Complex Multiplicative Calculus and Mixed Problems

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Abstract

In this thesis we consider multiplicative integrals and derivatives on the sets $\mathbb{R}_+$, $\mathbb{S} = \boldsymbol{e}^{\mathbb{C}}$ (the logarithmic Riemann surface) and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Additive structures on $\mathbb{R}$ and $\mathbb{C}$ are related to their multiplicative counterparts on $\mathbb{R}_+$ and $\mathbb{S}$ by defining exponential transition functors $\mathcal{T}_{\exp}$ and $\mathcal{T}_{\mathbf{exp}}$. We use a lift-projection method to transition multiplicative structures on $\mathbb{S}$ to their counterparts on $\mathbb{C}^*$. The process introduces potentially multivalued behaviour, which is the case for multiplicative integrals on $\mathbb{C}^*$-valued functions.

Some mixed problems, involving both additive and multiplicative structures, are also discussed. E.g. we consider the mixed differential equation $y' = y^*$, whose solution involves the Lambert W function. We extend the inequality of arithmetic and geometric means (AM-GM inequality) to the setting of non-negative random variables. The matrix AM-GM inequality (Theorem 10.9 in [6]) is also extended, and tweaked to an integral version which leads to a generalization of Hölder's inequality.

Acknowledgements

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Populärvetenskaplig sammanfattning

Vad är ett medelvärde? Om vi har två tal a och b, skulle nog de flesta säga att deras medelvärde är $A = \frac{a+b}{2}$, vilket vi kallar deras aritmetiska medelvärde. Men om $a, b \geq 0$ så har vi också ett annat rimligt medelvärde, nämligen det geometriska $G = \sqrt{ab}$. Redan i antikens Grekland kände man till AM-GM olikheten som säger att $A \geq G$ med likhet om och endast om $a = b$, och deras förståelse byggde på geometriska argument som i Figur 1 och 2. Dessutom gäller AM-GM olikheten för fler variabler en två, dvs om

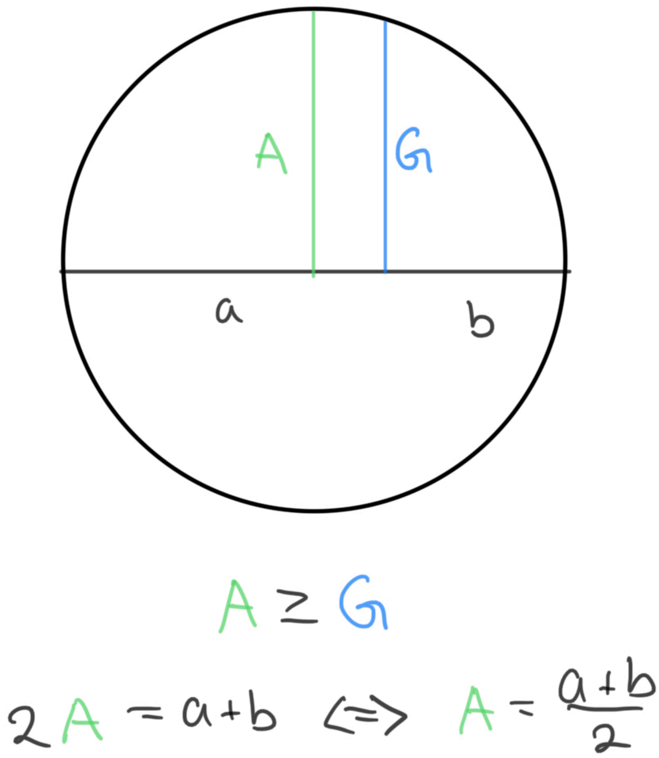
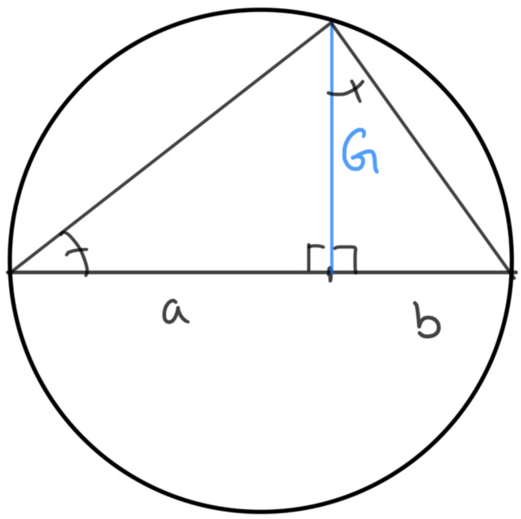


Figure 1: I grönt är det aritmetiska medelvärdet eftersom det är radien i cirkeln, och $a+b$ är diametern. Det geometriska medelärdet G är i blått och i nästa figur förklarar vi varför det är det geoemtriska medelvärdet. Notera även att $A = G$ i figuren om och endast om $a = b$.

$x_1, \ldots, x_n \geq 0$ så definierar vi det aritmetiska medelvärdet $A = \frac{x_1+x_2+\ldots+x_n}{n}$ och det geoemtriska medelvärdet $G = \sqrt[n]{x_1 x_2 \cdots x_n}$. Återigen gäller att $A \geq G$ med likhet om och endast om $x_1 = \ldots = x_n$. Samtidigt kan man även beräkna medelvärdet av ett kontinuum genom att använda integraler,



$$\frac{a}{G} = \frac{G}{b} \iff ab = G^2 \iff G = \sqrt{ab}$$

Figure 2: Vi använder likformighet för att visa att G faktiskt är det geometriska medelvärdet. Beviset bygger på att proportionen mellan a och G är densamma som den mellan G och b, från vilket är följer att $G = \sqrt{ab}$.

exempelvis om man ska beräkna tyngdpunkten av ett objekt. Frågan är då om det finns ett motsvarande geometriskt medelvärde för kontinuumet, och i så fall om AM-GM olikheten fortfarande gäller?

Det visar sig att svaret är ja på båda ovanstående frågor, och för att definiera det geometriska medelvärdet så använder vi oss av logaritmen och exponentialfunktionen. Detta på grund av att de kopplar ihop addition och multiplikation genom formlerna $\ln xy = \ln x + \ln y$ och $e^{x+y} = e^x e^y$. Exempelvis kan vi överföra det aritmetiska medelvärdet av a och b till det geometriska genomm att använda processen med logaritmer och exponentialfunktionen så som följer;

1. Först ersätter vi a och b med $\ln a$ och $\ln b$ så att multiplikation ersätts med addition.

2. Vi beräknar sedan deras medelvärde som blir $\frac{\ln a + \ln b}{2}$.

3. Slutligen så applicerar vi exponentialfunktionen för att återigen få något

multipllikativt, och vi erhåller $e^{\frac{\ln a + \ln b}{2}} = \sqrt{ab}$.

Processen med logaritmen och exponentialfunktion, applicerat på ett aritmetiskt medelvärde gav oss således ett geometriskt medelvärde, och vi applicerar samma teknik på kontinuum-aritmetiska medelvärden för att få motsvarande geoemtriska medelvärden. Dessutom applicerar vi även tekniken med logaritmen och exponentialfunktionen på derivator och integraler för att erhålla multiplivativa motsvarigheter.

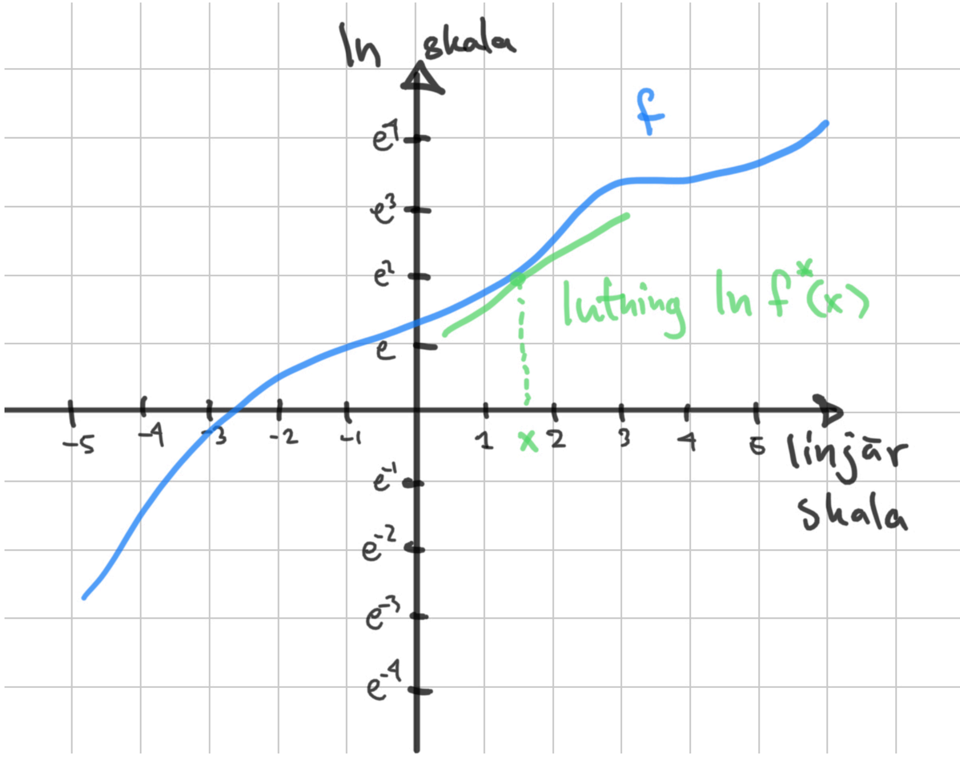


Figure 3: I blått ser vi en lin-ln plot of funktionen f, där x-axeln har linjär skala och y-axeln har logaritmisk skala. Lutningen (grön) i en punkt är då logaritmen av den multiplikativa derivatan $f^*(x)$.

