 39. Let $A, B \subset \mathbb{K}[\boldsymbol{x}]$ be vector spaces of polynomials of finite codimension such that $B \subseteq A$. Then

$$\dim(A/B) = |\mathrm{Lm}(A) \setminus \mathrm{Lm}(B)|$$

Proof. Let $F \subset A$ be a set of polynomials such that $\mathrm{Lm}(F) = \mathrm{Lm}(A) \setminus \mathrm{Lm}(B)$ and all leading monomials of the elements $f_i \in F$ are unique. Now let $f \in A$. Either $f \in B$ or $\mathrm{lm}(f) \in \mathrm{Lm}(F)$. Either way, denote the polynomial with the same leading monomial as f by g_1 and let $c_1 \in \mathbb{K}$ be such that $\mathrm{lt}(f) = \mathrm{lt}(c_1 g_1)$. Then $\mathrm{lm}(f - c_1 g_1) < \mathrm{lm}(f)$. Restart the same process with $f - c_1 g_1$ and denote the polynomial with the same leading term by g_2. We can continue in this way until we get

$$f = \sum_{i=1}^{m} c_i g_i$$

and since each $g_i \in B \cup F$, it follows that

$$A = B \oplus \mathrm{span}(F)$$

(where the sum is direct since $\mathrm{Lm}(F) \cap \mathrm{Lm}(B) = \emptyset \Rightarrow F \cap B = \emptyset$) and $A/B = \mathrm{span}(F)$. Since the polynomials of F have unique leading terms we have both $|F| = \dim \mathrm{span}(F)$ and $|F| = |\mathrm{Lm}(A) \setminus \mathrm{Lm}(B)|$, after which we are done. ☐

We are now ready to prove the first of two significant results of this section.

Theorem 40. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension where $\boldsymbol{\gamma} \not\sim \boldsymbol{\alpha}, \boldsymbol{\beta}$. Let L be an $\boldsymbol{\alpha}$-derivation or an $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction, and $A' = A \cap \ker\, L$. Then

$$\mathcal{D}_{\boldsymbol{\gamma}}(A') = \mathcal{D}_{\boldsymbol{\gamma}}(A)\big|_{A'}$$

Proof. Let $g \in M_{\boldsymbol{\alpha}}(A) \setminus M_{\boldsymbol{\alpha}}(A')$ be minimal with respect to the condition that $L(g) = 1$. Then $g(\boldsymbol{\alpha}) = 0$ and $g(\boldsymbol{\beta}) = -1$. By Lemmas 38 and 39, we will be done if we can show that

$$\left|\mathrm{Lm}(gM_{\boldsymbol{\gamma}}(A)) \setminus \mathrm{Lm}(M_{\boldsymbol{\gamma}}^2(A') \cap gM_{\boldsymbol{\gamma}}(A))\right| \leq 1.$$

Let B' be a vector space basis for $M_{\boldsymbol{\alpha}}(A)$ that contains g and where no polynomials in B' have the same leading monomials. Note that if $f \in B'$, then $f - L(f)g \in M_{\boldsymbol{\alpha}}(A')$ and $\mathrm{lm}(f - L(f)g) = \mathrm{lm}(f)$ since if $\mathrm{lm}(g) > \mathrm{lm}(f)$ we have $L(f) = 0$ by definition of g. Use B' to construct

$$B = \{f - L(f)g : f \in B' \setminus \{g\}\} \cup \{g\}.$$

Then $B \subset M_{\boldsymbol{\alpha}}(A)$ is a vector space basis where all elements have unique leading terms and $B \setminus \{g\} \subset M_{\boldsymbol{\alpha}}(A')$ and $\mathrm{Lm}(B) = \mathrm{Lm}(A)$.

Instead of splitting in to cases depending on whether L is an $\boldsymbol{\alpha}$-derivation or $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction, we just notice that for arbitrary $f_1, f_2 \in A$ we have in the former case

$$L(f_1 f_2) = f_1(\boldsymbol{\alpha})L(f_2) + f_2(\boldsymbol{\alpha})L(f_1),$$

and in the latter case

$$L(f_1 f_2) = f_1(\boldsymbol{\alpha})L(f_2) + f_2(\boldsymbol{\beta})L(f_1).$$

Thus, we can unify our two cases by assuming that

$$L(f_1 f_2) = f_1(\boldsymbol{\alpha})L(f_2) + f_2(\boldsymbol{\beta})L(f_1),$$

and let the $\boldsymbol{\beta} = \boldsymbol{\alpha}$ case correspond to when L is an $\boldsymbol{\alpha}$-derivation.

We claim that there exist $h \in B \setminus \{g\}$ such that $h(\boldsymbol{\beta}) \neq 0$. First of, at least one element $h \in B$ must satisfy $h(\boldsymbol{\beta}) \neq 0$ as $\boldsymbol{\gamma} \not\sim \boldsymbol{\beta}$ in A. Secondly, if g were the only element in B that didn't vanish under evaluation at $\boldsymbol{\beta}$, then we'd have $\boldsymbol{\gamma} \sim \boldsymbol{\beta}$ in A' as $B \setminus \{g\}$ is a vector space basis for $M_{\boldsymbol{\gamma}}(A')$, which would imply that L is a $\boldsymbol{\gamma}, \boldsymbol{\beta}$-evaluation subtraction by Lemma 3, and this is a contradiction to either Lemma 34 or 35.

Let h be the smallest polynomial in $B \setminus \{g\}$ such that $h(\boldsymbol{\beta}) \neq 0$ and let f be an arbitrary element in $B \setminus \{g, h\}$. Then we have the following inclusions

$$h \in M_{\boldsymbol{\gamma}}(A'),$$
$$f \in M_{\boldsymbol{\gamma}}(A'),$$

and,

$$fg - \frac{L(fg)}{L(g)}g = fg - \frac{L(f)g(\boldsymbol{\alpha}) + L(g)f(\boldsymbol{\beta})}{L(g)}g = fg - f(\boldsymbol{\beta})g \in M_{\boldsymbol{\gamma}}(A'),$$
$$hg - \frac{L(hg)}{L(g)}g = hg - \frac{L(h)g(\boldsymbol{\alpha}) + L(g)h(\boldsymbol{\beta})}{L(g)}g = hg - h(\boldsymbol{\beta})g \in M_{\boldsymbol{\gamma}}(A').$$

