Heuristic and Exact Evaluation of Two-Echelon Inventory Control Systems

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Abstract

The main theme of this project is inventory control with stochastic demand. In this thesis we consider a two-echelon inventory system which consist of one central warehouse and N retail stores. Customer demands at the retailers follow independent Poisson demand processes and each customer only demands one unit. Customer demands which are not satisfied directly from stock on hand are assumed to backordered, i.e., no lost sales exist. All transportation times are assumed to be constant. Replenishment, at the warehouse and at each retailer, are made according to so called order-up-to S policies (also denoted as $(S-1, S)$-policies). The first main goal is to derive an expected system cost function which consist of inventory holding costs and backorder costs. Secondly, we will optimize this cost function with respect to the base-stock levels S_i, $i = 0, \ldots, N$, where S_i represents the base-stock-level at retailer i (index 0 is for the warehouse).

We will first consider an exact method to optimize the expected system cost function. In this exact method the lead-times for the retail stores are stochastic due to possible delays when replenishing from the central warehouse. However, in practice, it is common to use an approximate method where the stochastic lead-times for the retailers are replaced by the corresponding mean values. Here, we will investigate the robustness of this approximate method in terms of changes in system parameters.

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1 Introduction

1.1 Historical background and introduction to the discipline of inventory control

Inventory control is a discipline within operation research, a subfield of mathematical sciences. The use of operation research is related back to military operations during the second World War. After the war, scientists and educational institutions, have tried to apply operation research, which provides mathematical techniques and algorithms to improve their efficiency when solving decisions problems. The essence of inventory control is to decide when we should order new item and how much we should order. The use of inventory control has become more essential given the fact that the industry is growing a lot. With an efficient inventory control model, companies can maximize their profits and avoid stocking up unnecessary amount of goods in warehouses [7].

1.2 Inventory control models

Inventory is the measured amount of items that are being stored, and in time the amount varies when the items are purchased according to a demand process. A common inventory policy, both in practice and in literature, is the (s, S)-policy. Here, s denotes the reorder point and S denotes the order-up-to level. Hence, according to this policy, when the inventory position (IP) ($IP =$ stock on hand + outstanding orders − backorders) declines to or below s, a new batch is ordered such that the inventory position jumps to the maximum level S. In this thesis we will use a special case of an (s, S)-policy, a so-called order-up to S-policy, where $s = S - 1$ is the reorder point and S is the order up to level. When the system is in a steady-state, the inventory position is equal S. When an item has been purchased, the inventory position declines to or below $S - 1$ and, we then order up to the maximum level S. That is, as soon as item has been purchased we immediately order another item [1]. Inventory control model is a model that describes the flow of goods from the warehouse and to the customers. The goal is to find an optimal reorder point and order quantity to minimize the total cost. The order quantity, in our case, is equal to one when using the $(S - 1, S)$-policy. The aim of this thesis is to find an algorithm to determine the optimal base-stock level, i.e., the level S for a two-echelon model. In order to do that we will first consider a single-echelon model. When considering the two-echelon model we find that the lead-times for the retailers are stochastic and that will make the optimization more complicated. Therefore, we will also consider an approximate method (METRIC approximation) when optimizing the two-echelon model. For each model we will describe how to calculate the probability function for the inventory level. This probability function will be used in order to calculate relevant costs. Finally, we will formulate a procedure for optimizing the inventory position S.

2 Single-echelon model

In this chapter we will consider an inventory control model that consists of one central warehouse and one retail store, where the warehouse has infinite capacity. We assume that the model follows a so-called order-up-to S policy and that there is a continuous review.

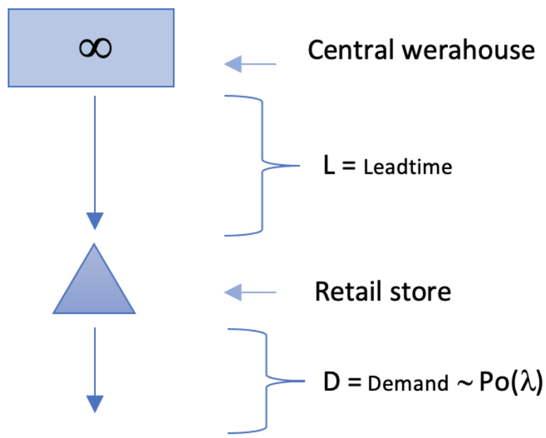

Figure 1: Single-echelon model

We assume that the customers arrive to the retail store according to a Poisson process and, that they only demand for one unit. As soon as a unit has been requested the retail store immediately places an order to the central warehouse. If the retail store has a unit in stock, the customer gets the unit, and the store is now short of one unit. In practice, it takes time for the new unit to arrive from the central warehouse to the retail store, we will call this time the lead-time. Now, let us assume that this has been a busy day at the retail store and there are no more units left when the next customer arrives. Since we are assuming that there are no lost sales in our model the customer is backorderd in a FIFO-queue (First in-First out-queue), see [7], until the customer gets the unit. Since the retail store requests a new unit directly after an order has been placed, a new unit is already on its way to the retail store. Finally, when the unit arrives to the store the customer that was next in line gets the unit.