Notera dock att vi hittils har begränsat oss enbart till funktioner som antar positiva värden, eftersom logaritmen enbart är definierad för positiva tal. Dock finns en komplex motsvarighet av logaritmen som även fungerar på negativa värden, men den är flervärd. Det beror på att $e^{2\pi in} = 1$ för varje heltal n, vilket kan tolkas som att vi snurrar runt i enhetscirkeln och där varje varv motsvarar vinkeln 2π. Vi använder den flervärda komplexa logaritmen och den komplexa exponentialfunktionen för att definiera komplexa multiplikativa integraler som också visar sig bli flervärda.

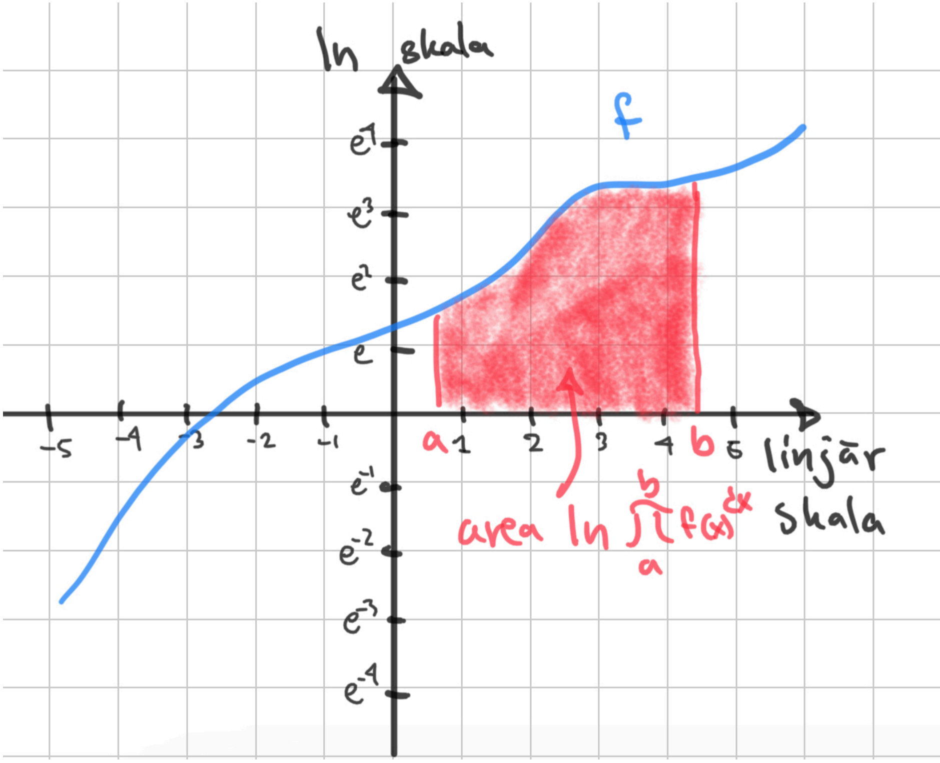


Figure 4: Arean under f i en lin-ln plot är logaritmen av produktintegralen $\prod_a^b f(x)^{\mathrm{d}x}$ (rött).

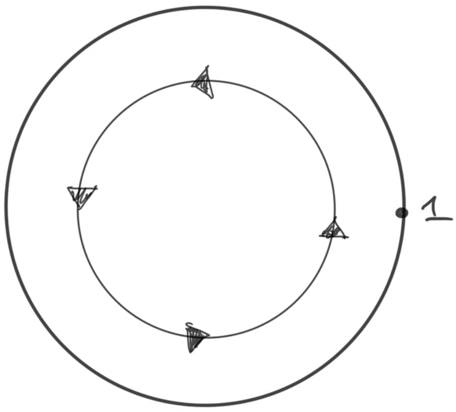


Figure 5: När vi snurrar runt på enhetscirkeln så återkommer vi till 1 efter varje varv.

Contents

1 Introduction

Arithmetic and geometric mean inequality

The arithmetic mean – geometric mean inequality (AM-GM) was already known by the ancient Greeks. In its simplest form it states that $\frac{a+b}{2} \geq \sqrt{ab}$ for non-negative numbers a and b, were equality holds if and only if $a = b$. This version has a very simple proof, yielding to the non-negativity of squares

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab},$$

from which the inequality follows. The equality condition follows from solving the equation $x^2 = 0 \iff x = 0$ for $x = \sqrt{a} - \sqrt{b}$. This version of the AM-GM inequality involves the arithmetic and geometric means of two variables a and b, and the inequality is often seen in its extended version involving the arithmetic and geometric means of finitely many variables. Given $x_1, x_2, \ldots, x_n \geq 0$, we define the arithmetic mean as

$$A = \frac{1}{n}\sum_{n=1}^{n} x_2 = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

and the geometric mean as

$$G = \left(\prod_{k=1}^{n} x_k\right)^{1/n} = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

The non-negativity of the variables is only required in the definition of the geometric mean, but in the AM-GM inequality both means need to be defined. Then the AM-GM inequality states that

$$A \geq G, \tag{1}$$

with equaltiy if and only if $x_1 = x_2 = \cdots = x_n$. The proof here is not quite as simple as the two-variable case, and relies on Jensen's inequality which is an indispensable tool for proving inequalities involving means and will be discussed more thoroughly later. However, the key part of the proof involves rewriting the geometric mean as

$$G = \exp\left(\frac{\ln x_1 + \ln x_2 + \ldots + \ln x_n}{n}\right) \tag{2}$$

for $x_1, \ldots, x_n > 0$ and then using the (strict) concavity of the logarithm and Jensen's inequality to show that $\ln A \geq \ln G$, with the strict concavity yielding the desired equality condition. If some $x_i = 0$, then $G = 0$ and $A \geq 0$ with equaltiy if and only if $\forall x_j = 0$.

Continuous AM-GM

With the discrete arithmetic and geometric means defined, one natural question to consider is whether it is possible to define continuous analogs, and whether the AM-GM inequality will still hold? To do so, lets first consider a continuous non-negative function $x : [0, T] \to [0, \infty)$ over which we would like to define these means. The arithmetic mean is given by

$$A = \frac{1}{T}\int_0^T x(t)dt,$$

and again x is not required to be a non-negative function, but will be so in the definition of the geometric mean. As we can see, upon going from the discrete arithmetic mean to the continuous one we replace the sum by an integral. The question then becomes what to correspondingly replace the multiplication with in the geometric mean? It would represent a continuous version of multiplication, or alternatively a multiplicative version of the integral and we shall represent it with the symbol $\prod\!\!\!\!\!\!\int$. In Table 1 we see the analogies

	additive	multiplicative
discrete	$\sum$	$\prod$
continuous	$\int$	$\prod\!\!\!\!\!\!\int$

Table 1: Analogies between $\sum$, $\prod$, $\int$ and $\prod\!\!\!\!\!\!\int$ in terms of discrete/continuous and additive/multiplicative. In particular $\prod\!\!\!\!\!\!\int$ can either be considered a continuous analogue of $\prod$ or a multiplicative analogue of $\int$.

between $\sum$, $\prod$, $\int$ and $\prod\!\!\!\!\!\!\int$, and how $\prod\!\!\!\!\!\!\int$ either can be viewed as a continuous version of multiplication or a multiplicative version of the integral.

Continuous products

The first approach, where $\prod\!\!\!\!\!\!\int$ is considered a continuous version of $\prod$, the transition from discrete to continuous corresponds to that from $\sum$ to $\int$. This means we could either take a Riemann or Lebesgue approach in going from $\prod$ to $\prod\!\!\!\!\!\!\int$. In the Riemann approach, one would consider Riemann products $\prod_{i=1}^{n} f(x_i)^{\Delta x_i}$ and limits [4]. Doing this suggests a suitable notation for Riemann multiplicative integrals, namely

$$\prod\!\!\!\!\!\!\int_a^b f(x)^{\,\mathrm{d}x}. \tag{3}$$

Similarly, one could get a measure theoretic multiplicative integral by mirroring the development of measure theory, with chosen notation

$$\prod f^{\,\mathrm{d}\mu} \tag{4}$$

for the multiplicative integral.

Connecting additive and multiplicative structures

However, other than replacing sums by products, we have yet to specify precisely what "mirrors" means? Doing so will require specification of the transition from additive to multiplicative, which can be used to directly transition from $\int$ to $\prod$. Recalling (2), we have already seen an example of such a transition and it suggests how to translate linearity. Addition $x + y$ is replaced by

$$\exp\left(\ln x + \ln y\right) = xy \tag{5}$$

multiplication, and scaling ax is replaced by

$$\exp\left(a \ln x\right) = x^a \tag{6}$$

exponentiation. Doing this allows us to transition the vector space $\mathbb{R}$ over itself to the multiplicative vector space $\mathbb{R}_+$ over $\mathbb{R}$.