Using these inclusions we see that

$$(fg - f(\boldsymbol{\beta})g)\, h = fgh - f(\boldsymbol{\beta})gh \in M^2_{\boldsymbol{\gamma}}(A'),$$
$$(hg - h(\boldsymbol{\beta})g)\, f = fgh - h(\boldsymbol{\beta})gf \in M^2_{\boldsymbol{\gamma}}(A'),$$

but then it follows that

$$f(\boldsymbol{\beta})gh - h(\boldsymbol{\beta})gf \in M^2_{\boldsymbol{\gamma}}(A') \cap gM_{\boldsymbol{\gamma}}(A),$$

and $\mathrm{lm}(f(\boldsymbol{\beta})hg - h(\boldsymbol{\beta})fg) = \mathrm{lm}(fg)$ since either $f > h \Rightarrow fg > hg$ or $f < h$ which means that $f(\boldsymbol{\beta}) = 0$ due to how we picked h. Thus we've show that $\mathrm{Lm}(M^2_{\boldsymbol{\gamma}}(A') \cap gM_{\boldsymbol{\gamma}}(A))$ contains all elements of $\mathrm{Lm}(gM_{\boldsymbol{\gamma}}(A))$ except possibly $\mathrm{lm}(gh), \mathrm{lm}(g^2)$. We can find the last missing monomial in a similar manner as above by noting that $g^2 - (g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))g \in M_{\boldsymbol{\gamma}}(A')$ and

$$h^2(g^2 - (g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))g) = h^2g^2 - (g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))h^2g \in M^2_{\boldsymbol{\gamma}}(A')$$
$$(hg - h(\boldsymbol{\beta})g)^2 = h^2g^2 - 2h(\boldsymbol{\beta})hg^2 + h^2(\boldsymbol{\beta})g^2 \in M^2_{\boldsymbol{\gamma}}(A'),$$

so

$$2h(\boldsymbol{\beta})hg^2 - (g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))h^2g - h^2(\boldsymbol{\beta})g^2 \in M^2_{\boldsymbol{\gamma}}(A').$$

We also have

$$h(hg - h(\boldsymbol{\beta})g) = h^2g - h(\boldsymbol{\beta})hg \in M^2_{\boldsymbol{\gamma}}(A'),$$
$$h(g^2 - (g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))g) = hg^2 - (g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))hg \in M^2_{\boldsymbol{\gamma}}(A'),$$

and combing the results we see that

$$2h(\boldsymbol{\beta})(g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))hg - h(\boldsymbol{\beta})(g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))hg - h^2(\boldsymbol{\beta})g^2 =$$
$$h(\boldsymbol{\beta})(g(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}))hg - h^2(\boldsymbol{\beta})g^2 \in M^2_{\boldsymbol{\gamma}}(A') \cap gM_{\boldsymbol{\gamma}}(A)$$

This yields our remaining leading term and we see that

$$\dim\big(\mathrm{Lm}(gM_{\boldsymbol{\gamma}}(A)) \setminus \mathrm{Lm}(M^2_{\boldsymbol{\gamma}}(A') \cap gM_{\boldsymbol{\gamma}}(A))\big) \leq 1$$

and we are done. □

Many interesting corrolaries follow from this. If we combine it with Theorem 27, then we immediately get the following promised result regarding trivial derivations.

Corollary 41. Let $A \subset \mathbb{K}[x]$ be a subalgebra of finite codimension such that $\boldsymbol{\alpha} \notin \mathrm{sp}(A)$. Then $\mathcal{D}_{\boldsymbol{\alpha}}$ is spanned by the $\boldsymbol{\alpha}$-derivations

$$D_i(f) = f'_{x_i}(\boldsymbol{\alpha})$$

for $x_i \in \boldsymbol{x}$.

Now we give the second of our two main results.

Theorem 42. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension where $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$. Let E be an $\boldsymbol{\alpha}, \boldsymbol{\beta}$-evaluation subtraction and denote $A' = A \cap \ker E$. Then

$$\mathcal{D}_{\boldsymbol{\alpha}}(A') = \mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'} \oplus \mathcal{D}_{\boldsymbol{\beta}}(A)\big|_{A'}$$

Proof. Let $g \in M_{\boldsymbol{\alpha}}(A) \setminus M_{\boldsymbol{\alpha}}(A')$ be minimal with respect to the condition that $E(g) = 1$. Then $g(\boldsymbol{\alpha}) = 0$ and $g(\boldsymbol{\beta}) = -1$.

Most of our efforts in this proof will be devoted to showing that

$$\dim\left(\left(gM_{\boldsymbol{\alpha}}(A') + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) \leq \dim\left(M_{\boldsymbol{\beta}}(A)/M^2_{\boldsymbol{\beta}}(A)\right), \qquad (1)$$

and we first explain how proving this identity proves the theorem statement. As codim $M_{\boldsymbol{\alpha}}(A) =$ codim $M_{\boldsymbol{\alpha}}(A') - 1$, we have that either

$$\dim\left(\left(gM_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) - 1 = \dim\left(\left(gM_{\boldsymbol{\alpha}}(A') + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right),$$

or

$$\dim\left(\left(gM_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) = \dim\left(\left(gM_{\boldsymbol{\alpha}}(A') + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right),$$

which when combined with Equation 1, would yield either

$$\dim\left(\left(gM_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) - 1 \leq \dim\left(M_{\boldsymbol{\beta}}(A)/M^2_{\boldsymbol{\beta}}(A)\right) \qquad (2)$$

or

$$\dim\left(\left(gM_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) \leq \dim\left(M_{\boldsymbol{\beta}}(A)/M^2_{\boldsymbol{\beta}}(A)\right). \qquad (3)$$

But we already know that all $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-derivations over A will be $\boldsymbol{\alpha}$-derivations over A'. Moreover, Lemma 33 tells us that these won't coincide when restricted to A' so

$$\mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'} \oplus \mathcal{D}_{\boldsymbol{\beta}}(A)\big|_{A'} \subseteq \mathcal{D}_{\boldsymbol{\alpha}}(A').$$

Hence

$$\begin{aligned}
\dim\left(M_{\boldsymbol{\beta}}(A)/M^2_{\boldsymbol{\beta}}(A)\right) &= \dim \mathcal{D}_{\boldsymbol{\beta}}(A) \\
&\leq \dim\left(\mathcal{D}_{\boldsymbol{\alpha}}(A')/\mathcal{D}_{\boldsymbol{\alpha}}(A)\big|_{A'}\right) \\
&= \dim\left(\left(gM_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) - 1.
\end{aligned}$$

Combining this inequality with those of Equation 2 and 3 yields that either

$$\begin{aligned}
\dim\left(\left(gM_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) &\leq \dim\left(M_{\boldsymbol{\beta}}(A)/M^2_{\boldsymbol{\beta}}(A)\right) \\
&\leq \dim\left(\left(gM_{\boldsymbol{\alpha}}(A) + M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right) - 1,
\end{aligned}$$

or

$$
\begin{aligned}
\dim\left(\left(gM_{\boldsymbol{\alpha}}(A)+M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right)-1 &\le \dim\left(M_{\boldsymbol{\beta}}(A)/M^2_{\boldsymbol{\beta}}(A)\right)\\
&\le \dim\left(\left(gM_{\boldsymbol{\alpha}}(A)+M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right)-1\\
&\Rightarrow\\
\dim\left(\left(gM_{\boldsymbol{\alpha}}(A)+M^2_{\boldsymbol{\alpha}}(A')\right)/M^2_{\boldsymbol{\alpha}}(A')\right)-1 &= \dim\left(M_{\boldsymbol{\beta}}(A)/M^2_{\boldsymbol{\beta}}(A)\right),
\end{aligned}
$$

holds. As the first of the resulting inequalities is contradictory, we see that the situation in Equation 3 is impossible and it must be that Equation 2 holds all the time. Combining our results with Lemma 38 and Theorem 20, gives us our theorem statement. Thus, we now direct our efforts to showing that Equation 1 holds.

Before we start, note that $M_{\boldsymbol{\alpha}}(A') = M_{\boldsymbol{\beta}}(A')$ since $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ in A' (but not in A of course). These different ways of writing the same space will be used interchangeably throughout the proof.