First, we could ask us the question; *Why not have an infinite, or at least a very large stock on hand. Then, the customer would never have to wait for an unit?* The answer to this question is that there is a substantial capital cost associated with holding inventory. We call this cost the holding cost. Considering this, we could also ask the extreme question; *Why then have any units stored at the retail store? Why not send the requested unit after demand?* By doing that, we would save all the extra cost on storing the units. The answer to this question is simply that there is also a cost for that a customer is waiting for a unit, we will call this cost the shortage cost and it is caused by what we will call backorders. Backorders are the units that have been demanded by the customers but have not yet been delivered. This lead us to the main question of this thesis; *what is the optimal inventory position, that is, the optimal number of units in the system, and is there a way to optimize this?* This is the question that we will try to answer in this thesis. We will begin analyzing the model described above and then later on we will consider a two-echelon model.

In order to do this optimization, we will consider two costs, the holding cost and the shortage cost, and balance them. The holding and shortage costs depends on the inventory level (IL), and the ordering decisions are based on the inventory position (IP). We define the inventory level and the inventory position in the following way:

$$Inventory\ level = Stock\ on\ hand - backorders$$

$$Inventory\ position = Stock\ on\ hand + outstanding\ orders - backorders.$$

Following arguments from [1], we note that there is an important relationship between the inventory level and the inventory position. Let us consider a time interval $[t, t+L]$ where t is arbitrary. We assume the inventory position is $IP(t)$ at time t and $D(t, t+L)$ is the demand in the interval $(t, t+L]$. Orders that were triggered at time t are on the way within the time interval $[t, t+L)$ and at time $t+L$ the orders have arrived. We can write the following relationship:

$$IL(t+L) = IP(t) - D(t, t+L) \tag{1}$$

In order to find the probability function of the inventory level we need to take the lead-time (L) into account. The lead-time is the time between when the retail store places an order to the central warehouse to send them a new unit and until that unit is available at that retail store. We will note here that the lead-time is not just the travel time from the central warehouse to the retail store. The lead-time can include the time, for example, the central warehouse to process the item or the transit delay. In many inventories control models, the lead-time is assumed to be exponentially distributed since the mathematical analysis becomes considerably simpler, due to the Markov property. However, in the real world the lead-time is often constant. We note that the exponential distribution is one of the distributions that has the greatest variance. That is, if one assumes that the lead time follows a exponential distribution, the variation between the lead-times could be very large. In practice, the variations are not so great.Therefor we will look at these two cases and analyze if the choice of a distribution has an effect on the probability function of the inventory level. Note that we have already established that the customers arrive to the retail store according to a Poisson process. Moreover, since the customers always order just one unit, we get that the number of requested units is equal to the number of customers. Hence, the demand during a time interval of t is Poisson distributed [1]:

$$P(D(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t},\ for\ k = 0, 1, 2, ... \tag{2}$$

2.1 Constant lead-time

We will begin by analyzing the probability function of the inventory level when the lead-time is deterministic and constant. In the case of an $(S-1, S)$-policy the inventory position is always S and the demand after time t is independent of the inventory position at time t [1].

Using the results from (1) and (2) we get that the probability function of the inventory level with a constant lead-time L is:

$$P(IL = k) = P(D(L) = S - k) = \frac{(\lambda L)^{S-k}}{(S-k)!} e^{-\lambda L}, \; for \; k \leq S. \tag{3}$$

2.2 Exponential distributed lead-time

We will now look at the distribution of the inventory level when the lead-time is exponential distributed. The system can now be interpreted as a $M/M/\infty$ - queuing system.

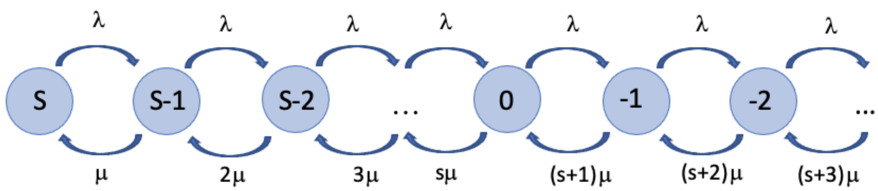

Figure 2: $M/M/\infty$-queuing system

We define P_k as the probability of being in state k, for $k = S, S-1, \ldots, 0, -1, -2, \ldots$, where the state variable is the inventory level. Hence, the probability function of the inventory level, $P(IL = k)$ is equal to the probability of being in state k. When determining the probability of being in state k we use the rate in-rate out method from Markov theory, see [7]. We find the balance equations for all states and then we solve the system of equations for the probability of being in state P_k, see [7]. We start by noting:

$$\lambda P_S = \mu P_{S-1} \;\Rightarrow\; P_{S-1} = \frac{\lambda P_S}{\mu}$$

By repeating this procedure we find that

$$\lambda P_{S-(k-1)} = k\mu P_{S-k} \;\Rightarrow\; P_{S-k} = \frac{\lambda P_{S-(k-1)}}{k\mu}$$

where

$$P_{S-(k-1)} = \frac{\lambda^{k-1} P_S}{\mu^{k-1}(k-1)!}$$

we then get that

$$P_{S-k} = \frac{(\lambda/\mu)^k}{k!} P_S$$

which means that

$$P_k = \frac{(\lambda/\mu)^{(S-k)}}{(S-k)!} P_S$$

we now solve for P_S

$$\sum_{k=0}^{\infty} P_{S-k} = \sum_{k=0}^{\infty} \frac{(\lambda/\mu)^{(k)}}{k!} P_S = 1$$

and we find that

$$P_S = e^{-\lambda/\mu}$$

Hence, we get that

$$P(IL = k) = P_k = \frac{(\lambda/\mu)^{S-k}}{(S-k)!} e^{-\lambda/\mu}, \ for\ k \leq S. \tag{4}$$

The probability function follows a Poisson distribution with mean λ/μ. We can conclude that whether the lead-time is constant, or exponential distributed we will have that the probability function of the inventory level is Poisson distributed with mean λ/μ. As a matter of fact, the choices of distribution for the lead-time does not matter, it will always be Poisson distributed according to *Palm's theorem* [3].