Multiplicative integrals

Inspired by (5) and (6) we may define the multiplicative integral as

$$\prod f^{\,\mathrm{d}\mu} := \exp\left(\int \ln f \,\mathrm{d}\mu\right). \tag{7}$$

The idea is that the logarithm transforms a multiplicative linear combination $f^\alpha g^\beta$ into a linear combination of the logarithms $\ln\left(f^\alpha g^\beta\right) = \alpha \ln f + \beta \ln g$. Then by linearity of the integral we get a linear combination of the logarithmic integrals

$$\int \ln\left(f^\alpha g^\beta\right)\,\mathrm{d}\mu = \alpha \int \ln f \,\mathrm{d}\mu + \beta \int \ln g \,\mathrm{d}\mu. \tag{8}$$

The exponentiation works in reverse, transforming a linear combination $\alpha u + \beta v$ to a multiplicative linear combination $e^{\alpha u + \beta v} = e^{\alpha u} e^{\beta v} = \left(e^u\right)^\alpha \left(e^v\right)^\beta$. Using this for $u = \int \ln f \,\mathrm{d}\mu$ and $v = \int \ln g \,\mathrm{d}\mu$ when taking exponents of both sides in (8) yields $e^{\int \ln\left(f^\alpha g^\beta\right) \mathrm{d}\mu} = \left(e^{\int \ln f \,\mathrm{d}\mu}\right)^\alpha \left(e^{\int \ln g \,\mathrm{d}\mu}\right)^\beta$. Using (7), this shows that the multiplicative integral is indeed multiplicative

$$\prod \left(f^\alpha g^\beta\right)^{\mathrm{d}\mu} = \left(\prod f^{\,\mathrm{d}\mu}\right)^\alpha \left(\prod g^{\,\mathrm{d}\mu}\right)^\beta . \tag{9}$$

The idea in (7) can be extended to other linear transformations such as integral transforms, to yield a multiplicative counterpart. The transition map from a linear transformation to its multiplicative counterpart will be denoted $\mathcal{T}_{\exp}$ and turns out to be a functor, which will be elaborated in the next section.

AM-GM inequality for non-negative random variables

With multiplicative integrals defined, we may return to the mission of defining the geometric mean G of the continuous function $x : [0, T] \to [0, \infty)$. If x is 0 on a set of non-zero measure then $G := 0$ and otherwise

$$G := \left(\prod_0^T x(t)^{\,\mathrm{d}\mu}\right)^{1/T} . \tag{10}$$

In fact, with the measure theoretic treatment one may define the arithmetic and geometric means by considering probability measures and expectation. The arithmetic mean of a random variable X is simply is expectation $\mathbb{E}X$ and will be denoted as $\mathbb{A}X$ when extra emphasis on the arithmetic nature is needed. The geometric mean is defined on non-negative random variable X in line with the previous development. If $P(X = 0) \neq 0$ then the geometric mean is $\mathbb{G}X := 0$ and otherwise it is defined as

$$\mathbb{G}X = e^{\mathbb{E} \ln X}. \tag{11}$$

In fact we will prove with Jensen's inequality that

$$\mathbb{A}X \geq \mathbb{G}X \tag{12}$$

for X non-negative random variable, with equality in (12) if and only if X is a point random variable, meaning it is constant almost surely. This version of AM-GM would generalize the discrete and continuous versions as well as the weighted version $\sum_{i=1}^n p_i x_i \geq \prod_{i=1}^n x_i^{p_i}$ where p_i is a discrete probability.

Extending the allowed range in multiplicative integrals

For multiplicative integrals, the need of positive functions can be motivated by use of the logarithm and its application to the function. With some modifications, non-negative functions have also been considered. One question then is how to extend the machinery of multiplicative integration to wider classes of functions? The allowed range of f in (7) is $\mathbb{R}_+ = e^{\mathbb{R}}$ which gives $\ln f$ a range of $\mathbb{R}$ which is well suited for integration. Also the modification where f is non-negative and consequently $\ln f$ takes values in $[-\infty, \infty)$ gives meaningful integration. Typically, the "next" extension to real-valued integration often considered is that of complex-valued integration, which in our setting would mean that "$\log f$" should be complex valued. The restriction on f is then that its values are in the logarithmic Riemann surface hereby denoted by $\mathbb{S}$. The idea is that on $\mathbb{S}$ the logarithm is a bijection onto $\mathbb{C}$, and on $\mathbb{C}$-valued function there is a notion of integration and this would extend to a corresponding multiplicative integration for $\mathbb{S}$-valued functions. Here $\mathbb{S} = \boldsymbol{e}^{\mathbb{C}}$ is an infinite spiral version of the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, where the argument is no longer 2π-periodic but instead behaves like an element of $\mathbb{R}$. For extra clarity, the logarithm on $\mathbb{S}$ and its inverse, the exponential function, will here be given distinctive notation $\mathbf{log} : \mathbb{S} \to \mathbb{C}$ and $\mathbf{exp} : \mathbb{C} \to \mathbb{S}$. With this notation, we may now define the multiplicative integral of $\boldsymbol{f} : X \to \mathbb{S}$ by

$$\prod \boldsymbol{f}^{\,\mathrm{d}\mu} = \mathbf{exp}\left(\int \mathbf{log}\, \boldsymbol{f} \,\mathrm{d}\mu\right) \tag{13}$$

where μ a measure on X.

Projections and lifts

Since the logarithmic Riemann surface $\mathbb{S}$ is related to $\mathbb{C}^*$, we want to use (13) to define a multiplicative integral for $\mathbb{C}^*$-valued functions. This requires us to first examine the relationship between $\mathbb{S}$ and $\mathbb{C}^*$ further. There is a natural projection $\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$, where the argument is taken modulo 2π. The converse of this projection will lift $z \in \mathbb{C}^*$ to all the points projecting onto it $\mathcal{L}z = \{\boldsymbol{w} \in \mathbb{S} \mid \mathcal{P}\boldsymbol{w} = z\}$. So far both $\mathcal{P}$ and $\mathcal{L}$ have been defined on points, but to be used in the desired setting they need to be extended for functions. Function projection behaves nicely when defined pointwise since it will preserve continuity. However, for function lifts we will have to restrict ourselves to continuous lifts of continuous functions on generalized strips (spaces of the form $\mathbb{R}^m \times [0,1]^n$ for $m, n \in \mathbb{N}$). With function lifts and projections defined, we may then move on to define multiplicative curve integrals for continuous $\mathbb{C}^*$-valued functions. This will be done by first lifting them to be $\mathbb{S}$-valued, then mutliplicatively integrate followed by a projection. The lift will be multivalued parameterized by $\mathbb{Z}$, and will typically stay so even after the multiplicative integration and projection. This explains the multivalued nature of complex multiplicative integrals. [2, 1]

When considering a probability measure and a continuous $\mathbb{C}^*$-valued random variable, the, multiplicative integral (then called geometric expectation) will not be multivalued. By characterising continuous lifts, this will be a consequence of the formula $e^{2\pi in} = 1$, which will collapse all possible geometric expectations.

Related inequalities

Going back to considering non-negative random variables and their arithmetic and geometric means, we may study similar mean inequalities to the AM-GM inequality in their probabilistic setting. In particular, we will extend the matrix form of the AM-GM inequality (Theorem 10.9 in [6]) to its probabilistic setting using an analogous proof. Upon inspection of the proof, it will be extended to an inequality involving integration with respect to a σ-finite measure and geometric means, and the order with which those operations are carried out. From this, Hölder's inequality will follow as a corollary and may in fact be extended to a version involving multiplicative integrals and norms instead of the standard version with products and norms.

2 Multiplicative function spaces

In this section, we aim to investigate the connection between additive and multiplicative structures of importance to the forthcoming topics. Initially, we shall consider the additive group $(\mathbb{R}, +)$ and the multiplicative group $(\mathbb{R}_+, \cdot)$, which are related through the exponential and logarithmic functions. However, due to the additional structure of $\mathbb{R}$, we may consider it as a real vector space over the field $\mathbb{R}$. This leads to the introduction of the corresponding notion for $\mathbb{R}_+$, which will be considered a real multiplicative vector space over the field $\mathbb{R}$. The introduced concept of a multiplicative vector space is algebraically still a vector space but written with a multiplicative notation inherited from the underlying Abelian group. The vector spaces considered will be real-valued function spaces and $\mathbb{R}_+$-valued function spaces in the multiplicative case. The exponential and logarithmic functions, extended to function spaces, will again be the link between the additive and multiplicative settings. Finally, the link between the linear and multiplicative transformations will be defined and will be of interest later as it allows us to translate integral transformations into their multiplicative counterparts.

Proposition 1. $\exp : (\mathbb{R}, +) \to (\mathbb{R}_+, \cdot)$ *is a group isomorhism with inverse* $\ln$.

Proof. Since $\exp : \mathbb{R} \to \mathbb{R}_+$ is a bijection we only need to verify that it is a homomorphism. The identity element is preserved since $e^0 = 1$ and the group operation is preserved since $e^{x+y} = e^x e^y$ for every $x, y \in \mathbb{R}$. □

Remark 2. We write $\mathbb{R}_+ = e^{\mathbb{R}}$ to emphasize the isomorphism in Proposition 1.

To extend Proposition 1 for vector spaces we shall first specify the meaning of a multiplicative vector space, which algebraically is a just vector space written with a multiplicative notation.

Definition 3 (Multiplicative vector space)**.** Let $(M, \cdot, 1)$ be a multiplicative abelian group and F a field, equipped with a function $M \times F \ni (u, \alpha) \mapsto u^\alpha \in M$. Then M is said to be a multiplicative vector space over F provided the following conditions are satisfied

1. $(u^\alpha)^\beta = u^{\alpha\beta}$
2. $u^{1_F} = u$
3. $(uv)^\alpha = u^\alpha v^\alpha$
4. $u^{\alpha+\beta} = u^\alpha u^\beta$

for every $u, v \in M$ and $\alpha, \beta \in F$ and where $1_F \in F$ is the identity of the field.