Let B' be a vector space basis for $M_{\boldsymbol{\alpha}}(A)$ that contains g and where no polynomials in B' have the same leading monomials. Note that if $f \in B'$, then $f - E(f)g \in M_{\boldsymbol{\alpha}}(A')$ and $\mathrm{lm}(f - E(f)g) = \mathrm{lm}(f)$ since if $\mathrm{lm}(g) > \mathrm{lm}(f)$ we have $E(f) = 0$ by definition of g. Use B' to construct

$$B = \{f - E(f)g : f \in B' \setminus \{g\}\} \cup \{g\}.$$

Then $B \subset M_{\boldsymbol{\alpha}}(A)$ is a vector space basis where all elements have unique leading terms, and $B \setminus \{g\} \subset M_{\boldsymbol{\alpha}}(A')$, and $\mathrm{Lm}(B) = \mathrm{Lm}(A)$. Let $\widehat{B} = B \cup \{\widehat{g} = g - g(\boldsymbol{\beta})\} \setminus \{g\}$. Then $\widehat{g}(\boldsymbol{\alpha}) = 1, \widehat{g}(\boldsymbol{\beta}) = 0$ and $\widehat{B}$ is a vector space basis for $M_{\boldsymbol{\beta}}(A)$.

Let $F \subset M_{\boldsymbol{\alpha}}(A')$ be a finite set of polynomials such that gF is a linearly independent set modulo $M^2_{\boldsymbol{\alpha}}(A')$. As $F \subset M_{\boldsymbol{\beta}}(A)$ as well (since $M_{\boldsymbol{\alpha}}(A') \subset M_{\boldsymbol{\beta}}(A)$), we are done if we can show that F is linearly independent modulo $M^2_{\boldsymbol{\beta}}(A)$.

We consider a linear relation among F modulo $M^2_{\boldsymbol{\beta}}(A)$. Assume towards a contradiction that there exist a set of scalars $\{a_i \in \mathbb{K} : f_i \in F\}$ such that some $a_i \neq 0$ and

$$\sum_{f_i \in F} a_i f_i = \sum_{(h_i, H_i) \in H} b_i h_i H_i$$

for some $H \subset \widehat{B} \times \widehat{B}$, and scalars $b_i \in \mathbb{K}$. Denote $p = \sum_{(h_i,H_i)\in H} b_i h_i H_i$. Then

$$g \sum_{f_i \in F} a_i f_i = gp,$$

and it follows that $gp \notin M^2_{\boldsymbol{\alpha}}(A')$ since gF is linearly independent modulo $M^2_{\boldsymbol{\alpha}}(A')$. I.e, we have p such that $p \in M^2_{\boldsymbol{\beta}}(A)$ but $gp \notin M^2_{\boldsymbol{\beta}}(A')$. Now consider each $h_i, H_i \in H \setminus (\widehat{g}, \widehat{g})$, then either h_i or H_i is not equal to $\widehat{g}$, say h_i, whence $h_i \in M_{\boldsymbol{\beta}}(A')$, and

$$gH_i - \frac{E(gH_i)}{E(\widehat{g})}\widehat{g} \in M_{\boldsymbol{\beta}}(A').$$

But if $H_i \in \widehat{B} \setminus \{\widehat{g}\}$, then $E(H_i) = 0$ together with $H_i \in M_{\boldsymbol{\beta}}(A)$ implies $H_i(\boldsymbol{\alpha}) = H_i(\boldsymbol{\beta}) = 0$, so $E(gH_i) = g(\boldsymbol{\alpha})H_i(\boldsymbol{\alpha}) - g(\boldsymbol{\beta})H_i(\boldsymbol{\beta}) = 0$ and we have $gH_i \in M_{\boldsymbol{\beta}}(A')$ in this case. If instead $H_i = \widehat{g}$, then we have $\widehat{g}(\boldsymbol{\alpha}) = 1, \widehat{g}(\boldsymbol{\beta}) = 0$ and $g(\boldsymbol{\alpha}) = 0, g(\boldsymbol{\beta}) = -1$ whence $E(g\widehat{g}) = 0$ and $gH_i \in M_{\boldsymbol{\beta}}(A')$ as well. Thus we have

$$gh_iH_i \in M^2_{\boldsymbol{\beta}}(A') \text{ for } (h_i, H_i) \in H \setminus (\widehat{g}, \widehat{g})$$

Let $s = 1$ if $(\widehat{g}, \widehat{g}) \in H$ and $s = 0$ otherwise. Then

$$\begin{aligned} gp &= \sum_{(h_i,H_i)\in H} b_i g h_i H_i \\ &= scg\widehat{g}^2 + \sum_{(h_i,H_i)\in H\setminus(\widehat{g},\widehat{g})} b_i g h_i H_i \end{aligned}$$

for some scalar $c \neq 0 \in \mathbb{K}$. We see that $gp \in M^2_{\boldsymbol{\beta}}(A')$ if $s = 0$ so s must be 1. Now note that the sum

$$\sum_{(h_i,H_i)\in H\setminus(\widehat{g},\widehat{g})} b_i g h_i H_i$$

lies in $gM_{\boldsymbol{\alpha}}(A)$, since for each h_i, H_i, either $E(h_i) = 0$ or $E(H_i) = 0$ whence $h_iH_i \in M_{\boldsymbol{\alpha}}(A') \subset M_{\boldsymbol{\alpha}}(A)$. Moreover, $g\widehat{g}^2 \notin gM_{\boldsymbol{\alpha}}(A)$ since $E(\widehat{g}^2) = \widehat{g}(\boldsymbol{\alpha})^2 - \widehat{g}(\boldsymbol{\beta})^2 = 1$, so the sum

$$g\widehat{g}^2 + \sum_{(h_i,H_i)\in H\setminus(\widehat{g},\widehat{g})} b_i g h_i H_i$$

doesn't lie in $gM_{\boldsymbol{\alpha}}(A)$, which is a contradiction to $gF \in gM_{\boldsymbol{\alpha}}(A)$ and we are done. □

6 $\boldsymbol{\alpha}$-Derivations as Derivative Evaluations.

For the univariate case we know that we can write $\boldsymbol{\alpha}$-derivations in A as linear combinations of derivative evaluations in the spectral elements that are equivalent to $\boldsymbol{\alpha}$ in A. Moreover, we saw in Corollary 41 that trivial derivations can be written as evaluations after partial derivatives. The purpose of this section is to explore how these statements generalize to multivariate polynomials.

We begin by describing the derivation spaces of subalgebras of codimension 1.

6.1 $\mathcal{D}_{\boldsymbol{\alpha}}(A)$ When A Has Codimension 1

First we investigate $\mathcal{D}_{\boldsymbol{\alpha}}(A)$ when $A = \mathbb{K}[x] \cap \ker D$ where $D(f) = f'_{\boldsymbol{u}}(\boldsymbol{\alpha})$ is an arbitrary $\boldsymbol{\alpha}$-derivation. According to Theorems 20 and 25, we know that

$$\dim\ \mathcal{D}_{\boldsymbol{\alpha}}(A) \leq 2n,$$

remember that n is the number of indeterminates $n = |\boldsymbol{x}|$. We still have that all linear functionals of the form $f \to f'_{x_i}(\boldsymbol{\alpha})$ form $\boldsymbol{\alpha}$-derivations over A. We have however introduced a linear dependence among them so they span a space of dimension $n-1$.