Palm's theorem:
When considering a $M/G/\infty$ queue with homogeneous Poisson input with rate λ, for $\lambda > 0$ and a stationary service distribution F with mean $1/\mu$. Then, as $t \to \infty$, the limiting(steady state) distribution of the number of arrivals still in service is Poisson withe mean λ/μ.

2.3 Constructing the cost function

As stated above, in order to find the optimal inventory position, we will consider the shortage cost and the holding cost and balance them. The optimization should be based on customer expectations, i.e, the underlying shortage costs, and on the costs for giving good service.

Notations:

$h =$ The holding cost per unit and time unit.

$b =$ shortage cost per unit and time unit.

The expected cost function is given by (see e.g. [1]):

$$C(S) = hE(IL^+) + bE(IL^-) = h\sum_{k=1}^{S} k \cdot P(IL = k) + b\sum_{k=1}^{\infty} k \cdot P(IL = -k) \tag{5}$$

We will us the following notation:

$$x^+ = max(x, 0)$$
$$x^- = max(-x, 0). \tag{6}$$

Noting that

$$x = x^+ - x^- \;\Rightarrow\; x^- = x^+ - x. \tag{7}$$

Using (7), we can rewrite the cost function in (5):

$$C(S) = hE(IL^+)+bE(IL^-) = h(IL^+)+b(E(IL^+)-E(IL)) = (h+b)E(IL^+)-bE(IL) \tag{8}$$

where

$$E(IL) = E(IP - D(L)) = S - E(D(L)) = (S - \lambda/\mu)$$

and

$$E(IL^+) = \sum_{k=1}^{S} k \cdot \frac{(\lambda/\mu)^{S-k}}{(S-k)!} e^{-\lambda/\mu}.$$

In order to find the optimal inventory position, we first prove that the cost function is convex. Since our function is discrete, we need to look at the difference of one step in discrete time. Therefore, we want to look at the cost difference, $C(S+1) - C(S)$. Following [1], we obtain:

$$\begin{aligned} C(S+1) - C(S) &= (h+b)E(IL^+ \mid S+1) - b(S+1-\lambda/\mu) - (h+b)E(IL^+ \mid S) - b(S-\lambda/\mu) \\ &= -b + (h+b)(E(IL^+ \mid S+1) - E(IL^+ \mid S)) \\ &= -b + (h+b)(\sum_{k=1}^{S+1} k \cdot P(IL = k|S+1) - \sum_{k=1}^{S} k \cdot P(IL = k|S)). \end{aligned} \tag{9}$$

From change of variables, the following holds

$$P(IL = k \mid S) = P(IL = k+1 \mid S+1)$$

Using this, we get that

$$\begin{aligned} \sum_{k=1}^{S} k \cdot P(IL = k|S) &= \sum_{k=1}^{S} k \cdot P(IL = k+1|S+1) \\ &= \sum_{k=2}^{S+1} (k-1)P(IL = k|S+1) \end{aligned}$$

by applying that to (9), we get

$$\begin{aligned} C(S+1)-C(S) &= -b+(h+b)(\sum_{k=1}^{S+1} k\cdot P(IL = k|S+1) - \sum_{k=1}^{S+1} k\cdot P(IL = k|S+1) + \sum_{k=1}^{S+1} P(IL = k|S+1)) \\ &= -b + (h+b)\sum_{k=1}^{S+1} P(IL = k|S+1). \end{aligned} \tag{10}$$

Since the sum of $P(IL = k|S+1)$ increases in S, it follows that the cost difference is increasing with S. We have then proven that the cost function is convex in S.

2.4 Optimization of the Single-echelon model

Since the cost function is convex from a discrete point of view, we can now determine the optimal inventory position, S. Our goal is to find the optimal S^* that minimizes the cost function $C(S)$. We start the process by setting $S = 0$ and look at the cost difference. When $C(S+1) - C(S) \leq 0$ we increase S by one, setting $S = S+1$ and we do this until $C(S+1)-C(S) > 0$ for the first time. We then set $S^* = S$, where S^* is the optimal inventory position.

Let us look at an example: Let $\lambda = 1, h = 2, b = 3$ and $L = 5$. We start the process by setting the inventory position to $S = 0$. We get that the cost is equal to $C(S) = 1.6240$ and $C(S+1) = 1.4060$, i.e., $C(S+1) - C(S) \leq 0$. We than set $S = S+1$, hence, $S = 1$ and get that $C(S) = 1.4060$ and $C(S) = 1.6240$, i.e., $C(S+1) - C(S) > 0$. Let us denote $S^* = 1$ as the optimal inventory position. See Table 1 and Figure 3, where we have plotted the cost function for an given value of S.