Remark 4. We shall also call multiplicative vector spaces *vector spaces.

The following claim is evident from the conditions in Definition 3, and is the multiplicative counterpart to the statement that $\mathbb{R}$ is a vector space over itself.

Proposition 5. $\mathbb{R}_+$ *is a multiplicative vector space over* $\mathbb{R}$.

We also consider vector isomorphisms which are bijective vector homomorphisms, defined below. These are algebraically linear transformations between two vector spaces over the same field, however notation is not necessarily written in additive notation.

Definition 6 (Vector homomorphism and isomorphism)**.** Let $(V, +)$ and $(W, *)$ be vector spaces over F, and let $\varphi : V \to W$. Then φ is a vector homomorphism if the following conditions are satisfied

1. $\varphi(u + v) = \varphi(u) * \varphi(v)$
2. $\varphi(\alpha v) = (\varphi(v))^\alpha$

for every $u, v \in V$ and $\alpha \in F$. Additionally, if φ is a bijection then we call it a vector isomorphism.

Remark 7. Note that the particular notation used is not necessary, but we implemented an additive notation for V and modified multiplicative notation for W. If both V and W employ an additive notation, then we call φ a linear transformation, and if both are multiplicative then φ is considered a multiplicative or *linear transformation. This will be written down for operators in Definition 14 and Remark 15.

Remark 8. The two conditions may be replaced by the joint condition that

$$\varphi(\alpha u + \beta v) = \big(\varphi(u)\big)^\alpha * \big(\varphi(v)\big)^\beta$$

is always satisfied.

Thus we may generalize Proposition 1 to a vector isomorphism.

Proposition 9. *The exponential map* $\mathbb{R} \ni x \mapsto e^x \in \mathbb{R}_+$ *is a vector isomorphism.*

Proof. The statement follows from Proposition 1 and the property $e^{\alpha x} = (e^x)^\alpha$ for $x, \alpha \in \mathbb{R}$. $\square$

Next we consider the function spaces $\mathbb{R}^X$ and $\mathbb{R}_+^X$. From linear algebra we now that $\mathbb{R}^X$ is a vector space over $\mathbb{R}$ when addition and scaling are defined pointwise, and the multiplicative counterpart is analogous and stated below.

Proposition 10. *Given $f, g \in \mathbb{R}_+^X$ and $\alpha \in \mathbb{R}$, define multiplication $fg \in \mathbb{R}_+^X$ and exponentiation $f^\alpha \in \mathbb{R}_+^X$ pointwise by*

1. $(fg)(x) = f(x)g(x)$
2. $(f^\alpha)(x) = (f(x))^\alpha$

*for any $x \in X$. Then $\mathbb{R}_+^X$ is a *vector space over $\mathbb{R}$.*

Proof. The proposition is verified by checking the conditions for *vector spaces in Definition 3 since we already know from group theory that $(\mathbb{R}_+^X, \cdot)$ is an Abelian group under pointwise multiplication. The properties hold pointwise anywhere since $\mathbb{R}_+$ is a *vector space, and therefore they will hold for $\mathbb{R}_+^X$ as well. □

Next we would like to extend the exponential (and logarithmic) function to transition between the function spaces $\mathbb{R}^X$ and $\mathbb{R}_+^X$.

Definition 11. We define the exponential map on function spaces $\exp : \mathbb{R}^X \to \mathbb{R}_+^X$ pointwise by $(\exp f)(x) = e^{f(x)}$ for every $f \in \mathbb{R}^X$ and $x \in X$. We also use notation $e^f = \exp f$, where $f \in \mathbb{R}^X$.

Remark 12. The logarithm $\ln : \mathbb{R}_+^X \to \mathbb{R}^X$ is defined analogously by $(\ln f)(x) = \ln(f(x))$ for every $f \in \mathbb{R}_+^X$ and $x \in X$, and it is the inverse of the exponential.

Proposition 9 may no be extended to function spaces.

Proposition 13. *The exponential map $\exp : \mathbb{R}^X \to \mathbb{R}_+^X$ is a vector isomorphism.*

Proof. We already know that exp is a bijection and only need to prove that $e^{f+g} = e^f e^g$ and $e^{\alpha f} = \left(e^f\right)^\alpha$ for every $f, g \in \mathbb{R}^X$ and $\alpha \in \mathbb{R}$. The first equation is proved by the chain of equalities

$$e^{f+g}(x) = e^{(f+g)(x)} = e^{f(x)+g(x)} = e^{f(x)}e^{g(x)} = e^f(x) \cdot e^g(x) = \left(e^f e^g\right)(x),$$

which hold for every $x \in X$. Similarly, the second statement is proved by

$$e^{\alpha f}(x) = e^{(\alpha f)(x)} = e^{\alpha \cdot f(x)} = \left(e^{f(x)}\right)^\alpha = \left(e^f(x)\right)^\alpha = \left(e^f\right)^\alpha(x),$$

which holds for any $x \in X$. □

With the relationships between the additive $\mathbb{R}^X$ and multiplicative $\mathbb{R}_+^X$ vector spaces examined, we now turn to their vector homomorphism operators (linear operators for $\mathbb{R}^X$ and multiplicative for $\mathbb{R}_+^X$) and the relationship between them.

Definition 14 (Linear operator)**.** Let V be a vector space over the field F, and let $\mathcal{A} : V \to V$. Then $\mathcal{A}$ is said to be a linear operotor on V provided that

1. $\mathcal{A}(\alpha v) = \alpha(\mathcal{A}v)$
2. $\mathcal{A}(u + v) = \mathcal{A}u + \mathcal{A}v$

for every $u, v \in V$ and $\alpha \in F$.

Remark 15 (Multiplicative operator)*.* Written in multiplicative notation, the two conditions in Definition 14 are

1. $\mathcal{A}\left(v^\alpha\right) = (\mathcal{A}v)^\alpha$
2. $\mathcal{A}(uv) = \mathcal{A}u \cdot \mathcal{A}v$

In the following definition, we will consider the set of all linear operators on a vector space.

Definition 16 ($\mathrm{End}(V)$)**.** Given the vector space V, we define $\mathrm{End}(V) := \{\mathcal{A} : V \to V | \mathcal{A} \text{ linear operator}\}$.

Remark 17*.* The endomorphism terminology originates from category theory [7] where one may consider the set of all morphisms of an object. In the category $\mathbf{Vect}_F$ the objects are vector spaces over the field F, and the morphisms are linear transformations between two such vector spaces. Thus an endomorphism is a linear transformation from a vector space to itself, or equivalently a linear operator on that space.

Example 18 (Endomorphisms of function spaces)**.** $\mathrm{End}(\mathbb{R}^X)$ consists of all linear operators on $\mathbb{R}^X$ and $\mathrm{End}(\mathbb{R}_+^X)$ consists of all multiplicative operators on $\mathbb{R}_+^X$.

From category theory we know that the endomorphism set forms a monoid (set equipped with binary operator satisfying associativity and having an identity), which in the context of vectors looks as follows.

Proposition 19 ($\mathrm{End}(V)$ is a monoid)**.** *Given a vector space V, then* $\mathrm{End}(V)$ *is a monoid under composition of linear transformations and where I is the identity. Composition is written as $\mathcal{AB}$ or alternatively $\mathcal{A} \circ \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in$* $\mathrm{End}(V)$*.*

Furthermore, from linear algebra we also know that the set of linear transformations forms a vector space when addition and scaling is defined pointwise. In multiplicative notation we consider multiplication and exponentiation instead of addition and scaling. We will write multiplication as $\mathcal{A}\cdot\mathcal{B}$ for $\mathcal{A},\mathcal{B} \in \mathrm{End}(V)$, which we hope the reader will not confuse with "dot products" as they will not be used in this context (and if they were an inner product notation would be utilized instead). We will use the straightforward notation $\mathcal{A}^\alpha$ for exponentiation.

Proposition 20. *Let V be a *vector space over the field F, and equip $\mathrm{End}(V)$ with multiplication $\mathcal{A}\cdot\mathcal{B}$ and exponentiation $\mathcal{A}^\alpha$ for $\mathcal{A},\mathcal{B} \in \mathrm{End}(V)$ and $\alpha \in F$ defined pointwise by*

1. $(\mathcal{A}\cdot\mathcal{B})v = (\mathcal{A}v)(\mathcal{B}v)$
2. $(\mathcal{A}^\alpha)v = (\mathcal{A}v)^\alpha$

*for every $v \in V$. Then $\mathrm{End}(V)$ is a *vector space over F.*

With additive and multiplicative operators defined we now turn to the topic of transitioning between them.

Definition 21 ($\mathcal{T}_{\exp}$)**.** We define the exponential transition map

$$\mathcal{T}_{\exp} : \begin{cases} \mathrm{End}(\mathbb{R}^X) \to \mathrm{End}(\mathbb{R}_+^X) \\ \mathcal{A} \mapsto \mathcal{A}_{\exp} \end{cases}$$

by demanding that

$$\mathcal{A}_{\exp} f = e^{\mathcal{A}(\ln f)}, \tag{14}$$

for any $f \in \mathbb{R}_+^X$.

