H_∞ controller for smoothing the coupling error in co-simulated technical systems.

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Chapter 1

Introduction

In this thesis I present an innovative approach related with the simulation of dynamical systems described by differential equations, in particular, with linear time invariant systems. This systems can be so complex in general that it turns out advantageous to consider them in a decoupled way, so that the subsystems are solved separately, e.g.: co-simulation of the two masses of a dual mass spring damper system, or the co-simulation og MBD vehicle model and an EPAS system model. This separation introduces an error because the initial system is not decoupled. In current literature the simulation of these decoupled systems is known as co- simulation and has been studied extensively [1, 2, 5, 7, 10, 11]...

From this different studies I selected a co- simulation approach from a recent PhD thesis [1]. Here, it is applied a control theory technique to handle and minimize the coupling error. The purpose of my thesis is to follow the lines of this phD thesis in order to understand its chosen approach, the H_∞ synthesis method. To conclude with, I attempted to reproduce the results of the phD thesis with a similar experiment set up.

The central chapter of my work chapter 6. However, to reach there I had to introduce essential proofs and concepts from control theory and co- simulation. I suggest that the reader who is only interested in co-simulation jumps directly to chapter 5 and takes the first chapters as a reference. Personally, I found important to keep this additional chapters with this structure because they were of great relevance for my understanding of the core of this thesis.

The thesis has been structured in the following way: on chapter 2 I give the mathematical background related to the control theory results we will be using. On chapter 3 I give a broad description of the required control theory notions to keep in mind when reading the next chapters. chapter 4 introduces the H_∞ norm and H_∞ methods in robust control theory. chapter 5 presents formally the notion of coupled system in co-simulation and its different schemes and configurations. Finally, like described above. In chapter 6 I present and explain the

core of this thesis, the application of the H_∞ synthesis method to a decoupled system with the purpose of minimizing the coupling error derived from the co-simulation. Lastly, on chapter 7 we set up the dynamics of a double mass spring damper where I attempted to reproduce the approach we described in chapter 6.

Chapter 2

Mathematical Background.

Control theory strongly depend on the frequent use and application of many mathematical techniques. One of the main purposes of the study of control systems is to develop a set of analytical tools so that reasonable, predictable and trustworthy designs can be attained with no dependency on an extensive computer simulation or experimentation.

The mathematical background required in the study of control theory include among other mathematical fields and notions: complex variable analysis, differential equations or Laplace and Z transforms.

Furthermore, modern control theory requires an even broader mathematical support that also relies on mathematical fields such as matrix theory, set theory, linear algebra, variational calculus or probability theory.

2.1 Preliminaries

2.1.1 Complex Variable

We can divide a complex variable $s = \sigma + j\omega \in \mathbb{C}$ in two components, its real part σ represented in the x- axis and its imaginary part $j\omega$ represented in the vertical axis.

Consider a complex variable function $G : \mathbb{C} \to \mathbb{C}$, $s \mapsto G(s)$. Because s can be divided in two parts, $G(s)$ can also be represented as

$$G(s) = \mathrm{Re}(G(s)) + j\ \mathrm{Im}(G(s))$$

where Re $(G(s))$ represents the real part of $G(s)$ and Im $(G(s))$ the imaginary part.

2.1.2 Analytical Functions

A function $G : \mathbb{C} \to \mathbb{C}$ is analytic in a region Ω if all its derivatives exist in that region.

Example 2.1.1 (Analytical Functions). *Consider the functions G_1, G_2*

- *The function*
$$G_1(s) = \frac{1}{s(s+1)}$$
is analytical for all s except for $s = 0$ and $s = -1$.

- *The function $G_2(s) = s + 2$ is analytical for all s.*

In the study of analytical functions the notion of singularity is specially important.

Definition 1 (Singularity). *The singularities of a function are the points on the plane where the function or its derivatives does not exist.*

We present now a key type of singularity for a complex function, the poles of the function.

Definition 2 (Pole). *Consider an analytic function in the neighbourhood of $s_i \in \mathbb{C}$. We say that $G(s)$ has a pole of order r in $s = s_i$ if the limit*
$$\lim_{s \to s_1} [(s - s_i)^r G(s)]$$
is finite and different from 0.

Example 2.1.2 (Pole). *Consider the function*
$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^2}$$
It has a pole of order 2 in $s = -3$ and two poles of order 1 in $s = 0, -1$

Definition 3 (Zero). *Consider an analytical function $G : \mathbb{C} \to \mathbb{C}$. We say that G has a zero of order r in $s = s_i$ if the limit*
$$\lim_{s \to s_i} \left[(s - s_i)^{-r} G(s)\right]$$
has a finite value different from zero. In other words $G(s)$ has a zero of order r in $s = s_i$ if $1/G(s)$ has a pole of order r in $s = s_i$.

Example 2.1.3 (Zero). *Consider again the function*
$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^2}$$
This function has 4 finite poles in $s = 0, -1, -3$ and one zero in $s = -2$. Moreover, we consider the existence of three infinite zeros since
$$\lim_{s \to \infty} G(s) = \lim_{s \to \infty} \frac{10}{s^3} = 0$$

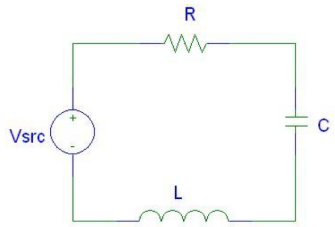


Figure 2.1: RLC circuit scheme

2.2 Ordinary Differential Equations. Linear systems. Laplace Transform.

A great variety of engineering and physical systems are modelled by differential equations. Even though a big part of this physical systems are not linear, in this thesis we restrict our attention to the linear and time invariant case, the so- called LTI systems.

As an example of physical systems described by a system of ODE's we can use the representation of the RCL electric circuit in Figure 2.1, which states that the sum of the voltages equal zero, i.e.

$$0 = V_L + V_R + V_C - V_{src}$$

Denoting $i(t)$, the current of the circuit, we can rewrite the equation of the first equation in terms of $i(t)$ so that

$$Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int i(t)\mathrm{d}t = V_{src} \tag{2.1}$$

$$\frac{R}{L}i(t) + \frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{LC}\int i(t)\mathrm{d}t = \frac{V_{src}}{L} \tag{2.2}$$

We transform now Equation 2.2,to a linear ODE system. Considering x_1 the charge of the capacitor, x_2 the current flowing through the circuit and that the charge is the integral of current over time, then

$$x_1 = \int i\mathrm{d}t = \int x_2\mathrm{d}t$$

$$x_2 = \dot{x_1} = \frac{\mathrm{d}x_1}{\mathrm{d}t}$$

using this definitions we can rewrite equations Equation 2.2, as the following set of differetial equations

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \frac{1}{LC}x_1 + \frac{R}{L}x_2 - \frac{1}{L}V_{src}$$

Because we have now two different states in the system, it is no longer explicit which state is the output of our linear set of ODE's. Therefore, by our given definition of the current $i(t)$ our parameter of interest is x_2 so that we must explicitly define the output of our system as such

$$y = 0x_1 + 1x_2 = x_2$$

In general we can describe a linear ordinary differential equation of order n as

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}x(t)}{\mathrm{d}t^{n-1}} + ... + a_1\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_0 y(t) = f(t) \tag{2.3}$$

We write a differential equation of order n in n differential equations of order 1. This is a common procedure when solving differential equations with big order because, as we stated in the RLC circuit example, solving equations of order 1 is normally easier than solving differential equations of higher order.

Thus, we can break our differential equation of order n, Equation 2.3, into n differential equations of first order in the following way

$$\begin{aligned} \tfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= x_2(t) \\ \tfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_3(t) \\ &\vdots \\ \tfrac{\mathrm{d}x_n(t)}{\mathrm{d}t} &= -a_0x_1(t) - a_1x_2(t) - ... - a_{n-2}x_{n-1}(t) - a_{n-1}x_n(t) + f(t) \end{aligned} \tag{2.4}$$

We can also express the state equations Equation 2.3 in matrix form as

$$\frac{\mathrm{d}x_n(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \tag{2.5}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_n \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Definition 4 (State variables)**.** *The state variables of a system are the minimal set of variables $x_1(t), x_2(t), ..., x_n(t)$ whose knowledge for any initial time t_0 and of the input information completely determines the state of the system for every time $t > t_0$.*

The state of a system refers to the past, present and future conditions of the system.

2.2.1 State Space representation

The state space representation is a formalized method for representing linear systems of differential equations, developed to make it easier working with them. This formalization utilizes matrix notation to represent linear sets of ODEs in a mathematically relevant way.

In order to explain this representation let us consider again the system of differential equations of the RLC circuit example represented in Figure 2.1

$$\dot{x}_1 = 0x_1 + 1x_2 + 0V_{src}$$
$$\dot{x}_2 = \frac{1}{LC}x_1 + \frac{R}{L}x_2 - \frac{1}{L}V_{src}$$

taking all the factors out we get

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{1}{LC} & \frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{L}\end{bmatrix}V_{src}$$

where we consider the 2×2 matrix

$$A := \begin{bmatrix}0 & 1\\ \frac{1}{LC} & \frac{R}{L}\end{bmatrix}$$

the state matrix of the system, and the 2×1 matrix

$$B := \begin{bmatrix}0\\ -\frac{1}{L}\end{bmatrix}$$

the input matrix.

Since we are using multiple states to describe the system, it is also necessary to specify the output explicitly. Recalling the output of the LRC system as

$$y = 0x_1 + 1x_2$$

We can convert this into matrix notation such that

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

where

$$C := \begin{bmatrix}0 & 1\end{bmatrix}$$

is considered the output matrix.

Now that we have a particular idea in mind of the state space representation we can generalize our previous example with the system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned} \tag{2.6}$$

where A is the state matrix, B the input matrix, C the output matrix, $\vec{x}$ the state vector and u the system input. This will also be the general system we will take as a reference when we refer to a general LTI system during our paper.

2.2.2 Laplace Transform

Among many other applications in many different fields, the Laplace transform is one of the most important analytic tools we use to solve linear ordinary differential equations.

Definition 5 (Laplace transform)**.** *We define the Laplace transform as*

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty y(t)e^{-st}\mathrm{d}t < \infty$$

where $f(t)$ is a real function that satisfies

$$\int_0^\infty \left|f(t)e^{-st}\right| \mathrm{d}t < \infty$$

where s is a complex variable $s = \sigma + j\omega$ with real numbers σ and ω.

This transform has two main characteristics of our interest:

Property 6. *The homogeneous solution and the particular solution are obtained by just one operation.*

Property 7. *The Laplace transform turns a differential equation into an algebraic equation in $s \in \mathbb{C}$.*

Therefore, once the differential equation is converted into an algebraic equation, in order to find a solution for our differential equation in the s domain, is enough to just manipulate the algebraic equation through basic algebraic rules, so that the final solution is obtained by taking the inverse of the Laplace transform.

Definition 8 (Inverse of the Laplace tranasform)**.** *Given the Laplace transform $F(s)$, we denote its inverse $f(t)$ as*

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\mathrm{d}s$$

where c is a real constant, higher than the real parts of every singularity Definition 1 of $F(s)$.

Laplace Transform Properties

$$f(t) \qquad F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t \qquad \text{(Definition)} \qquad (2.7)$$

$$af(t) + bg(t) \qquad aF(s) + bG(s) \qquad \text{(Linearity)} \qquad (2.8)$$

$$\mathrm{e}^{at}f(t) \qquad F(s-a) \qquad \text{(s} - \text{ shift)} \qquad (2.9)$$

$$f'(t) \qquad sF(s) - f(0) \qquad (2.10)$$

$$f''(t) \qquad s^2F(s) - sf(0) - f'(0) \qquad (2.11)$$

$$f^{(n)}(t) \qquad s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0) \qquad (2.12)$$

$$tf(t) \qquad -F'(s) \qquad (2.13)$$

$$t^nf(t) \qquad (-1)^nF^{(n)}(s) \qquad (2.14)$$

$$u(t-a)f(t-a) \qquad \mathrm{e}^{-as}F(s) \qquad \text{(t} - \text{ translation or t} - \text{ shift)} \qquad (2.15)$$

$$u(t-a)f(t) \qquad \mathrm{e}^{-as}\mathcal{L}(f(t+a)) \qquad \text{(t} - \text{ translation)} \qquad (2.16)$$

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau \qquad F(s)G(s) \qquad (2.17)$$

$$\int_0^t f(\tau)\mathrm{d}\tau \qquad \frac{F(s)}{s} \qquad \text{(integration rule)} \qquad (2.18)$$

We consider now an example for the application of the Laplace transform when solving a linear ordinary differential equation.

Example 2.2.1 (Laplace Transform)**.** *Consider the differential equation*

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2x(t) = 5u_s(t)$$

with initial conditions $x(0) = -1, x^{(1)}(0) = 2$ *and* $u_s(t)$ *defined as*

$$u_s(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases}$$

Firstly we want to apply the Laplace transform on both sides of the equation such that

$$s^2X(s) - sx(0) - x^{(1)}(0) + 3sX(s) - 3Y(0) + 2X(s) = 5/s \tag{2.19}$$

Now, substituting the initial conditions we get

$$X(s) = \frac{-s^2 - s - 5}{s(s^2 + 3s + 2)} = \frac{-s^2 - s - 5}{s(s+1)(s+2)} \tag{2.20}$$

expanding the latter

$$X(s) = \frac{5s}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$$

Finally, taking the inverse Laplace transform, we get the complete solution of our differential equation

$$x(t) = \frac{5}{2} - 5\mathrm{e}^{-t} + \frac{3}{2}\mathrm{e}^{-2t} \quad t \geq 0$$

2.3 Difference Equations. Z transform.

Frequently differential equations represent systems of continuous signals. Difference equations systems aim to represent these systems in discrete time, with the advantage that they are easier to implement. Thus, difference systems are commonly used to approximate differential equations.

We can write a difference equation of order n and constants coefficients $a_0, ..., a_{n-1}$ as

$$x(k+n) + a_{n-1}x(k+n-1) + \cdots + a_1x(k+1) + a_0x(k) = f(k) \tag{2.21}$$

Analogous to the differential equations case it is convenient to turn this difference equation of order n to n difference equations of first order, such that

$$\begin{gathered} x_1(k) = x(k) \\ x_2(k) = x_1(k+1) = x(k+1) \\ \vdots \\ x_{n-1}(k) = x_{n-2}(k+1) = x(k+n-2) \\ x_n(k) = x_{n-1}(k+1) = x(k+n-1) \end{gathered} \tag{2.22}$$

so that

$$x_n(k+1) = -a_0 x_1(k) - a_1 x_2(k) - \cdots - a_{n-1} x_n(k) + f(k) \tag{2.23}$$

Once more, we can also write these n state equations in matrix form as

$$x(k+1) = Ax(k) + Bu(k) \tag{2.24}$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.25}$$

2.3.1 Z Transform

The Z-transform of a time series is the analogue to the Laplace transform for a time dependent function.

Definition 9 (Z transform)**.** *Consider the sequence $y(k)$ with $k = 0, 1, 2, ...$ We define the Z transform of $y(k)$ by*

$$Y(z) = \mathcal{Z}\{y\}(k) = \sum_{k=0}^{\infty} y(k) z^{-k} \tag{2.26}$$

where z is a complex variable.

2.3.1.1 Relationship between Laplace and Z transform

We can represent a sequence, or signal, $y(kT)$, $k = 0, 1, 2, ...$ as an impulse train separated by an interval of time T, also called sampling period.

Definition 10 (Impulse of the kth instant)**.** *The impulse of the k-th instant is denoted as $\delta(t - kT)$, where δ is the Dirac function.*

In digital control one often samples a signal $y(t)$ every T seconds so to form a time sequence that represents the samples instants. Therefore, we can relate the sequence $y(kT)$ with a signal that can be expressed as

$$y^*(t) = \sum_{k=0}^{\infty} y(kT)\delta(t - kT) \tag{2.27}$$

Taking the Laplace transform on both sides of the equation we get

$$Y^*(s) = \mathcal{L}[y^*(t)] = \sum_{k=0}^{\infty} y(kT) e^{-kTs} \tag{2.28}$$

Thus, we can see how the Z transform is related to the Laplace transform through $z = e^{Ts}$.

Below we present an example of the application of the Z transform to a function in the time domain.

Example 2.3.1 (Z transform)**.** *Consider the function*

$$y(t) = e^{\alpha t} u_s(t); \qquad u_s(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases} \tag{2.29}$$

To get the Z transform first we need to represent the values of $y(t)$ on the instants $t = kT$, $k = 0, 1, 2...$ to get the function

$$y^*(t) = \sum_{k=0}^{\infty} e^{-\alpha kT} \delta(t - kT) \tag{2.30}$$

Then we obtain the Laplace transform on both sides of the equation

$$Y^*(s) = \sum_{k=0}^{\infty} e^{-\alpha kT} e^{-kTs} = \sum_{k=0}^{\infty} e^{-(s+\alpha)kT} \tag{2.31}$$

expressing the previous equation in a more compact way and applying $z = e^{Ts}$ we obtain the z transform of our initial function

$$Y(z) = \frac{z}{z - e^{-\alpha T}} \tag{2.32}$$

Z- Transform Properties

$$af(k) + bg(k) \qquad aF(z) + bG(z)\mathrm{e}^{-st}\mathrm{d}t \qquad \text{(Linearity)} \tag{2.33}$$

$$f(k+1) \qquad zF(z) - zf(0)\mathrm{e}^{-st}\mathrm{d}t \qquad \text{(Left Shift by 1)} \tag{2.34}$$

$$f(k+2) \qquad z^2F(z) - z^2f(0) - zf(1)\mathrm{e}^{-st}\mathrm{d}t \qquad \text{(Left Shift by 2)} \tag{2.35}$$

$$f(k+n) \qquad z^nF(z) - z^n \sum_{k=0}^{n-1} f(k)z^{-k} \tag{2.36}$$

$$= z^n \left(F(z) - \sum_{k=0}^{n-1} f(k)z^{-k} \right) \mathrm{e}^{-st}\mathrm{d}t \text{ (Left Shift by n)} \tag{2.37}$$

$$f(k-n) \qquad z^{-n}F(z) \qquad \text{(Right Shift by n)} \tag{2.38}$$

$$kf(k) \qquad -z\frac{\mathrm{d}F(z)}{\mathrm{d}z} \qquad \text{(Multiplication by time)} \tag{2.39}$$

$$a^k f(k) \qquad F\left(\frac{z}{a}\right) \qquad \text{(Scale in z)} \tag{2.40}$$

$$f\left(\frac{k}{n}\right) \qquad F(z^n);\ \ n \text{ integer } n \geq 1 \qquad \text{(Scale in time)} \tag{2.41}$$

$$f(k) * g(k) \qquad F(z)G(z) \qquad \text{(Convolution)} \tag{2.42}$$

$$f(0) = \lim_{z\to\infty} F(z) \qquad \text{(Initial Value Theorem)} \tag{2.43}$$

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)F(z) \qquad \text{(Final Value Theorem (if exists))} \tag{2.44}$$

Let us now introduce another example, this time illustrating the application of the Z transform to solve linear difference equations.

Example 2.3.2 (Z transform)**.** *Consider the difference equation*

$$y(k+1) + y(k) = 0 \tag{2.45}$$

to solve this equation we are going to first take the z transform on both sides of the equation such that

$$\sum_{k=0}^{\infty} y(k+1)z^{-k} + \sum_{k=0}^{\infty} y(k)z^{-k} = 0 \quad k = 0, 1, 2\cdots \tag{2.46}$$

applying the definition of $Y(z)$ and one of the translation theorem from Equation 2.3.1.1 we can write the previous equation as

$$z\left[Y(z) - y(0)\right] + Y(z) = 0 \tag{2.47}$$

solving for $Y(z)$ we obtain

$$Y(z) = \frac{z}{z+1}y(0) \tag{2.48}$$

we can expand $Y(z)$ in the power serie

$$Y(z) = (1 - z^{-1} + z^{-2} - z^{-3}\cdots)y(0) \tag{2.49}$$

so that

$$y(k) = (-1)^k y(0) \quad k = 0, 1, 2, ... \tag{2.50}$$

Chapter 3

Control Theory Prerequisites

Control theory is an interdisciplinary subfield of science whose main purpose is to influence the behaviour of a dynamical system, commonly denominated plant of the control system, so that the output of the system follows a reference signal i.e. desired control signal, with a fixed or changing value. In order to fulfill this, it is needed to design a controller that checks the output of the dynamical system and compares it with the reference. The difference between output of a system and the reference signal is called error signal. This difference is commonly used to modify the input of the system through feedback Definition 17 to bring the actual output closer to the reference.
The main focus of Control theory is the study of the stability section 3.4 of a system, understood as the convergence of the output of a system to the reference signal, its controllability Definition 28 and its observability Definition 30.

We divide the control theory field into two sections: linear control theory, governed by linear differential equations, and nonlinear control theory, governed by nonlinear differential equations. As we said in chapter 2, in this master thesis we will focus our attention on a major subclass of linear control theory systems, linear time invariant systems (LTI), amenable to the frequency domain mathematical techniques, which include tools such as the Laplace or the Z transform, and can always be expressed in a more compact way thank to the notion of transfer function Definition 20 and block diagram representation subsection 3.2.3.

The results presented and definition presented on this chapter have been inspired mostly in [4, 9, 12].

3.1 Preliminaires

3.1.1 Process

Definition 11 (Process)**.** *We define process as a continuous time physical system to be controlled.*

Example 3.1.1 (Process)**.** *Mathematical models that describe the swinging of a clock pendulum (Figure 3.1) or the flow of water in a pipe are examples of processes.*

Choosing $x1(t)$ *as the angular position and* $x_2(t)$ *as the angular velocity of*

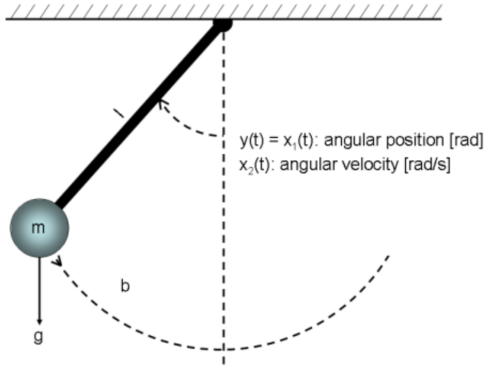


Figure 3.1: Scheme view of a classical pendulum

the pendulum, the previous scheme can be mathematically modelled with a state-space structure as follows

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_2(t)$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -(g/l) \cdot sin(x_1(t)) - (b/(m * l^2)) \cdot x_2(t)$$

$$y(t) = x_1(t)$$

3.1.2 Controller. Feedback vs Feedforward.

Definition 12 (Controller)**.** *Is the component of a control system that observes some of the input variables and adjusts some other output variables to achieve a desired operation. In control theory we have two main types of controllers: feedback and feedforward.*

Example 3.1.2. *The heating system of many houses have a thermostat which can be consider the controller of a control heating system. The thermostat will notice when the temperature (output) in the house is too cold, and then it will turn on the heater (controlled output) so the that the temperature reaches back*

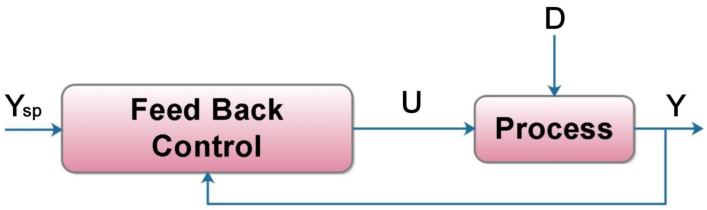


Figure 3.2: Block diagram with a feedback controller where, Y is the controlled variable , U the manipulated variable, D the disturbance (D) and Y_{sp} the desire set point so that once the controlled variable Y is measured the information is sent to the controller which compares the obtained measured variable value with the desired set point ((Y_{sp})

its desired value. Once the heater has been on for a while the thermostat will sense when the temperature is too hot, and shut off the heater.

As we mentioned before, in this example, the controller of our system is clearly represented by the thermostat, which indicates the heater how to behave. The heater is the processor that warms the air inside the house to the desired temperature, the air temperature reading mechanism inside the house is the feedback. And finally, the house is the environment in which the heating system operates.