Table 1: Values of the total cost and reorder points for a given value of S

$\lambda = 1, h = 2, b = 1, L = 2$		
S	C(S)	C(S+1)
0	1.6240	1.4060
1	1.4060	1.6240
2	1.6240	

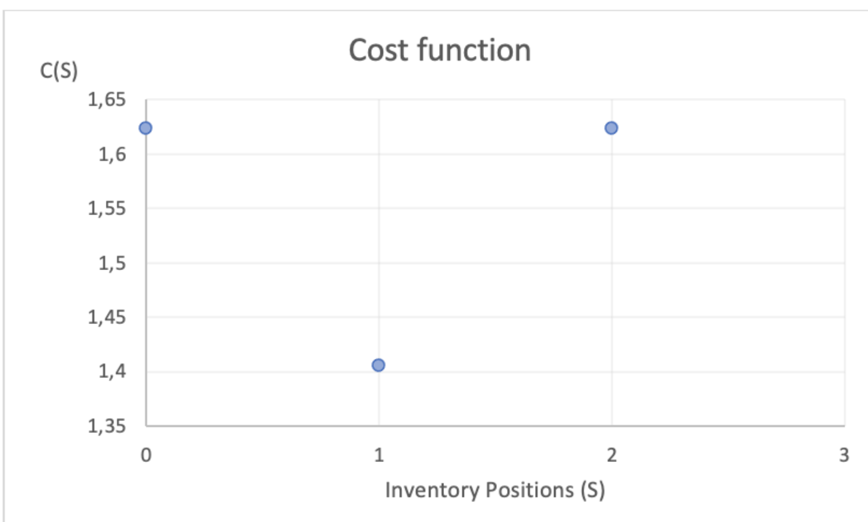

Figure 3: Here we have plotted the values of the total cost for a given value of S. Note the convex structure of the cost function

3 Two-echelon model

In this chapter we will consider a two-echelon model that consists of one central warehouse and N retailers. Just as for the previous model, we assume that the ordering decisions follow an $(S-1,S)$-policy and that there is a continuous review. We also assume that each retail store i faces a Poisson distribution and that all retailers are independent of each other. However, for a two-echelon model we now assume that the central warehouse does not have infinite capacity. The units arrive to the central warehouse according to some lead-time. We will begin analyzing the structure of the two-echelon model using the exact method and then move on to analyze the model using the approximate method (the so called METRIC approximation).

Notations:

L_0 = The lead-time for an unit to arrive at the central warehouse.

L_i = The transportation time from the central warehouse to the retail store i.

S_0 = Inventory position at the central warehouse.

S_i = Inventory position at the retails store i.

λ_i = the demand intensity at the retail store i.

$\lambda_0 = \sum_{i=1}^{N} \lambda_i$ = the demand intensity at the central warehouse.

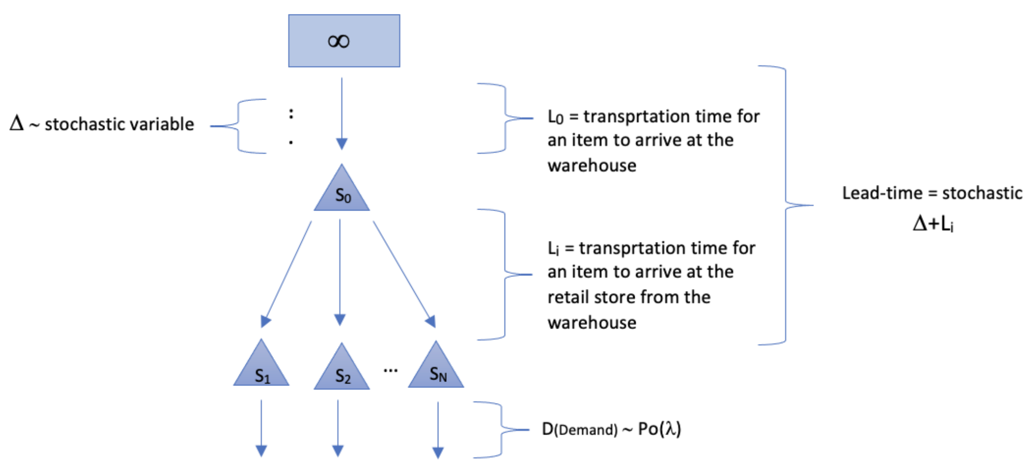

Figure 4: Two-echelon model

3.1 Structure of the two-echelon model using the exact method

In a two-echelon model, a customer arrives to a retail store i according to a Poisson process with rate λ_i, and only demands for one unit according to an $(S-1,S)$-policy. When the customer has placed an order, the retail store will be short of one unit and immediately send a request to the central warehouse to send them a new unit. At the same time the central warehouse will now be short of one unite and will request for one unit as well. As in the previous model, when the retail store is empty the customers are backorderd in a FIFO-queue. Furthermore, we assume that backorders at the central warehouse are filled according to the FIFO-queue. Now, if the demand occurs when the retail store and the warehouse is empty, the order to the retail store is delayed. We define $\tilde{L_i} = L_i + \Delta$ as the lead-time for the retail store i, where L_i is now the transportation time from the central warehouse to the retail store i and Δ is the stochastic delay. This means that the lead-times for the retail stores are now stochastic. We define Δ as the random delay encountered at the central warehouse when the central warehouse is out of stock, where $0 \leq \Delta \leq L_0$. When a unit is back in stock at the central warehouse, the unit will be sent to the retail store according to a FIFO-policy. That is, when a unit is requested at a retail store, and both the retail store and the central warehouse is empty, a unit, that we know is already on its way to the central warehouse has already been assigned to a specific retail store before it arrives at the central warehouse [2].