Remark 22. We now show that $\mathcal{T}_{\exp}$ is well-defined by checking that $\mathcal{A}_{\exp} \in \mathrm{End}(\mathbb{R}_+^X)$, or equivalently that $\mathcal{A}_{\exp}$ is a multiplicative operator. Since $\mathcal{A} \in \mathrm{End}(\mathbb{R}^X)$ we know that $\mathcal{A}$ is a linear operator. We shall check the (joint) condition for multiplicativity, namely that $\mathcal{A}_{\exp}\left(f^\alpha g^\beta\right) = (\mathcal{A}_{\exp} f)^\alpha (\mathcal{A}_{\exp} g)^\beta$ for $f, g \in \mathbb{R}_+^X$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} \mathcal{A}_{\exp}\left(f^\alpha g^\beta\right) &= e^{\mathcal{A}\left(\ln\left(f^\alpha g^\beta\right)\right)} = e^{\mathcal{A}(\alpha \ln f + \beta \ln g))} = e^{\alpha\mathcal{A}(\ln f) + \beta\mathcal{A}(\ln g)} \\ &= \left(e^{\mathcal{A}(\ln f)}\right)^\alpha \left(e^{\mathcal{A}(\ln g)}\right)^\beta = (\mathcal{A}_{\exp} f)^\alpha (\mathcal{A}_{\exp} g)^\beta \end{aligned}$$

Thus we have verified that $\mathcal{T}_{\exp}$ transforms a linear operator into a multiplicative one.

Remark 23 (Implicit notation)*.* We will frequently just write $\mathcal{A}_{\exp}$ instead of $\mathcal{T}_{\exp}\mathcal{A}$ or $\mathcal{T}_{\exp}(\mathcal{A})$.

Remark 24 ($\mathcal{T}_{\ln}$)*.* The inverse of $\mathcal{T}_{\exp}$ is

$$\mathcal{T}_{\ln} : \begin{cases} \mathrm{End}(\mathbb{R}_+^X) \to \mathrm{End}(\mathbb{R}^X) \\ \mathcal{A} \mapsto \mathcal{A}_{\ln} \end{cases}$$

and satisfies

$$\mathcal{A}_{\ln} f = \ln\left(\mathcal{A}e^f\right)$$

for every $f \in \mathbb{R}^X$.

The role of $\mathcal{T}_{\exp}$ and $\mathcal{T}_{\ln}$ is to transition between additive and multiplicative linear operators, much like how exp and ln transitioned between the corresponding vectors. Just like those turned out to be (vector) isomporphisms, the same holds for the exponential and logarithmic transitions which turn out to be both vector isomorphisms and monoid isomorphisms.

Proposition 25. *$\mathcal{T}_{\exp}$ is a monoid isomorphism.*

Proof. We already know that $\mathcal{T}_{\exp}$ is a bijection, and must therefore only show that it is a monoid homomorphism.

1. First we check that the identity is preserved namely that $I_{\exp}$ acts as an identity in $\mathrm{End}(\mathbb{R}_+^X)$. Then, for any $f \in \mathbb{R}_+^X$, we have by (14) that

$$I_{\exp} f = e^{I(\ln f)} = e^{\ln f} = f,$$

which proves that $I_{\exp}$ is the identity $\mathbb{R}_+^X$.

2. Next we check that composition is preserved, meaning that $(\mathcal{A}\mathcal{B})_{\exp} = \mathcal{A}_{\exp}\mathcal{B}_{\exp}$ should hold for every $\mathcal{A}, \mathcal{B} \in \mathrm{End}(\mathbb{R}_+^X)$. Again, using (14) for any $f \in \mathbb{R}_+^X$ we find that

$$(\mathcal{A}\mathcal{B})_{\exp} f = e^{(\mathcal{A}\mathcal{B})(\ln f)} = e^{\mathcal{A}(\mathcal{B}(\ln f))} = e^{\mathcal{A}\left(\ln e^{\mathcal{B}(\ln f)}\right)},$$

and note that we find $e^{\mathcal{B}(\ln f)} = \mathcal{B}_{\exp} f$ within the expression. We then get

$$(\mathcal{A}\mathcal{B})_{\exp} f = e^{\mathcal{A}(\ln \mathcal{B}_{\exp} f)},$$

on which we may again use (14) with the function $\mathcal{B}_{\exp} f \in \mathbb{R}_+^X$. This finally gives us that

$$(\mathcal{A}\mathcal{B})_{\exp} f = \mathcal{A}_{\exp}(\mathcal{B}_{\exp} f) = (\mathcal{A}_{\exp}\mathcal{B}_{\exp}) f,$$

which proves the desired statement.

□

Proposition 26. *$\mathcal{T}_{\exp}$ is a vector isomorphism.*

Proof. We only need to show that $\mathcal{T}_{\exp} : \mathrm{End}(\mathbb{R}^X) \to \mathrm{End}(\mathbb{R}_+^X)$ is a vector homomorphism since it has already been established to be a bijection. The proof will use (14), Theorem13 and the properties of the operator vector spaces (both in additive and multiplicative notation).

1. First we show that $(\mathcal{A} + \mathcal{B})_{\exp} = \mathcal{A}_{\exp} \cdot \mathcal{B}_{\exp}$ for $\mathcal{A}, \mathcal{B} \in \mathrm{End}(\mathbb{R}^X)$. Consider any $f \in \mathbb{R}_+^X$, and then

$$\begin{aligned}(\mathcal{A} + \mathcal{B})_{\exp} f &= e^{(\mathcal{A}+\mathcal{B})(\ln f)} = e^{\mathcal{A}(\ln f)+\mathcal{B}(\ln f)} = e^{\mathcal{A}(\ln f)} e^{\mathcal{B}(\ln f)} \\ &= (\mathcal{A}_{\exp} f)(\mathcal{B}_{\exp} f) = (\mathcal{A}_{\exp} \cdot \mathcal{B}_{\exp}) f,\end{aligned}$$

   which proves the desired statement.

2. What remains to show is that $(\alpha\mathcal{A})_{\exp} = \mathcal{A}_{\exp}^{\alpha}$ for any $\mathcal{A} \in \mathbb{R}^X$ and $\alpha \in \mathbb{R}$. For any $f \in \mathbb{R}_+^X$ we have that

$$(\alpha\mathcal{A})_{\exp} f = e^{\alpha\mathcal{A}(\ln f)} = \left(e^{\mathcal{A}(\ln f)}\right)^{\alpha} = (\mathcal{A}_{\exp} f)^{\alpha} = \mathcal{A}_{\exp}^{\alpha} f,$$

   as desired.

□

We summarize the major takeaways from the section thus far (Proposition 13, 25 and 26) in the following theorem, which gives strong structural preserving properties in the transition between additive multiplicative vector spaces. Essentially it states that the vector structure of the function spaces and their linear operators are isomorphically preserved and that composition of such operators is also isomorphically preserved.

Theorem 27 (Isomorphisms between additive and multiplicative function spaces and their operators)**.** *The exponential map on function spaces* $\exp : \mathbb{R}^X \to \mathbb{R}_+^X$ *is a vector isomorphism and the exponential transition* $\mathcal{T}_{\exp} : \mathrm{End}(\mathbb{R}^X) \to \mathrm{End}(\mathbb{R}_+^X)$ *is a vector and monoid isomorphism. Their inverses are* $\ln$ *and* $\mathcal{T}_{\ln}$*, respectively.*

2.1 Extensions of $\mathcal{T}_{\exp}$

We have thus far taken an algebraic perspective on transitioning from additive to multiplicative structures, however we are primarily interested in the multiplicative integrals and derivatives and as such will need to consider topological and analytical aspects as well. The following examples serve to motivate some extensions to the presentation given thus far, so that $\mathcal{T}_{\exp}$ can be used in more situations.

Example 28 (Multiplicative Riemann integral operator)**.** Integration is not defined for every function $f : [a, b] \to \mathbb{R}$. However, a sufficient condition is to demand that $f \in C_{\mathbb{R}}([a, b])$ is continuous. In that case the Riemann integral operator $f \mapsto F$ defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

for every $x \in [a, b]$, is indeed an operator on $C_{\mathbb{R}}([a, b])$ since $F \in C_{\mathbb{R}}([a, b])$ is also continuous. Thus in this case, the (linear) function space of interest is $C_{\mathbb{R}}([a, b])$ with its associatied endomorphism space $\mathrm{End}\,(C_{\mathbb{R}}([a, b]))$, in which the Riemann integral operator belongs. The first is a subspace of $\mathbb{R}^{[a,b]}$ and the second one is structurally a subspace of $\mathrm{End}\left(\mathbb{R}^{[a,b]}\right)$ which will motivate a minor modification to the definition of $\mathcal{T}_{\exp}$. We still define $\mathcal{T}_{\exp}$ by (14) but restricted to continuous functions. To find the multiplicative counterpart of the Riemann integral integral operator, we then apply the modified $\mathcal{T}_{\exp}$ to $\mathrm{End}\,(C_{\mathbb{R}}([a, b]))$, the image of which will be $\mathrm{End}\left(C_{\mathbb{R}_+}([a, b])\right)$. The multiplicative continuity space $C_{\mathbb{R}_+}([a, b])$ is also the image of exp restricted to $C_{\mathbb{R}}([a, b])$. The multiplicative Riemann integral operator (or Riemann *integral operator) $f \mapsto F_*$ then satisfies

$$F_*(x) = e^{\int_a^x \ln f(t)\,\mathrm{d}t} =: \prod_a^x f(t)^{\,\mathrm{d}t}$$

for $x \in [a, b]$. From[4] we now that the logarithm of $\prod_c^d f(t)^{\,\mathrm{d}t} = F_*(d) - F_*(c)$ is the area under the lin-log plot of f restricted to $[c, d] \subseteq [a, b]$.

From Example 28 we therefore extend the exponential map and transition to subspaces of function spaces.