Definition 13 (Feedback controller)**.** *This controllers can only act on the result of a disturbance, once the system is affected by it, so that the controlled variable is "fed back" into the controller. Its main drawback comes from the fact that the controlled variable might not be at the desired set point so that the feedback controller will not react until there has been a significant deviation from the reference.*

Example 3.1.3. *Suppose the thermostat of the heating system of a house is a example feedback controller. If the door of the house is opened on a cold day, the house cools down. This way, after the temperature fells below the desired temperature (reference), the heater will then be turned on, but there would inevitably be a period when the house was colder than desired.*

Definition 14 (Feedforward controller)**.** *With these controllers, the disturbances are measured and accounted before they can affect the system fighting the slowness of the feedback controllers. However, in order to achieve this the effect of the disturbances on the system must be perfectly predicted, and there must not be any surprise disturbances.*

Example 3.1.4. *Consider the thermostat of the heating system of a house to be a feedback controller. If the door of the house is is opened, with this controller the heater will be automatically turned on before the house can get too cold. Nevertheless, if a window is opened without being measured, the thermostat, might miss it letting the house cool down.*

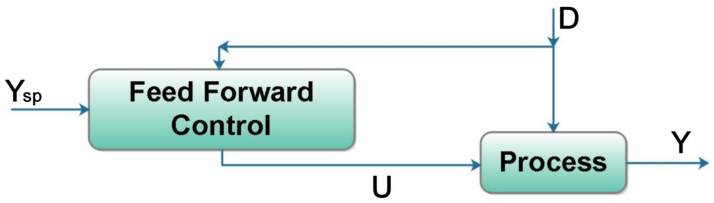


Figure 3.3: This time the disturbance D is measured instead of the controlled variable Y. The meausred information is again sent to the controller which manipulated U to prevent the effect which may be caused by the disturbance during the process.

Frequenctly, the feedforward controller takes care of the major disturbance, and the feedback controller takes care of everything else that might cause the process variable to deviate from its set point.

3.1.3 Open Loop vs. Closed loop systems

Definition 15 (Open loop system)**.** *Open loop systems are control systems in which the output does not have an effect on the control signal. Thus, in this type of control systems there is no reason to apply feedback or measure the output to compare it with the input or any other reference signal of the system.*

This leaves us with a situation where, for every reference signal there is a fixed operation. Therefore, the accuracy of the system depends only on how we calibrate the relationship established between the input and output of the system in order to obtain a desired output.

Definition 16 (Closed loop systems)**.** *In this type of control systems the output has a direct effect over the control of the system. This implies that the closed loop systems are feedback control systems.*

As we mentioned before, we refer to the difference between the input and output signal as the error signal of the system, in closed loop systems we will make use of feedback in order to reduce the error of the system.

3.1.3.1 Feedback control system

Definition 17 (Feedback system)**.** *We consider as a feedback system to the systems that compare the output of any variable of the system with its input or any other variable, so to establish the most adequate control design.*

Definition 18 (Gain)**.** *We define the gain of a control system as the value that relates magnitude of the input to the magnitude of the output signal at steady state, i.e. the ratio of the change in input to the change in output.*

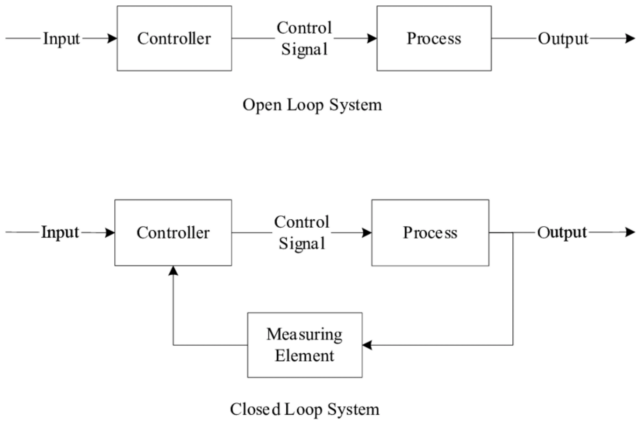


Figure 3.4: General Open Loop vs Closed Loop Systems

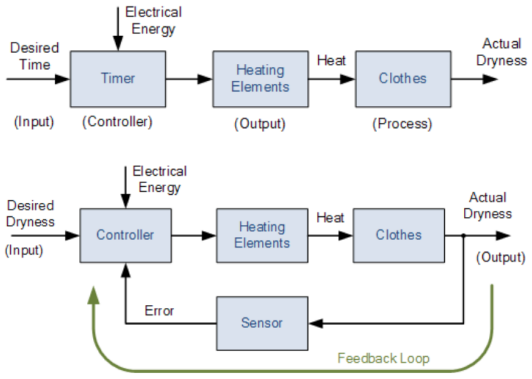


Figure 3.5: Example of a textile dryer as an open loop system

Definition 19 (Feedback Control)**.** *Feedback control is an operation that, in presence of disturbances, tries to reduce the difference between the output and input of the system under the premises of this difference (or error signal). We can distinguish two different feedback control systems:*

- ***Positive*** *feedback control systems: systems where the set point and output values are added together by the controller. The effect of this type of feedback control is to increase the systems gain, so that the overall gain is greater than the gain without feedback.*

- ***Negative*** *feedback control system: system where the set point and output values are subtracted from each other. The effect of negative feedback control is to "reduce" the overall gain.*

The majority of feedback control systems are negative, reducing the effects of the gain. This is because negative feedback control produces stable circuit responses, improves stability and increases the operating bandwidth Definition 44 of a given system.

Recall that we use feedback control when we consider only unpredictable disturbances; Predictable disturbances can be included inside the system, in absence of control.

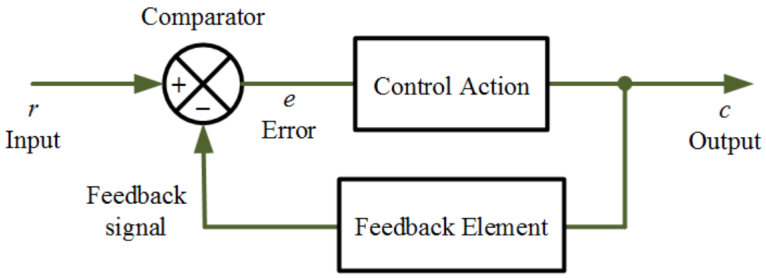


Figure 3.6: Feedback control

3.2 Transfer Function and Block diagrams

We commonly represent dynamical systems in the form of time domain systems of ordinary differential equations. However, there is another way to represent LTI systems using the Laplace transform Definition 5.

Transfer functions describe how an incoming input signal is modified when going through a system, they can be used to describe the response of the system to an arbitrary input function. In this section we describe how to convert a system represented in state space form into a transfer function and vice versa, and introduce some techniques for using block diagrams to create composite transfer functions of multiple systems.

3.2.1 Transfer Function

Definition 20 (Transfer Function)**.** *The transfer function of a linear system is the Laplace transform of the impulse response with initial conditions equal to zero.*

Consider a system with input $u(t)$, output $y(t)$ and impulse response $g(t)$, then we define the transfer function $G(s)$ as

$$G(s) = \mathcal{L}\left[g(t)\right] \tag{3.1}$$

This transfer function is related to the Laplace transform of the input ($u(t)$) $U(s)$ and the Laplace transform of the output ($y(t)$) $Y(s)$ of the system through the following equality

$$G(s) = \frac{Y(s)}{U(s)} \tag{3.2}$$

and all the inital conditions equal to zero.

Consider now the input- output equations of a general linear time invariant system of order n and constant coefficients

$$\begin{gathered} \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \cdots + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = \\ b_m\frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + b_{m-1}\frac{\mathrm{d}^{m-1} y(t)}{\mathrm{d}t^{m-1}} + \cdots + b_1\frac{\mathrm{d}u(t)}{\mathrm{d}t} + b_0 u(t) \end{gathered} \tag{3.3}$$

In order to obtain the transfer function of our system we will take the Laplace transform on both sides of the equation and impose that our initial conditions are zero:

$$\begin{gathered} (s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = \\ (b^m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)U(s) \end{gathered} \tag{3.4}$$

Thus, from this expression, the definition of the transfer function comes straightforward,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b^m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)} \tag{3.5}$$

3.2.1.1 Laplace Transfer Function from State Space equations

Consider the LTI system in its state space form Equation 2.2.1

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

taking the Laplace transform of this system we obtain

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

and since the general form of the transfer function is Y(s)/U(s), we can rewrite the previous equations so that

$$sX(s) = AX(s) + BU(s)$$

$$sX(s) - AX(s) = BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$

Substituting this in the output equation

$$Y(s) = CX(s) + DU(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Recall the similarities between the expression sI - A with the equation used to find the eigenvalues of a matrix, $det(\lambda I - A)$. This illustrates why the characteristic equation of a system described by the state matrix eigenvalue structure also describes the pole structure in the Laplace domain.

3.2.1.2 Properties of the transfer function

1. The transfer function is only defined for linear and time invariant systems. it is not defined for non linear systems.

2. The transfer function between the input and output variables of a system is defined by the Laplace transform of the impulse response. Thus, the transfer function relates the Laplace transform of the output variables and the Laplace transform of the input variables.

3. To obtain the transfer function of a system we need its initial conditions to be zero.

4. The transfer function is independent of the input of the system.

5. The transfer function of a continuous system is defined as a function of s, with s complex variable. The transfer function of discrete time difference equation systems (subsection 3.2.4) is defined as a function of z where the Z transform is used, and which we will present later in this chapter.

Definition 21 (Proper and Strictly Proper Transfer Function)**.** *We say that the transfer function of the Equation 3.5 is strictly proper if* $deg(U(s)) = n > m = deg(Y(s))$*. If* $n = m$ *the we say that the transfer function is proper.*

Example 3.2.1 (Strictly Proper, Proper and Non Proper Transfer functions)**.** *Here we present one example of a strictly proper, a proper and a non proper transfer function:*

1. *Strcitly proper:*

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

2. Proper:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

3. Non proper:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

Definition 22 (characteristic equation)**.** *Consider again our reference linear system Equation 3.3 with its reference transfer function Equation 3.5. We define the characteristic equation of our transfer function as*

$$U(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

3.2.2 Transfer Function of Multivariable systems (MIMO)

Generally, if a linear system has p inputs and q outputs we can define the transfer function that relates the j-th input with the i-th output as

$$G_{ij}(s) = \frac{Y_i(s)}{R_j(s)} \tag{3.6}$$

with $R_k(s) = 0, \quad k = 1, 2, \cdots, p, \quad k \neq j.$

Hence, when all the p entries are considered at a time, the i-th output transfer function can be written as

$$Y_i(s) = G_{i1}(s)R_1(s) + G_{i2}(s)R_2(s) + \cdots + G_{ip}(s)R_p(s) \tag{3.7}$$

we can express the latter in matrix form as

$$Y(s) = G(s)R(s) \tag{3.8}$$

where

$$Y(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_q(s) \end{bmatrix}; \quad R(s) = \begin{bmatrix} R_1(s) \\ R_2(s) \\ \vdots \\ R_p(s) \end{bmatrix}; \quad \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{1p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{q1}(s) & G_{q2}(s) & \cdots & G_{qp}(s) \end{bmatrix} \tag{3.9}$$

3.2.3 Block Diagrams

Block diagrams are often used by control engineers to model every kind of systems. A block diagram, together with transfer function, can describe how the system components are connected without further mathematical details. Consequently, if we know the mathematical and functional relationship of each of

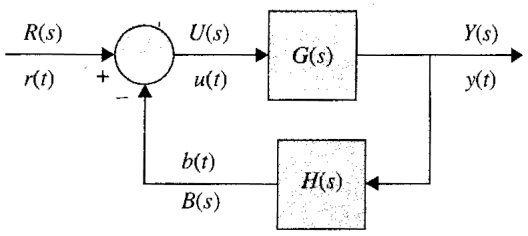


Figure 3.7: Block diagram of a SISO (i.e. single input single output) system.

the elements of the system, the block diagram serves as a tool that enable us to obtain the analytical solution of the system and to compute it.

In Figure 3.7 it is shown the block diagram of a linear feedback control system, whose main components are:

$r(t), R(s)$ = command or reference input.
$y(t), Y(s)$ = output or controlled variable.
$b(t), B(s)$ = feedback signal
$u(t), U(s)$ = acting signal which is the error signal e(t), E(s) when $H(s) = 1$
$H(s)$ = feedback transfer function
$G(s)H(s)$ = L(s) loop transfer function
$G(s)$ = transfer function of the direct path
$M(s) = Y(s)/R(s)$ = transfer function of the closed loop or the system.

We can express the transfer function of the closed loop $M(s)$ using $G(s), H(s)$ such that

$$Y(s) = G(s)U(s) \tag{3.10}$$

and

$$B(s) = H(s)Y(s) \tag{3.11}$$

Furthermore, we can define the acting signal as

$$U(s) = R(s) - B(s) \tag{3.12}$$

substituting Equation 3.12 on Equation 3.10 we get

$$Y(s) = G(s)R(s) - G(s)B(s) \tag{3.13}$$

and substituting Equation 3.11 on Equation 3.13 and solving for $Y(s)/R(s)$ we get the transfer function

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{3.14}$$

Recall that in general a control system can contain more than a closed loop.

Let us now address the multivariable case. We will designate input and output signals individually.

From Figure 3.8 we can express the relationships of the different components of the system as

$$\begin{aligned} Y(s) &= G(s)U(s) \\ U(s) &= R(s) - B(s) \\ B(s) &= H(s)Y(s) \end{aligned} \tag{3.15}$$

where $Y(s)$ is the output vector with dimensions $q \times 1$; $U(s), R(s), B(s)$ are vectors of with dimension $p \times 1$, and $G(s), H(s)$ are the matrix of two transfer functions with dimension $q \times p$ and $p \times q$ respectively.

Substituting the third equation of Equation 3.15 on the second one and then the second one to the first one we obtain

$$Y(s) = G(s)R(s) - G(s)H(s)Y(s) \tag{3.16}$$

solving $Y(s)$ from Equation 3.16 we get

$$Y(s) = [I + G(s)H(s)]^{-1} G(s)R(s) \tag{3.17}$$

and considering that the determinant of $I + G(s)H(s)$ is not zero, the matrix of the closed loop can be defined as

$$M(s) = [I + G(s)H(s)]^{-1} G(s) \tag{3.18}$$

accordingly, we can rewrite Equation 3.17 as

$$Y(s) = M(s)R(s) \tag{3.19}$$

Example 3.2.2. *In figure Figure 3.9 it is illustrated a reduction of a multiple loop system represented as block diagram subsection 3.2.3 and the derivation of its transfer function.*

3.2.4 Transfer Function on Discrete dynamical systems

When we consider a discrete- time dynamical system, inputs and outputs can be represented as samples of sequences separated by a sample period h. Assume the continuous time data $r(t)$ is sampled through an ideal sampler $r^*(t)$ with a sample period T. The output of this ideal sampler is basically a impulse train (i.e. train of action potentials spaced over time, with time varying intervals between them) with the values of $r(t)$ at a sample period T driven by the impulses. Intuitively this ideal sampler cannot be found in the physical world, we just use it as an assumption to represent mathematically the time discrete

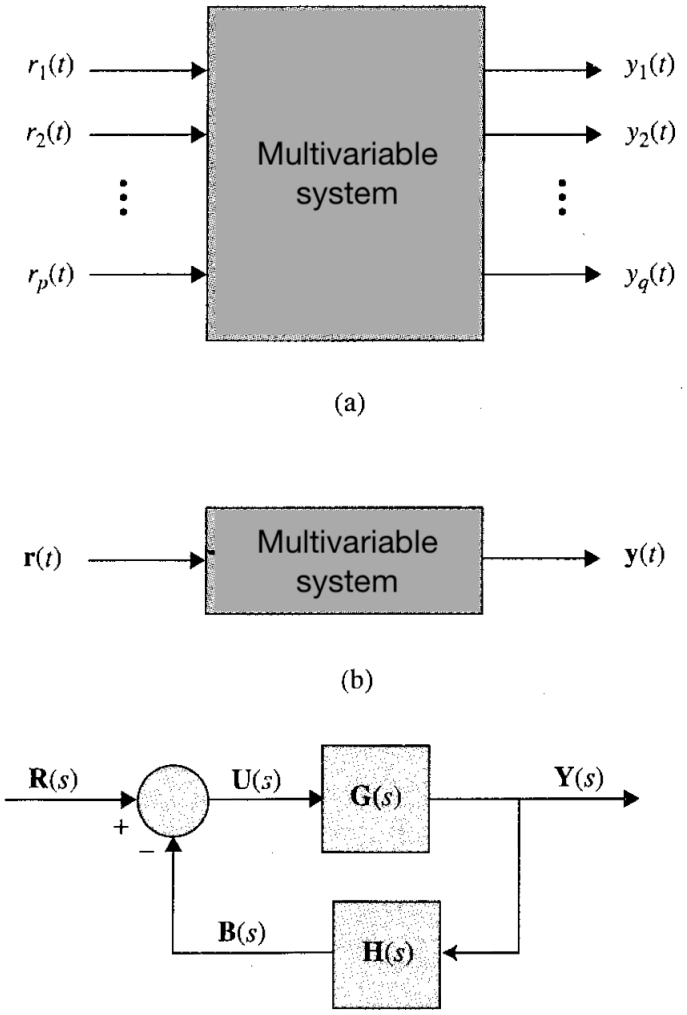


Figure 3.8: Block diagram of a MIMO (i.e. multiple input multiple output) system

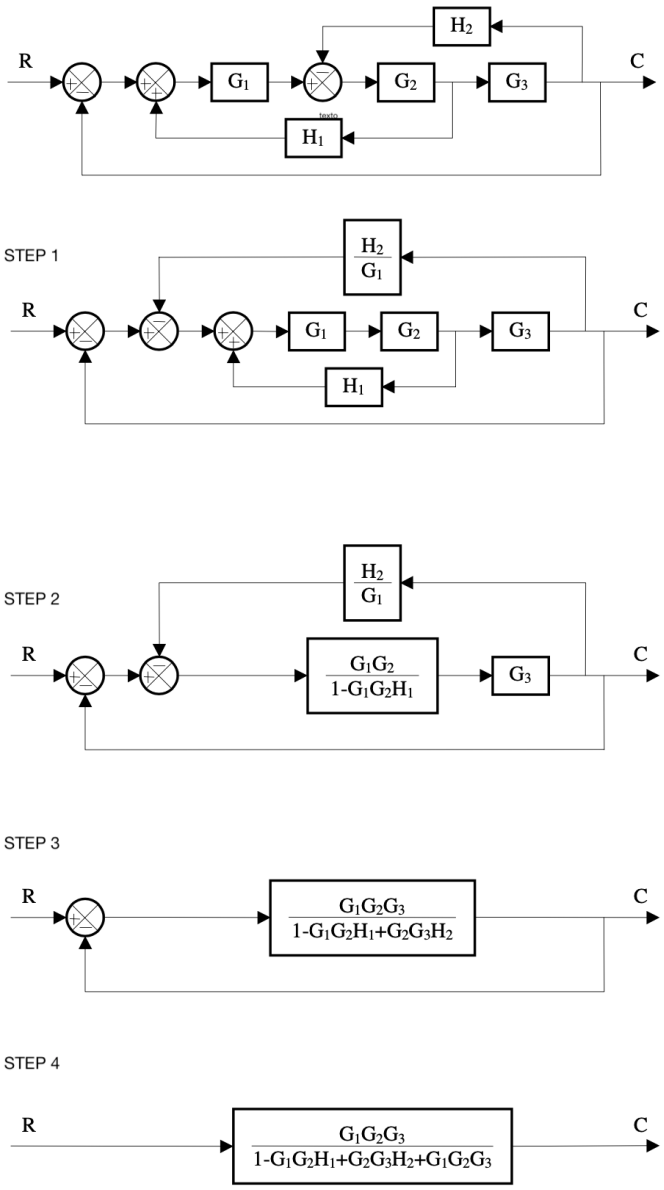


Figure 3.9: Successive reduction of a multiple loop Block diagram

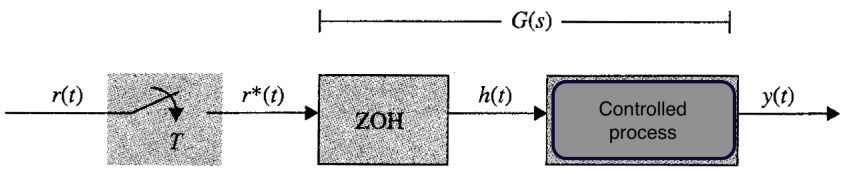


Figure 3.10: Block diagram of a discrete- time system

signal. In order to reconstruct the missing signal between the sampling period points we will use a hold operator. This operator determines the process input until a new number from the sequence (sample) is delivered, making the input continuous.

As a reference example of hold operator we will consider the simplest one, the zero order hold, ZOH, which is also the one applied in [1] for its simplicity given the results of more complex methods do not make a big difference when approximating the coupling variables.

The ZOH operator is defines the value of the signal at $t = kT$ to be constant until the next impulse arrives at $t = (k + 1)T$.

To obtain the transfer function of the system in Figure 3.10 we use the Fourier representation of the signal $r^*(t)$.

$$r^*(t) = r(t)\delta_T(t) \tag{3.20}$$

where $\delta_h(t)$ is the train of unitary impulses

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(kT) \tag{3.21}$$

Moreover, $\delta_h(t)$ is a periodic function of period h that can be described as the Fourier serie

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{-j2\pi nt/T} \tag{3.22}$$

where C_n is the Fourier coefficient

$$C_n = \frac{1}{T}\int_0^T \delta_T(t)e^{-jn\omega_s t}\mathrm{d}t \tag{3.23}$$

and $\omega_s = 2\pi/T$ is the sampling frequency.

Since the unitary impulse is defined as a pulse with width δ and height $1/\delta$

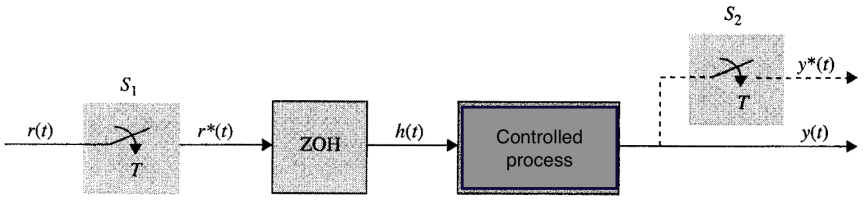


Figure 3.11: Block diagram of a time discrete system with a ficticious sampler

when $\delta \to 0$, C_n can be written as

$$C_n = \lim_{\delta \to 0} \frac{1}{T\delta} \int_0^{\delta} e^{-jn\omega_s t} \mathrm{d}t = \lim_{\delta \to 0} \frac{1 - e^{-jn\omega_s t}}{jn\omega_s T\delta} = \frac{1}{T} \tag{3.24}$$

Substituting Equation 3.24 in Equation 3.20 we get

$$r^*(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} r(t) e^{-jn\omega_s t} \tag{3.25}$$

and taking the Laplace transform of this last expression and using its translation property (Equation 2.16) we get

$$R^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} R(s - jn\omega_s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} R(s + jn\omega_s) \tag{3.26}$$

where equation Equation 3.26 represent the Laplace transform of the sample signal $r^*(t)$. We can rewrite this equation as

$$R^*(s) = \sum_{k=0}^{\infty} r(kT) e^{-kTs} \tag{3.27}$$

and since the limits of the sum of $R^*(s)$ go from $-\infty$ to ∞, replacing s with $s + jn\omega_s$ in Equation 3.26 we get

$$R^*(s + jm\omega_s) = R^*(s) \tag{3.28}$$

3.2.4.1 Pulse Transfer Function

The Laplace transform of the output $y(t)$ of the system can be described as

$$Y(s) = G(s)R^*(s) \tag{3.29}$$

We denote the sample form of $y(t)$ by $y^*(t)$. Applying Equation 3.28 to Equation 3.29 and the translation property of the Laplace transform (Equation 2.16) we get

$$Y^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} G(s + jn\omega_s) + R^*(s + jn\omega_s) \tag{3.30}$$

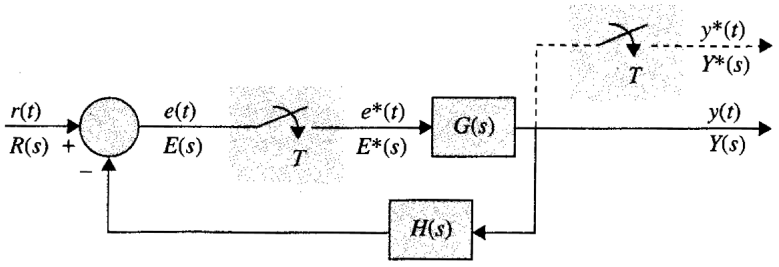


Figure 3.12: Block diagram of a time discrete closed loop system

using the equality Equation 3.26 we can rewrite Equation 3.30 as

$$Y^*(s) = R^*(s)\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jn\omega_s) = R^*(s)G^*(s) \tag{3.31}$$

where $G^*(s)$ is defined in the same way as $R^*(s)$ does in Equation 3.26 and is called **Pulse Transfer Function** of $G(s)$.