Let us now define X_i as the limiting age of the oldest unit at retail store i that has not been assigned to any waiting customer. The age of an unit is assumed to start when the unit is ordered from the central warehouse. Also, let us define X_0 as the limiting age of the oldest unit at the central warehouse that has not been assigned to any waiting customer. The age is assumed to start when the central warehouse requests for that unit, see [6] for further information. Since all retailers face a Poisson process with rate λ_i and are independent of other retail stores, according to the property *Superposition* of a Poisson process, see [7], we can conclude that the warehouse faces a Poisson process with rate $\lambda_0 = \sum_{i=1}^{N} \lambda_i$. Now, X_i is the time that the unit i spends in the system, hence, sum of independent exponentially distributed variables, we note that X_i, for $i = 0, \ldots, N$, is Erlang(λ_i, S_i) distributed, see [6]:

$$f_{X_i}(t) = \frac{\lambda_i^{S_i} t^{S_i - 1}}{(S_i - 1)!} e^{-\lambda_i t}, \ for \ 0 \leq t. \tag{11}$$

and

$$F_{X_i}(t) = 1 - \sum_{k=0}^{S_i - 1} \frac{(\lambda_i t)^k}{k!} e^{-\lambda_i t}, \ for \ 0 \leq t. \tag{12}$$

Note that we can write, $\Delta = L_0 - X_0$ for $0 \leq X_0 \leq L_0$. And by using (11) we get that the density of the stochastic delay, Δ is:

$$f_{\Delta}(t) = f_{X_0}(L_0 - t) = \frac{\lambda_0^{S_0} (L_0 - t)^{S_0 - 1}}{(S_0 - 1)!} e^{-\lambda_0 (L_0 - t)}, \ for \ 0 \leq t \leq L_0. \tag{13}$$

and

$$P(\Delta = 0) = P(X_0 > L_0) = 1 - F_{X_0}(L_0) = \sum_{k=0}^{S_0 - 1} \frac{(\lambda_0 L_0)^k}{k!} e^{-\lambda_0 L_0} \tag{14}$$

see [6]. Let us now continue determining the probability function of the inventory level at retail store i. As stated above $L_i + \Delta$ is now the lead-time for the retail store i. Then, given that the delay $\Delta = s$ and using (3) we find that conditional stationary probability function of IL_i is:

$$P(IL_i = k \mid \Delta = s) = \frac{(\lambda_i^{S_i}(L_i + s))^{S_i - k}}{(S_i - k)!} e^{-\lambda_i(L_i+s)}.$$

By the law of total probability we get that the probability function of the inventory level is:

$$P(IL_i = k) = P(IL_i \mid \Delta = 0)P(\Delta = 0) + \int_0^{L_0} P(IL_i = k \mid \Delta = s) f_\Delta(s)\, ds$$

[6]. We now construct the cost function. We remember that all retailers are independent of each other. Also, the inventory position at the retail stores does not affect the holding cost at the central warehouse since the demand at the warehouse is not affected if we change the inventory position at the retail stores. Hence, the expected total cost is given by:

$$E(C) = h_0 E(IL_0^+) + \sum_{i=1}^{N} h_i E(IL_i^+) + E(B) \tag{15}$$

where

$$E(IL_0^+) = \sum_{k=1}^{S_0} k \cdot \frac{(\lambda_0 L_0)^{S_0 - k}}{(S_0 - k)!} e^{-\lambda_0 L_0}$$

is the average stock on hand at the central warehouse, and

$$E(IL_i^+) = \sum_{k=1}^{S} k \cdot P(IL_i = k)$$

is the average stock on hand at the retail store i. $E(B)$ is the total expected backorder cost, per unit and time unit:

$$E(B) = \sum_{i=1}^{N} E(B_i)$$

where

$$E(B_i \mid \Delta = s) = \lambda_i b_i \int_0^{L_i+s} (L_i + s - t) f_{X_i}(t) dt. \tag{16}$$

In (16) we have used the well known *Little's theorem*, see [5], and from (16), we obtain the unconditional expected backorder cost for retail store i as:

$$E(B_i) = E(B_i \mid \Delta = 0) \cdot P(\Delta = 0) + \int_0^{L_i+s} E(B_i \mid \Delta = s) f_\Delta(s) ds$$

[6]