We can find n more linearly independent derivations by considering

$$f \to f''_{\boldsymbol{uv}}(\boldsymbol{\alpha}) = \sum_{i=0}^{n}\sum_{j=0}^{n} u_i v_j f''_{x_i x_j}(\boldsymbol{\alpha})$$

since

$$(fg)''_{\boldsymbol{uv}} = f''_{\boldsymbol{uv}}(\boldsymbol{\alpha})g(\boldsymbol{\alpha}) + f'_{\boldsymbol{u}}(\boldsymbol{\alpha})g'(\boldsymbol{\alpha})_{\boldsymbol{v}} + f'_{\boldsymbol{v}}(\boldsymbol{\alpha})g'(\boldsymbol{\alpha})_{\boldsymbol{u}} + f(\boldsymbol{\alpha})g''(\boldsymbol{\alpha})_{\boldsymbol{uv}}$$

and

$$f'_{\boldsymbol{u}}(\boldsymbol{\alpha})g'(\boldsymbol{\alpha})_{\boldsymbol{v}} = f'_{\boldsymbol{v}}(\boldsymbol{\alpha})g'(\boldsymbol{\alpha})_{\boldsymbol{u}} = 0$$

for $f, g \in A$.

Finally, we have our last $\boldsymbol{\alpha}$-derivation given by $f \to f^{(3)}_{\boldsymbol{uuu}}$ since

$$(fg)^{(3)}_{\boldsymbol{uuu}} = f^{(3)}_{\boldsymbol{uuu}}g + 3f^{(2)}_{\boldsymbol{uu}}g^{(1)}_{\boldsymbol{u}} + 3f^{(1)}_{\boldsymbol{u}}g^{(2)}_{\boldsymbol{uu}} + fg^{(3)}_{\boldsymbol{uuu}},$$

and

$$3f^{(2)}_{\boldsymbol{uu}}(\boldsymbol{\alpha})g^{(1)}_{\boldsymbol{u}}(\boldsymbol{\alpha}) = 3f^{(1)}_{\boldsymbol{u}}(\boldsymbol{\alpha})g^{(2)}_{\boldsymbol{uu}}(\boldsymbol{\alpha}) = 0$$

for $f, g \in A$.

We have found $2n$ linearly independent $\boldsymbol{\alpha}$-derivations over A and in turn a basis for $\mathcal{D}_{\boldsymbol{\alpha}}(A)$.

Now let $A = \ker E$ where $E(f) = f(\boldsymbol{\alpha}) - f(\boldsymbol{\beta})$. Applying Theorem yields 42

$$\mathcal{D}_{\boldsymbol{\alpha}}(A) = \mathcal{D}_{\boldsymbol{\alpha}}(\mathbb{K}[\boldsymbol{x}])\big|_A \oplus \mathcal{D}_{\boldsymbol{\beta}}(\mathbb{K}[\boldsymbol{x}])\big|_A,$$

after which we can use Corollary 41 to see that $\mathcal{D}_{\boldsymbol{\alpha}}(A)$ is a $2n$-dimensional space consisting of functionals of the form $f \to af'_{\boldsymbol{u}}(\boldsymbol{\alpha}) + bf'_{\boldsymbol{v}}(\boldsymbol{\beta})$.

6.2 Notation and General Leibniz Rule for Directional Derivatives

As we see in the exploration above, we are going to need to deal with higher order directional derivatives. Notation can be a bit involved so we introduce any necessary conventions here.

When $\boldsymbol{u}$ is a vector that is meant to describe a directional derivative, we write $f'_{\boldsymbol{u}}$ and to avoid redundant degrees of freedom, we can assume that $|\boldsymbol{u}| = 1$.

We will use multisets in what follows as partial and directional derivatives can exist with multiplicity, whence sets will not do, and the operators commute on polynomials, so tuples are not quite right either. Multisets will be written with bracket [] notation.

When we have a multiset $U = [\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n]$, and we want to compose the directional derivatives after each other, we write $f_U^{(n)}$. If n is small, we may just write $f^{(n)}_{\boldsymbol{u}_1\boldsymbol{u}_2\dots\boldsymbol{u}_n}$.

The General Leibniz Rule is then given by

$$(fg)_U^{(n)} = \sum_{U' \in \mathcal{P}(U)} f_{U'}^{(|U'|)} g_{U \setminus U'}^{(|U|-|U'|)}$$

where $\mathcal{P}(U)$ is the power-multiset of the multiset U. Note that $\mathcal{P}(U)$ will contain duplicates if U does. We introduce one more notation. We write $\boldsymbol{d}^j$ to be the set of all multisets of combinations of j elements from $\boldsymbol{x} = \{x_1, x_2, \dots x_n\}$ drawn with repetitions. Basically, we use $\boldsymbol{d}^j$ to represent all possible combinations of j pure (as in not directional) partial derivatives. I.e if $n = 3$ then

$$\begin{aligned}\boldsymbol{d}^3 =& \{[x_1, x_1, x_1], [x_1, x_1, x_2], [x_1, x_1, x_3], [x_1, x_2, x_2], [x_1, x_2, x_3],\\ & [x_1, x_3, x_3], [x_2, x_2, x_2], [x_2, x_2, x_3], [x_2, x_3, x_3], [x_3, x_3, x_3], \}\end{aligned}$$

Each $\boldsymbol{d}^j$ will be of size

$$|\boldsymbol{d}^j| = \binom{n+j-1}{n}$$

as any element of $\boldsymbol{d}^j$ corresponds to a non-negative integer solution of $y_1 + y_2 + \dots + y_n = j$ where y_i is the count of x_i in a given $d^j \in \boldsymbol{d}^j$

6.3 Main Theorem of α-Derivations

From the exploration of the codim = 1 situation, it seems reasonable to hypothesize the following generalization of the Main Theorem.

Theorem 43 (Main Theorem of $\boldsymbol{\alpha}$-Derivations)**.** Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a subalgebra of finite codimension. Then there exist some integer N such that any $\boldsymbol{\alpha}$-derivation over A can be written in the form

$$f \to \sum_{\boldsymbol{\alpha}_i \sim \boldsymbol{\alpha}} \sum_{j=1}^{N-1} \sum_{d_k^j \in \boldsymbol{d}^j} c_{i,j,k} f_{d_k^j}^{(j)}(\boldsymbol{\alpha}_i) \tag{4}$$

where each $c_{i,j,k} \in \mathbb{K}$.

Throughout this section, we will assume that A only has one cluster. This will make things a whole lot easier, and we can easily recover the general case via Theorem 40. We will need a few constructions before we are able to prove Theorem 43. We shall just like in the univariate case, show that there exist some $N' \in \mathbb{N}$ such that A contains a subalgebra which is obtained by kerneling by all evaluation subtractions that hold in A, and all possible derivations of the form

$$D(f) = \sum_{\boldsymbol{\alpha}_i \sim \boldsymbol{\alpha}} \sum_{j=1}^{N'-1} \sum_{d_k^j \in \boldsymbol{d}^j} c_{i,j,k} f_{d_k^j}^{(j)}(\boldsymbol{\alpha}_i).$$

for some $N' \in \mathbb{N}$. We will call this subalgebra $Q_{N'}(\text{sp}(A)) \subset \mathbb{K}[\boldsymbol{x}]$ (proper definition below) and it is the generalization of the algebra $\pi_A^N \mathbb{K}[\boldsymbol{x}] + \mathbb{K}$ from the proof of the Main Theorem in the univariate case. For the purpose of being clear, we will introduce this subalgebra and its components within a numbered definition, but note that it won't have much use beyond this section (as of right now).