Now that all functions in Equation 3.31 are sampled, where $G^*(s), R^*(s), Y^*(s)$ are of the form Equation 3.27, we take the Z transform Equation 2.26 on both sides of their equations using that $z = \mathrm{e}^{Ts}$ obtaining

$$Y(z) = G(z)R(z) \tag{3.32}$$

where $G(z)$ is defined as the Z transfer function Equation 2.26 of G(s).

Example 3.2.3 (Closed loop Transfer Function of a discrete time systems)**.** *Consider the Figure 3.14, where the output of the sampler is the input of the system, so that the system has inputs $R(s)$ and $E^*(s)$, and outputs $E(s)$, $Y(s)$ we can write the cause effect equations of the system as follows*

$$\begin{aligned} E(s) &= R(s) - G(s)H(s)E^*(s) \\ Y(s) &= G(s)E^*(s) \end{aligned} \tag{3.33}$$

Taking the pulse transfer function of the first member in Equation 3.33 and solving for $E^*(s)$ we get

$$E^*(s) = \frac{R^*(s)}{1 + [G(s)H(s)]^*} \tag{3.34}$$

now, if we substitute Equation 3.34 in the second memeber of Equation 3.33 we get

$$Y(s) = \frac{G(s)R^*(s)}{1 + [G(s)H(s)]^*} \tag{3.35}$$

taking the pulse transform in both sides of the equation Equation 3.35 and using Equation 3.28 we get the close loop transfer function

$$\frac{Y^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + [G(s)H(s)]^*}$$

Finally taking the Z transform on both sides of the equation we obtain

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} \tag{3.36}$$

3.2.4.2 Z Transfer Function of ZOH

The transfer function of the zero order hold operator is defined as

$$G_{zoh}(s) = \mathcal{L}\left[g_{zoh}(t)\right] = \frac{1 - \mathrm{e}^{-Ts}}{s} \tag{3.37}$$

Hence, the combination of the transfer function of the hold operator and the process becomes

$$G(z) = \mathcal{L}\left[G_{zoh}(s)G_P(s)\right] = \tilde{\mathcal{L}}\left(\frac{1 - \mathrm{e}^{-Ts}}{s}G_P(s)\right) \tag{3.38}$$

now, applying the time delay property (Equation 2.3.1.1) of the Z transform to Equation 3.38 we obtain

$$G(z) = (1 - z^{-1})\tilde{\mathcal{L}}\left(\frac{(G_P(s)}{s}\right) \tag{3.39}$$

which corresponds to the Z Transform of the zero order hold operator.

Example 3.2.4 (ZOH Transfer Function combined with the Transfer Function of a system)**.** *Consider the system in Figure 3.10 with transfer function*

$$G_p(s) = \frac{1}{s(s + 0.5)} \tag{3.40}$$

with sample period equal to 1. Applying Equation 3.39 we obtain the Z transform of the system

$$\begin{aligned} G(z) &= (1 - z^{-1})\tilde{\mathcal{L}}\left(\frac{1}{s^2(s + 0.5)}\right) \\ (1 - z^{-1})\tilde{\mathcal{L}}\left(\frac{2}{s^2} - \frac{4}{s} + \frac{4}{s + 0.5}\right) &= \frac{0.426z + 0.361}{z^2 - 1.606z + 0.606} \end{aligned} \tag{3.41}$$

3.2.5 State matrix and transition state equation of a continuous system

Consider the state equation

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.42}$$

Consider now the initial state vector $\mathbf{x}(t_0)$ and the input vector $\mathbf{u}(t)$ for $t \geq t_0$. We define the first term of Equation 3.42 as the homogeneous part of the state equation.

Definition 23. *We define $\phi(t)$, the transition state matrix with dimensions $n \times n$ of the state equation Equation 3.42. $\phi(t)$ satisfies*

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \boldsymbol{A}\phi(t) \tag{3.43}$$

In particular, given the initial state $\boldsymbol{x}(0)$ at $t = 0$, we define $\phi(t)$ as

$$\boldsymbol{x}(t) = \phi(t)\boldsymbol{x}(0) \tag{3.44}$$

the solution of the homogeneous state equation at $t \geq 0$.

One way to determine $\phi(t)$ is to take the Laplace transform Definition 5 on both sides of the equation $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ Equation 3.42

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) \tag{3.45}$$

By solving this equation for $\mathbf{X}$ we obtain

$$\mathbf{X}(s) = (\mathbf{sI\text{-}A})^{-1}\mathbf{x}(0) \tag{3.46}$$

where we need $(\mathbf{sI\text{-}A})$ not to be singular (i.e. its determinant has to be different from 0). Now taking the inverse Laplace transform in both parts of the equation we get

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(\mathbf{sI\text{-}A})^{-1}\right]\mathbf{x}(0) \quad t \geq 0 \tag{3.47}$$

Comparing Equation 3.44 with Equation 3.47 the transition state matrix can be defined as

$$\phi(t) = \mathcal{L}^{-1}\left[(\mathbf{sI\text{-}A})^{-1}\right] \tag{3.48}$$

Now consider the whole state equation Equation 3.42, taking the Laplace transform on both sides we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{3.49}$$

where $\mathbf{x}(0)$ is the initial state vector at $t = 0$, solving Equation 3.49 for $\mathbf{X}$(s)

$$\mathbf{X}(s) = (\mathbf{sI\text{-}A})^{-1}\mathbf{x}(0) + (\mathbf{sI\text{-}A})^{-1}\left[\mathbf{B}\mathbf{U}(s)\right] \tag{3.50}$$

Thus, the transition state equation of Equation 3.42 is given taking the inverse Laplace transfer function on both sides of Equation 3.50

$$\begin{aligned}\mathbf{x}(t) &= \mathcal{L}^{-1}\left[(\mathbf{sI\text{-}A})^{-1}\right]\mathbf{x}(0) + \mathcal{L}^{-1}\left[(\mathbf{sI\text{-}A})^{-1}\left[\mathbf{BU}(s)\right]\right] \\ &= \phi(t)\mathbf{x}(0) + \int_0^1 \phi(t-\tau)\mathbf{BU}(\tau)\mathrm{d}\tau \quad t \geq 0\end{aligned} \tag{3.51}$$

3.2.6 State Matrix and transition state equation of a discrete system

Consider the discrete time control system in Figure 3.11. The output signal is a continuous time signal that corresponds to the output of the sampler device, $f(t)$, a sequence of steps that can be described as

$$f(t) = f(kT); \quad kT \leq t < (k+1)T \quad , k = 0, 1, 2... \tag{3.52}$$

Consider the process G with the equations

$$\begin{aligned}\frac{d\mathbf{x}(t)}{\mathrm{d}t} &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}f(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}f(t)\end{aligned} \tag{3.53}$$

where $\mathbf{x}$(t) is a $n \times n$ state vector, $f(t)$ and $y(t)$ are the scalar input and output of the system respectively, and **A, B, C, D** the coefficient matrices.

Definition 24 (State transition equation)**.** *Given the system Equation 3.53 we define the state transition equation as*

$$\boldsymbol{x}(t) = \phi(t-t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \phi(t-\tau)\boldsymbol{B}f(\tau)\mathrm{d}\tau \; t \geq t_0 \tag{3.54}$$

Particularly, if our interest is restricted to the response of the sample instants we can apply $t = [(k+1)T]$ and $t_0 = kT$ so that Equation 3.54 turns to

$$\boldsymbol{x}\left[(k+1)T\right] = \phi(T)\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T} \phi\left[(k+1)T - \tau\right]\boldsymbol{B}f(\tau)\mathrm{d}\tau \tag{3.55}$$

Since by the definition of $f(t)$ given in Equation 3.52 $f(t)$ is partially constant, from the input $f(\tau)$ in Equation 3.55 we can take the sign out of the integral so that

$$\mathbf{x}\left[(k+1)T\right] = \phi(T)\mathbf{x}(kT) + \int_{kT}^{(k+1)T} \phi\left[(k+1)T - \tau\right]\mathbf{B}\mathrm{d}\tau f(kT) \tag{3.56}$$

or

$$\mathbf{x}\left[(k+1)T\right] = \phi(T)\mathbf{x}(kT) + \theta(T)f(kT) \tag{3.57}$$

where

$$\theta(h) = \int_{kT}^{(k+1)T} \phi\left[(k+1)T - \tau\right] \mathbf{B} \mathrm{d}\tau = \int_{0}^{T} \phi(T-\tau)\mathbf{B}\mathrm{d}\tau \tag{3.58}$$

Moreover, with nh ($n \in \mathbb{Z}^+$) considered the initial time, the transition state equation can be rewritten as

$$\mathbf{x}\left[(n+N)T\right] = \phi^N(T)\mathbf{x}(nT) + \sum_{i=0}^{N-1} \phi^{N-i-1}(h)\theta(T) f\left[(n+i)T\right] \tag{3.59}$$

where $N \in \mathbb{Z}^+$.

On the other hand, the output of the system on the sample instants is obtained substituing $t = nh$ and Equation 3.57 on Equation 3.53

$$\begin{aligned} y(nT) &= \mathbf{C}\mathbf{x}(nh) + \mathbf{D}f(nT) \\ = \mathbf{C}\phi(nh)\mathbf{x}(0) + \mathbf{C}\sum_{i=0}^{n-1} &\phi\left[(n-i-1)T\right]\theta(T) f(iT) + \mathbf{D}f(nh) \end{aligned} \tag{3.60}$$

Definition 25 (Equations of system described only with discrete signals)**.** *When a linear system has only discrete signals, its dynamic can be described with the equations*

$$\begin{aligned} \boldsymbol{x}\left[(k+1)T\right] &= \boldsymbol{A}\boldsymbol{x}(kT) + \boldsymbol{B}\boldsymbol{r}(kT) \\ \boldsymbol{y}(kT) &= \boldsymbol{C}\boldsymbol{x}(kT) + \boldsymbol{D}\boldsymbol{r}(kT) \end{aligned} \tag{3.61}$$

We can also find a solution for these discrete state equations making use of the Z transform.

Consider

$$\mathbf{x}\left[(k+1)T\right] = \mathbf{A}\mathbf{x}(kT) + \mathbf{B}\mathbf{r}(kT) \tag{3.62}$$

taking the Z transform on both sides of the equation we get

$$z\mathbf{X}(z) - z\mathbf{x}(0) = \mathbf{A}\mathbf{X}(z) + \mathbf{B}\mathbf{R}(z) \tag{3.63}$$

solving for $\mathbf{X}(z)$

$$\mathbf{X}(z) = (z\mathbf{I}\text{-}\mathbf{A})^{-1} z\mathbf{x}(0) + (z\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{B}\mathbf{R}(z) \tag{3.64}$$

finally, taking the inverse z transform in both sides of the previous equation we get

$$\mathbf{x}(nh) = \mathcal{L}^{-1}\left[(z\mathbf{I}\text{-}\mathbf{A})^{-1} z\right]\mathbf{x}(0) + \mathcal{L}^{-1}\left[(z\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{B}\mathbf{R}z\right] \tag{3.65}$$

3.2.7 Transfer Function and characteristic equation of a discrete system

Once our discrete system is modelled with Equation 3.61 and the initial state $\mathbf{x}(0)$ is established as zero so that Equation 3.64 is turned into

$$\mathbf{X}(z) = (z\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{B}\mathbf{R}(z) \tag{3.66}$$

and substituting this previous equation in the second equation of Equation 3.61we get

$$\mathbf{Y}(z) = \left[\mathbf{C}(z\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{R}(z) = G(z)\mathbf{R}(z) \tag{3.67}$$

from where we can define the transfer function as

$$\mathbf{G}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{3.68}$$

and we can define the characteristic equation (see Definition 22) as

$$|z\mathbf{I} - \mathbf{A}| = 0 \tag{3.69}$$

Definition 26 (LTI discrete system)**.** *We can describe a general, linear, time invariant (LTI), discrete system with the difference equation of constant coefficients*

$$\begin{aligned} y\left[(k+n)h\right] + a_{n-1}y\left[(k+n-1)T\right] + a_{n-2}y\left[(k+n-2)T\right] \\ +\ldots + a_1 y\left[(k+1)T\right] + a_0 y(kT) \\ = b_m r\left[(k+m)T\right] + b_{m-1}r\left[(k+m-1)T\right] \\ +\ldots + b_1 r\left[(k+1)T\right] + b_0 r(kT) \end{aligned} \tag{3.70}$$

with $n \geq m$.

Taking the Z transform in both sides of Equation 26 and establishing zero initial conditions, we can write the transfer function as

$$\frac{Y(z)}{R(z)} = \frac{b_m z^m + b_{m-1}z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} \tag{3.71}$$

where the characteristic equation is obtained from the denominator

$$z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0 \tag{3.72}$$

3.3 Controlability and Observability of a System

Now we will characterize a control system analysing its controllability and observability, two crucial properties of the behaviour of control system proposed by Kalman.

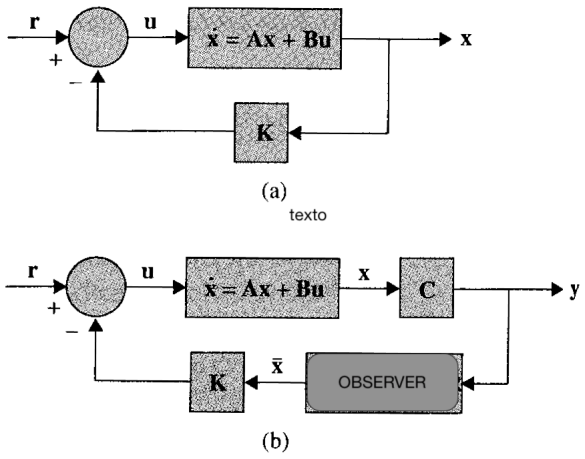


Figure 3.13: Control system with feedback and control system with feedback and an observer

To understand the motivation under its investigation and the key conditions that characterize whether a control system is conrollable let us look at the systems in Figure 3.13 where the process is described by the state equation

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.73}$$

and the close loop system of the figure arises from applying feedback to the state variables through the feedback gain matrix $\mathbf{K}$ so that

$$u(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{r}(t) \tag{3.74}$$

so that, the closed loop system can be described by

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{(A\text{-}BK)}\mathbf{x}(t) + \mathbf{B}\mathbf{r}(t) \tag{3.75}$$

The objective of this design is to find a feedback matrix $\mathbf{K}$ thank to which the eigenvalues of **A-BK** (or the closed loop system) have specific desired values. The existence of solution assigning poles arbitrarily (i.e. the eigenvalues of the transfer function of the closed loop system **A-BK**) through the feedback of the states, is directly based on the controllability of the states of the system.

Thus, we can characterize the controllability of our system Equation 3.73 with the following proposition:

Proposition 27. *If the system represented in the Equation 3.73 is controllable, then it exists a feedback matrix that allows the eigenvalues of* ***A-BK*** *to be assigned arbitrarily.*

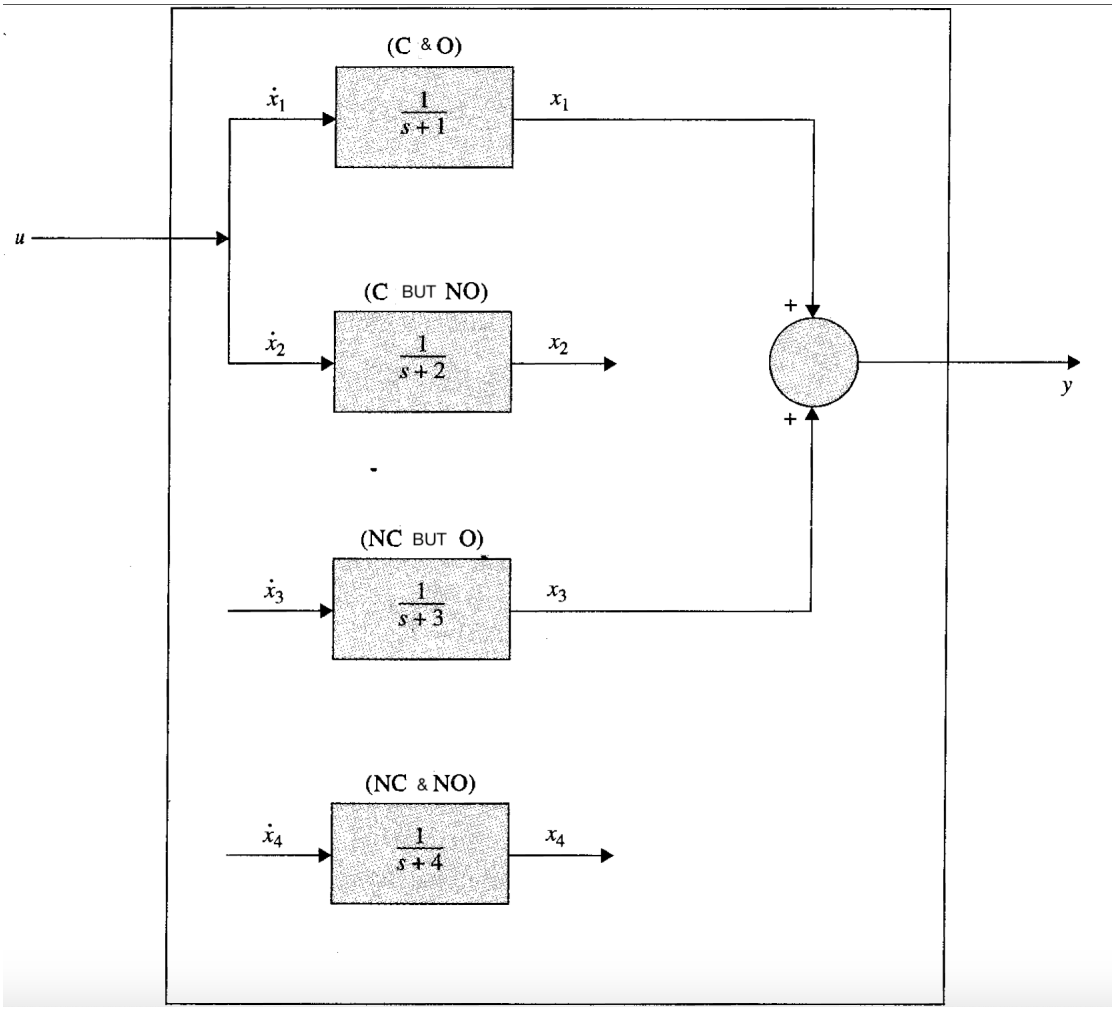


Figure 3.14: Block diagram where we can find controllable (C), non- controllable (NC), observable (O) and non- observable (NO) components

Now that we have an intuitive notion of what controllability means in control theory let us formally define this concept.

3.3.1 Controllability

Definition 28 (Completely Controllable system)**.** *We say that a process is completely controllable if each state variable of the process can be controlled to reach a certain objective in a finite amount of time, through some unrestricted control of* $\boldsymbol{u}(t)$*.*

Definition 29 (Controllability of the states)**.** *Consider the LTI system*

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned} \tag{3.76}$$

We say that the state $\boldsymbol{x}(t)$ *is controllable at* $t = t_0$ *if it exists a continuous input through* $\boldsymbol{u}(t)$ *intervals that will move the state to some final state* $\boldsymbol{x}(t_f)$ *in a finite time* $(t_f - t_0) \geq 0$*.*

If each state $x(t_0)$ *of the system is controllable in a finite interval of time we say that the system is completely controllable.*

3.3.2 Observability

Essentially, a system is completely observable if each state variable of the system has an effect on some of the outputs.

Definition 30 (Observability)**.** *Considering the LTI system in Equation 3.76 we say that the sate* $\boldsymbol{x}(t_0)$ *is observable if for each input* $\boldsymbol{u}(t)$ *exists a time* $t_f \geq t_0$ *such that all the knowledge we have from* $\boldsymbol{u}(t)$*,* $\boldsymbol{A}$*,* $\boldsymbol{B}$*,* $\boldsymbol{C}$*,* $\boldsymbol{D}$*, and the output* $\boldsymbol{y}(t)$ *at* $t_0 \leq t \leq t_f$ *, are enough to determine* $\boldsymbol{x}(t_0)$*.*
If every state of the system is observable for a certain t_f *finite, we say that the system is completely observable*

Example 3.3.1 (Illustrated controllability and observability with different block diagrams)**.** *Controllability and observability is illustrated in a block diagram basis on Figure 3.14*

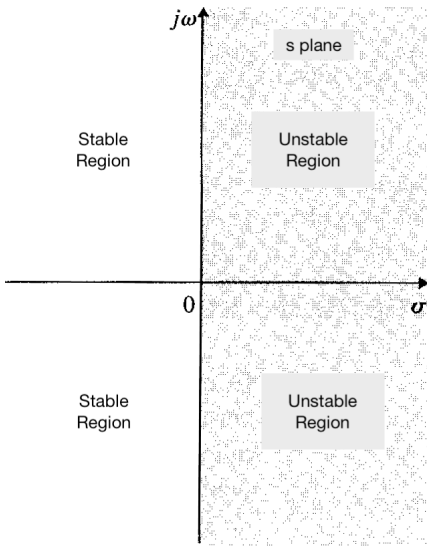


Figure 3.15: Stable and unstable regions in the s plane

3.4 Stability

Intuitively a system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. In control theory the most important form of stability for systems and signals that take inputs is BIBO stability, where a stable system produces a bounded output for a given bounded input.

3.4.1 BIBO stability for continuous systems

Consider $u(t), y(t), g(t)$ the input, output and impulse response of a SISO LTI system with initial conditions equal to 0; We say that a system under this premises is a bounded- input/ bounded- output (BIBO) system if its output $y(t)$ is bounded for a bounded input $u(t)$. The integral that relates $u(t), y(t), g(t)$ is

$$y(t) = \int_0^\infty u(t-\tau)g(\tau)\mathrm{d}\tau \tag{3.77}$$

taking the absolute value of this expression on both sides

$$|y(t)| = \left| \int_0^\infty u(t-\tau)g(\tau)\mathrm{d}\tau \right| \tag{3.78}$$

or

$$|y(t)| \leq \int_0^\infty |u(t-\tau)|\,|g(\tau)|\,\mathrm{d}\tau \tag{3.79}$$

where if $u(t)$ is bounded

$$|u(t)| \leq M \tag{3.80}$$

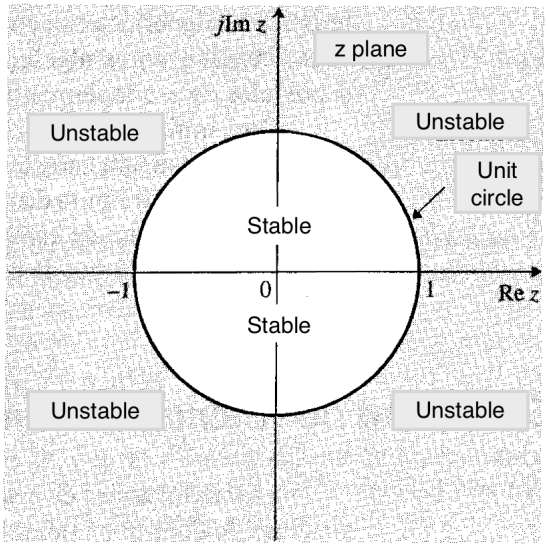


Figure 3.16: Stable and unstable regions in the z plane

and being M finite and positive definite we have that,

$$|y(t)| \leq M \int_0^\infty |g(\tau)| \, \mathrm{d}\tau \tag{3.81}$$

thus, if $y(t)$ is bounded or $|y(t)| \leq N < \infty$ with N finite positive number, the following condition needs to remain

$$M \int_0^\infty |g(\tau)| \, \mathrm{d}\tau \leq N < \infty \tag{3.82}$$

or for any finite positive Q

$$\int_0^\infty |g(\tau)| \, \mathrm{d}\tau \leq Q < \infty \tag{3.83}$$

which means that the surface under the curve $|g(\tau)|$ has to be finite.

Let us now explain how the roots of the characteristic equation Definition 22 are related with BIBO stability. Firstly, let us define the transfer function $G(s)$ of a system in the Laplace domain as

$$G(s) = \mathcal{L}\left[g(\tau)\right] = \int_0^\infty g(t) \mathrm{e}^{-st} \mathrm{d}t \tag{3.84}$$

taking the absolute value of this expression

$$G(s) = \left| \int_0^\infty g(t) \mathrm{e}^{-st} \mathrm{d}t \right| \leq \int_0^\infty |g(t)| \left| \mathrm{e}^{-st} \right| \mathrm{d}t \tag{3.85}$$

since $|e^{-st}| = |e^{-\sigma t}|$ where σ is the real part of s, if it is the case that s adopts a pole of $G(s)$ then $G(s) = \infty$ so that

$$\infty \leq \int_0^\infty |g(t)| \left|e^{-st}\right| dt \tag{3.86}$$

and if one or more roots of the characteristic equation are in the half right plane s or in the $j\omega$ axis so that $\omega \geq 0$ then

$$\left|e^{-\sigma t}\right| \leq M = 1 \tag{3.87}$$

so that Equation 3.86 turns into

$$\infty \leq \int_0^\infty M |g(t)| dt = \int_0^\infty |g(t)| dt \tag{3.88}$$

which violets the BIBO stability requirements.