3.2 METRIC approximation

METRIC approximation is an approximate method where the real stochastic lead-times for the retailers are approximated by their mean values. When using METRIC approximation, we first need to look at the central warehouse [1]. We start by determining the distribution of the inventory level for the central warehouse. As we know from Section 2.1, there is a relationship between the inventory position and the inventory level. For the central warehouse, we now have:

$$IL_0(t + L_0) = S_0 - D_0(L_0).$$

That is, the inventory level at time $t + L_0$ is equal to the inventory position at the central warehouse minus the demand in the time interval $(t, t + L_0]$, see [1]. Also, we know that the demand in that time interval follows a Poisson distribution. Hence, we have the probability function of the inventory level at the central warehouse:

$$P(IL_0 = k) = P(D_0(L_0) = S_0 - k) = \frac{(\lambda_0 L_0)^{S_0 - k}}{(S_0 - k)!} e^{-\lambda_0 L_0},\ for\ k \leq S_0. \tag{17}$$

It follows, as stated before, that the average number of backorders is

$$E(IL_0^-) = E(IL_0^+) - E(IL_0) \tag{18}$$

where the average stock on hand is

$$E(IL_0^+) = \sum_{k=1}^{S_0} k \cdot P(IL_0 = k)$$

and

$$E(IL_0) = (S_0 - \lambda_0 L_0)$$

see [1]. The result of this tells us that the central warehouse can be modeled as an $M/G/\infty$ queue according to Palm's theorem. Then, according to *Little's theorem*, we can express the average delay at the central warehouse as the average number of backorders divided by the mean of the demand to the central warehouse [5]. Hence,

$$E(\Delta) = \frac{E(IL_0^-)}{\lambda_0} \tag{19}$$

Since the demand at the central warehouse is Poisson distributed and follows the FIFO-queuing system, the average delay is the same for all retailers. Using METRIC approximation we replace the stochastic lead-time by its mean and we now have that the lead-time for the retail store i is constant:

$$\hat{L_i} = L_i + E(\Delta)$$

We can now determine the distribution of the inventory level just as we did for the central warehouse [1]:

$$P(IL_i = k) = P(D_i(\hat{L}_i) = S_i - k) = \frac{(\lambda_i \hat{L}_i)^{S_i - k}}{(S_i - k)!} e^{-\lambda_i \hat{L}_i}, \; for \; k \leq S_i. \tag{20}$$

We now construct the cost function. We note that the retail stores do not affect the holding cost at the central warehouse. However, backorders at the the central warehouse indirectly influence the costs at the retailers, due to the possible delay for when the central warehouse is empty [1]. Hence, the expected total cost is given by:

$$E(C) = h_0 E(IL_0^+) + \sum_{i=1}^{N} h_i E(IL_i^+) + b_i E(IL_i^-). \tag{21}$$

3.3 Optimization using the approximated method

When using METRIC approximation to optimize the two-echelon model we can decompose the model and treat all installations as a single-echelon system [1].

Notations:

C = average system costs per time unit.

S_0^* is the optimal value of S_0

S_i^* is the optimal value of S_i

$\overline{S_0^*}$ is the upper bound for the optimal value of S_0

$\underline{S_0^*}$ is the lower bound for the optimal value of S_0

$\overline{S_i^*}$ is the upper bound for the optimal value of S_i

$\underline{S_i^*}$ is the lower bound for the optimal value of S_i

$S_i^*(S_0)$ is the optimal value of S_i for a given S_0

h_i = holding cost per unit and time unit at installation i, $(i = 0, 1, 2, \ldots, N)$.

b_i = backorder cost per unit and time unit at retailer i, $(i = 0, 1, 2, \ldots, N)$.

We will use the following notation for the cost function:

$$EC(S) = C_0(S_0) + \sum_{i=1}^{N} C_i(S_0, S_i) \tag{22}$$

where

$$C_0(S_0) = h_0 E(IL_0^+)$$

is the average holding costs per time unit at the central warehouse, and

$$C_i(S_0, S_i) = h_i E(IL_i^+) + b_i E(IL_i^-)$$

is the average holding and backorder costs per time unit at retail store i, $(i = 1, 2, \ldots, N)$. Note here, the lead-time for the retail store i is now $\hat{L}_i = L_i + E(\Delta)$. Since the lead-time is constant, it follows from the results in Section 2.3 that, for a given S_0 the retailer costs is convex in S_i. That is, $C_i(S_0, S_i)$ is convex in S_i. However, the total cost function is not necessarily convex in S_0, see [1].

We start the optimization process by obtaining the lower and upper bound for the optimal inventory position at retail store i, S_i^*. We obtain $\underline{S_i^*}$, by minimizing $C_i(S_0, S_i)$ with respect to S_i given that $S_0 = \infty$. Note that when $S_0 = \infty$ we have that the lead-time for retail store i is L_i, i.e., the shortest possible. We then find $\overline{S_i^*}$ in the same way, only now $S_0 = 0$, that is the lead-time for retail store i is now $L_0 + L_i$, i.e., the longest possible. When we have determined $\underline{S_i}$ and $\overline{S_i}$, we find the lower and upper bound for S_0, i.e., $\underline{S_0}$ and $\overline{S_0}$. $\underline{S_0^*}$ is found by optimizing EC(S_0, $\overline{S_1^*}$, $\overline{S_2^*}$, $\ldots$, $\overline{S_N^*}$) with respect to S_0. In the same way we find $\overline{S_0^*}$ by optimizing EC(S_0, $\underline{S_1^*}$, $\underline{S_2^*}$,$\ldots$, $\underline{S_N^*}$) with respect to S_0. Since the cost function is not necessarily convex in S_0, S_0^* is found by evaluating the cost function for all values of S_0 within the boundaries of the upper and lower bound for S_0, i.e., we choose $S_0 \in \{$ $\underline{S_0^*}$, $\ldots$, $\overline{S_0^*}$ $\}$ such that:

$$EC(S_0) = C_0(S_0) + \sum_{i=1}^{N} C_i(S_0, S_i^*(S_0)) \tag{23}$$

is minimized [6]. Before we look at an example, we will first optimize the same model using the exact method.