Definition 29 (Subspace exp and $\mathcal{T}_{\exp}$)**.** Let $V \subseteq \mathbb{R}^X$ be a subspace and consider the restriction of the exponential map $\exp|_V : V \to \exp(V) = e^V =: V_*$, which will still be denoted exp. We extend Definition 21 by letting

$$\mathcal{T}_{\exp} : \begin{cases} \mathrm{End}(V) \to \mathrm{End}(V_*) \\ \mathcal{A} \mapsto \mathcal{A}_* \end{cases}$$

satisfy

$$\mathcal{A}_* f = e^{\mathcal{A}(\ln f)}.$$

for every $f \in V_*$.

Remark 30. The inverses of exp and $\mathcal{T}_{\exp}$ are denoted ln and $\mathcal{T}_{\ln}$, respectively.

Remark 31. Note that the restricted exponential map is a vector isomorphism by inheritance from its unrestricted counterpart, and by repeating previous calculations the exponential transition is still both a vector and a monoid isomorphism.

The extended definition of $\mathcal{T}_{\exp}$ in Definition 29 may be used on derivative operators as well, to gain a multiplicative counterpart.

Example 32 (Multiplicative derivative operator)**.** Let $U \subseteq \mathbb{R}$ be open and consider the function space $\mathbb{R}^U$. While the derivative is a linear transformation on the (linear) space of differentiable functions in $\mathbb{R}^U$, it is only an operator on the (linear) subspace of infinitely differentiable functions $C^\infty_{\mathbb{R}}(U)$. The derivative operator $D : C^\infty_{\mathbb{R}}(U) \to C^\infty_{\mathbb{R}}(U)$ can thus be transformed by $\mathcal{T}_{\exp}$ to the multiplicative derivative operator $D_* : C^\infty_{\mathbb{R}_+}(U) \to C^\infty_{\mathbb{R}_+}(U)$. Note that $\exp\left(C^\infty_{\mathbb{R}}(U)\right) = C^\infty_{\mathbb{R}_+}(U)$ consists of infinitely differentiable positive functions. Its action on $f \in C^\infty_{\mathbb{R}_+}(U)$ is given by

$$D_* f =: f^* = e^{(\ln f)'} = e^{f'/f}.$$

From [4] we know that the logarithm of $f^*(x)$ finds the slope of f at $x \in U$ in the lin-log plot of $f : U \to \mathbb{R}$.

However, we may also be interested in differentiating functions which are not C^∞ smooth, and their multiplicative counterparts. To do this we fix some open subset X of $\mathbb{R}$ and define the subspace chain of smoothness classes of the function space $\mathbb{R}^X$, and we will surpress X in the notation for brevity. We note that we can also let X be a closed interval and then define the derivative at endpoints to be given by right/left derivatives. Such modifications will not alter the definition of the chain of spaces. We will also not include $\mathbb{R}$ in the notation, but when there will be need for distinction with the multiplicative counterparts on $\mathbb{R}_+$, these will be indexed.

Definition 33 (Smoothness chain spaces)**.** Define the chain of smoothness spaces $C^0 \supset C^1 \supset C^2 \supset \ldots \supset C^n \supset C^{n+1} \supset \ldots \supset C^\infty$ by letting $C^0_{\mathbb{R}} = C(X, \mathbb{R})$ denote the continuous functions, $C^1_{\mathbb{R}} = \{f \mid f' \text{ exists and } f' \in C^0_{\mathbb{R}}\}$ and $C^{n+1}_{\mathbb{R}} = \{f \in C^1 \mid f' \in C^n\}$. Finally we let $C^\infty = \bigcap_{n=1}^\infty C^n$.

Remark 34. Note that all these spaces are indeed linear subspaces of the function space, and the derivative $D : C^1 \to C^0$ restricted to a subspace in the chain is a linear transformation to the previous space. This can be verified with standard formulas of scaling and addition with respect to derivatives.

The reason for considering such a chain of smoothness spaces is that this does not only allow us to transfer the definition of the derivative to the multiplicative setting, but also composition properties as well as smoothness information. This will ultimately result in a greater level of recycling of the additive theory of derivatives to its multiplicative counterpart.

While we could define $\mathcal{T}_{\exp}$ on the smoothness chain, and this would be an invertible composition chain isomorphism, the following proposition and remark suggests that we should consider categories instead for a more general treatment, which will also allow us to transfer other interesting analysis topics to their multiplicative counterparts.

Proposition 35. *The composition chain $C^0 \supset C^1 \supset \ldots \supset C^n \supset \ldots$ forms a subcategory $\mathscr{C}_{\mathbb{R}}(X)$ of $\mathbf{Vect}_{\mathbb{R}}$, the category of vector spaces over $\mathbb{R}$.*

Proof. The objects in $\mathscr{C}$ are C^n for every $n \in \mathbb{N}$ and the morphisms are linear transformations $\mathcal{A} : C^n \to C^m$. Composition of such $\mathcal{A}$ and $\mathcal{B} : C^p \to C^n$ is the linear transformation $\mathcal{AB} : C^p \to C^m$ and satisfies associativity. $\square$

Remark 36. Since the derivative operator (and its restrictions) $D : C^{n+1} \to C^n$ are linear then any restriction of D is a morphism in $\mathscr{D}$. We also note that the iterated derivative $D^k : C^k \to C^0$ and its restrictions are also morphisms.

It is therefore natural to consider the category whose objects are all subspaces of $\mathbb{R}^X$ and whose morphisms are all linear transformations between these subspaces. This category, written $\mathscr{F}_{\mathbb{R}}(X)$, thus is a subcatgory of $\mathbf{Vect}_{\mathbb{R}}$. Defining $\mathcal{T}_{\exp}$ on this category would include all examples given thus far by considering subcategories. However, we have restricted ourselves to function spaces with fixed input, which may not be suitable for some integral transforms or for non-square matrices. Therefore we shall allow any domain in the function spaces, and the corresponding construction will be iterated in the multiplicative case.

Definition 37. We define the categories $\mathscr{F}_{\mathbb{R}}$ and $\mathscr{F}_{\mathbb{R}_+}$ as follows.

($\mathscr{F}_{\mathbb{R}}$) Let $\mathscr{F}_{\mathbb{R}}$ be the category whose objects are subspaces $V \subseteq \mathbb{R}^S$ of function spaces for any set S. The morphisms are inherited from the supcategory $\mathbf{Vect}_{\mathbb{R}}$ and are linear transformations between objects.

($\mathscr{F}_{\mathbb{R}_+}$) Similarly, we define the category $\mathscr{F}_{\mathbb{R}_+}$ whose objects are subspaces of $\mathbb{R}_+$-valued function spaces, with morphisms inherited from $\mathbf{Vect}_{\mathbb{R}}$ and given by multiplicative transformations.

Remark 38. Note that $\mathscr{F}_{\mathbb{R}}$ and $\mathscr{F}_{\mathbb{R}_+}$ are full subcategories of $\mathbf{Vect}_{\mathbb{R}}$.

We may now define the functor $\mathcal{T}_{\exp} : \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}_+}$, which will include both the exponential map and transition. This will be an extension of previous definitions, and the functorial statement will contain Theorem 27, which is the special case for a one-object full subcategory of $\mathscr{F}_{\mathbb{R}}$. The functorial treatment of $\mathcal{T}_{\exp}$ will not only translate operators such as derivatives and integrals to their multiplicative counterpart, but additionally since the compositional structure is preserved statements such as the fundamental theorem of calculus will also be naturally transferred to the multiplicative setting. We will also define the inverse functor $\mathcal{T}_{\ln} : \mathscr{F}_{\mathbb{R}_+} \to \mathscr{F}_{\mathbb{R}}$ of $\mathcal{T}_{\exp}$, which translates from a multiplicative structure to the additive counterpart.

Definition 39. We define $\mathcal{T}_{\exp} : \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}_+}$ and $\mathcal{T}_{\ln} : \mathscr{F}_{\mathbb{R}_+} \to \mathscr{F}_{\mathbb{R}}$ as follows.

($\mathcal{T}_{\exp}$)

1. For objects, we let $\mathcal{T}_{\exp} : V \mapsto V_* = e^V$.
2. For morphisms, we let $\mathcal{T}_{\exp} : \mathcal{A} \mapsto \mathcal{A}_*$ be defined by

$$\mathcal{A}_* f = e^{\mathcal{A}(\ln f)}$$

for every $f \in V_*$.

($\mathcal{T}_{\ln}$)

1. For objects, we let $\mathcal{T}_{\ln} : V \mapsto V_\dagger := \ln V$.
2. For morphisms, we let $\mathcal{T}_{\ln} : \mathcal{A} \mapsto \mathcal{A}_\dagger$ be defined by

$$\mathcal{A}_\dagger f = \ln\left(\mathcal{A} e^f\right)$$

for every $f \in V_\dagger$.

Theorem 40. *$\mathcal{T}_{\exp}$ is a functor with inverse functor $\mathcal{T}_{\ln}$.*

Proof. First we show that $\mathcal{T}_{\exp}$ is a functor by verifying that identities and composition are preserved. The identity I of any object V in $\mathscr{F}_{\mathbb{R}}$ maps to its counterpart identity since $I_* f = e^{I(\ln f)} = e^{\ln f} = f$ for any $f \in V_*$. Furthermore, composition is preserved since for any linear transformations $\mathcal{A} : V' \to W$ and $\mathcal{B} : V \to V'$ acting between $\mathbb{R}$-valued linear function spaces V, V' and W, we have that

$$(\mathcal{A}\mathcal{B})_* f = e^{(\mathcal{A}\mathcal{B})(\ln f)} = e^{\mathcal{A}\left(\ln e^{\mathcal{B}(\ln f)}\right)} = e^{\mathcal{A}(\ln \mathcal{B}_* f)} = \mathcal{A}_* \mathcal{B}_* f$$

for every $f \in V_*$. Consequently, we have $(\mathcal{A}\mathcal{B})_* = \mathcal{A}_*\mathcal{B}_*$ and it concludes the proof that $\mathcal{T}_{\exp}$ is a functor. By analogous calculations, we may also deduce that $\mathcal{T}_{\ln}$ is a functor.