Proposition 31 (BIBO Stability)**.** *We say BIBO stability requirements are fullfilled when the roots of the characteristic equation (i.e. the poles of $G(s)$) are not located in the half right plane or in the $j\omega$ axis, i.e. all of them need to be located on the left half plane s.*

3.4.2 BIBO stability for discrete systems

Consider $u(kh), y(kh), g(kh)$ the input, output and the impulse sequence of a discrete time SISO LTI system, with zero initial conditions; we say that a system under this premises is BIBO stable if its output sequence $y(kh)$ is bounded for a bounded input $u(kh)$.

Property 32. *If a discrete time system is BIBO stable it satisfies*

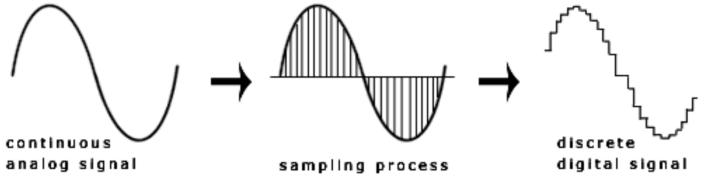
$$\sum_{k=0}^{\infty} |g(kh)| < \infty \tag{3.89}$$

Proposition 33. *BIBO stability for discrete time systems require that the roots of the characteristic equation are located inside the unit circle $|z| = 1$*

3.5 Sampling and Reconstruction

Our computers can deal with discrete- time signals, but they cannot directly handle the continuous-time signals that are prevalent in the physical world. In this section we present the interface between the continuous and discrete time signals.

A discrete-time signal is constructed by sampling a continuous-time signal, and a continuous-time signal is reconstructed by interpolating a discrete-time signal.



3.5.1 Sampling

Before presenting the main results of this section, let us introduce a set of basic definitions regarding the sampling process and the reconstruction of the signal of our control system.

Definition 34 (Sampling process)**.** *We call sampling process to the conversion of the output signal of a process into a finite number of numbers (signals).*

Usually in signal theory an Analogue- Digital (A-D) converter is used to get a finite precise digital number from an analog output depending on how many bits or levels are used in the conversion.

Example 3.5.1 (Audio Sampling)**.** *When we are recording music we want to capture audio covering the entire range of human hearing, i.e. 20–20000 Hz. In this specific cases, audio wave- forms are typically sampled at 44.1 kHz, 48 kHz, 88.2 kHz, or 96 kHz. This approximately double-rate requirement is a consequence of the Nyquist theorem (Sampling theorem 65). Indeed, sampling rates higher than about 50 kHz to 60 kHz cannot supply more usable information for human listeners.*

Example 3.5.2. *Another, more general, sampling process example will be functions that vary with time. If we set a continuous function or signal $f(t)$ to be sampled, and assume the sampling is performed by measuring the value of the continuous function every h seconds, where we say h is the sampling period, then the sampled function is given by $f(nT)$ where $n \in \mathbb{Z}$.*

Definition 35 (Sampling Period)**.** *The time between two sampling instants is called the sampling period. Thus, when periodic sampling is applied, the output is measured and the control signal is applied at each k- th time unit.*

Definition 36 (Sampling Frequency)**.** *We define the Sampling frequency as $w_s = 2\pi/T$ where T is the sampling period or interval.*

Example 3.5.3 (Sampling Frequency)**.** *Since the units of the sampling frequency are samples per second or hertz, 48 kHz is equivalent to 48,000 samples per second ($1Hz = s^{-1}$).*

Property 37. *The sampling period is the inverse of the sampling frequency.*

Example 3.5.4 (Sampling period)**.** *In signal theory the sampling period can be seen specifically as the time difference between two consecutive samples in a*

$$\mathit{Sampler}_T\colon [\mathit{Reals} \to \mathit{Complex}] \to [\mathit{Integers} \to \mathit{Complex}],$$

Figure 3.17: Sampler for complex values signals

Sound. Consider a situation where the sampling frequency is 44100 Hz, then the sampling period is $1/44100 = 2.2675736961451248e^{-5}$ *seconds. Therefore, the samples are spaced approximately 23 microseconds apart.*

Example 3.5.5 (Time dependance)**.** *Programs that solve problems over time need to use data that are time dependent, i.e., information that changes with time. E.g* y_t *representing the temperature of the day* t *with* $t = 0, 1, 2...$*;* y_t *representing piece of a stock at a day* t*; or* y_{it} *representing weight ratio* i *at a day* t*.*

Definition 38 (Sampler)**.** *A sampler is a device that converts continuous- time signals into a sequence of numbers.*

Example 3.5.6 (Sampler)**.** *A theoretical ideal sampler would produce samples which are equivalent to the instant value of the continuous signal at desired points.*

3.6 Frequency Domain

We know from previous sections that the solution of a control system in the time domain is frequently harder to determine analytically, specially for systems with high order. On the other hand, in the frequency domain we can use graphical methods that are not limited to low order systems. Furthermore, the frequency domain is also more convenient when measuring the sensibility of the noise of a system or when studying the variation of parameters.

To start with, we develop the idea of transfer function in the frequency domain.

Definition 39 (Input and output of a system with sinusoidal time)**.** *We define the input of a linear system with sinusoidal time, amplitude* R *and frequency* ω_0

$$r(t) = R \cdot \operatorname{sen}(\omega_0 t) \tag{3.90}$$

The output in a stable state $y(t)$ *of the system will also be sinusoidal with frequency* ω_0 *but possibly with different amplitude and phase, so that*

$$y(t) = Y \cdot \operatorname{sen}(\omega_0 t + \phi) \tag{3.91}$$

where Y *is the sinusoidal wave amplitude and* ϕ *is the phase shift.*

Consider now the transfer function M(s) of the SISO linear system Figure 3.7,

$$Y(s) = M(s)R(s)$$

If we are in a permanent sinusoidal state, we replace s for $\jmath\omega$ so that

$$Y(j\omega) = M(j\omega)R(j\omega)$$

We can also write $Y(j\omega) = |Y(j\omega)| \angle Y(j\omega)$ with analogous definitions for $M(j\omega)$ and $R(j\omega)$. From this definitions and the definition of the transfer function of. Figure 3.7, $Y(s) = M(s)R(s)$, we can define both, the magnitude relation of our system

$$|Y(j\omega)| = |M(j\omega)|\,|R(j\omega)|$$

and its phase relation

$$\angle Y(j\omega) = \angle M(j\omega) + \angle R(j\omega)$$

Thus, for the input $r(t) = R \cdot \text{sen}(\omega_0 t)$ and the output $y(t) = Y \cdot \text{sen}(\omega_0 t + \phi)$ we define the sinusoidal amplitude of the output as $Y = R\,|M(j\omega_0)|$ and the sinusoidal output phase of the output as $\phi = \angle M(j\omega_0)$.

Therefore, once we know the transfer function $M(s)$ of a linear control system, its amplitude ($|M(j\omega)|$) and phase characterization ($\angle M(j\omega)$) will describe completely their steady state performance.

3.6.1 Frequency output of a closed loop system

Consider the transfer function of the closed loop system Figure 3.7

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{3.92}$$

which in a permanent sinusoidal state i.e. $s = j\omega$ turns into

$$M(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} \tag{3.93}$$

we can express then the transfer function in a permanent sinusoidal state $M(j\omega)$ in terms of its magnitude and phase characterization, so that

$$M(j\omega) = |M(j\omega)| \angle M(j\omega) \tag{3.94}$$

where the magnitude of $M(j\omega)$ is defined as

$$|M(j\omega)| = \left| \frac{G(j\omega)}{1 + G(j\omega)h(j\omega)} \right| = \frac{|G(j\omega)|}{1 + G(j\omega)H(j\omega)} \tag{3.95}$$

and the phase of $M(j\omega)$

$$\angle M(j\omega) = \phi_M(j\omega) = \angle G(j\omega) - \angle\left[1 + G(j\omega)H(j\omega)\right] \tag{3.96}$$

Now that we have introduced a new way of representing the transfer function of a control system when they belong to a permanent sinusoidal state, we will introduce the main concepts we will need to study in the frequency domain to analyse the behaviour of our control system.

Definition 40 (Cutting frequency)**.** *It is a limit in the frequency response of a system where the energy that flows through starts decreasing.*

Next we briefly introduce a tool that will be used in our our main chapter, chapter 6, to smooth the high frequency signals of our coupling error.

Definition 41 (Low- pass filter)**.** *A low pass filter only allows low frequency signals from 0Hz to the cutting frequency point ω_c to pass and blocks any higher signal.*

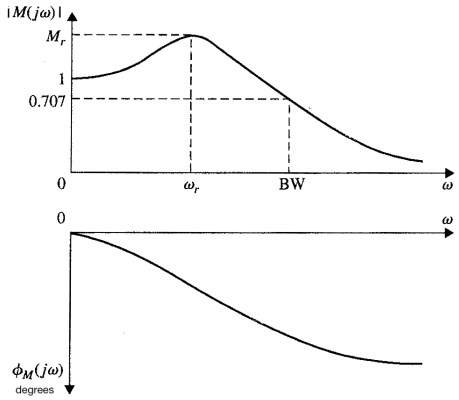


Figure 3.18: Characterization of a feedback control system

As it can be seen in the figure Figure 3.18 if the cutting frequency ω_c is infinitely increased, the output $Y(j\omega)$ could be identical to the input $R(j\omega)$ for all frequencies. Thus, a system could follow exactly a step input in the time domain.

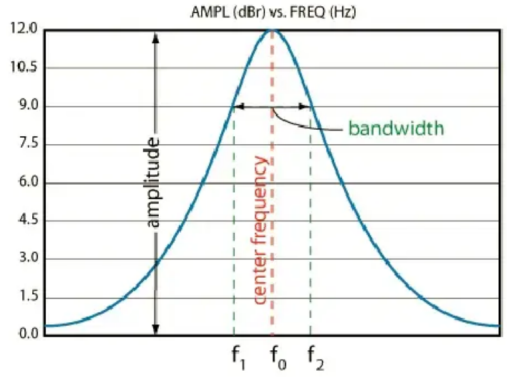
From equation Equation 3.95 it can be observed that $|M(j\omega)|$ can only be one for all frequencies if $G(j\omega) = \infty$, which is impossible to attain in real life. Indeed, it is undesirable since the majority of the control systems become unstable when their close loop is really high.

Furthermore, every control system are subject to noise during their operation, thus, apart from responding to the input, the system has to be able to reject and suppress the noise and the non-desirable signals.

Property 42. *Control systems with high frequency noise need to have a finite cutting frequency* ω_c

Definition 43 (Resonance Frequency)**.** *The resonance frequency* ω_r *is the frequency where the peak of resonance occur (see Figure 3.18), i.e.* M_r*, where* M *is the transfer function of the closed loop of a system.*

Definition 44 (Bandwidth)**.** *The bandwidth (BW) is the frequency where the magnitude of the closed loop system* $|M(j\omega)|$ *does not drop under -3 decibels.*



Proposition 45. *In general the bandwidth of a control system, gives an indication of the properties of the transient response (i.e. the response of the control system during the transient state, state where the output stays after applying an input to the control system until reaching the steady state) in the time domain and it is proportional to the speed of the response of the system to an input signal.*

Property 46. *A big bandwidth corresponds to a short increase of time in the response of the system given that the highest frequency pass more easily through the system. On the contrary, if the bandwidth is small only low frequency signals can pass the system and the answer will be slow.*

Property 47. *The bandwidth indicates the noise filtering and system robustness characterization of the system.*

Definition 48 (Robustness)**.** *Robustness represents a sensibility measure of a system with variation of parameters. A system is consider to be robust if it is not sensible to parameters variation.*

We will now illustrate the the concepts of cutting frequency, resonance frequency, and bandwidth studying how are they determined for a general control system of second order in the frequency domain.

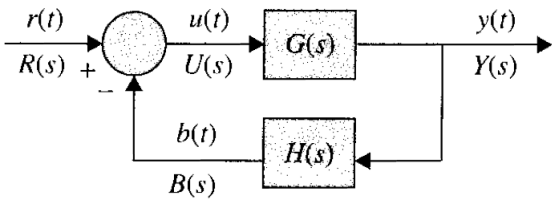


Figure 3.19: Control system with non- unitary feedback

3.6.2 M_r, ω_r and BW of a reference control system of second order

Let us now consider the system represented in Figure 3.19, M_r the resonance peak, ω_r resonance frequency and the bandwidth BW are related in a unique way with the damping factor ζ and with the natural frequency (undamped) of the system. The transfer function of the closed loop of the second order reference system Figure 3.19, M_r is

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3.97}$$

which in permanent sinusoidal state $s = j\omega$ is

$$\begin{aligned} M(s) = \frac{Y(j\omega)}{R(j\omega)} &= \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \\ &= \frac{1}{1 + j2(\omega/\omega_n)\zeta - (\omega/\omega_n)^2} \end{aligned} \tag{3.98}$$

which can be simplified by doing $u = \omega/\omega_n$ as

$$M(ju) = \frac{1}{1 + j2u\zeta - u^2} \tag{3.99}$$

where the magnitude and phase of $M(ju)$ are

$$\begin{aligned} |M(ju)| &= \frac{1}{[(1-u^2)^2 + (2\zeta u)^2]^{1/2}} \\ \angle M(ju) = \phi_M(ju) &= -\tan^{-1}\frac{2\zeta u}{1-u^2} \end{aligned} \tag{3.100}$$

Reasonance frequency

We can compute the resonance frequency by taking the derivative of $|M(ju)|$ with respect to u and matching it to zero. From this we obtain

$$4u^3 - 4u + 8u\zeta^2 = 4u(u^2 - 1 + 2\zeta^2) = 0 \tag{3.101}$$

whose roots are $u_r = 0$ and

$$u_r = \sqrt{1 - 2\zeta^2} \tag{3.102}$$

and where the solution of $u_r = 0$ indicated that the slope of $|M(ju)|$ against ω is zero when $\omega = 0$ and it is not a true maximum if $\zeta \leq 0.707$. This way the equation Equation 3.102 gives the resonance response

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \tag{3.103}$$

so that the last equation is meaningful only when $2\zeta^2 \leq 1$ or $\zeta \leq 0.707$, which implies that for all values of ζ higher than 0.707 the resonanse response is $\omega_r = 0$ and $M_r = 1$.
Substituting Equation 3.102 on Equation 3.100 and simplifying we obtain

$$M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{3.104}$$

where $\zeta \leq 0.707$.

From this latter formula it is important to notice that M_r is just a function that depends on the relative damping ζ, and ω_r is a function that depends both on ζ and ω_n

Bandwidth
According to the bandwidth definition we need to make $|M(ju)| = 1/\sqrt{2}$

$$|M(ju)| = \frac{1}{\left[(1 - u^2)^2 + (2\zeta u)^2\right]^{1/2}} = 1/\sqrt{2} \tag{3.105}$$

thus,

$$\left[(1 - u^2)^2 + (2\zeta u)^2\right]^{1/2} = \sqrt{2} \tag{3.106}$$

which take us to

$$u^2 = (1 - 2\zeta^2) \pm \sqrt{4\zeta^4 - 4\zeta^2 + 2} \tag{3.107}$$

Were we need to pick the + sign so u can be a positive real number for all possible ζ. Therefore, we can define the bandwidth of our reference system of second order as

$$BW = \omega_n \left[(1 - 2\zeta^2) \pm \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{1/2} \tag{3.108}$$

Addition of a pole to the transfer function of the trajectory

Property 49. *Adding a pole to the transfer function of the trajectory makes the closed loop system less stable, and decreases its bandwidth.*

Example 3.6.1. *Consider again the transfer function*

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{3.109}$$

If we add a pole at $s = -1/T$ the transfer function turns into

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)(1 + Ts)} \tag{3.110}$$

Addition of a zero to the transfer function of the trajectory

Property 50. *The general effect of adding a zero to the transfer function of the trajectory is to increase the closed loop system bandwidth.*

Example 3.6.2. *Consider the transfer function*

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{3.111}$$

If we add a zero at $s = -1/T$ the transfer function turns into

$$G(s) = \frac{\omega_n^2(1 + Ts)}{s(s + 2\zeta\omega_n)} \tag{3.112}$$

so the transfer function of the closed loop is

$$M(s) = \frac{\omega_n^2(1 + Ts)}{s^2 + (2\zeta\omega_n + T\omega_n^2)s + \omega_n^2} \tag{3.113}$$

where the bandwidth is

$$BW = (-b + 1/2\sqrt{b^2 + 4\omega_n^4})^{1/2} \tag{3.114}$$

3.7 Nyquist Stability Criteria

The Nyquist criteria is a method that determines the stability of a closed loop system in the frequency domain.

Given the transfer functions G(s), H(s) in Figure 3.20, we would like to study the stability of this general system applying the Nyquist stability criteria. For this purpose, we need some concepts and results that conform this relevant graphical technique.

From Figure 3.20, consider the open-loop transfer function $G(s)H(s)$, which

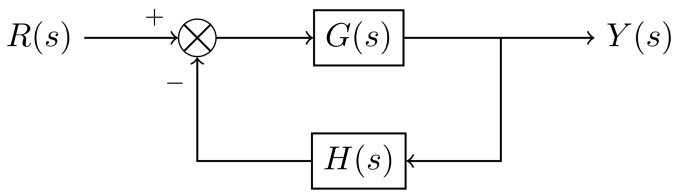


Figure 3.20: Caption

maps s to the GH- plane.

Definition 51 (Close path)**.** *Consider the path $\gamma : [a, b] \to \mathbb{C}$ we say that γ is a closed path if $\gamma(a) = \gamma(b)$.*

Definition 52 (Contour)**.** *A closed path on the s- plane is considered a contour.*

Definition 53 (Plot)**.** *A closed path on the GH- plane is considered a plot.*

Property 54. *Given Figure 3.20, every countour on the s-plane can be mapped to $w = G(s)H(s)$, plot on the GH-plane.*

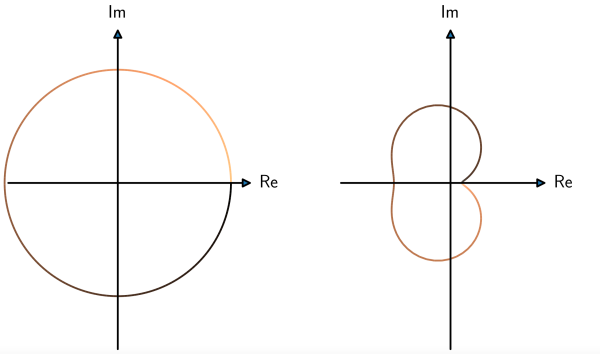


Figure 3.21: Contour on the s-domain mapped to a plot on the GH- plane

Considering now, the zeros Definition 3 and poles Definition 2 of the transfer function $G(s)H(s)$ we have the following properties:

• If a clockwise contour does not encircle zeros nor poles, then the plot will not encircle the origin.

• If a clockwise contour encircles a zero, then the plot will encircle the origin clockwise once.

• If a clockwise contour encircles a pole, then the plot will encircle the origin counterclockwise once.

Theorem 55 (Cauchy's Argument Principle)**.** *If a clockwise contour encircles Z zeros and P poles, then the number of clockwise encirclements of the origin N, is given by*

$$N = Z - P \to Z = N + P$$

Looking back at our closed loop system example Figure 3.20, we know that its transfer function is of the form

$$\frac{G(s)}{1 + G(s)H(s)}$$

Having analyzed the values of the poles and zeros of $F(s) = 1 + G(s)H(s)$ Cauchy's argument principle stated that for a clockwise contour Γ on the s-plane $Z = N + P$ where

• Z is the number of zeros of F i.e., number of closed-loop poles in the contour.

• P is the number of poles of F, i.e., number of open-loop poles in the contour.

• N is the number of clockwise encirclements of the origin for the plot $F(\Gamma)$

• N is also the number of clockwise encirclements of -1 for the plot $GoH(\Gamma)$

Example 3.7.1. *Consider the transfer function*

$$G(s)H(s) = \frac{s}{s^2 + 2s + 2} = \frac{s}{(s + 1 + j)(s + 1 - j)}$$

with open loop poles $(-1+j)$, $(-1-j)$ *then by definition of* $F(s) = 1+G(s)H(s)$

$$F(s) = \frac{s^2 + 3s + 2}{s^2 + 2s + 2}$$

F(s) has two zeros (closed loop poles) at $s = -1$, $s = -2$. *Accordingly, it can be seen in Figure 3.22 how the contour plot encircles 2 poles (crosses),* $Z = 1$, *and 1 zero (circles),* $P = 2$ *so that* $(N = Z - P = -1)$. *Thus, F plot encircles the origin counterclockwise i.e.* $N = -1$ *by the Cauchy Argument Theorem 55.*

Property 56. *The stability of system is related to whether there exists any closed-loop poles (or zeros of F(s)) on the Right Half Plane.*

Definition 57 (Nyquist Contour Γ_n)**.** *The Nyquist contour consist on 3 segments:*

1) *The imaginary axis from 0 to* $+j\infty$.

2) *A semicircle of infinite radius that encloses the entire right half s -plane.*

3) *The imaginary axis from* $-j\infty$ *to 0.*

Therefore, the Nyquist Contour is a 'big' semicircle that encloses the right half plane, and the direction of the encirclement is clockwise.

Proposition 58 (Nyquist Stability criterion)**.** *By the Cauchy's principle of argument (Theorem 55)*

$$Z = N + P$$

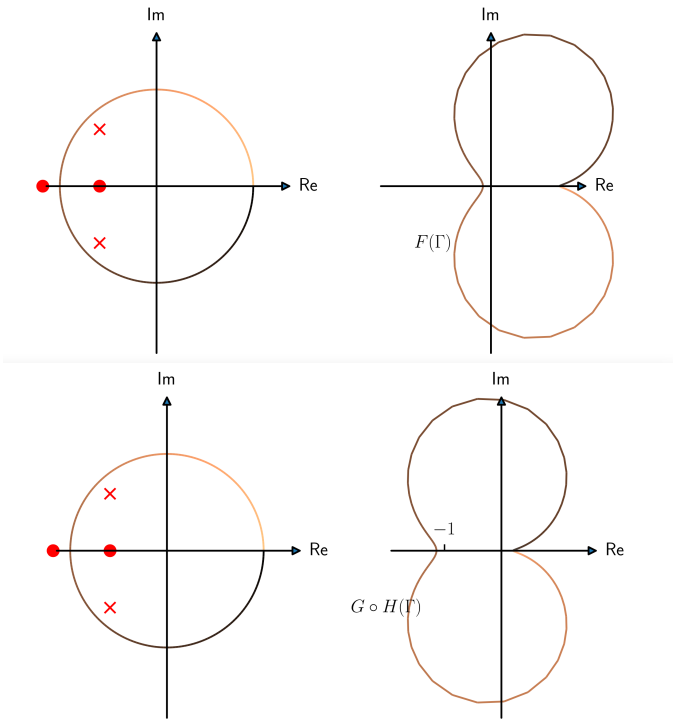


Figure 3.22: Example Contour and Plot

where Z is the number of zeros of F i.e. the number of unstable closed-loop poles; P is the number of poles of F i.e. the number of unstable open- loop poles, and N is the number of clockwise encirclements of -1 on the GH- plane for the plot $GoH(\Gamma)$.

The Nyquist Stability criterion states that:

- *The closed loop system Figure 3.20 is stable, (i.e.* $Z = 0$*) when* $N = -P$.

- *A feedback system is stable if and only if* $N = -P$ *i.e. the number of the counterclockwise encirclements of the point –1 by the Nyquist plot in the GH-plane is equal to the number of the unstable poles of the open-loop transfer function.*

The implications of the Nyquist stability criterion can be summarized with the following properties:

Property 59. *If the open-loop system is stable (*$P = 0$*), the closed-loop system is stable if and only if the Nyquist plot does not encircle –1 point*

Property 60. *If the open-loop system has P unstable poles, the closed-loop*

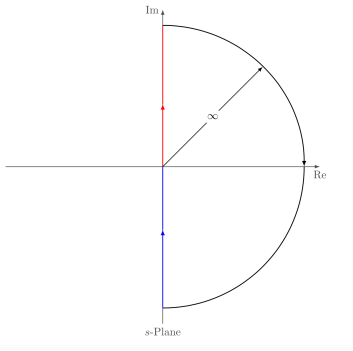


Figure 3.23: The Nyquist Contour is a 'big' semicircle that encloses the RHP. The direction of the encirclement is clockwise.

system is stable if and only if the Nyquist plot encircles –1 point P times counterclockwise.