Definition 44. Let S be a set of spectral elements and $\boldsymbol{\alpha}$ be some element in S. We define $\mathcal{E}(S, \boldsymbol{\alpha})$ to be the set of evaluation subtractions

$$\mathcal{E}(S, \boldsymbol{\alpha}) = \{f \to f(\boldsymbol{\alpha}) - f(\boldsymbol{\beta}) : \boldsymbol{\beta} \in S \setminus \{\boldsymbol{\alpha}\}\}.$$

The purpose of this definition is to have $\mathcal{E}(\text{sp}(A))$ encapsulate an independent but spanning set of evaluation subtractions which hold in A. Therefore we fixed the first evaluation to avoid redundancies, and since we assume A to have a single cluster, the construction above will collect all the evaluation subtractions we need to describe A. For our applications, the actual value of $\boldsymbol{\alpha}$ will never matter, thus we will simply write $\mathcal{E}(S)$.

We also define $\mathcal{D}_N(S)$ to be the set of linear functionals

$$\mathcal{D}_N(S) = \left\{ f \to f_{d_k^j}^{(j)}(\boldsymbol{\alpha}_i) : \boldsymbol{\alpha}_i \in S, j \in [1..N-1], d_k^j \in \boldsymbol{d}^j \right\}.$$

Lastly we use these two sets of conditions to define the subalgebra

$$Q_N(S) = \bigcap_{L \in \mathcal{E}(S) \cup \mathcal{D}_N(S)} \ker\ L.$$

If you don't believe that $Q_N(S)$ is an algebra, just note that $f_{d_k^j}^{(j)}(\boldsymbol{\alpha}_i)$ is a $\boldsymbol{\alpha}_i$-derivation in any algebra where $f_{d_k^l}^{(j)}(\boldsymbol{\alpha}_i) = 0$ for all d_k^l where $l < j$ (this follows from the Leibniz rule). Now imagine a construction of $Q_N(S)$ where we start by kerneling $\mathbb{K}[\boldsymbol{x}]$ with all evaluation subtractions of $\mathcal{E}(S)$ and trivial derivations in elements of S, and work our way up inductively by kerneling by all higher order derivations which are available, until we reach $Q_N(S)$.

Now, as we will soon see, the Main Theorem will follow fairly easily as long as we can show that A contains some $Q_N(\mathrm{sp}(A))$, and that the derivations over any Q_N can be written as in equation 4. I.e we need the following two lemmas.

Lemma 45. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a single cluster subalgebra of finite codimension. Then there exist some $N \in \mathbb{N}$ such that

$$Q_N(\mathrm{sp}(A)) \subseteq A.$$

Lemma 46. Let S be a set of spectral elements, $\boldsymbol{\alpha} \in S$, and $N \in \mathbb{N}$. Then

$$\mathcal{D}_{\boldsymbol{\alpha}}(Q_N(S)) = \langle \mathcal{D}_{2N}(S) \setminus \mathcal{D}_N(S) \rangle$$

Unfortunately, proving these two lemmas will be pretty difficult though. We worry about this later and assume their validity for now in giving a proof of the Main Theorem.

Proof of The Main Theorem of $\boldsymbol{\alpha}$-Derivations. Consider first the case when A is a single cluster subalgebra. Let $\boldsymbol{\alpha} \in \mathrm{sp}(A)$ and D be an arbitrary $\boldsymbol{\alpha}$-derivation on A. By the Lemma 45, there exist some N' such that $Q_{N'}(\mathrm{sp}(A)) \subseteq A$. Moreover, we know from Lemma 46 that $\mathcal{D}_{\boldsymbol{\alpha}}(Q_{N'}(\mathrm{sp}(A))) \subset \mathcal{D}_{2N'}(\mathrm{sp}(A))$. But as D must be an $\boldsymbol{\alpha}$-derivation when restricted to $Q_{N'}(\mathrm{sp}(A))$ it follows $D\big|_{Q_{N'}(\mathrm{sp}(A))}$ can be written as in equation (4) with $N = 2N'$. Moreover, since $Q_{N'}(\mathrm{sp}(A))$ is obtained from A as the intersections of kernels of linear functionals which also can be written as (4), it follows from Lemma 4 that D can be written as (4) as well.

Generalization to arbitrary subalgebras $A \subset \mathbb{K}[\boldsymbol{x}]$ of finite codimension (i.e not necessarily single cluster), can be done with induction and Theorem 40.

We have two nested induction processes. First an outer one on the number of clusters. Let A be an algebra and $C \subset \mathbb{K}$ a set of spectral elements which form one of the clusters in A. Let $\mathcal{D}', \mathcal{E}'$ be all derivations and evaluation subtractions which vanish on A that don't pertain to the cluster C. Let A' be the algebra obtained as the kernel of $\mathcal{D}'$ and $\mathcal{E}'$. Similarly, let $\mathcal{D}'', \mathcal{E}''$ be the conditions which only pertain to C and let A'' be the algebra obtained as the kernel of these conditions. Note that we then have

$$A = \bigcap_{L \in \mathcal{D}' \cup \mathcal{D}'' \cup \mathcal{E}' \cup \mathcal{E}''} \ker L$$

We proved above that the Main Theorem holds over single cluster algebras such as A''. Let the induction hypothesis of the outer induction process be that the Main Theorem holds over A'. Then an inner induction process can be applied with Theorem 40 on the conditions of $\mathcal{D}' \cup \mathcal{E}'$ to obtain that the C-derivations of A are the same as those of A'', and similarly, we can obtain that the $\mathrm{sp}(A) \setminus C$-derivations of A are the same as those of A'. This gives the Main Theorem over A. $\square$

The remainder of this section will be dedicated to proving lemmas 45 and 46. In the interest of being structured, we will decompose the constructions and logic into various definitions and lemmas, even if they won't be used for any other sections. The route we will take is to first define a subalgebra $Q'_N(\mathrm{sp}(A))$ which is easy to show is contained in A for some $N \in \mathbb{N}$. After this we will spend quite a bit of effort to show that $Q'_N(\mathrm{sp}(A)) = Q_N(\mathrm{sp}(A))$ which will result in a proof of lemma 45. We will obtain a proof of lemma 46 along the way as well.