3.4 Optimization using the exact method

We will now optimize the exact model. The optimization process is similar to the process when optimizing the approximated model. However, the lead-time is now stochastic. The optimal inventory position at retail store i depends on the inventory position at the central warehouse. When S_0=∞, the lead-time for retail store i becomes L_i, i.e. the shortest lead-time the retail stores can have. On the other hand, when $S_0 = 0$ the lead-time for retail store i becomes $L_0 + L_i$, i.e. the longest lead-time possible that retail store i can have. When using the exact method, we also have that for a given S_0, that $C_i(S_0, S_i)$ is convex in S_i [6].

We start the process in the same way that we did when using the approximated method. First, we find the lower and the upper bound for the optimal inventory position at retail store i. We obtain $\underline{S_i^*}$, by minimizing $C_i(S_0, S_i)$ with respect to S_i given that $S_0 = \infty$. However, when $S_0 = \infty$ we have that the lead-time is L_i, i.e., the lead-time is constant. That implies that the holding cost at retail store i is increasing in S_i. Therefore, for a given S_0 we have that, $\underline{S_i^*} \leq S_i^*(S_0) \leq \overline{S_i^*}$. Then, the next step is to find the lower and upper bound for S_0^*. Similarly to the approximated method, $\underline{S_0^*}$ is found by optimizing EC(S_0, $\overline{S_1^*}$, $\overline{S_2^*}$, $\ldots$, $\overline{S_N^*}$) with respect to S_0, and $\overline{S_0^*}$ is found by optimizing EC(S_0, $\underline{S_1^*}$, $\underline{S_2^*}$, $\ldots$, $\underline{S_N^*}$) with respect to

S_0. Just as for the approximated method, the total cost function is not necessarily convex in S_0. Therefore, we need to evaluate the cost function for all values of $S_0 \in \{ \underline{S_0^*}, \ldots, \overline{S_0^*} \}$ such that:

$$EC(S_0) = C_0(S_0) + \sum_{i=1}^{N} C_i(S_0, S_i^*(S_0)) \tag{24}$$

is minimized. In the next section we will look at few examples and analyze them for when using the different methods [6].

4 Analysis

In this section we will look at few examples and compare the result from when using the approximated method and the exact method when optimizing the two-echelon model. We are interested to see if there is a pattern when optimizing the same model using these two different methods. If we find that there is a noticeable pattern and we are able to define why, we could, in practical, use the approximated method instead of the exact method. The reason why we are interested in doing this is, because using the approximated is less complicated than using the exact method, due to the stochastic lead-time. In the following examples we will assume that there are only two retail stores and that they are identical.

Example 1:

$\lambda_1 = \lambda_2 = 0.5,\ h_0 = 2,\ h_1 = h_2 = 2,\ b_1 = b_2 = 1,\ L_0 = 2,\ L_1 = L_2 = 2$		
Optimization	Approximated method	Exact method
$\underline{S_i^*},\ \overline{S_i^*}$	$0, 1$	$0, 1$
$\underline{S_0},\ \overline{S_0}$	$1, 1$	$1, 1$
S_0^*	1	1
$S_1^*,\ S_2^*$	$1, 1$	$1, 1$
$EC(S_0^*, S_1^*, S_2^*)$	2.4959	2.7313

In example 1, the system evaluates the lower and the upper bound for S_i^* and S_0 to be the same when using the different methods. We then find that the optimal values for the central warehouse and the retail stores are the same when we use the different method, i.e., $S_0^* = 1$, $S_1^* = S_2^* = 1$. What is interesting about this is that the approximate method finds the true optimal policy. We note that, even though we get the same optima values when using the different methods, the total cost is not same. When using the approximated method, we get that $EC(S_0^*, S_1^*, S_2^*) = 2.4959$ and, when using the exact method, we get that $EC(S_0^*, S_1^*, S_2^*) = 3.4091$, i.e., the total cost is less than the total cost when using the approximated method, respectively when using the exact method.

Example 2:

$\lambda_1 = \lambda_2 = 0.5,\ h_0 = 2,\ h_1 = h_2 = 2,\ b_1 = b_2 = 3,\ L_0 = 2,\ L_1 = L_2 = 2$		
Optimization	Approximated method	Exact method
$\underline{S_i^*},\ \overline{S_i^*}$	$1, 2$	$1, 2$
$\underline{S_0},\ \overline{S_0}$	$1, 1$	$1, 2$
S_0^*	1	1
$S_1^*,\ S_2^*$	$1, 1$	$2, 2$
$EC(S_0^*, S_1^*, S_2^*))$	4.7011	5.2989, 5.6308

In example 2 we have only change the value of $b_1 = b_2 = 3$ from example 1. We now get a slightly different lower and upper bound for S_0 when using these two different methods. When using the approximated method we get that $\underline{S_0} = \overline{S_0} = 1$, and when using the exact method we get that $\underline{S_0} = 1$ and $\overline{S_0} = 2$. For a given value of S_0, we find that the expected total cost is minimized when $S_i^*(1) = 2$. We therefor set the optimal inventory position to $S_0^* = 1$ at the central warehouse and $S_i^* = 2$ at the retail stores. Note that these are not exactly the same optima values when using the approximated method. The optimal inventory positions at the retail stores are slightly underestimated when using the approximated method.