Property 61. *If the Nyquist plot passes through -1, then the system has a closed-loop pole on the imaginary axis (critically stable).*

The procedure for determining the stability using the Nyquist stability criteria can be divided in 4 steps

1) Draw the Nyquist plot

2) Determine the clockwise encirclement N.

3) From the open-loop transfer function, find the number of unstable open-loop poles (P).

4) Stable if $N = -P$.

The procedure to sketch the Nyquist plot is easy with the use of a computer, let us illustrate this with an example:

Example 3.7.2 (Nyquist plot)**.** *Consider*

$$G(s)H(s) = \frac{s}{s^2 + 2s + 2}$$

We will now divide the sketching according to the 3 segments the Nyquist contour Definition 57 has:

Segment 1*(The imaginary axis from 0 to $+j\infty$): to plot this segment we make use of Bode plot.*

We need to find 4 different points:
∘ $\omega = 0$*;*
∘ $phase = 180°C$ *(Real intersection);*
∘ $phase = 180N + 90°C$ *(Imaginary intersection);*
∘ $\omega = \infty$

Around this points we can also deduce the trend in a way such that if the phase is decreasing, the plot goes clockwise and if the phase is increasing, the plot goes counterclockwise. With this information we can already plot these points on the GH- plane and draw a smooth line to connect them.

Segment 2 *(A semicircle of infinite radius that encloses the entire right half s- plane): to be able to plot segment 2 we need to take into consideration that non-proper transfer functions are not physically realizable.*
It is easy to see that our transfer function $G(s)$ is proper since

$$\lim_{s\to\infty} \frac{s}{s^2 + 2s + 2} = \lim_{s\to\infty} \frac{s}{s^2} = 0$$

Segment 3 *(the imaginary axis from $-j\infty$ to 0): To plot this segment we just need to applied that $G(s) = \overline{G(s)}$, thus, segment 3 is the mirror reflection of segment 1 around the real axis.*

Let us know do a full study of the Nyquist stability criterion applied to two different close loop systems systems.

Example 3.7.3 (Nyquist Stability Criterion I)**.** *Consider the system with open-loop transfer function*

$$G(s)H(s) = \frac{1}{(s+1)(0.1s+1)}$$

We would like to determine the stability of the closed-loop system using the Nyquist stability criterion:

1) *To sketch the Nyquist Plot we follow the procedure explained at (see 3.7.2) and make use of the Bode plot in Figure 3.7.3:*

Segment 1:
∘ *when $\omega \to 0$, $G(j\omega)H(j\omega) \to 1$;*
∘ *No intersection on the real axis since the phase is never $180°C$ for $0 < \omega < \infty$*
∘ *There is an intersection on the imaginary axis when $\omega = \sqrt{10}$*
∘ *when $\omega \to \infty$, $G(j\omega)H(j\omega) \to 0$*
∘ *since the phase is always decreasing the plot goes clockwise.*

Segment 2: *Since the system is strictly proper, Segment 2 is the origin.*

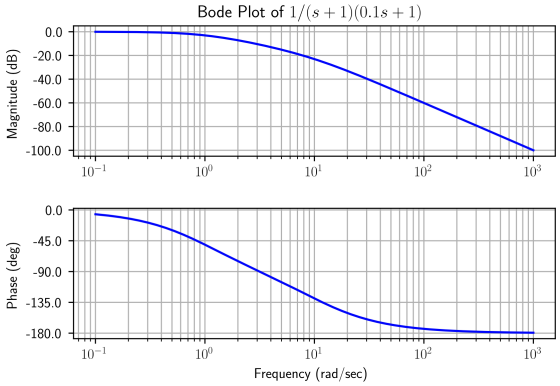


Figure 3.24: Bode Plot for $G(s) = \frac{1}{(s+1)(0.1s+1)}$

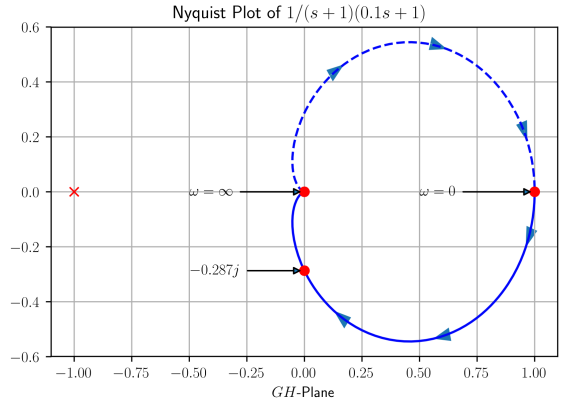


Figure 3.25: Nyquist plot for $G(s) = \frac{1}{(s+1)(0.1s+1)}$

Segment 3: *Mirror reflection of segment 1.*

- *The Nyquist plot is sketched in Figure 3.25*

2) *Find N and P to analyse stability:*
∘ *The Nyquist plot does not encircle -1. Therefore $N = 0$*
∘ *The open-loop poles are -1, -10. Therefore, $P = 0$.*
3), **4**) *Since $Z = N + P = 0 \Rightarrow$ the closed-loop system is stable by the Nyquist stability Criterion (Proposition 58).*

Example 3.7.4 (Nyquist Stability Criterion II)**.** *Consider a feedback system with open-loop transfer function*

$$G(s)H(s) = \frac{a(s-1)}{s^2+s+4}, \quad a > 0$$

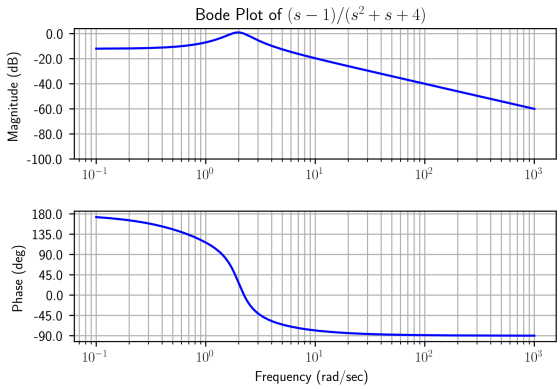


Figure 3.26: Caption

We would like to determine the values of a such that the system is stable.

To analyse the stability we want to first fix $a = 1$.

1) *For this value of a the Bode plot is illustrated in Figure 3.26*

Segment 1:
◦ *when* $\omega \to 0$, $G(j\omega)H(j\omega) \to -0.25$
◦ *There is a real intersection when* $\omega \approx 2$, *the instersection is around 1.*
◦ *There is an intersection with the imaginary axis when* $1 < \omega < 2$, *the intersection is between* $0.1j$ *and* j
◦*when* $\omega \to \infty$, $G(j\omega)H(j\omega) \to 0$
◦ *since the phase is always decreasing the plot goes clockwise.*
Segment 2: *The system is strictly proper, thus, Segment 2 is the origin.*
Segment 3: *Mirror reflection of segment 1.*

2) *Find the values for N and P:*
◦ *The poles are at* $(-0.5 - 1.94j)$, $(-0.5 + 1.94j)$. *Therefore, P=0.*
• *If* $a < 4$ *then* $-0.25a < -1$ *so the Nyquist plot does not encircle* -1. *Therefore,* $N = 0 \Rightarrow$ *the closed loop system is stable for these values of a*
• *If* $a > 4$ *then* $-0.25a > -1$ *so the Nyquist plot encircles* -1 *once. Therefore,* $N = 1$ *and* $Z = 1 \Rightarrow$ *There is one unstable pole for the closed-loop system.*

Property 62. *The Nyquist stability criteria fosters the absolute stability of a system and gives information about the relative stability of a stable system and about the instability of an unstable system. It also gives us information about how to improve the stability of the system.*

Property 63. *The Nyquist trace gives us information about the characteristics of the frequency domain such that,* M_r, ω_r, *the bandwidth, and more.*

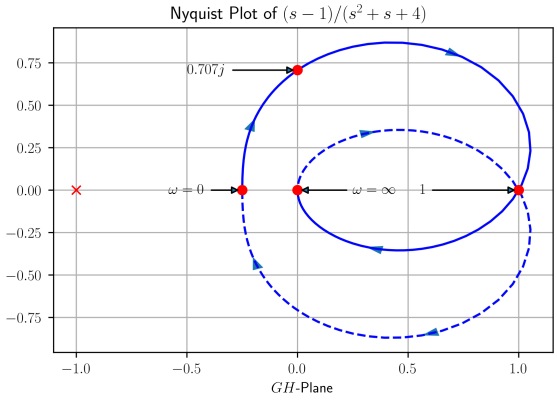


Figure 3.27: Nyquist plot of Example II

3.8 Aliasing and the Sampling theorem

In the next subsection we are going to introduce and explain one of the main potential problems that can appear during the sampling process, the aliasing effect.

Definition 64 (Aliasing)**.** *The effect that causes distinct continuous signals to become indistinguishable when digitally sampled is called aliasing. When this phenomenon occurs the original signal cannot be reconstructed uniquely from the digital signal.*

One way of constituting a sample- data control system is to make first a continuous time design and then approximate it into a discrete- time controller. This way, given an enough small sampling period the system should behave as the continuous system that was firstly provided.

In the following example it can be appreciated the great importance of choosing a small enough time step in order to construct the sample- data control system.

Example 3.8.1 (Aliasing)**.** *If we consider two sinusodial signals*

$$f_1(t) = \sin((1.8t - 1)\pi), \quad f_2(t) = \sin(0.2\pi t)$$

with the sampling period $h = 1$ we observe that there is no way to distinguish $f_1(t), f_2(t)$ from the sampled values between the two signals, see Figure 3.28. This is because it is impossible to determine if the sampled values are obtained from a low or high frequency signal. As a consequence, the continuous- time signal cannot be recovered from the sample- data signal, which implies the loss of information during a sampling process.

We would like to point out that during the sample process not only information can be lost due to aliasing, but also due to interferences between the

sampled continuous-time signal and the sampling frequency. Indeed, the introduction of new frequencies can also occur when sampling, causing fading and beating in the sample- data. To avoid the aliasing effect Whittaker discovers

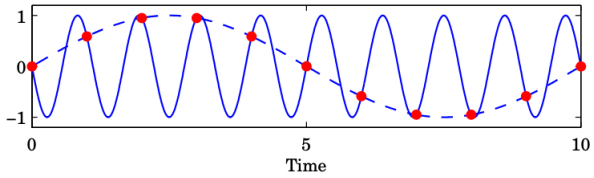


Figure 3.28: Example of aliasing

a result that he published in 1915 and Shannon will later cite when presenting his Sampling theorem in 1949.

Theorem 65 (Sampling theorem; 1949 Shannon)**.** *If a signal contains no frequencies above ω_{max} (maximum frequency of our signal), then the continuous-time signal can be uniquely reconstructed from a periodically sampled sequence where the sampling frequency is greater that $2\omega_{max}$.*

Definition 66 (Nyquist Rate)**.** *In accordance with the Sampling theorem, the Nyquist rate denotes the specific sampling rate (in Hz) that equals twice the highest frequency or bandwidth of a given function or signal.*
Therefore, with an equal or higher rate than the Nyquist rate we can say that the resulting discrete- time sequence is free of aliasing effect.

Definition 67 (band-limited signal)**.** *Signal whose value is non-zero between some $-\omega_{max}$ and ω_{max} Hz.*

In order to understand the sampling theorem let us consider a band limited signal $x(t)$. We can represent the latter band-limited signal in the frequency domain as shown in Figure 3.29 To avoid the aliasing effect we need a sampling

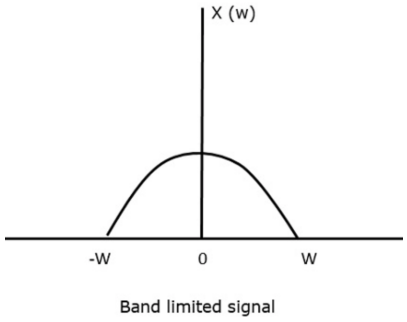


Figure 3.29: Band limited signal

frequency at which there is no loss of information, not even after sampling. For

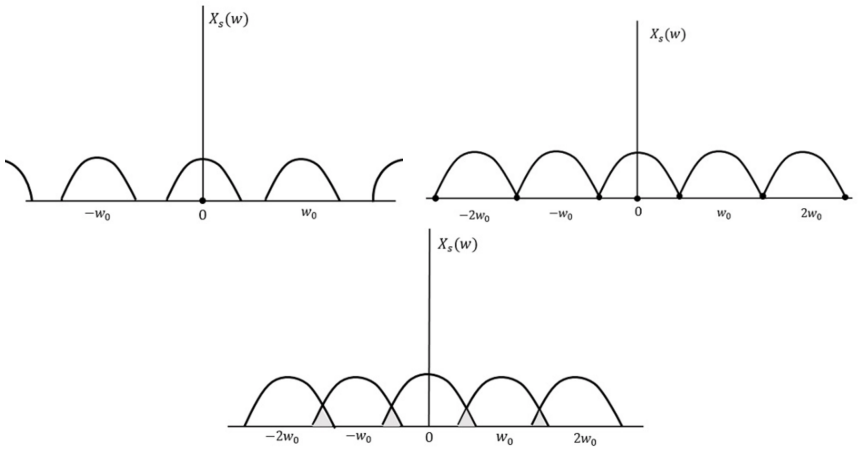


Figure 3.30: Signal when the sampling rate is higher, equal and lower than twice its highest frequency

this purpose, we will apply the sampling theorem and the Nyquist rate so that the sampling frequency picked should be two times the maximum frequency of our signal.

Definition 68 (Fourier transform of the signal $x_s(t)$)**.**

$$X_s(w) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(w - nw_{max}) \tag{3.115}$$

where T_s is the sampling period and $w_{max} = \frac{2\pi}{T_s}$

In Figure 3.30 it is illustrated the different cases we face depending on how we pick the sampling frequencies we apply to our signal $x(t)$, in terms of its Fourier transform Equation 3.115: in the first figure we see our signal sampled at a higher rate than the Nyquist rate, so that the signal is fully recovered, in the second figure our signal is sampled at a rate that is equal to twice the highest frequency, thus, according to the sampling theorem, is also fully recovered, on the last figure the sampling rate is lower than twice the highest frequency of our signal, therefore it can be seen how the signal is not fully recovered.

Chapter 4

H_∞ methods in Control Theory

In chapter 3 we studied the notions of control theory and signal processing required to understand the main chapter of this thesis, chapter 6. In this section we will describe what the H_∞ norm and the H_∞ methods are in a control theory background and its corresponding controllers. In chapter 6 we apply the H_∞ synthesis method in co- simulation to reduce the coupling error.

4.0.1 H_∞ norm

Definition 69 (H_∞ norm)**.** *Let the transfer function of a stable linear system be given by*

$$G(s) = C(sI - A)^{-1}B + D$$

where A, B, C, D are its state space matrices. The H_∞ norm refers to the largest singular value of the transfer function in the frequency domain on a permanent sinosoidal state ($s = j\omega$)

$$\|G(s)\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$$

4.1 H_∞ methods

We can find different methods for controller design such as PID (proportional–integral–derivative controller [9, Pages 45-49]) or pole-placement controller design [9, Pages 52-57]), in this thesis we are interested in LTI feedback optimization methods, in particular in H- infinity optimization.

H_∞ methods are optimization methods that attain to synthesize controllers that stabilize control systems. Indeed, the H_∞ problem can be described mathematically as an optimization problem that aims to find a controller that solves an optimization problem.

The main advantages of H_∞ techniques over classical control techniques rely on the fact that they can be applicable to multivariate systems with cross- coupling between channels, in a way that can be used to minimize the closed loop impact of a perturbation. On the contrary, its most important disadvantage is the need for a reasonably good model of the system to be controlled. Hence, the optimal controller we get is only optimal for a specified cost function, but it does not have to be the best in terms of other performance measures.

4.1.1 Representation of the feedback control system

Let us first introduce a general notation for linear systems that are controlled and derive the equations of the final closed loop system.

Consider Figure 4.1, where G denotes the transfer function of our LTI system with state vector x, and the input and output of the system is splitted in two so that, u is the input vector with the disturbance signal, u_c is the control signal, y is the output vector with the measurement signal and $Error$ is the performance signal. The state representation of our system G is given by

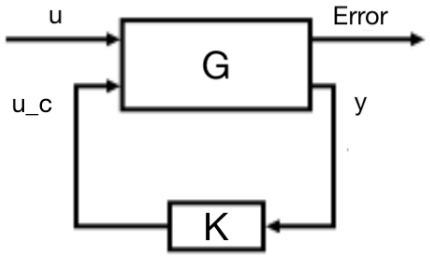


Figure 4.1: Feedback control system with an H_∞ controller

$$G: \begin{pmatrix} \dot{x} \\ Error \\ y \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x \\ u \\ u_c \end{pmatrix} \tag{4.1}$$

where $D_{22} = 0$ so that our system is strictly proper from u_c to y. If this is not possible we should be able to find a controller $\widetilde{K}$ where $D_{22} = 0$ and then contruct a controller for the system Equation 4.1 such that

$$K = \widetilde{K}(I + D_{22}\widetilde{K})^{-1}$$

The linear controller in Figure 4.1 is denoted with the letter K. K takes y as an input and outputs the control signal u_c. The state space representation of our controller K is given by

$$K: \begin{pmatrix} \dot{x}_K \\ u \end{pmatrix} = \begin{pmatrix} K_A & K_B \\ K_C & K_D \end{pmatrix} \begin{pmatrix} x_K \\ y \end{pmatrix} \tag{4.2}$$

where x_K is the state vector of the controller.

Now we can derive the expressions that characterize the closed loop system represented in Figure 4.1. Denoting G_{cl}, the closed loop system, we can derive its state- space representation combining Equation 4.1 and Equation 4.2

$$u_c = K_C x_K + K_D y = K_C x_K + K_D C_2 x + K_D D_{21} u$$
$$\dot{x}_K = K_A + x_K + K_B y = K_A x_K + K_B C_2 x + K_B D_{21} u$$
$$\dot{x} = Ax + B_1 u + B_2 u_c = (A + B_2 K_D C_2)x + B_2 K_C x_K + (B_1 + B_2 K_D D_{21})u$$
$$Error = C_1 x + D_{11} u + D_{12} u_c = (C_1 + D_{12} K_D C_2)x + D_{12} K_C x_K + (D_{11} + D_{12} K_D D_{21})u$$

from this equations we have

$$G_{cl}: \quad \begin{pmatrix} \dot{x} \\ \dot{x}_K \\ z \end{pmatrix} = \begin{pmatrix} A + B_2 K_D C_2 & B_2 K_C & B_1 + B_2 K_D D_{21} \\ K_B C_2 & K_A & K_B D_{21} \\ C_1 + D_{12} K_D C_2 & D_{12} K_C & D_{11} + D_{12} K_D D_{21} \end{pmatrix} \begin{pmatrix} x \\ x_K \\ u \end{pmatrix} \tag{4.3}$$

where the closed loop system states are $\begin{pmatrix} x \\ x_K \end{pmatrix}$ and we can partition the matrix in Equation 4.3 as

$$A_{cl} = \begin{pmatrix} A + B_2 K_D C_2 & B_2 K_C \\ K_B C_2 & K_A \end{pmatrix}; \quad B_{cl} = \begin{pmatrix} B_1 + B_2 K_D D_{21} \\ K_B D_{21} \end{pmatrix}$$
$$C_{cl} = \begin{pmatrix} C_1 + D_{12} K_D C_2 & D_{12} K_C \end{pmatrix} \quad D_{cl} = D_{11} + D_{12} K_D D_{21}$$

so that

$$G_{cl}: \quad \begin{pmatrix} \dot{x}_{cl} \\ error \end{pmatrix} = \begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} \begin{pmatrix} x_{cl} \\ u \end{pmatrix} \tag{4.4}$$

From Equation 4.4 we can define the transfer function of G_{cl} in terms of $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ as

$$G_{cl} = C_{cl}(sI - A_{cl})^{-1} B_{cl} + D_{cl}$$

4.1.2 H_∞ Optimization problem

The H infinity optimization problem in control theory is the task of minimizing the H_∞ norm of a stable linear system, in our case the closed loop system in Figure 4.1, represented by the transfer function G_{cl}.

H_∞ Problem

min γ

subjected to

$$\begin{bmatrix} A_{cl}^T G + G A_{cl} + C_{cl}^T C_{cl} & C_{cl}^T D_{cl} + G B_{cl} \\ B_{cl}^T G + D_{cl}^T C_{cl} & D_{cl}^T D_{cl} - \gamma I \end{bmatrix} \prec 0$$

where $G_{cl}(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl} < \gamma$.

In order to solve this optimization problem we want to find out the controller that minimizes the coupling error of the error plant of a concrete co- simulation system.
To solve this problem one can use the Matlab control toolbox and the python control library. Moreover, in order to understand briefly how this problem is solved analytically It is recommended to read in [7, Chapter 2]

Chapter 5

Coupled systems and co-simulation

This chapter and its respective sections follows the structure of [1, Chapter 2]. Co- simulation, along with the H_∞ synthesis method is one of the core concepts we are aiming to understand in this thesis. Here it is introduced a broad background and characterization and classification of how to simulate decoupled systems in a distributed way.

5.1 Coupled systems

In a co- simulated system, the different subsystems that form a coupled problem are simulated in a distributed way. During the simulation, the subsystems are solved in what is called a black box manner, i.e. the systems are seen in terms of its inputs and outputs without any knowledge of how they work internally.

Some co- simulation schemes allow us to execute the integration of each subsystem in parallel. With this scheme the resources needed for the computation are distributed between the different cores of the same computer.

The numerical integration of the subsystems its coordinated so that the coupling variables introduce information in a subsystem regarding its environment (inputs) or take the information of a subsystem to share it with the rest of the simulation elements (outputs). The exchange of information of these coupling variables is done in a discrete manner, i.e. at a certain instants of communication that belong to what we refer to as the communication interval.

Definition 70 (Coupling problem)**.** *Refers to the problem that raises from the innacuracy of the coupling variables when exchanged in a distributed manner. For example, this can be the case when we have multiple time rates and the coupling variables are not updated fast enough. As a consequence, the unavoidable*

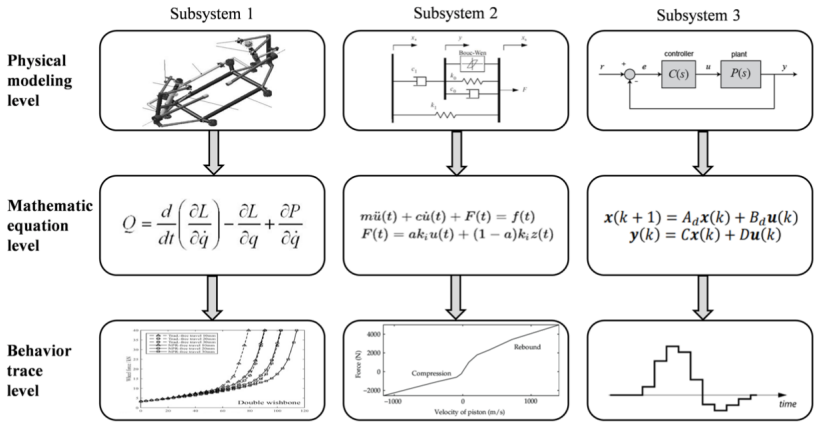


Figure 5.1: Coupled dynamic system at different levels of coupling.

coupling error degrades the accuracy of the result and can even destabilise the co- simulation.

Furthermore, since the coupling variables are transmitted by signals or enforced by control systems, the associated communication delay, measured noises and control errors may degrade as well the accuracy and stability of the co- simulated system.

Definition 71 (coupling methods)**.** *Refers to the means by which the coupling variables must be approximated.*

5.2 Co-simulation

Co-simulation results convenient when simulating multi-domain problems. Some of its multiple advantages include a better numerical efficiency that enables software tolerance integration, and its implementation using multiprocessors and distributed computer systems.

Definition 72 (Co-simulation)**.** *Co-simulation is the cooperation of simulation, where the solution of a system of linear (or nonlinear) differential equations is distributed.*

However, it is not easy to find a general method that is adequate for all its possible applications. Indeed, one of the main disadvantages of this technique results from the exchange of information through the discrete temporal interface that creates coupling and discretization errors. This errors might degrade the accuracy of the integration and its stability.