What remains to show is that $\mathcal{T}_{\exp}$ and $\mathcal{T}_{\ln}$ are inverses, which means that we need to show that $\mathcal{T}_{\ln}\mathcal{T}_{\exp}$ and $\mathcal{T}_{\exp}\mathcal{T}_{\ln}$ are the identity functors on $\mathscr{F}_{\mathbb{R}}$ and $\mathscr{F}_{\mathbb{R}_+}$ respectively. We will show the first statement and note that the second one follows from analogous calculations. For objects we have that $\mathcal{T}_{\ln}\mathcal{T}_{\exp} : V \mapsto V_* \mapsto (V_*)_\dagger$, where $(V_*)_\dagger = \ln\left(e^V\right) = V$. Furthermore, for morphisms we have that $\mathcal{T}_{\ln}\mathcal{T}_{\exp} : \mathcal{A} \mapsto \mathcal{A}_* \mapsto (\mathcal{A}_*)_\dagger$where

$$(\mathcal{A}_*)_\dagger f = \ln\left(\mathcal{A}_* e^f\right) = \ln\left(e^{\mathcal{A}\left(\ln e^f\right)}\right) = \ln\left(e^{\mathcal{A}f}\right) = \mathcal{A}f$$

for every $f \in V$. Therefore $\mathcal{T}_{\ln}\mathcal{T}_{\exp} : \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}}$ acts as an idenity on both objects and morphisms, which concludes the proof that it is an identity functor $\square$

Example 41 (Multiplicative integral)**.** Let $(X, \mathscr{A}, \mu)$ be a measure space and consider the space of integrable functions $\mathcal{L}^1(X, \mu, \mathbb{R})$. We also consider the real numbers viewed as a trivial function space over itself $\mathbb{R}^{\{1\}} \cong \mathbb{R}$. Then the integral is an operator $\mathcal{L}^1(X, \mu, \mathbb{R}) \to \mathbb{R}$ given by $f \mapsto \int f \, \mathrm{d}\mu$. Upon transformation by $\mathcal{T}_{\exp}$, the space of integrable functions transforms into its multiplicative counterpart, which in this case is the space of positively valued integrable functions $\mathcal{L}^1(X, \mu, \mathbb{R}_+)$. Furthermore, the multiplicative integral operator $\mathcal{L}^1(X, \mu, \mathbb{R}_+) \to \mathbb{R}_+$ with $f \mapsto \prod f^{\,\mathrm{d}\mu}$ satisfies

$$\prod f^{\,\mathrm{d}\mu} = e^{\int \ln f \, \mathrm{d}\mu},$$

and we call it the multiplicative integral (or *integral) of f.

Example 42 (Multiplicative matrix algebra)**.** Consider the finite dimensional function spaces $\mathbb{R}^{\{1,\dots,n\}} \cong \mathbb{R}^n$ for $n \in \mathbb{N}$, in which case linear transformations $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ are given by matrix multiplication with $m \times n$ matrices A. The multiplicative counterpart $\mathcal{A}_* : \mathbb{R}^n_+ \to \mathbb{R}^m_+$ acts on $\boldsymbol{x} \in \mathbb{R}^m_+$ such that $\boldsymbol{y} = \mathcal{A}_*\boldsymbol{x} \in \mathbb{R}^m_+$, and its entries are given by

$$y_i = \prod_{j=1}^{n} x_j^{a_{ij}}.$$

Note that since $(\mathcal{A}\mathcal{B})_* = \mathcal{A}_*\mathcal{B}_*$, composition is still respresented by the matrix AB calculated through ordinary matrix multiplication of the matrix representations A and B.

Remark 43 (Multiplicative integral transforms). While the two previous examples could most likely be generalized to the discussion of integral transforms and their multiplicative counterparts, this will be omitted due to the technical nature of the topic.

3 Complex multiplicative spaces

In this section we will consider complex multiplicative function spaces and their relation to their additive counterparts. After defining the canonical 1-dimensional *vector space $\mathbb{S}$ over $\mathbb{C}$, the development will essentially be a streamlined translation of the topics discussed in the preceding section. Most important will be the introduction of the invertible functor $\mathcal{T}_{\mathbf{exp}} : \mathscr{F}_{\mathbb{C}} \to \mathscr{F}_{\mathbb{S}}$ and its categories, which will correspond to their real counterparts $\mathcal{T}_{\exp} : \mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}_+}$. We then turn to the topic of relating $\mathbb{S}$ to the (group) multiplicative structure $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, which provides a link to the complex numbers. This will be achieved by introducing projections $\mathcal{P}$ and lifts $\mathcal{L}$ between $\mathbb{S}$ and $\mathbb{C}^*$, and extending them to continuous functions. Then a lift-projection scheme for transferring multiplicative transformations to the $\mathbb{C}^*$-setting will be considered, which will potentially introduce multi-valued behaviour.

We begin by providing a geometric definition of $\mathbb{S}$, in which it will be considered a helicoid in 3-space.

Definition 44 ($\mathbb{S}$)**.** We define $\mathbf{exp} : \mathbb{C} \to \mathbb{C}^* \times \mathbb{R}$ by $z \mapsto (e^z, \Im z)$, and let $\mathbb{S} := \mathbf{exp}(\mathbb{C})$ be the image.

Remark 45. Onwards we shall consider $\mathbf{exp} : \mathbb{C} \to \mathbb{S}$ a bijection.

Remark 46. In real coordinates the exponential bijection takes the form $x + iy \mapsto (e^{x+iy}, y)$ and $\mathbb{S}$ can therefore be considered a surface (helicoid) in 3-space. The inverse, $\mathbf{log} : \mathbb{S} \to \mathbb{C}$, acts by $(w, y) \mapsto \ln\left(\frac{w}{e^{iy}}\right) + iy$ for $(w, y) \in \mathbb{S} \subset \mathbb{C}^* \times \mathbb{R}$.

We shall let $\mathbb{S}$ become a 1-dimensional multiplicative vector space over $\mathbb{C}$ by transforming the additive structure of $\mathbb{C}$ with $\mathbf{exp}$, which will also ensure that $\mathbf{exp}$ is a vector isomorphism.

Definition 47 (Multiplication and exponentiation on $\mathbb{S}$)**.** Given $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{S}$ and $\alpha \in \mathbb{C}$, define multiplication by

$$\boldsymbol{z}\boldsymbol{w} := \mathbf{exp}(\mathbf{log}\,\boldsymbol{z} + \mathbf{log}\,\boldsymbol{w})$$

and exponentiation by

$$\boldsymbol{z}^\alpha := \mathbf{exp}(\alpha\,\mathbf{log}\,\boldsymbol{z}).$$

Remark 48. If we let $\boldsymbol{e} := \mathbf{exp}(1)$, then $\boldsymbol{e}^z = \mathbf{exp}(z\,\mathbf{log}\,\boldsymbol{e}) = \mathbf{exp}(z)$ for every $z \in \mathbb{C}$. We shall more frequently use the notation $z \mapsto \boldsymbol{e}^z$ for the exponential function.

Proposition 49. *$\mathbb{S}$ is a 1 dimensional *vector space over $\mathbb{C}$.*

Remark 50. In real coordinates $\mathbb{S}$ is a 2-dimensional *vector space over $\mathbb{R}$. An example of a basis is given by the *vectors $\boldsymbol{e} = (e, 0)$ and $\boldsymbol{e}^i = (e^i, 1)$. Therefore $\mathbb{S}$ is a direct *sum (multiplication) of their respective spans $\boldsymbol{e}^{\mathbb{R}}$ and $\boldsymbol{e}^{i\mathbb{R}}$. Since $\boldsymbol{e}^{\mathbb{R}}$ behaves identical to $\mathbb{R}_+$ we shall use them interchangeably. The direct multiplication decomposition is thus given by $\mathbb{S} = \mathbb{R}_+ \cdot \boldsymbol{e}^{i\mathbb{R}}$ when considered as *vector spaces over $\mathbb{R}$.

We extend the exponential and logarithmic functions to act on function spaces, and their subspaces.

Definition 51. Define $\mathbf{exp} : \mathbb{C}^X \to \mathbb{S}^X$ and $\mathbf{log} : \mathbb{S}^X \to \mathbb{C}^X$ on function spaces by composition namely $\boldsymbol{e}^f(x) = \boldsymbol{e}^{f(x)}$ and $(\mathbf{log}\, \boldsymbol{g})(x) = \mathbf{log}(\boldsymbol{g}(x))$ for every $x \in X$ and $f \in \mathbb{C}^X$ and $\boldsymbol{g} \in \mathbb{S}^X$. For subspaces $V \subset \mathbb{C}^X$ and $W \subset \mathbb{S}^X$ we let restrictions $\mathbf{exp}\,|_V$ and $\mathbf{log}\,|_W$ induce images $V_* = \boldsymbol{e}^V$ and $W_\dagger = \mathbf{log}\, W$ respectively.