Definition 47. Let $P(\boldsymbol{\alpha}) = \{x_i - \alpha_i : \alpha_i \in \boldsymbol{\alpha}\}$ and $P_N(\boldsymbol{\alpha})$ be the subset of $P(\boldsymbol{\alpha})_{\mathrm{mon}}$ consisting of all monomials in $P(\boldsymbol{\alpha})$ of total degree N. For example,

$$P_3(2,1) = \{(x_1-2)^3, (x_1-2)^2(x_2-1), (x_1-2)(x_2-1)^2, (x_2-1)^3\}.$$

I.e if $|\boldsymbol{\alpha}| = n$, then

$$|P_N(\boldsymbol{\alpha})| = \binom{n+N-1}{N}.$$

Elaborating further, if S is a set of spectral elements we will use the notation $\Pi_N(S)$ to denote the set consisting of all possible product combinations of elements from the sets $P_N(\boldsymbol{\alpha})$ for every $\boldsymbol{\alpha} \in S$. For example, if $S = \{(2,1),(0,0),(1,3)\}$, then every element of $\Pi_N(S)$ will be a product of one polynomial in $P_N(2,1)$, one polynomial in $P_N(0,0)$, and one polynomial in $P_N(1,3)$, and every possible combination of polynomials drawn from the three sets, will exist as a product in $\Pi_N(S)$. I.e, the set will have magnitude

$$|\Pi_N(S)| = \binom{n+N-1}{N}^s,$$

where $s = |S|$ and all polynomials in $\Pi_N(S)$ will have total degree Ns. A quick example is given by

$$\begin{aligned}\Pi_2(\{(0,0),(0,1)\}) = \{&x_1^4, x_1^3(x_2-1), x_1^2(x_2-1)^2,\\ &x_1^3x_2, x_1^2x_2(x_2-1), x_1x_2(x_2-1)^2\\ &x_2^2x_1^2, x_2^2(x_2-1)x_1, x_2^2(x_2-1)^2\}.\end{aligned}$$

If the reader is familiar with the theory presented in [2], note that in the univariate case, we have $\Pi_N(\mathrm{sp}(A)) = \{\pi_A^N(x)\}$, where π_A is notation reused from Theorems 19 and 20 in [2].

Finally, we define Q'_N to be the ideal subalgebra generated by Π_N as

$$Q'_N(S) = \Pi_N(S)\mathbb{K}[\boldsymbol{x}] + \mathbb{K}.$$

To be clear, here we write $\Pi_N(S)\mathbb{K}[\boldsymbol{x}]$ to be the ideal generated by $\Pi_N(S)$ in $\mathbb{K}[\boldsymbol{x}]$.

Our end goal here is to show that $Q'_N = Q_N$. We constructed Q'_N as aobve because we want all of the conditions that were used to define Q_N to vanish on Q'_N, and as we will soon see, being divisible by N linear factors that all vanish on some of the elements of $\boldsymbol{\alpha}$ ensures that a polynomial will be annihilated by $\mathcal{D}_N(\{\boldsymbol{\alpha}\})$. Also we want the largest such algebra, indeed we need for $Q'_N = Q_N$. This is why we don't just multiply all polynomials of the $P_N(\boldsymbol{\alpha})$ into one big polynomial and consider the algebra it generates. By the end of this section, we will have shown that we Q'_N is indeed the largest such algebra (if you have doubts and thoughts regarding least common multiples, see the remark after the following lemma).

Lemma 48. Let $f \in \mathbb{K}[\boldsymbol{x}], \pi \in P_N(\boldsymbol{\alpha})$ and $D \in \mathcal{D}_N(\{\boldsymbol{\alpha}\})$. Then $D(f\pi) = 0$.

Proof. Let D be given as $D(f) = f^{(j)}_{d^j_k}(\boldsymbol{\alpha})$. Note that $j < N$ by the construction of $\mathcal{D}_N(\{\boldsymbol{\alpha}\})$. By the generalized Leibniz rule we have

$$D(f\pi) = \sum_{d^i_l \in \mathcal{P}(d^j_k)} (f)^{(j-i)}_{d^j_k \setminus d^i_l}(\boldsymbol{\alpha})(\pi)^{(i)}_{d^i_l}(\boldsymbol{\alpha}).$$

We now claim that $(\pi)^{(i)}_{d^i_l}(\boldsymbol{\alpha}) = 0$ for all $i < N$. To see this fix some $x_k \in \boldsymbol{x}$. If d^i_l has a higher count of x_k than π contains factors of $x_k - \alpha_k$, then the d^i_l-th derivative will completely annihilate π. If instead d^i_l has at most the same count of x_k as the amount of factors $x_k - \alpha_k$ in π, for all $x_k \in \boldsymbol{x}$, and a lower count for at least one $x_k \in \boldsymbol{x}$, then $(\pi)^{(i)}_{d^i_l}$ will be divisible by some $x_k - \alpha_k$, after which the $\boldsymbol{\alpha}$-evaluation will annihilate $(\pi)^{(i)}_{d^i}$. Finally one of these cases must occur since π consists of N factors and $i < N$. It follows that all terms in the sum above are zero and $D(h\pi) = 0$. □

It follows that $Q'_N \subset Q_N$.

Corollary 49. Let S be a set of spectral elements. Then $Q'_N(S) \subset Q_N(S)$.

Proof. It's trivial to see that $\mathcal{E}(S)$ kills all of $Q'_N(S)$ since the generators of $Q'_N(s)$ all have roots in every element of S and it follows from the prior lemma that all elements in $\mathcal{D}_N(S)$ vanish on $Q'_N(s)$. □

Remark. You might have noticed that we can obtain a *seemingly* larger subalgebra on which the conditions that define Q_N vanish. If we instead would have combined the elements of the various P_N via taking the least common multiple as opposed the product, then all of the proofs above would still hold and such a construction would result in a subalgebra which is contained in Q_N. But as said before, we will see that $Q_N = Q'_N$ so it must be the case that these 'LCM combinations' are obtainable as linear combinations of elements in Π_n.

To give some evidence of this, consider the example $S = \{(0,0),(0,1)\}$. Then

$$\begin{aligned}
P_2(0,0) =& \{x_1^2, x_1x_2, x_2^2\}, \\
P_2(0,1) =& \{x_1^2, x_1(x_2-1), (x_2-1)^2\}, \\
\Pi_2(S) =& \{x_1^4, x_1^3(x_2-1), x_1^2(x_2-1)^2, \\
& x_1^3x_2, x_1^2x_2(x_2-1), x_1x_2(x_2-1)^2 \\
& x_2^2x_1^2, x_2^2(x_2-1)x_1, x_2^2(x_2-1)^2\}.
\end{aligned}$$

We now write some of the 'LCM combinations' of $P_1(0,0), P_2(0,1)$ as linear combinations of elements in $\Pi_2(S)$,

$$\begin{aligned}
\operatorname{lcm}(x_1^2, x_1^2) = x_1^2 &= x_1^2x_2^2 - x_1^2(x_2-1)^2 - 2x_1^2x_2^2 + 2x_1^2x_2(x_2-1), \\
\operatorname{lcm}(x_1^2, x_1(x_2-1)) = x_1^2(x_2-1) &= x_1^2x_2(x_2-1) - x_1^2(x_2-1)^2, \\
\operatorname{lcm}(x_1x_2, x_1(x_2-1)) = x_1x_2(x_2-1) &= x_1x_2^2(x_2-1) - x_1x_2(x_2-1)^2,
\end{aligned}$$

and so on... Of course we could have defined Q'_N as the set of these LCM combinations. We don't do this as it would complicate some of the upcoming proofs, and the current definition works.

We now show that some Q'_N is contained in A.

Lemma 50. Let $A \subset \mathbb{K}[\boldsymbol{x}]$ be a single cluster subalgebra of finite codimension. Then there exist some $N \in \mathbb{N}$ such that $Q'_N(\operatorname{sp}(A)) \subset A$.

Proof. We will prove the lemma by induction on the codimension of A. For our induction step, we will only consider kerneling by $\boldsymbol{\alpha}$-derivations. We can do this since we can kernel by all evaluation subtractions before we start kerneling by $\boldsymbol{\alpha}$-derivations. This will require our base case to pertain to an algebra obtained by kerneling with a certain amount of evaluation subtractions.