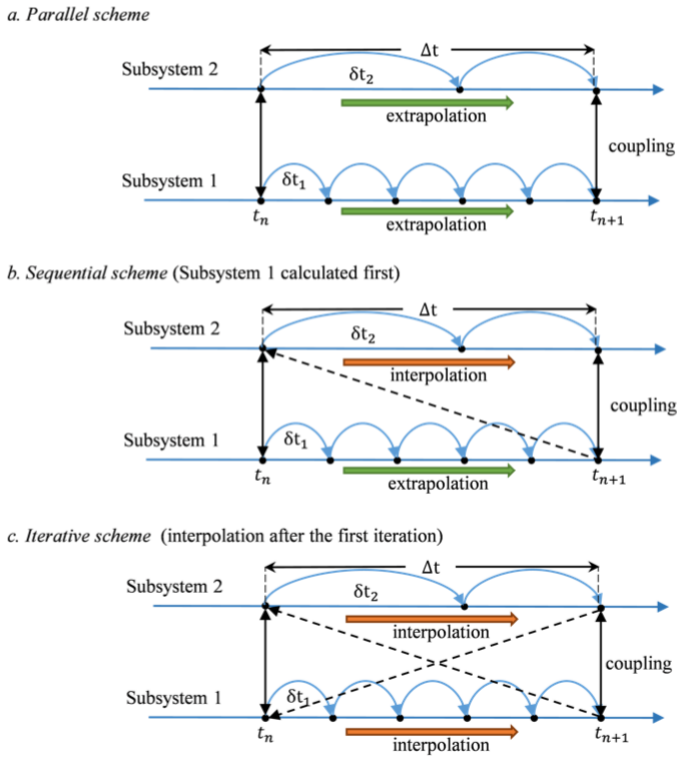


Figure 5.2: Co- simulation of 2 coupled subsystems with 3 different caculation schemes

5.3 Calculation schemes

In co- simulation each subsystem is calculated by a local solver at a fixed or variable micro- step δt. The coupling input- output communication occurs at every macro- step Δt and $\Delta t \geq \delta t$. Depending on the calculation scheme we apply to our co- simulated environment we will be prioritizing stability, accuracy or computational efficiency. In the next section we classify the different calculation schemes we could apply to out co- simulated system.

■ **Parallel or Jacobi Scheme** In this scheme each subsystem is calculated simultaneously and the input during the communication interval is updated according to previous values.

■ **Sequential or Gauss Seidel scheme** in this scheme coupled systems are calculated in a specific sequence. Subsystems calculated first or *a priori*, update the input by extrapolation, the subsystem calculated 'afterwards', or *a posteriori* in the sequence will then approximate the output of the first ones (i.e. its

input) by interpolation.

■ **Iterative scheme** This scheme starts with one of the previous calculation schemes (Parallel or Sequential scheme) and continues integrating the subsystems by iterations. As a consequence, each subsystem can roll back to a previous macro- step and the input can be interpolated by previous iterations. The iterative process will terminate when we get a specific error tolerance or an iterative time bound.

Generally, given the same size of Δt it has been observed that with respect to stability and accuracy the *Iterative Scheme* is the most accurate and stable, while the *Parallel Scheme* is the worst in these two categories. Nevertheless, with what concerns to computational efficiency, the *Parallel Scheme* is the fastest and *Iterative Scheme* the slowest [5].

Furthermore, in terms of the difficulty of implementation the *Parallel Scheme* results to be the easiest with commercial software, while advance features of main- auxiliary, such as *pyFMI*[2] are required for the other schemes.

In order to have a good balance of all of these important categories a combination of the calculation schemes has been proposed as the ideal choice. In this thesis we focus our attention on the study of the parallel approach and how to minimize its accuracy and stability error.

5.4 Off- line and Online co- simulation

Depending on how physical parts (i.e. sensors) are involved in the exchange of information of our co- simulated system we can also classify co- simulation in: Off- line and online co- simulation.

• **Off-line co- simulation**: in this type of co simulation models are calculated separately by domain specific solvers and coupled by exchange variables at specific communicative instants t_K, as it is shown in (Figure 5.3).

• **Online co- simulation**: in this type of co simulation models we usually have physical parts in the loop where measured variables are exchanged to the virtual models at the sensor sampling rate.

In both cases the system has the same integration problem.

Definition 73 (The integration problem)**.** *The main problem the integration of a coupled co- simulated system has is caused by the exchange variables being available at specified communicative instants or sampling time, but unknown at local integration steps* δt_1, δt_2.

Thus, we will need to make use of extrapolation methods that approximate

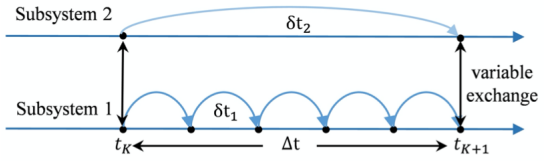


Figure 5.3: The integration of 2 coupled systems under off- line co- simulation.

this variables when needed during the integration steps.

5.5 Coupling configurations

Depending on how the system is divided and coupled, every couple dynamical system can have different configurations. In order to understand its main differences let us illustrate these configurations on a dual mass- spring- damper Figure 5.4 (linear oscillator with two degrees of freedom), where it is possible to study the masses separately. In this example each subsystem with a mass can be coupled by two different approaches: the applied- force approach or by the algebraic constraint.

In order to study our specific system we can set the following inital conditions: $m_1 = m_2 = 1$ kg, $k_1 = 10N/m$, $k_2 = 1000N/m$, $k_c = 100N/m$, $d_1 = d_2 = d_c = 0Ns/m$, $\dot{x}_{1,0} = 100m/s$, $\dot{x}_{2,0} = -100m/s$ taking the null initial displacements so that the system starts from a resting point. With this initial conditions set up we can obtain the following movement equations

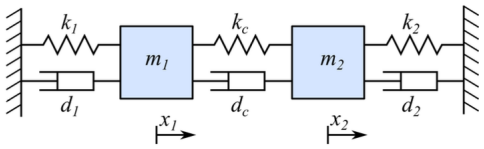


Figure 5.4: Reference example: Dual mass spring damper system

$$\begin{aligned} m_1\ddot{x}_1 &= -k_c(x_1 - x_2) - d_c(\dot{x}_1 - \dot{x}_2) - k_1x_1 - d_1\dot{x}_1 \\ m_2\ddot{x}_2 &= -k_c(x_2 - x_1) - d_c(\dot{x}_2 - \dot{x}_1) - k_2x_2 - d_2\dot{x}_2 \end{aligned} \tag{5.1}$$

with matrix representation

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} d_1 + d_c & -d_c \\ -d_c & d_c + d_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_c & -k_c \\ -k_c & k_c + k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{5.2}$$

or in a more compact way

$$M\ddot{X} + D\dot{X} + KX = f \tag{5.3}$$

where M is the mass matrix, $\dot{X}$ the velocity matrix, X the position vector, f the vector of generalized forces, D the damping matrix and K the rigidity matrix.

In Equation 5.2 f_1, f_2 represent the applied forces for each corresponding mass, for simplicity of our study we assume this forces are zero. We can obtain the analytical solution of the system transforming the second order differential equation system Equation 5.5 into a first order differential equation. With this purpose, let set the vectors

$$Z = \begin{pmatrix} x_1 \\ x_2 \\ \dot{x_1} \\ \dot{x_2} \end{pmatrix}; \quad \dot{Z} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \ddot{x_1} \\ \ddot{x_2} \end{pmatrix} \tag{5.4}$$

from where we can obtain the first order differential equation

$$\dot{Z} = AZ \tag{5.5}$$

where matrix A is obtained from Equation 5.2

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_c)}{m_1} & \frac{k_1}{m_1} & \frac{-(d_1+d_c)}{m_1} & \frac{d_c}{m_1} \\ \frac{k_c}{m_2} & \frac{-(k_2+k_c)}{m_2} & \frac{d_c}{m_2} & \frac{-(d_2+d_c)}{m_2} \end{pmatrix} \tag{5.6}$$

and

$$Z(t) = e^{At} \cdot Z_0 \tag{5.7}$$

is the solution of Equation 5.5 with Z_0 a vector that contains the initial positions and velocities of the subsystems.

Now that we have integrated the analytical solution of the reference non- decoupled system, let's compare its results with the results we obtain when we decouple its masses with different coupling configurations.

5.5.1 Applied force

■ **Force Displacement Coupling** In this coupling configuration the system has been separated in a way that the intermediate spring damper is embedded in Subsystem 1, and subsystem 2 has a force input

$$F_c = k_c(x_1 - x_2) + d_c(\cdot x_1 - \cdot x_2)$$

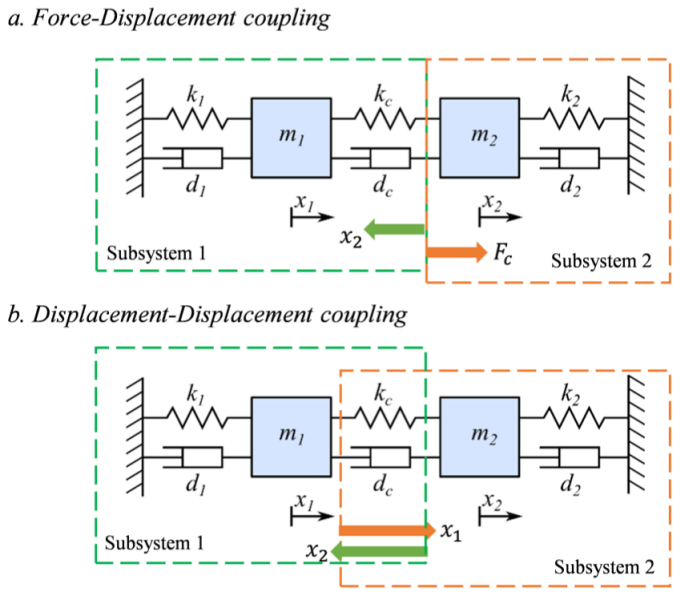


Figure 5.5: Double mass spring damper coupled with a FD and a DD configuration

We see in Figure 5.5 how the subsystem 1 contains the information of k_1, d_1, k_c, d_c, m_1 while the Subsystem 2 only contains the informaton of k_2, d_2 and m_2.

Thus, we have the following equations for m_1 and m_2:

$$\begin{aligned} m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_c\dot{x}_2 + (k_1 + k_c)x_1 - k_c x_2 &= f_1 \\ m_2\ddot{x}_2 - c_c\dot{x}_1 + (c_c + c_2)\dot{x}_2 - k_c x_1 + (k_c + k_2)x_2 &= f_2 \end{aligned} \tag{5.8}$$

To find the dynamical response we need to compute the acceleration of the system:

$$\begin{aligned} \ddot{x}_1 &= \frac{c_c[\dot{x}_2] + k_c[x_2] + (c_1 + c_c)\dot{x}_1 - (k_1 + k_c)x_1 + f_1}{m_1} \\ \ddot{x}_2 &= \frac{[c_c(\dot{x}_1 - \dot{x}_2) + k_c(x_1 - x_2)] - c_2\dot{x}_2 - k_2 x_2 + f_2}{m_2} \end{aligned} \tag{5.9}$$

In this equations we can find the parameters that each subsystem is missing in square brackets. This parameters are going to be the exchange information between the subsystems, i.e. their coupling variables. Hence, the resultant force of the spring and damper connecting the two systems are going to be the outputs of subsystem 1 and the inputs of subsystem 2

$$F_c = y_1 = c_c(\dot{x}_1 - \dot{x}_2) + k_c(x_1 - x_2) \tag{5.10}$$

and the velocity and displacement of the second mass are be the outputs of subsystem 2

$$y_2 = \begin{pmatrix} x_2 \\ \dot{x}_2 \end{pmatrix} \tag{5.11}$$

and the inputs of subsystem 1.

■ **Displacement Displacement coupling** In this configuration both subsystems share the intermediate spring- damper with a displacement input.

This time the acceleration of the subsystems is given by:

$$\begin{aligned} \ddot{x}_1 &= \frac{c_c[\dot{x}_2] + k_c[x_2] - (c_1 + c_c)\dot{x}_1 - (k_1 + k_c)x_1 + f_1}{m_1} \\ \ddot{x}_2 &= \frac{c_c[\dot{x}_1] + k_c[x_1] - (c_c + c_2)\dot{x}_1 - (k_c + k_2)x_2 + f_2}{m_2} \end{aligned} \tag{5.12}$$

where again we find in square brackets the information that needs to be exchanged among the subsystems during the communication interval. As a result, both subsystems give as outputs its velocity and displacement

$$y_1 = \begin{pmatrix} x_1 \\ \dot{x}_1 \end{pmatrix}; \quad y_2 = \begin{pmatrix} x_2 \\ \dot{x}_2 \end{pmatrix} \tag{5.13}$$

which are the respective input of the other subsystem.

In [8] it has been already studied the greater accuracy and stability of Displacement Displacement coupling over Force Displacement coupling.

5.5.2 Algebraic constraint

This approach will normally be applied to the coupling of rigid bodies and can be divided in two different configurations: the Force- Force coupling and the Displacement- Displacement coupling (see Figure 5.6). For both configuration exists a bi- directional dependency of the inputs and the outputs of each subsystem at the same t. This creates an algebraic loop problem Definition 76 that needs to be solved.

5.5.3 The algebraic Loop

Whether an algebraic loop appears inside a closed loop interconnection or not is determined by the *feed through* and *non feed- through* connections of the closed-loop subsystem.

Definition 74 (Feed through)**.** *The feed through connection has the output related to the input at the same time, e.g.* $y = f(t, x, u)$.

Definition 75 (Non Feed- through)**.** *If the output is not dependent on the input at the same instant, e.g.* $y = f(t, x)$ *then the connection is non- feed through.*

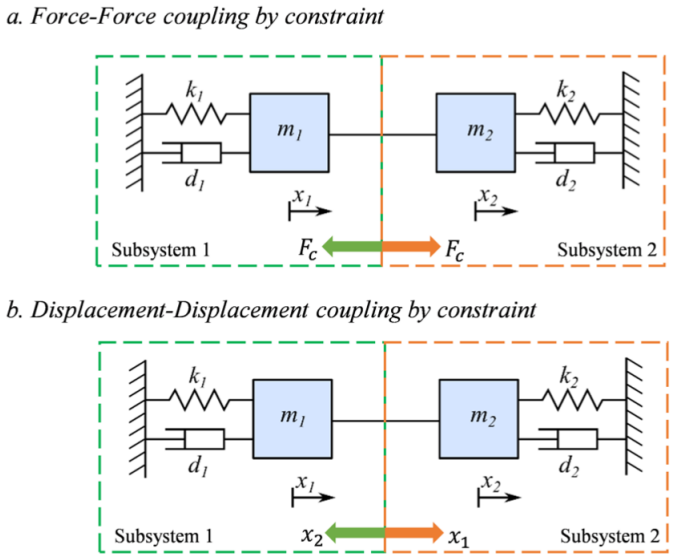


Figure 5.6: Different coupling configurations under the algebraic constraint

Definition 76 (Algebraic Loop)**.** *We say that an algebraic loops appear when a closed loop interconnection is only constitued by feed- through connections, meaning that there is no explicit calculation sequence.*

The algebraic loop can be broken just by adding non- feed through connections (e.g. a low pass filter). Consequently, the main drawback of the algebraic loop is the need of adding additional dynamics to our original system.

Property 77. *MIMO systems can have both feed through and non feed through connections, see Figure 5.7. However, unless we have access to the feed- through information of the subsystems, what might seem initially an algebraic loop may just be a software issue, this is considered a fake algebraic loop.*

5.5.4 Qualitative study of the coupling configurations

For each system exists different coupling configurations depending on how the components interact with each other. The coupling configuration we choose will influence the accuracy and stability of the system response. In order to find out which is the best coupling configuration for our system we need to observe the equations of our system:

$$\begin{bmatrix} \dot{z} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \tag{5.14}$$

where

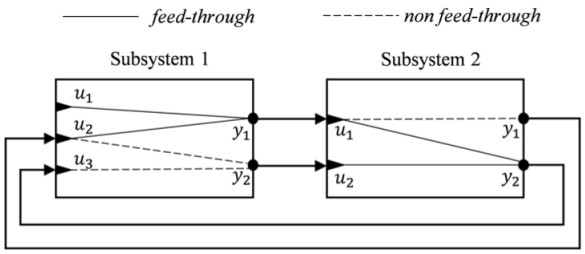


Figure 2.5: *MIMO systems can have both feed-through and non feed-through connections. There is no algebraic loop exists in this case.*

Figure 5.7: MIMO with feed through and non feed though connentions. No algebraic loop.

- z: state variable
- y: output
- u: input
- $\dot{z}$: state derivative
- A: state matrix
- B: input matrix
- C: output matrix
- D: feedthrough matrix

In Figure Figure 5.8 we can see two subsystems being co- simulated. Taking subsystem 1 as a reference, we should follow the following steps to obtain the system response:

1. Exchange the necessary amount of information to obtain inputs and outputs when we are at the communication interval ($t = t_n$).
2. Evaluate the state- derivative vector $\dot{x}_n = Az_n + Bu_n$.
3. Integration step. The subsystem is solved and we move to the next time step $t = t_{n+1}$
4. Compute the output at $t = t_{n+1}$, $y_{n+1} = Cx_{n+1} + Du_{n+1}$

Each of these steps have an effect on the final response of the system. Observing step 4 we can realize how in order to compute y_{n+1} we need missing data (u_{n+1}). There are few ways to solve this situation.

If the feed-through matrix is of the form $D = 0$, the output of the subsystem does not depend on the input e.g. this is the case of the displacement-displacement coupling configuration of two masses joined by a spring. In this case, step 4 only consists on evaluating $y_{n+1} = Cx_{n+1}$.

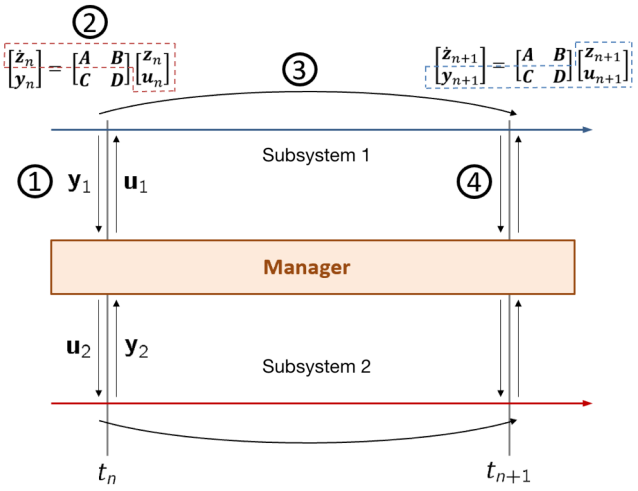


Figure 5.8: Evaluation of the state space equation during a time step of a SISO system co simulated with a Jacobi configuration

On the other hand if $D \neq 0$ the output does depend on the input, and needs to receive updated information at the communication interval, e.g. force displacement coupling configuration of two masses joined by a spring. In this latter case we will need to make use of an extrapolation method that approximates the input value for the current time instant i.e. u_{n+1} using the available information from previous steps.

Therefore, it is expected that couplings configurations where $D = 0$ are more stable and accurate than configurations with $D \neq 0$.

5.6 Stability of co- simulation

Stability of co- simulation, mainly refers to the convergence property of the coupling error, in the numerical analysis field this concept is related directly to the spectral radius (the largest absolute value of its eigenvalues of a square matrix) of the difference equations.

5.6.1 Numerical Analysis Framework

In order to make the analysis simpler we can consider a LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= Ly \end{aligned} \tag{5.15}$$

where x is the state vector and L is the coupling matrix mapping the output vector y to the corresponding system input u. If we are in a mono- simulation case the equality $u = Ly$ is satisfied for every value of time, thus, the two first equations of the previous system can be rewritten as

$$\dot{x} = \left(A + BL\left(I - DL\right)^{-1} C\right) x \tag{5.16}$$

On the other hand, if we are in a co-simulation context, the coupling matrix L occurs only at the communicative instant t_n, thus, we can describe the coupled dynamical system as

$$\begin{aligned} \dot{\tilde{x}}_n &= A\tilde{x}_n + B\Psi(\tilde{u}(\tau)) \\ \tilde{y}_n &= C\tilde{x}_n + D\Psi(\tilde{u}(\tau)) \\ \tilde{u}_n &= L\tilde{y}_n \end{aligned} \tag{5.17}$$

where Ψ is the extrapolation operator, $\tau \in [t_n, t_{n+1})$, and the subsystem input is

$$\Psi(\tilde{u}(\tau)) = \tilde{u}_n \quad \tau \in [t_n, t_{n+1}) \tag{5.18}$$

In the ideal case where the system can be exactly solved, the updated state $\tilde{x}_n$ at the next communicative instant t_{n+1} is derived as

$$\tilde{x}_{n+1} = e^{A\Delta t}\tilde{x}_n + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)} B\tilde{u}_n d\tau = e^{A\Delta t}\tilde{x}_n + K(\Delta t)BL\tilde{y}_n \tag{5.19}$$

with $K(\Delta t) = \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)} d\tau$

Indeed, we can express the dynamical coupled system with constant extrapolation (ZOH) as

$$\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{A\Delta t} & K(\Delta t)BL \\ Ce^{A\Delta t} & CK(\Delta t)BL + DL \end{bmatrix}}_{A^*} \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} \tag{5.20}$$

Definition 78 (Spectral radius)**.** *Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$. We define the spectral radius of A as*

$$\rho(A) = max\left\{|\lambda_1|, ..., |\lambda_n|\right\}$$

Proposition 79. *The numerical stability of co- simulation depends on the spectral radius $\rho(A^*)$, so that if $\rho(A^*) \leq 1$ and there is no more than one eigenvalue on the unit circle the system is stable and the numerical result convergent.*

Property 80. *The spectral radius of A^*, $\rho(A^*)$ depends on the macro- step Δt, the extrapolation operator Ψ (see (Figure 5.9)), and the system dynamics, i.e. matrices A,B,C,D and L based on the coupling configuration and system parameters.*

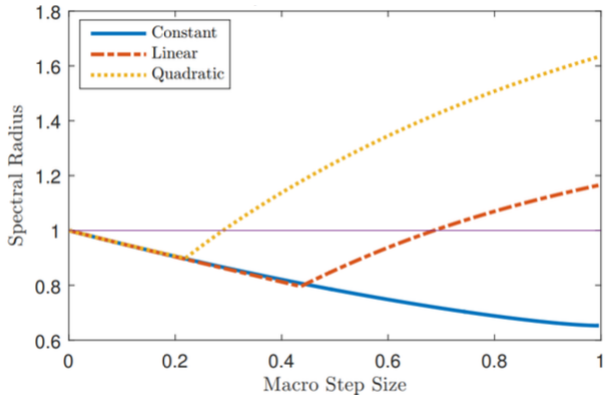


Figure 5.9: Spectral radius of co- simulation with different extrapolation methods: ZOH, FOH, SOH.

Example 5.6.1. *Consider a linear time invariant coupled system of two subsystems with a single state x, a single input u, a single output y and for which the coupling is determined by the value of two parameters d_1 and d_2.*

Subsystem 1

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_1 \\ y_1 &= x_1 + d_1 u_1 \end{aligned} \tag{5.21}$$

Subsystem 2

$$\begin{aligned} \dot{x}_2 &= -x_2 + 3u_2 \\ y_2 &= -5x_2 + d_2 u_2 \end{aligned} \tag{5.22}$$

and the coupling

$$y \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{L} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{5.23}$$

Now, discretizing the coupled system using constant extrapolation, i.e. rewriting it such that

$$\begin{aligned} x_{n+1} &= \Phi(x_n, u_n) = \Phi(x_n, Ly_n) \\ y_{n+1} &= Cx_{n+1} + DLy_n \end{aligned} \tag{5.24}$$

and letting $\Delta t \to 0$ we obtain

$$\begin{bmatrix} y_{n+1_{(1)}} \\ y_{n+1_{(2)}} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_{n+1_{(1)}} \\ x_{n+1_{(2)}} \end{bmatrix} + \underbrace{\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{L} \begin{bmatrix} y_{n_{(1)}} \\ y_{n_{(2)}} \end{bmatrix} \tag{5.25}$$

where we can state that the coupled system is ***stable*** *if the spectral radius* $\rho(DL) \leq 1$ *where* $\lambda_{1,2} = \pm\sqrt{d_1 d_2}$

5.6.2 Control Theory Framework

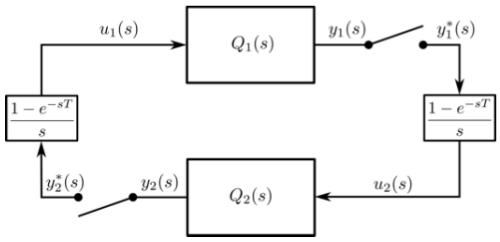


Figure 5.10: Co- simulation emulated by a closed loop sample- data with constant extrapolation, (ZOH operator)

Assuming the solver is accurate enough and restricting our subsystems to the LTI class with zero initial condition represented by a transfer function in the frequency domain, the coupled dynamical system can be emulated by a close-loop sample data system, as in Figure 5.10, where co- simulation stability is highly related to the stability of the closed loop interconnection (see section 3.4) and the coupling error is seen as the multiplicative disturbance.