Example 3:

$\lambda_1 = \lambda_2 = 0.5$, $h_0 = 2$, $h_1 = h_2 = 2$, $b_1 = b_2 = 1$, $L_0 = 3$, $L_1 = L_2 = 2$		
Optimization	Approximated method	Exact method
$\underline{S_i^*}$, $\overline{S_i^*}$	$0, 2$	$0, 2$
$\underline{S_0}$, $\overline{S_0}$	$1, 1$	$1, 2$
S_0^*	1	1
S_1^*, S_2^*	$1, 1$	$1, 1$
$EC(S_0^*, S_1^*, S_2^*)$	2.5306	3.0245, 3.0575

In example 3 we have only changed the value of $L_0 = 3$ from example 1. From the table we can see that we get the same optima values when using the different methods, i.e., $S_0^* = 1$ and $S_i^* = 1$. That is, the approximate method finds the true optimal policy.

Example 4:

$\lambda_1 = \lambda_2 = 0.7$, $h_0 = 2$, $h_1 = h_2 = 2$, $b_1 = b_2 = 1$, $L_0 = 2$, $L_1 = L_2 = 2$		
Optimization	Approximated method	Exact method
$\underline{S_i^*}$, $\overline{S_i^*}$	$1, 2$	$1, 2$
$\underline{S_0}$, $\overline{S_0}$	$1, 1$	$1, 2$
S_0^*	1	2
S_1^*, S_2^*	$1, 1$	$1, 1$
$EC(S_0^*, S_1^*, S_2^*)$	2.8706	3.4222, 3.4133

In example 4 we have change the values of $\lambda_1 = \lambda_2 = 0.7$ from example 1. In this example, the optimal inventory positions is not the same. We get that $S_0^* = 2$ and $S_i^* = 1$ when using the exact method and $S_0^* = 1$ and $S_i^* = 1$ when using the approximated method. Hence, the approximated method underestimates the optimal value for the central warehouse.

As we can see from these examples, the approximated method often finds the true optimal policy. However, in some cases the optima values are underestimated when using the approximated method.

5 Conclusions and discussion

We have now formulated a procedure for optimizing the inventory position S. When using METRIC approximation, the analysis of the central warehouse is rather similar to when using the exact method, while all retailers are handled approximately. That is, the optimal values are sometimes underestimated. We also note that in these examples the evaluated total cost is always lower when using the approximated methods. The reason for this is that, when using METRIC approximation we usually underestimate the values of $E(IL^+)$ and $E(IL^-)$.

Example 1:

$S_0 = 1,\ S_1 = S_2 = 1,\ \lambda_1 = \lambda_2 = 0.5,\ L_0 = 2,\ L_1 = L_2 = 2$		
Optimization	Approximated method	Exact method
$E(IL_0^+)$	0.1353	0.1353
$E(IL_1^+) = E(IL_2^+)$	0.2085	0.2209
$E(IL_1^-) = E(IL_2^-)$	0.7762	0.7886

Example 2:

$S_0 = 1,\ S_1 = S_2 = 1,\ \lambda_1 = \lambda_2 = 0.5,\ L_0 = 3,\ L_1 = L_2 = 2$		
Optimization	Approximated method	Exact method
$E(IL_0^+)$	0.0498	0.0498
$E(IL_1^+) = E(IL_2^+)$	0.1320	0.1459
$E(IL_1^-) = E(IL_2^-)$	1.1569	1.1707

Example 3:

$S_0 = 1,\ S_1 = S_2 = 1,\ \lambda_1 = \lambda_2 = 0.7,\ L_0 = 2,\ L_1 = L_2 = 2$		
Optimization	Approximated method	Exact method
$E(IL_0^+)$	0.0608	0.0608
$E(IL_1^+) = E(IL_2^+)$	0.0973	0.1066
$E(IL_1^-) = E(IL_2^-)$	1.4277	1.4370

One might wonder why we would use the approximate method if the estimation is sometimes underestimated. However, in practice, using the exact method can be rather difficult and hard to understand. Companies that want to use inventory control to maximize their profits and avoid stocking up unnecessary amount of goods in their warehouses, do not always have the understanding of how the exact method works due to the stochastic lead-time. Then, if we know that the optimal inventory position at the retailers is sometimes underestimated when using the approximated method, keeping these factors in mind, companies can use that in their advantages, allowing them to use the approximated method. METRIC approximation has been used successfully in practice, and in general, using this method works well as long as the demand at each retail store is low relative to the total demand [4].

It could be interesting to study how much the optimal value, obtained by using METRIC approximation, variate from the optimal value using the exact method and perhaps see if the variation follows a particular distribution. Companies that use the METRIC approximation, would then have a better idea how accurate the optimal values are. However, we will not look into this in this thesis.

6 References

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