Consider the case of a single cluster algebra A which is obtained from $\mathbb{K}[\boldsymbol{x}]$ by kerneling by evaluation subtractions only. Let $N = 1$ and note that any $\pi \in \Pi_N(\operatorname{sp}(A))$ has a root in every spectral element. Hence $E(f\pi) = 0$ for all $\pi \in \Pi_N(\operatorname{sp}(A)), f \in \mathbb{K}[\boldsymbol{x}]$ and any evaluation subtraction E that holds in A. Membership of linear combinations $f_1\pi_1 + f_2\pi_2$ follows by linearity of E.

Moving on to the induction step, let A' be a single cluster subalgebra of finite codimension such that the statement of the lemma holds in A' with N'. Let A be obtained from A' as the kernel of some non-trivial $\boldsymbol{\alpha}$-derivation D. We set $N = 2N'$. Note that for any $\boldsymbol{\beta} \in \operatorname{sp}(A) = \operatorname{sp}(A')$, we have that each polynomial in $P_N(\boldsymbol{\beta})$ can be written as a product of two polynomials in $P_{N'}(\boldsymbol{\beta})$. Thus each $\pi \in \Pi_N(\operatorname{sp}(A))$ can be written $\pi = \pi_1\pi_2$ for $\pi_1, \pi_2 \in \Pi_{N'}(\operatorname{sp}(A'))$. It follows that $f\pi = (f\pi_1)(\pi_2) \in M^2_{\boldsymbol{\alpha}}(A')$ whence $D(f\pi) = 0$ and $f\pi \in A$ for all $\pi \in \Pi_N(\operatorname{sp}(A)), f \in \mathbb{K}[\boldsymbol{x}]$. Membership of linear combinations follows by linearity of D. □

We now have everything we need to prove Lemma 46.

Proof of Lemma 46. Our first ambition is to show that the statement of the lemma holds when $S = \{\boldsymbol{\alpha}\}$, I.e that

$$\mathcal{D}_{\boldsymbol{\alpha}}(Q_N(\{\boldsymbol{\alpha}\})) = \langle \mathcal{D}_{2N}(\{\boldsymbol{\alpha}\}) \setminus \mathcal{D}_N(\{\boldsymbol{\alpha}\})\rangle. \tag{5}$$

As it turns out, it will be easier to deduce the derivations space of $Q'_N(\{\boldsymbol{\alpha}\})$. Thus we begin by showing that $Q_N(\{\boldsymbol{\alpha}\}) = Q'_N(\{\boldsymbol{\alpha}\})$. We can see this by a dimensional argument. The codimension of $Q_N(\{\boldsymbol{\alpha}\})$ is given as the amount of linear functionals in $\mathcal{E}(\{\boldsymbol{\alpha}\}) \cup \mathcal{D}_N(\{\boldsymbol{\alpha}\})$, but $\mathcal{E}(\{\boldsymbol{\alpha}\})$ is empty so

$$\begin{aligned}
\text{codim } Q_N(\{\boldsymbol{\alpha}\}) &= |\mathcal{D}_N(\{\boldsymbol{\alpha}\})| \\
&= \sum_{j=1}^{N-1} |\boldsymbol{d}^j| \\
&= \sum_{j=1}^{N-1} \binom{n+j-1}{n} \\
&= \binom{n+N-1}{n},
\end{aligned}$$

where we used the Hockeystick identity for the last equality. As for the codimension of $Q'_N(\{\boldsymbol{\alpha}\})$, note that $Q'_N(\{\boldsymbol{\alpha}\})$ contains polynomials of all total degrees greater than or equal to N. There are $\binom{n+N-1}{n}$ non-constant polynomials of total degree less than N. Thus we get the bound

$$\text{codim } Q'_N(\{\boldsymbol{\alpha}\}) \leq \binom{n+N-1}{n} = \text{codim } Q_N(\{\boldsymbol{\alpha}\})$$

But we know from Corollary 49 that $Q'_N(\{\boldsymbol{\alpha}\}) \subset Q_N(\{\boldsymbol{\alpha}\})$. Combining this with the derived codimensional inequality yields $Q'_N(\{\boldsymbol{\alpha}\}) = Q_N(\{\boldsymbol{\alpha}\})$ (The bound above is only tight when $|S| = 1$ and can unfortunately not be used to prove Lemma 45).

Now we perform a change of variables so that $\boldsymbol{\alpha} \mapsto \mathbf{0}$. In this setting, a minimal SAGBI basis for $Q'_N(\{\mathbf{0}\})$ is given by all monomials of total degrees between N and $2N-1$ inclusively. There are

$$\binom{n+2N-1}{n} - \binom{n+N-1}{n}$$

such monomials and it follows from Theorem 20 that

$$\dim\ \mathcal{D}_{\boldsymbol{\alpha}}(Q'_N(\{\mathbf{0}\})) \leq \binom{n+2N-1}{n} - \binom{n+N-1}{n}.$$

We also have that

$$|\mathcal{D}_{2N}(\{\mathbf{0}\}) \setminus \mathcal{D}_N(\{\mathbf{0}\})| = \binom{n+2N-1}{n} - \binom{n+N-1}{n},$$

so equality (5) would follow in the $\boldsymbol{\alpha} = \mathbf{0}$ case if we could show that

$$\langle \mathcal{D}_{2N}(\{\mathbf{0}\}) \setminus \mathcal{D}_N(\{\mathbf{0}\})\rangle \subset \mathcal{D}_{\boldsymbol{\alpha}}(Q'_N(\{\mathbf{0}\})), \tag{6}$$

since $\mathcal{D}_{2N}(\{\mathbf{0}\}) \setminus \mathcal{D}_N(\{\mathbf{0}\})$ is a linearly independent set, even when considered as elements of the dual of $Q'_N(\{\mathbf{0}\})$.

To see that the inclusion (6) holds, consider the generalized Leibniz rule and Lemma 48. Indeed, given $f, g \in \mathbb{K}[\boldsymbol{x}]$, $\pi_1, \pi_2 \in \Pi_N(\{\mathbf{0}\})$ and $D = f \mapsto f^{(j)}_{d^j_k} \in \mathcal{D}_{2N}(\{\mathbf{0}\}) \setminus \mathcal{D}_N(\{\mathbf{0}\})$ we have

$$D(fg\pi_1\pi_2) = \sum_{d^i_l \in \mathcal{P}(d^j_k)} (f\pi_1)^{(i)}_{d^i_l} (g\pi_2)^{(j-i)}_{d^j_k \setminus d^i_l}$$

and since $j < 2N$ either i or $j-i$ must be less than N whence every term is zero by Lemma 48 as π_1, π_2 both contains factors from $P_N(\boldsymbol{\alpha})$. Thus D vanishes on $Q'_N(\{\boldsymbol{\alpha}\})$ and

$$\mathcal{D}_{\boldsymbol{\alpha}}(Q'_N(\{\mathbf{0}\})) = \langle \mathcal{D}_{2N}(\{\mathbf{0}\}) \setminus \mathcal{D}_N(\{\mathbf{0}\})\rangle.$$

Undoing our change of variables yields

$$\mathcal{D}_{\boldsymbol{\alpha}}(Q'_N(\{\boldsymbol{\alpha}\})) = \langle \mathcal{D}_{2N}(\{\boldsymbol{\alpha}\}) \setminus \mathcal{D}_N(\{\boldsymbol{\alpha}\})\rangle.$$