5.7 Stability of weakly coupled systems

The stability and accuracy of the weakly coupled systems i.e. subsystems that are not coupled by the exact values, changes according to the communication step size Δt. In this section we will analyse the different types of errors that will help us computing the approximation order of our co- simulation systems.

5.7.1 Local Error

To analyze the error in one step of our numerical approximation, we assume the state $\tilde{x}_{n-1}$ and input $\tilde{u}_{n-1}$ from the previous macro- step t_{n-1} are error free.

Thus, the local error ε_{x_n} can be derived as

$$\begin{aligned} x\varepsilon_{x_n} = x_n - \tilde{x}_n = \\ \mathrm{e}^{A\Delta t}x_{n-1} + \int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)}Bu(\tau)\mathrm{d}\tau - \mathrm{e}^{A\Delta t}\tilde{x}_{n-1} \\ - \int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)}B\Psi(u(\tau))\mathrm{d}\tau \\ = \int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)}B(u(\tau) - \Psi(u(\tau)))\mathrm{d}\tau \\ \leq \left| B\int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)}\mathrm{d}\tau \right| \varepsilon_{u_n,max} \end{aligned} \tag{5.26}$$

where $\varepsilon_{u_n,max} = \|u(\tau) - \Psi(u(\tau))\|_\infty$, $\tau \in [t_{n-1}, t_n)$ the maximum norm of the input error on the communicative interval. The corresponding Taylor expansion can be described as

$$\begin{aligned} \varepsilon_{x_n} \leq \left| \frac{B}{A}(-I + \mathrm{e}^{A\Delta t}) \right| \varepsilon_{u_n,max} \\ = \left| \frac{B}{A}(A\Delta t + \frac{(A\Delta t)^2}{2} + \frac{(A\Delta t)^3}{6} + ...) \right| \varepsilon_{u_n,max} \\ = |B\,O(\Delta t)|\,\varepsilon_{u_n,max} \end{aligned} \tag{5.27}$$

Property 81. *When a k degree extrapolation method is used to approximate the input the local error ε_{x_n} is bounded by*

$$|B|\,\mathrm{O}(\Delta t^{k+2})$$

Thus, the output local error can be derived as

$$\begin{aligned} \varepsilon_{y_n} \leq |C\varepsilon_{x_n} + D\varepsilon_{u_n}| \\ \leq \left|C\mathrm{O}(\Delta t^{k+2}) + D\mathrm{O}(\Delta t^{k+1})\right| \end{aligned} \tag{5.28}$$

so that for a feed- through system ($D \neq 0$), the output global error ε_{y_n} is bounded by $\mathrm{O}(\Delta t^{k+1})$, whereas for a non feed- through system ($D = 0$), ε_{y_n} is bounded by $\mathrm{O}(\Delta t^{k+2})$.

5.7.2 Global Error

The global error is the error made by the numerical approximation during the whole simulation time. We can derive this error by removing the error- free assumption from the local error,

$$\xi_{x_n} = x_n - \tilde{x}_n = \mathrm{e}^{A\Delta t}x_{n-1} + \int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)}Bu(\tau)\mathrm{d}\tau - \mathrm{e}^{A\Delta t}\tilde{x}_{n-1} -$$

$$
\begin{aligned}
&- \int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)} B\Psi(u(\tau))\mathrm{d}\tau \\
&= \mathrm{e}^{A\Delta t}\xi_{x_{n-1}} + \int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)} B\left[u(\tau) - \Psi(u(\tau)) + \Psi(u(\tau)) - \Psi(\tilde{u}(\tau))\right]\mathrm{d}\tau \\
&\leq \mathrm{e}^{A\Delta t}\xi_{x_{n-1}} + \varepsilon_{x_n} + \left| \int_{t_{n-1}}^{t_n} \mathrm{e}^{A(n\Delta t-\tau)} B\left[\Psi(u(\tau)) - \Psi(\tilde{u}(\tau))\right]\mathrm{d}\tau \right|
\end{aligned}
$$

And taking Ψ as a linear operator we have that

$$\Psi(u(\tau)) - \Psi(\tilde{u}(\tau)) = \Psi\left[u(\tau) - \tilde{u}(\tau)\right] = O(\Delta t^{k+1})$$

We can bound our global as

$$
\begin{aligned}
\xi_{x_n} &\leq \mathrm{e}^{A\Delta t}\xi_{x_{n-1}} + |B|\left(O(\Delta t^{k+2}) + O(\Delta t^{k+2})\right) \\
&\leq \mathrm{e}^{A2\Delta t}\xi_{x_{n-2}} + |B|\, O(\Delta t^{k+2})(1 + \mathrm{e}^{A\Delta t}) \\
&\leq |B|\, O(\Delta t^{k+2})(1 + \mathrm{e}^{A\Delta t} + ... + \mathrm{e}^{A(n-1)\Delta t}) = |B|\, O(\Delta t^{k+2})\frac{1 - \mathrm{e}^{An\Delta t}}{1 - \mathrm{e}^{A\Delta t}} \\
&= |B|\, O(\Delta t^{k+1})(1 - \mathrm{e}^{At_n})
\end{aligned}
$$

thus, the output global error during the simulation time $[0, t_n]$ is

$$\xi_{y_n} = y_n - \tilde{y}_n = C\xi_{x_n} + D\xi_{u_n} \leq O(\Delta t^{k+1}) \tag{5.29}$$

Property 82. *For both, feedthrough and non feed through systems, the global error of the state and the output global errors are of order* $\mathrm{O}(\Delta t^{k+1})$

5.7.3 Error from the solver

In [1] it is argued the reasons why we could just neglect the solver error in co-simulation.

Consider that the subsystems are solved with a m- step, explicit or implicit numerical method where δt is the solver integration step (micro step),

$$\tilde{x}_n = f(t_n; \tilde{\tilde{x}}_{n-m}, \tilde{\tilde{x}}_{n-m+1}, ..., \tilde{\tilde{x}}_{n-1}; \delta t) \quad \textit{explicit method} \tag{5.30}$$

$$\tilde{x}_n = f(t_n; \tilde{\tilde{x}}_{n-m}, \tilde{\tilde{x}}_{n-m+1}, ..., \tilde{\tilde{x}}_{n}; \delta t) \quad \textit{implicit method} \tag{5.31}$$

we can define the approximation error from a p order explicit (or implicit) method as

$$\varepsilon_{x_n}^n = \tilde{x}_n - \tilde{\tilde{x}}_n = \tilde{x}_n - f(t_n; \tilde{x}_{n-m}, \tilde{x}_{n-m+1}, ..., \tilde{x}_{n-1}; \delta t) \tag{5.32}$$

whose local and global error are, respectively

$$\begin{aligned} \varepsilon_{x_n}^n &= O(\delta t^{p+1}); \\ \xi_{x_n}^n &= O(\delta t^p) \end{aligned} \tag{5.33}$$

hence, the local error in co- simulation can be derived as

$$\begin{aligned} \varepsilon_{x_n}^{'} &= x_n - \tilde{\tilde{x}}_n \\ &= x_n - \tilde{x}_n + \tilde{x}_n - \tilde{\tilde{x}}_n \\ &= \varepsilon_{x_n} + \varepsilon_{x_n}^n \leq O(\Delta t^{k+2}) + O(\delta t^{p+1}) \end{aligned} \tag{5.34}$$

Now, if we remove the error free condition of the local error from previous steps condition, the global error of co- simulation is derived as

$$\begin{aligned} \xi_{x_n}^{'} &= x_n - \tilde{\tilde{x}}_n \\ &= x_n - \tilde{x}_n + \tilde{x}_n - \tilde{\tilde{x}}_n \\ &= \xi_{x_n} + \xi_{x_n}^n \leq O(\Delta t^{k+1}) + O(\delta t^p) \end{aligned} \tag{5.35}$$

Property 83. *The error from the solver is bounded by the sum of the coupling error with order* $\mathrm{O}(\Delta t^{k+1})$ *and the approximation error from the solver of order* $\mathrm{O}(\delta t^p)$.

Property 84. *Given that the macro step normally is larger than the micro step (i.e.* $\Delta t \geq \delta t$*), and that high order and multi- step numerical methods are quite common while extrapolation of high degree is rarely used because of stability issues, we have that* $p \geq k$.

Proposition 85. *From the results of this section we can state given that the coupling error is dominant over the error from the solver it is reasonable to neglect the latter error from our analysis and focus on the coupling error.*

We will focus our analysis in the coupling error when improving the co simulation behaviour.

Chapter 6

H_∞ in co-simulation

Co-simulation is widely used in industry for the simulation of multi-domain systems. However, as we have seen in the co- simulation chapter, due to the not continuous communication between our coupling variables, simulation results end up being unstable and inaccurate.

Among the different strategies that have been developed in the recent years to control co-simulation drawbacks, the aim of this thesis is focused in application of the H_∞ synthesis method.

When we are in a mono-simulation scenario where the dynamic equations are solved together by one solver, accuracy and stability of the simulation relies just on the time-stepping method and the step size. Nevertheless, when we are in a co-simulation stage, stability and accuracy results also dependent on the discrete communication between the different subsystems.

We can distinguish our co- simulation systems according to: the time- stepping method of the main, which can be explicit, semi explicit, and implicit (iterative); the slave subsystems section 5.3, and the coupling configurations section 5.5.

We will follow [1] and study the explicit-parallel co-simulation scenario in the frequency domain. This scheme is argued to be chosen for being easy to implement and the most common of all the calculation schemes. Nevertheless, as we saw in chapter 5, it will also lead us to accuracy and stability drawbacks. Its main disadvantage is the need of using extrapolation methods to approximate the inputs of each subsystem, which are unknown during the communication interval Δt. In this thesis we choose the zero- order hold (ZOH) (subsubsection 3.2.4.2) as the main holder operator of our decoupled system.

Recall that because in an explicit co-simulation approach we cannot go back and recalculate the input, the coupling error of our co- simulation system can

be notable compared to iterative schemes. In order to improve the explicit co-simulation result and reduce the extrapolation error we will study the H_∞ controller design presented in [1, Paper III].

6.1 Closed loop interconnection formulation

To show how by coupling a system, co-simulation deteriorates stability, we consider a mono- simulation system and decouple it as a closed loop interconnection of two subsystems.

In order to focus on the analysis of the deterioration of the coupling error we assume that the coupled subsystems are linear, time invariant, single input-single output, with homogeneous initial condition, and accurately solve by an appropriate solver so that the coupling error is greater than the integration error.

We denote our subsystems with the transfer functions $Q_1(s)$ and $Q_2(s)$ were s denotes a variable in the complex plane (see chapter 2). Recall that depending on whether $Q_1(s)$ and $Q_2(s)$ are proper (the degree of the numerator is less than the degree of the denominator) or strictly proper [13] (the degree of the numerator is less or equal than the degree of the denominator) the connection is, respectively, feed- through (proper) or non feed-through (strictly proper), see Definition 74 Definition 75.

We will see the communication of our input and output variables every macro-step as a sample data closed loop interconnection Figure 5.10 where stability issues arise.

Definition 86 (Sampled input)**.** *We define the sampled input $u^*(t)$, and its Laplace transform $u^*(s)$ as the product of the continuous input $u(t)$ and a periodic impulse train (see Equation 3.20).*

$$u^*(t) = \sum_{n=-\infty}^{\infty} u(t)\delta(t - n\Delta t), \quad u^*(s) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} u(s - jn\omega_s) \tag{6.1}$$

where $\omega_s = 2\pi/\Delta t$ is the sampling frquency.

Definition 87 (Approximated input)**.** *We define the approximated input as the Laplace transform of our sample input $u^*(s)$ held with an extrapolation operator $H(s)$, in our case the zero- order- hold (subsubsection 3.2.4.2)*

$$\begin{aligned}\tilde{u}(s) = \frac{H(s)}{\Delta t}\sum_{n=-\infty}^{\infty} u(s - jn\omega_s) \\ = H(s)u^*(s)\end{aligned} \tag{6.2}$$

We could have used different holders $H(s)$ for each of the subsystems. In Figure 6.1, as in [1, Paper III] we will assume that the same $H(s)$ is applied in

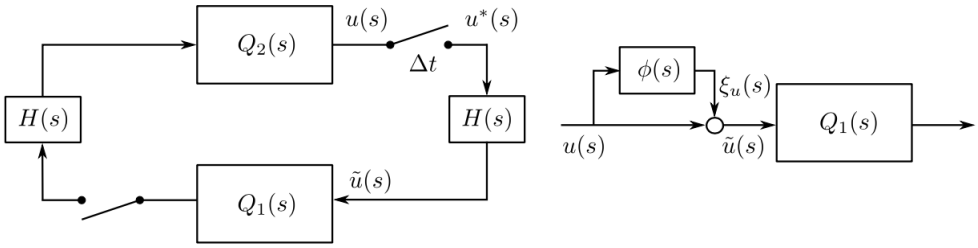


Figure 6.1: co-simulated system as a closed-loop interconnection, and a truncated subsystem with coupling error ξ_u as an input multiplicative disturbance

the whole interconnection.

The stability analysis of the present co-simulated system Figure 6.1 is characterized by two main factors:

1.Input coupling error ξ_u caused by extrapolation and its influence on simulation accuracy.

2.Numerical stability and robustness of co-simulation.

6.2 Analysis of the coupling error

Definition 88 (Coupling error ξ_u)**.** *We define the input coupling error ξ_u as the difference of the continuous input and its approximation*

$$\begin{aligned}\xi_u(s) = \tilde{u}(s) - u(s) &= \frac{H(s)}{\Delta t}\sum_{n=-\infty}^{\infty} u(s - jn\omega_s) - u(s) \\ &= \underbrace{\frac{H(s)}{\Delta t}\sum_{n=1}^{\infty} u(s \pm jn\omega_s)}_{Higher\ \ frequency} + \underbrace{\left(\frac{H(s)}{\Delta t} - 1\right) u(s)}_{Lower\ \ frequency}\end{aligned} \tag{6.3}$$

Since there is insufficient data to know how the original signal looks like, we need to give special attention to the case where the frequency of the input signal $u(s)$ denoted by ω is high so that the sampling frequency from $\tilde{u}(s)$ denoted by $\omega_s = 2\pi/\Delta t$ is not high enough to fulfill the Nyquist criterion causing aliasing effect (Definition 64); in this case, the lower and higher frequency part of ξ_u are not possible to decouple and the problem is not treatable to our knowledge. In order to avoid the aliasing effect we need to ensure that Δt is small enough. Following this condition, in [1, Paper III] Δt is chosen according to the subsystem bandwidth (Definition 44). However, this will not guarantee the stability of our co-simulated system.

Now, in order to introduce the definition of the different coupling errors you get depending on the extrapolation method that is used to approximate the coupling variables during the communication interval, we first define the most basic three holder operators: zero order hold (ZOH), first order hold (FOH), second order hold (SOH).

$$H_{ZOH}(s) = \frac{1 - e^{-s\Delta t}}{s} \tag{6.4}$$

$$H_{FOH}(s) = \frac{1 + \Delta t}{\Delta t}\left(\frac{1 - e^{-s\Delta t}}{s}\right)^2 \tag{6.5}$$

$$\begin{aligned} H_{SOH}(s) = &\left(\frac{0.5 - e^{-s\Delta t} + 0.5e^{-2s\Delta t}}{\Delta t^2}\right)\left(\frac{2 - \Delta t^2 s^2 e^{-s\Delta t} - 2\Delta t s e^{-s\Delta t} - 2e^{-s\Delta t}}{s}\right) \\ &+ \left(\frac{1.5 - 2e^{-s\Delta t} + 0.5e^{-2s\Delta t}}{s^2}\right) + \frac{1 - e^{-s\Delta t}}{s} \end{aligned} \tag{6.6}$$

Computing the coupling error ξ_u combined with the different holder operators $H(s)$ (see Equation 3.38) we obtain the following Taylor series expansions

$$\begin{aligned} \xi_{u,ZOH}(s) &= u(s)\left(\frac{1 - e^{-s\Delta t} - s\Delta t}{s\Delta t}\right) + \frac{1 - e^{-s\Delta t}}{s\Delta t}\sum(s) \\ &= u(s)\left(\frac{1 - \left(-s\Delta t + \frac{(-s\Delta t)^2}{2} + \frac{(-s\Delta t)^3}{6} + ...\right) - s\Delta t}{s\Delta t}\right) \\ &\quad + \frac{1 - \left(-s\Delta t + \frac{(-s\Delta t)^2}{2} + \frac{(-s\Delta t)^3}{6} + ...\right)}{s\Delta t}\sum(s) \\ &= \left[-\frac{1}{2}s\Delta t + s^2\mathcal{O}(\Delta t^2)\right]u(s) + [1 + s\mathcal{O}(\Delta t)]\sum(s) \end{aligned} \tag{6.7}$$

with the notation

$$\sum(s) = \sum_{n=1}^{\infty} u(s \pm jn\omega_s) \tag{6.8}$$

Analogously,

$$\xi_{u,FOH}(s) = \left[-\frac{5}{12}(s\Delta t)^2 + s^3\mathcal{O}(\Delta t^3)\right]u(s) + \left[1 + s^2\mathcal{O}(\Delta t^2)\right]\sum(s) \tag{6.9}$$

$$\xi_{u,SOH}(s) = \left[-\frac{3}{8}(s\Delta t)^3 + s^4\mathcal{O}(\Delta t^4)\right]u(s) + \left[1 + s^3\mathcal{O}(\Delta t^3)\right]\sum(s) \tag{6.10}$$

As a consequence of this equations if Δt is small enough $\xi_u(s)$ can be correctly approximated by its low frequency component [14] which enables to extract the following results.

Theorem 89. *A k- degree extrapolation method gives an error with order* $O(\Delta t^{k+1})$, *everytime* $\sum(s)$ *is negligible [1, paper III].*

Property 90 (Frequency property)**.** *The error grows with the input frequency.*

The co- simulated system represented in Figure 6.1 and its respective nominal system satisfy that

$$\tilde{u}(s)Q(s) = \tilde{y}(s) \;\; and \;\; u(s)Q(s) = y(s) \tag{6.11}$$

thus, the output error can be seen as a linear map from the input error

$$\xi_u(s)Q(s) = (\tilde{u}(s) - u(s))Q(s) = \tilde{y}(s) - y(s) = \xi_y(s) \tag{6.12}$$

Definition 91 (Output coupling error)**.** *We define the output coupling error of our LTI coupled system as the linear projection of our coupling error* $\xi_u(s)$ *in the frequency domain with the same error that characterizes* $\xi_u(s)$, *i.e.*

$$\xi_y(s) = \tilde{y}(s) - y(s)$$

6.3 Stability and Robustness

In this section we describe how the different components of our decouple system need to behave to guarantee the stability of our system. Assuming that the nominal system is designed to be stable, we present two main results that ensure the stability of the closed loop of the co-simulation system.

In co-simulation stability is characterized by the convergence of the error ξ_y. We represent the closed loop interconnection of our the co- simulated system Figure 6.1as

$$\tilde{y}_1 = (1+\phi)Q_1\tilde{y}_2 \;\; and \;\; \tilde{y}_2 = (1+\phi)Q_2\tilde{y}_1 \tag{6.13}$$

where ϕ is an operator that denotes multiplicative disturbance that represents the coupling error of our system.

Now, considering the nominal system we can extend its closed loop interconnection $y_1 = Q_1y_2$, $y_2 = Q_2y_1$ to

$$y_1 = (1+\phi)Q_1y_2 - \phi Q_1y_2 \;\; and \;\; y_2 = (1+\phi)Q_2y_1 - \phi Q_2y_1 \tag{6.14}$$

from where we can derive the output error $\xi_y = [\xi_{y_1}, \xi_{y_2}]$ by computing the difference

$$\xi_{y_1} = \tilde{y_1} - y_1 = (1+\phi)Q_1\xi_{y_2} + \phi Q_1y_2 \tag{6.15}$$

$$\xi_{y_2} = \tilde{y_2} - y_2 = (1+\phi)Q_2\xi_{y_1} + \phi Q_2y_1 \tag{6.16}$$

In a more compact representation, by definition of $\xi_y(s)$ Definition 91 in Figure 6.1, the output approximated output of our closed loop system becomes

$$\tilde{y} = \frac{1}{1-(1+\phi)^2Q_1Q_2}\begin{bmatrix}(1+\phi)Q_1 & (1+\phi)^2Q_1Q_2\\ (1+\phi)^2Q_1Q_2 & (1+\phi)Q_2\end{bmatrix}\tilde{u} \tag{6.17}$$

On the other hand we can express the error free output of the nominal system as

$$y = \frac{1}{1 - Q_1 Q_2} \begin{bmatrix} Q_1 & Q_1 Q_2 \\ Q_1 Q_2 & Q_2 \end{bmatrix} u \tag{6.18}$$

Thus, we can derive ξ_y from the difference $\xi_y(s) = \tilde{y}(s) - y(s)$ so that

$$\xi_y = \frac{1}{1 - (1+\phi)^2 Q_1 Q_2} \frac{1}{1 - Q_1 Q_2} \begin{bmatrix} \phi Q_1 + \phi(1+\phi) Q_1^2 Q_2 & \phi(2+\phi) Q_1 Q_2 \\ \phi(2+\phi) Q_1 Q_2 & \phi Q_2 + \phi(1+\phi) Q_1 Q_2^2 \end{bmatrix} \xi_u$$

$$\xi_y = \frac{1}{1 - Q_1(1+\phi) Q_2(1+\phi)} \begin{bmatrix} \phi(1+\phi) Q_1 Q_2 & \phi Q_1 \\ \phi Q_2 & \phi(1+\phi) Q_1 Q_2 \end{bmatrix} \xi_u \tag{6.19}$$

where $\xi_y = \begin{bmatrix} \xi_{y_1} & \xi_{y_2} \end{bmatrix}$ and $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$.

Since Q_1, Q_2 and the terms cascaded with ϕ are designed to be stable the nominal closed- loop system is stable. Therefore, in order to analyze the convergence of the output error ξ_y we just need to study the transfer function of the closed loop system $-(1+\phi)^2 Q_1 Q_2$, which needs to follow the Nyquist criterion Theorem 58 in order to be stable, i.e. not encircle the point $-1 + j0$ in the complex plane.

Now we introduce the main two theorems [1, Paper III](and some of their related properties) that characterize the stability of the presented closed loop system. Both of them based their formulation on the Nyquist stability criterion.

Theorem 92 (Small gain theorem)**.** *Given that $\left\|(1+\phi)^2 Q_1 Q_2\right\|_\infty$ is the maximum gain Definition 18 of our SISO system represented in Figure 6.1. The closed loop interconnection in Figure 6.1 defined by the equations Equation 6.13 is input-output stable Definition 31, if the maximum norm of its loop gain satisfies*

$$\left\|(1+\phi)^2 Q_1 Q_2\right\|_\infty < 1 \tag{6.20}$$

This theorem implies that the system is stable if $-(1+\phi)^2 Q_1 Q_2$ is bounded within the unit circle, and since ZOH does not amplify the system gain, it is guaranteed a stable co- simulation if the nominal system satisfies

$$\left\|Q_1 Q_2\right\|_\infty < 1 \tag{6.21}$$

with no aliasing effect.

Property 93. *The smaller the loop gain $-(1+\phi)^2 Q_1 Q_2$ of the system is, the better the rejection to disturbance (or coupling error).*

Property 94. *Although scaling down the coupling variables gives incorrect simulation results, the loop gain will also be reduced making the system more stable, which can be useful for a stable initial setup.*

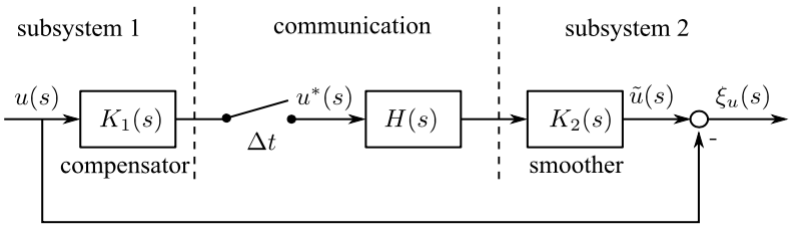


Figure 6.2: Formulation of an error system for one coupling variable

Theorem 95 (Passivity theorem)**.** *The closed loop interconnection in Figure 6.1 defined by the equations Equation 6.13 is stable if its subsystems* $Q_1(1+\phi)$ *and* $-Q_2(1+\phi)$ *are strictly passive (in the context of the delay or phased lag 39 led by extrapolation) or output strictly passive with zero- state observable (i.e. generalization for a nonlinear system).*

Geometrically, this theorem says that the system is stable if it has a phase angle, section 3.6 of less than 180 degrees, so the Nyquist plot (Figure 3.25) does not encircle the $-1+j0$ on the complex plane and the Nyquist stability criterion is satisfied. Physically speaking this theorem implies that an additional energy flow into the interconnection of the system, if it is not sufficiently dissipated, might make the system unstable.