As we've shown that $Q'_N(\{\boldsymbol{\alpha}\}) = Q_N(\{\boldsymbol{\alpha}\})$, we have

$$\mathcal{D}_{\boldsymbol{\alpha}}(Q_N(\{\boldsymbol{\alpha})\}) = \langle \mathcal{D}_{2N}(\{\boldsymbol{\alpha}\}) \setminus \mathcal{D}_N(\{\boldsymbol{\alpha}\})\rangle.$$

If we now let $\boldsymbol{\alpha}_i$ be an arbitrary element in S, Theorem 40 and induction yields that

$$\mathcal{D}_{\boldsymbol{\alpha}_i}\left(\bigcap_{\boldsymbol{\alpha} \in S} Q_N(\{\boldsymbol{\alpha}\})\right) = \langle \mathcal{D}_{2N}(\{\boldsymbol{\alpha}_i\}) \setminus \mathcal{D}_N(\{\boldsymbol{\alpha}_i\})\rangle,$$

after which using Theorem 42 inductively yields

$$\mathcal{D}_{\boldsymbol{\alpha}}(Q_N(S)) = \bigoplus_{\boldsymbol{\alpha} \in S} \langle \mathcal{D}_{2N}(\{\boldsymbol{\alpha}\}) \setminus \mathcal{D}_N(\{\boldsymbol{\alpha}\})\rangle = \langle \mathcal{D}_{2N}(S) \setminus \mathcal{D}_N(S)\rangle$$

which completes our proof. □

We shall require one more lemma before we are able to prove Lemmas 45 and 46.

Lemma 51. Let S be a set of spectral elements and $N \in \mathbb{N}$. Then $Q'_N(S) = Q_N(S)$.

Proof. We already know from Corollary 49 that $Q'_N(S) \subset Q_N(S)$. Our strategy for this proof is to show that any derivations or evaluation subtractions which hold over $Q'_N(S)$ also hold over $Q_N(S)$, after which equality ensues. I.e, we need to show that no extra derivations or evaluation subtractions hold over $Q'_N(S)$.

If it were the case that $Q'_N(S) \subsetneq Q_N(S)$, then by Theorem 24, $Q'_N(S)$ would be contained in some algebra which can be obtained as the kernel of some evaluation subtraction or $\boldsymbol{\alpha}$-derivations. Thus we only need to consider evaluation subtractions, trivial $\boldsymbol{\alpha}$-derivations, and $\boldsymbol{\alpha}$-derivations in any of the $\mathcal{D}_{\boldsymbol{\alpha}}(Q'_N(S))$ for $\boldsymbol{\alpha} \in S$.

We begin with the case when $D \in \mathcal{D}_{\boldsymbol{\alpha}}(Q_N(\mathrm{sp}(A)))$ is a non-trivial $\boldsymbol{\alpha}$-derivation given as

$$D(f) = f^{(j)}_{d^j_k}(\boldsymbol{\alpha})$$

for some $d^j_k \in \boldsymbol{d}^j$ and $j \in [N..2N-1]$. We will construct a polynomial in $Q'_N(\mathrm{sp}(A))$ on which D is not zero. Let

$$\pi_1(\boldsymbol{x}) = \prod_{x_i \in d^j_k} (x_i - \alpha_i).$$

Then $\pi_1 = f\widehat{\pi}_1$ for some $\widehat{\pi}_1 \in P_N(\boldsymbol{\alpha}), f \in \mathbb{K}[\boldsymbol{x}]$ and also $D(\pi_1) = M$ for some integer M. For every spectral element in $\boldsymbol{\beta} \in \mathrm{sp}(A) \setminus \{\boldsymbol{\alpha}\}$ there must exist some index $i(\boldsymbol{\beta})$ such that $\beta_{i(\boldsymbol{\beta})} \neq \alpha_{i(\boldsymbol{\beta})}$. Construct

$$\pi_2(\boldsymbol{x}) = \prod_{\boldsymbol{\beta} \in \mathrm{sp}(A) \setminus \{\boldsymbol{\alpha}\}} (x_{i(\boldsymbol{\beta})} - \beta_{i(\boldsymbol{\beta})})^N.$$

Then $\pi_2(\boldsymbol{\alpha}) \neq 0$ and $\pi = \pi_1\pi_2 \in \Pi_N(\mathrm{sp}(A)) \subset Q'_N(\mathrm{sp}(A))$. Applying our derivation, we get

$$\begin{aligned}
D(\pi) &= (\pi_1\pi_2)^{(j)}_{d^j_k}(\boldsymbol{\alpha}) \\
&= \sum_{d^i \in \mathcal{P}(d^j_k)} (\pi_1)^{(i)}_{d^i}(\boldsymbol{\alpha})(\pi_2)^{(j-i)}_{d^j_k \setminus d^i}(\boldsymbol{\alpha}) \\
&= (\pi_1)^{(j)}_{d^j_k}(\boldsymbol{\alpha})\pi_2(\boldsymbol{\alpha}) \\
&= M\pi_2(\boldsymbol{\alpha}) \\
&\neq 0.
\end{aligned}$$

For arbitrary $\boldsymbol{\alpha}$-derivations, we can combine linear combinations of polynomials like π above, one for each derivative evaluation, and since $\mathbb{K}$ is infinite, we can chose scalars in such a way that the given $\boldsymbol{\alpha}$-derivation doesn't vanish on the linear combination of polynomials.

To see that no extra trivial derivations hold, let $\boldsymbol{\gamma} \notin \mathrm{sp}(A)$. Just like we constructed π_2, we can construct a polynomial $\pi_3 \in Q'_N(\mathrm{sp}(A))$ such that π_3 is a product of Ns linear factors and $\pi_3(\boldsymbol{\gamma}) \neq 0$. Then if we multiply π_3 by some $x_i - \gamma_i$ we get that $(\pi_3\gamma_i)'_{x_i} = \pi_3(\gamma_i) \neq 0$. We extend the result to deal with arbitrary $\boldsymbol{\gamma}$-derivations in exactly the same way as we did for non-trivial $\boldsymbol{\alpha}$-derivations above.

To see that no evaluation subtractions hold, let $\boldsymbol{\gamma} \notin \text{sp}(A)$. Just like we constructed π_2 and π_3, we can construct a polynomial $\pi_4 \in Q'_N(\text{sp}(A))$ such that π_4 is a product of Ns linear factors and $\pi_4(\boldsymbol{\gamma}) \neq 0$. Let $\boldsymbol{\delta} \in \mathbb{K}^n$ such that $\boldsymbol{\delta} \neq \boldsymbol{\gamma}$. Then there exist some index i such that $\delta_i \neq \gamma_i$. Let $h = \pi_4(x_i - \delta_i) \in Q'_N(\text{sp}(A))$. Then $h(\boldsymbol{\gamma}) - h(\boldsymbol{\delta}) = h(\boldsymbol{\gamma}) \neq 0$, and $\boldsymbol{\gamma} \not\sim \boldsymbol{\delta}$ over $Q'_N(\text{sp}(A))$.

□

Lemma 45 now follows trivially,

Proof of Lemma 45. Follows immediately from Lemmas 50 and 51, □

and we can finally considered the Main Theorem as settled.

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