As a consequence of the small gain theorem and the passitivity theorem we can conclude that to improve the stability of the co- simulated system Figure 6.1 the loop gain should be reduced (Theorem 92) or the phase delay compensated (Theorem 95).

6.4 H_∞ Problem Formulation

We know from Proposition 85 that the sample- hold process is the main source of error of co- simulation. To deal with this error and minimize it we want to study the application of a particular H infinity controller design, presented in [1, paper III] Recall that the input coupling error is described by Equation 6.3 where $u(s)$ is a coupling variable that is not accessible and therefore, ξ_u is unknown.

The error system plant presented in [1], is based on the formulation of an input coupling error system presented in the research work in modern signal reconstruction [6], this is illustrated in Figure (Figure 6.2). Here K_1 is a feedback controller that compensates the error to correct the output of the system, K_2 is a low pass filter Definition 41

$$K_2(s) = \frac{1}{\left(\frac{s}{2\pi f_{K_2}}\right)^2 + 1.4142\frac{s}{2\pi f_{K_2}} + 1} \tag{6.22}$$

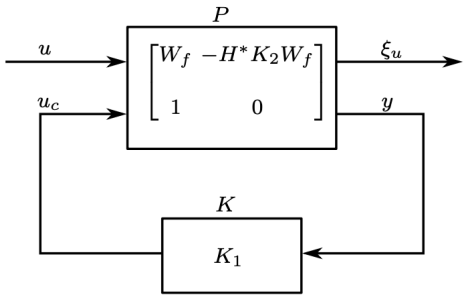


Figure 6.3: Generalized plant P and undetermined controller K as an equivalence to the error system

that smooths the input signal. $H^*(s)$ is the resulting holder operator (ZOH) after applying a Padé approximation

$$H^*(s) \approx \frac{120000}{s^2 + 600s + 1.2e05}$$

According to this the problem consists on finding the best pair of K_1, K_2 to reduce $\xi_u(s)$. Given that we do not have access to the exact input $u(s)$ nor $\xi_u(s)$ one could try to tackle this problem considering that $K_1(s)K_2(s)H^*(s) = 1$ (i.e. the original signal is unchanged, thus $\xi_u(s) = 0$) so that $K_1(s)K_2(s) = H^*(s)^{-1}$. However, because $H^*(s)^{-1}$ is improper 21, this transfer function has always at least one pole at infinity, meaning that not all the poles are inside the unit circle, which makes it unstable and a not valid nor implementable solution.

Instead, [1, Paper III] proposes to reformulate the problem as a minimization of the L_2 norm of the coupling error $\|\xi\|_2$, such that

$$\|\xi_u\|_2 = \|T_{ue}u\|_2 \leq \|T_{ue}\|_\infty \|u\|_2 \tag{6.23}$$

where T_{ue} is the transfer function of the error system. The L_2 norm of the error $\|\xi_u\|_2$ is upper bounded by the supremum norm of the error system $\|T_{ue}\|_\infty \|u\|_2$ (i.e. $\|\xi_u\|_2 \leq \|T_{ue}\|_\infty \|u\|_2$).

According to this, if we design $K_1(s)$ and $K_2(s)$ properly, they should give us a minimal energy gain $\|T_{ue}\|_\infty$, so that the energy of coupling error is minimized. To implement this problem, we apply the H_∞ synthesis optimization method (chapter 5).

6.4.1 H_∞ synthesis for the coupling design

To apply H_∞ synthesis method to our problem we need to transfer the error system in Figure 6.2 to a generalized plant P, (see Figure 6.3) where u is the disturbance input to the plant and ξ_u is the input error. Once this is done we can describe our H_∞ problem as follows:

Theorem 96 (H_∞ Problem:)**.** *Given a LTI system G, find a feedback gain K such that the closed loop system is stable and the following is satisfied:*

$$\|\xi_u\|_2 < \gamma \|u\|_2 , \qquad \|T_{ue}\|_\infty := \sup_{Re\{s\}>0} \|T_{ue}(s)\| < \gamma \tag{6.24}$$

where γ is the level of L_2 gain performance to be minimized.

This problem requires that the synthesized control law K is stable, proper and casual (i.e. the degree of the numerator is smaller than the degree of the denominator), so it only depends on previous inputs.

W_f is defined in [1] as a weighting function,

$$W_f = \frac{1}{\frac{1}{2\pi f_{W_f}+1} + 1} \tag{6.25}$$

designed to be another low- pass filter, this time with regards to the low frequency component of the coupling error. Even though the sampling and holder $H^*(s)$ is Padé approximated, and the highest component of ξ_u results from its high frequency part, the minimization of $\|\xi_u\|_2$ when we apply the smoother K_2 in all frequency range will, in particular, distort the low- frequency component. This is the reason why it is required to include in the plant of our error system another low-pass filter, which this time is meant to decrease the error in the low frequency range

$$\left(\frac{H(s)}{\Delta t} - 1\right) u(s)$$

Furthermore, in order to gain computational efficiency, the optimization could also bound the poles of the error system plant adding pole- placement constraints to the arguments of our minimization problem.

Chapter 7

Implementation

In this chapter we aim to reproduce the results of the studied approach. With this purpose, let us consider a double mass spring damper decoupled through a displacement-displacement configuration, see Figure 7.1.

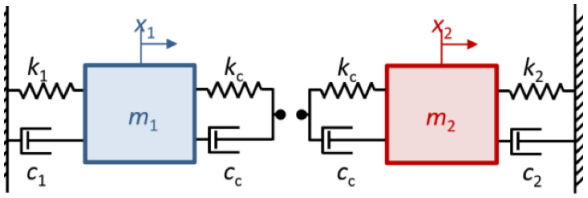


Figure 7.1: Caption

From chapter 5 we know the acceleration of this dynamical system equations is defined by:

$$\ddot{x}_1 = \frac{c_c[\dot{x}_2] + k_c[x_2] - (c_1 + c_c)\dot{x}_1 - (k_1 + k_c)x_1}{m_1}$$

$$\ddot{x}_2 = \frac{c_c[\dot{x}_1] + k_c[x_1] - (c_c + c_2)\dot{x}_1 - (k_c + k_2)x_2}{m_2}$$

and that the exchange of information during the macrostep, i.e. the coupling variables of the decoupled system are:

$$y_1 = \begin{pmatrix} x_1 \\ \dot{x}_1 \end{pmatrix} \quad y_1 = \begin{pmatrix} x_2 \\ \dot{x}_2 \end{pmatrix}$$

In Figure 7.2, Figure 7.3 the non- decoupled system is simulated with a RK4 solver, $\Delta t = 50\ ms$ and the following parameters:

Masses	Underdamped	Overdamped
$m_1 = 8.6856$, $m_2 = 6.7863$	$k_1 = 176.6611$, $k2 = 895.8553$, $k_c = 171.7$,	$d_1 = 3.6585$, $d_2 = 16.4162$, $d_c = 0.0857$
$m_1 = 3.2365$, $m_2 = 8.9549$	$k_1 = 120.5520$, $k2 = 995.0490$, $k_c = 701.9907$,	$d_1 = 20.2187$, $d_2 = 231.6654$, $d_c = 12.5520$,

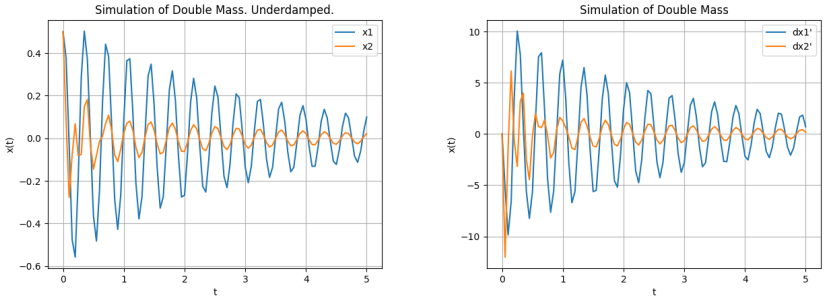


Figure 7.2: Simulation with parameters: $d_1 = 3.6585$, $d_2 = 16.4162$, $d_c = 0.0857$, $k_1 = 176.6611$, $k2 = 895.8553$, $k_c = 171.7$, $m_1 = 8.6856$, $m_2 = 6.7863$; over time.

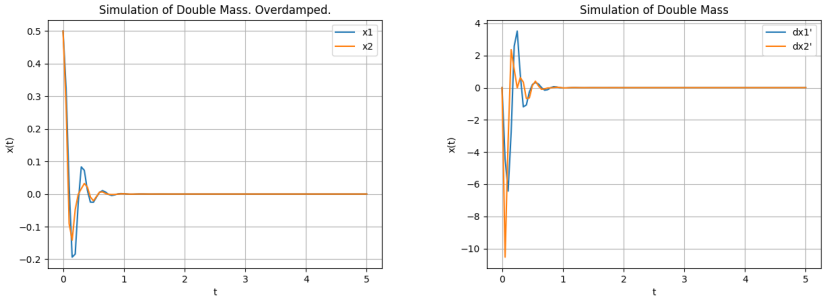


Figure 7.3: Simulation with parameters: $d_1 = 20.2187$, $d_2 = 231.6654$, $d_c = 12.5520$, $k_1 = 120.5520$, $k2 = 995.0490$, $k_c = 701.9907$, $m_1 = 3.2365$, $m_2 = 8.9549$; over time.

7.1 Co-simulation

For the co-simulation test we decoupled our double mass spring damper as we presented in Figure 7.1 and simulate the subsystems using again RK4 method as a solver, a micro step of $\delta t = 1ms$, a macro step of $\Delta t = 50ms$ and the

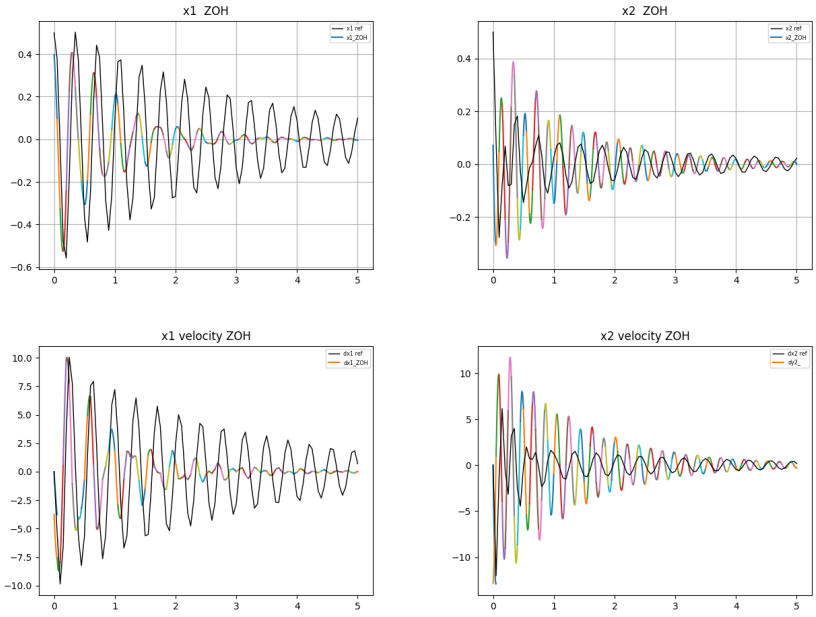


Figure 7.4: $x_1, x_2, \dot{x}_1, \dot{x}_2$ with ZOH extrapolation for the underdamped system over time.

zero-order hold to approximate the coupling variables during the communication interval. We also use the same two sets of parameters we used for the reference (mono-simulated) system.

The system parameters, the micro-step and macro-step values are all taken from the experimental results in [1]. However, we did not picked the same coupling configuration, which in [1, Paper III, p 13] is a force- displacement configuration and here a displacement-displacement configuration.

In Figure 7.4 and Figure 7.5 it is plotted the simulation of the reference variables in black in contrast to the resulting variables of the decoupled systems over time (x-axis). These are drawn in different colours illustrating that distinguish the communication instants, where the macrostep is represented by the change of colour. To implement the co- simulated system I used the MPI module in python with two different cores, two for Subsystems 1 and 2 (secondary), and a third one for the update of the coupling (primary).

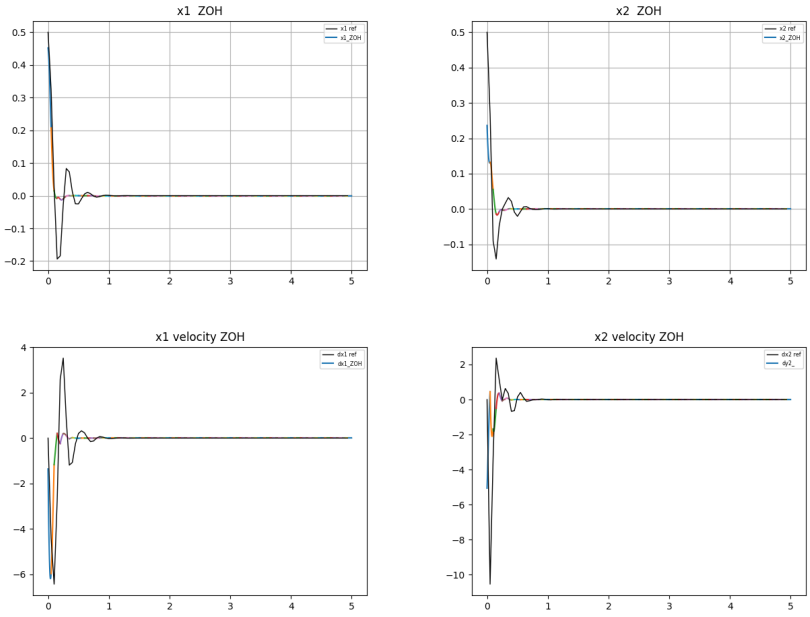


Figure 7.5: $x_1, x_2, \dot{x}_1, \dot{x}_2$ with ZOH extrapolation for the overdamped system over time.

7.2 Approximation of the H_∞ method

Now, it is of our interest, given the difference in the mono-simulation and co-simulation results illustrated in the plots, to find and synthesized a H_∞ controller according to the designed plant of our error system Figure 6.3, where we recall

$$K_2(s) = \frac{1}{\left(\frac{s}{2\pi f_{K_2}}\right)^2 + 1.4142\frac{s}{2\pi f_{K_2}} + 1} \tag{7.1}$$

$$W_f = \frac{1}{\frac{1}{2\pi f_{W_f}+1} + 1} \tag{7.2}$$

$$H^*(s) \approx \frac{120000}{s^2 + 600s + 1.2e05} \tag{7.3}$$

that smoothes the coupling error and brings the coupling variables closer to the reference system.

In order to do this I used the matlab control toolbox as suggested in [1, Paper III]. In particular, I used

`h2hinfsyn.m`

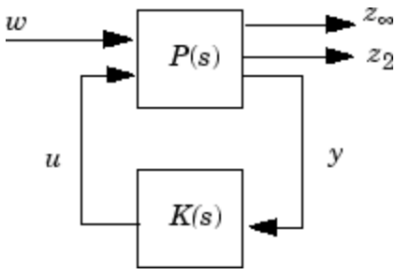


Figure 7.6: Matlab h2hinfsyn.m

a function that need to be given a plant, see P in Figure 7.6, the number of measurement signals (1 in our closed loop error system), the number of control signals (1), the number of signals subjected to the H2 constraint (0), the weights of performance of $H2$ and H_∞, $[W_\infty, W2]$ which since we do not have fixed ($[0, 0]$) and a name value argument that will permit us to restrict the controller to the H_∞ norm and put pole placement constraints (restrict the value of our eigenvalues so all of them behave inside the stability region of our method). This function outputs the state space representation of our controller K_1 and the state space representation of the closed loop system.

In order to set the plant the matlab function needs as a parameter, we define the missing values of our two low pass filters with the same values that are

given for the force- decoupled double mass spring damper test in [1, Paper III], i.e. $f_{K_2} = 10Hz$, $f_{W_f} = 0.2$. Now, computing the transfer function inside the matrix of our error system Figure 6.3 we have all the required to implement the mentioned matlab function.

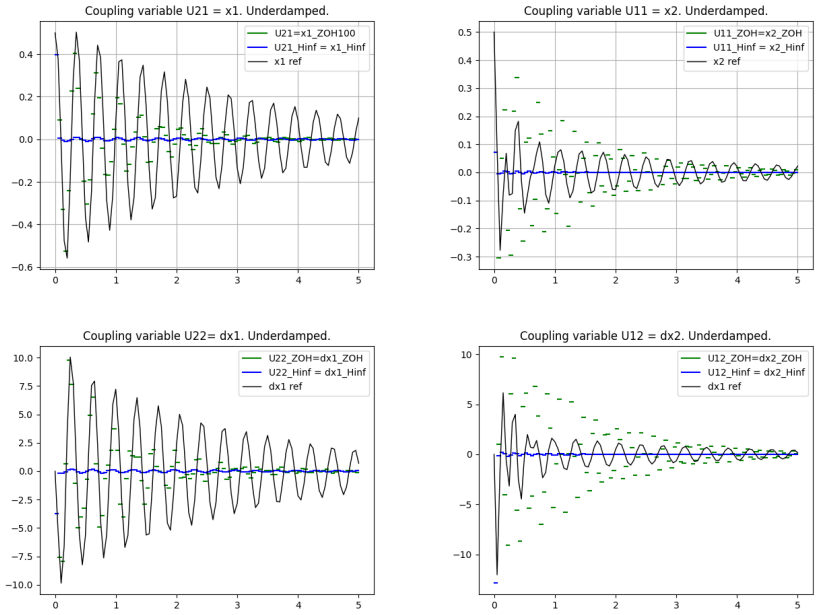


Figure 7.7: Simulation of coupling the variables with parameters $f_{k_2} = 10$ and $f_{w_f} = 0.2$, with the H_∞controller, over time.

In Figure 7.7 we plotted in blue, over time, the approximation of the coupling variables once applied our synthesized the controller. We compared this results plotting the coupling variables approximated with the ZOH approximation in green and the reference in black. In Figure 7.7 It is clear that the obtained results smooth our coupling signal excessively, making it further away from the reference than when we extrapolated it with the ZOH. I relate to this two main possible scenarios: a bad selection of the low pass filter parameters f_{K_2}, f_{W_f}, which are specific for each controller desinged depending on our dynamical system, or a problem with the synthesis of the controller.

In order to study both scenarios at the same time we studied the behaviour of a different controller this time synthesized with a bigger value for the parameters f_{K_2}, f_{W_f}, whose transform function is given in [1, Paper III]

$$K_1(s) = \frac{(s+1682.9)(s+6.2825)(s2+600s+1.2e05)(s2+266.6s+3.553e04)}{(s+1470.1)(s+6.2833)(s2+1870s+9.674e05)(s2+617.6s+5.111e05)} \tag{7.4}$$

where $f_{K_2} = 30\ Hz$, $f_{W_f} = 1\ Hz$.

this time in order to synthesized the controller we just needed to use the Matlab control toolbox function

`tf2ss.m`

which converts a continuous-time or discrete-time single-input transfer function into an equivalent state-space representation.

In figures Figure 7.8 we plotted the results for the approximated coupling variables with our new synthesized controller Equation 7.4 in blue, the coupling variables approximated with the ZOH approximation in green and the reference in black, over time.

In this case we also plotted the solutions. In both cases could see a bigger improvement of the approximation of the H_∞ controller.

Even though it could be first appreciated an improvement of the coupling variables approximation for our second controller, when computing the normalized root mean square (NRMS) error of the coupling variables and the solutions

$$\epsilon_{NRMS,x} = \frac{\sqrt{\sum_{t=0}^{T} \left(x(t) - x_{ref}(t)\right)^2 / T}}{x_{ref,max} - x_{ref,min}}$$

the evidence does not seem consistent.

`0.14095558240065628 error X1ZOH`

`0.14449389607213403 error X1HINF`

`0.1285915260887583 error DX1ZOH`

`0.1305777235335553 error DX1HINF`

`0.12397044587346658 error X2ZOH`

`0.1098653753680789 error X2HINF`

`0.13852078615214833 error DX2ZOH`

`0.11364227132142762 error DX2HINF`

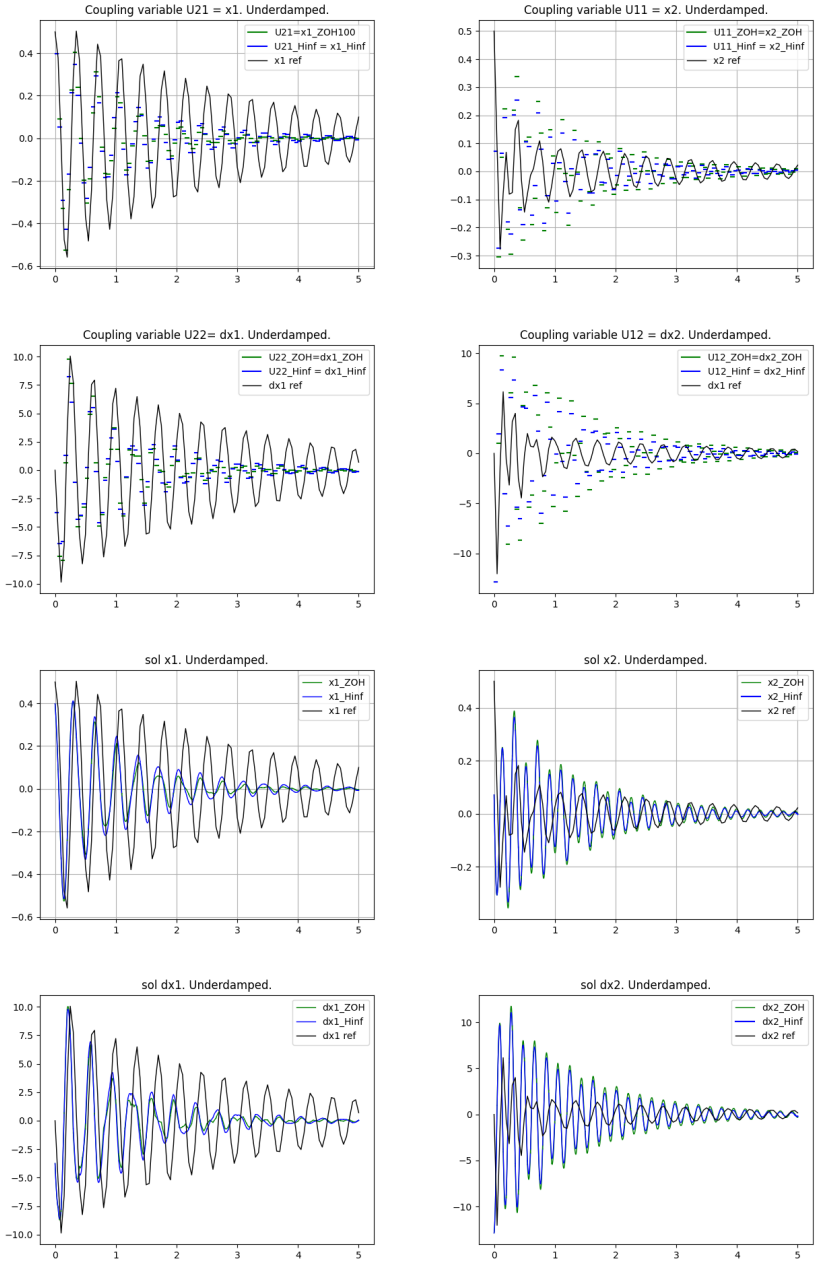


Figure 7.8: Simulation of the underdamped system when applying our synthesized H_∞ controller to our coupling variables and when using ZOH extraplation method, over time.

Furthermore, when checking if the controller the matlab function synthesises with this parameters matches the same state matrix that characterizes Equation 7.4 we get different results, which clears up the fact that the synthesis of the controller of our experiments is not well computed for reasons that, we still cannot explain.

Chapter 8

Conclusions

In this thesis we presented a co- simulation approach for linear time invariant systems based on the control theory method, H_∞ synthesis.

This papers aims to decompressed the detailed and required theoretical fact one needs to understand our reference paper [1, Paper III].

After this theoretical part, a couple of experiments were performed in Python to demonstrate the method. A couple of spring damper systems were used to show the effect of the controller. Not all the results in [1] could be reproduced, in particular the synthesis of the controller using the Matlab Control Toolbox. Nevertheless, I believe this is a located issue, based on a misunderstanding of the Matlab controlbox functions that synthesize H_∞ controllers according to an specific plant. I still cannot explain what has been the confusion when implementing these control functions, nor how to specify the parameters that are involved in the error plant.

This thesis opens up two fascinating fields of study, control theory and decoupled simulation. If I would have more time to continue with this project I would have tried to inquire even deeper in the Matlab control Toolbox platform, the python control library, and the analytical procedure that solves the H_∞ minimization problem.

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