Remark 52. Note that notation is abused by recycling **exp** and **log** for their function space counterparts, and again abused when the restriction are given the same notation.

Proposition 53. **exp** *and* **log** *are vector isomorphisms and each others inverses.*

We shall now define the categories $\mathscr{F}_{\mathbb{C}}$ and $\mathscr{F}_{\mathbb{S}}$, as well as the functors $\mathcal{T}_{\mathbf{exp}}$ and $\mathcal{T}_{\mathbf{log}}$ between them.

Definition 54 ($\mathscr{F}_{\mathbb{C}}$)**.** We let the objects of $\mathscr{F}_{\mathbb{C}}$ be (linear) subspaces $V \subseteq \mathbb{C}^S$ for any set S, and its morphisms be any linear transformation between two such spaces.

Definition 55 ($\mathscr{F}_{\mathbb{S}}$)**.** We let the objects of $\mathscr{F}_{\mathbb{S}}$ be (multiplicative) subspaces $V \subseteq \mathbb{S}^S$ for any set S, and its morphisms be any *linear transformation between two such spaces.

Remark 56. Both $\mathscr{F}_{\mathbb{C}}$ and $\mathscr{F}_{\mathbb{S}}$ are full subcategories of $\mathbf{Vect}_{\mathbb{C}}$ generated by their objects.

Definition 57 ($\mathcal{T}_{\mathbf{exp}}$)**.** We define $\mathcal{T}_{\mathbf{exp}} : \mathscr{F}_{\mathbb{C}} \to \mathscr{F}_{\mathbb{S}}$ to act by

1. $V \mapsto V_*$ for every object V
2. $\mathcal{A} \mapsto \mathcal{A}_*$ satisfying $\mathcal{A}_* \boldsymbol{f} = \boldsymbol{e}^{\mathcal{A}(\mathbf{log}\, \boldsymbol{f})}$ for every $\boldsymbol{f}$ in the domain of any induced morphism $\mathcal{A}_*$.

Definition 58 ($\mathcal{T}_{\mathbf{log}}$)**.** We define $\mathcal{T}_{\mathbf{log}} : \mathscr{F}_{\mathbb{S}} \to \mathscr{F}_{\mathbb{C}}$ to act by

1. $W \mapsto W_\dagger$ for every object W

2. $\mathcal{A} \mapsto \mathcal{A}_\dagger$ satisfying $\mathcal{A}_\dagger g = \mathbf{log}\left(\mathcal{A}e^g\right)$ for every g in the domain of any morphism $\mathcal{A}_\dagger$.

Remark 59. Note that $\mathcal{T}_{\mathbf{exp}} : \mathscr{F}_{\mathbb{C}} \to \mathscr{F}_{\mathbb{S}}$ is indeed well defined since V_* is a *vector space and $\mathcal{A}_*$ is a *linear transformation. Similarly, $\mathcal{T}_{\mathbf{log}}$ is also well-defined and is furthermore the inverse of $\mathcal{T}_{\mathbf{exp}}$.

The complex counterpart of Theorem 40 is stated below, and since the proof is identical it will be omitted.

Theorem 60. *$\mathcal{T}_{\mathbf{exp}}$ is an invertible functor with inverse $\mathcal{T}_{\mathbf{log}}$.*

Example 61 (Multiplicative derivative)**.** Given some open subset $U \subset \mathbb{C}$ the derivative is an operator on the space on the space of holomorphic functions on U, namely $D : H(U) \to H(U)$. By applying $\mathcal{T}_{\mathbf{exp}}$ we get the multiplicative counterpart $D_* : H_*(U) \to H_*(U)$ where $H_*(U) = \left\{\boldsymbol{e}^f \mid f \in H(U)\right\}$ and where

$$D_*\boldsymbol{f} = \boldsymbol{f}^* = \boldsymbol{e}^{(\mathbf{log}\,\boldsymbol{f})'}.$$

Example 62 (Multiplicative Riemann integral operator)**.** The Riemann integral operator for $\mathbb{C}$-valued functions is an operator on $C(\mathbb{R}, \mathbb{C})$ given by $f \mapsto F$ where $F(x) = \int_0^x f(t)\,\mathrm{d}t$ for every $x \in \mathbb{R}$. Its multiplicative counterpart acts by $\boldsymbol{f} \mapsto \boldsymbol{F}_*$ where

$$\boldsymbol{F}_*(x) = \prod_0^x \boldsymbol{f}(t)^{\,\mathrm{d}t} = e^{\int_0^x \mathbf{log}\,\boldsymbol{f}\,\mathrm{d}t}$$

is the Riemann *integral operator. It acts on the space $\boldsymbol{e}^{C(\mathbb{R},\mathbb{C})}$ of functions of the form $\boldsymbol{e}^f$, where $f \in C(\mathbb{R}, \mathbb{C})$. When considering the topology of $\mathbb{S}$ (defined in the next subsection), which is locally given by the topology of $\mathbb{C}^*$, the continuity of $\mathbf{exp}$ then yields that $\boldsymbol{e}^{C(\mathbb{R},\mathbb{C})} = C(\mathbb{R}, \mathbb{S})$. This is the space of continuous functions from $\mathbb{R}$ to $\mathbb{S}$.

Example 63 (Fundamental theorem of calculus on space of entire functions)**.** On the space of entire functions H the derivative D and integral operator $J : H \to H$ are operators where $Jf(z) = \int_0^z f(\zeta)\,\mathrm{d}\zeta$. We have the fundamental theorem of calculus on the space of entire functions given by $DJ = I$ and $JDf(z) = f(z) - f(0)$ for every $f \in H$ and $z \in \mathbb{C}$. The multiplicative counterparts $D_*, J_* : H_* \to H_*$ satisfy $D_*J_* = I_* = I$ and $J_*D_*\boldsymbol{f}(z) = \boldsymbol{e}^{\mathbf{log}\,\boldsymbol{f}(z) - \mathbf{log}\,\boldsymbol{f}(0)} = \frac{\boldsymbol{f}(z)}{\boldsymbol{f}(0)}$ for every $\boldsymbol{f} \in H_*$ and $z \in \mathbb{C}$. Note that the condition on $\boldsymbol{f}$ is equivalent to $\mathbf{log}\,\boldsymbol{f} \in H$ being an entire function, and H_* is the space of *entire functions. Furthermore the *integral satisfies

$$J_*\boldsymbol{f}(z) = \prod_0^z \boldsymbol{f}(\zeta)^{\mathrm{d}\zeta} = \boldsymbol{e}^{\int_0^z \mathbf{log}\,\boldsymbol{f}(\zeta)\,\mathrm{d}\zeta},$$

which is well-defined like J since integration is path-independent for entire functions. In more explicit notation, the fundamental theorem of *calculus for *entire functions may be written as

$$\left(\prod_0^z \boldsymbol{f}(\zeta)^{\mathrm{d}\zeta}\right)^* = \boldsymbol{f}(z) \tag{15}$$

$$\prod_0^z \boldsymbol{f}^*(\zeta)^{\mathrm{d}\zeta} = \frac{\boldsymbol{f}(z)}{\boldsymbol{f}(0)} \tag{16}$$

for any $\boldsymbol{f} \in H_*$ and $z \in \mathbb{C}$.

Example 64 (Multiplicative integral)**.** Recall Example 41 on multiplicative integrals for real-valued integrable functions. This time, however, we consider $\mathbb{S}$-valued functions $\boldsymbol{f} = \boldsymbol{e}^g$ where $g = \mathbf{log}\,\boldsymbol{f} \in \mathcal{L}^1(X, \mu, \mathbb{C}) = \mathcal{L}^1(X, \mu, \mathbb{R}) \oplus \mathcal{L}^1(X, \mu, i\mathbb{R})$ is integrable. The integral $g \mapsto \int g\,\mathrm{d}\mu \in \mathbb{C}$ then has the multiplicative counterpart

$$\boldsymbol{f} \mapsto \prod \boldsymbol{f}^{\mathrm{d}\mu} = \boldsymbol{e}^{\int \mathbf{log}\,\boldsymbol{f}\,\mathrm{d}\mu} \in \mathbb{S}.$$

3.1 Lifts and Projections

With some investigations of $\mathbb{S}$ undertaken we next consider how it relates to the punctured complex plane $\mathbb{C}^*$, which is a group multiplicative structure. To do so ,we will consider the projection $\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$, defined below, which preserves important structure. Not only will $\mathcal{P}$ be a group homomorphism, but also a covering which will lead us to consider topological properties as well.

Definition 65 ($\mathcal{P}$)**.** Let the projection $\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$ be defined by $\mathcal{P}\boldsymbol{z} = e^{\mathbf{log}\,\boldsymbol{z}}$

Remark 66. The projection satisfies $\mathcal{P}\boldsymbol{e}^z = e^z$ for every $z \in \mathbb{C}$.

Remark 67. Restricted to the set $\mathbb{S}_\theta := \{\boldsymbol{e}^z \in \mathbb{S} \mid \theta < \Im z < \theta + 2\pi\}$, the projection $\mathcal{P}_\theta : \mathbb{S}_\theta \to \mathbb{C}_\theta$ is a bijection to the slit complex plane $\mathbb{C}_\theta = \mathbb{C} \backslash e^{i\theta}\mathbb{R}$. This can be seen by using a locally defined logarithm $\log : \mathbb{C}_\theta \to \mathbb{R} + i(\theta, \theta + 2\pi)$ on the slit plane, which yields the formula $\mathcal{P}_\theta^{-1} z = \boldsymbol{e}^{\log z}$.

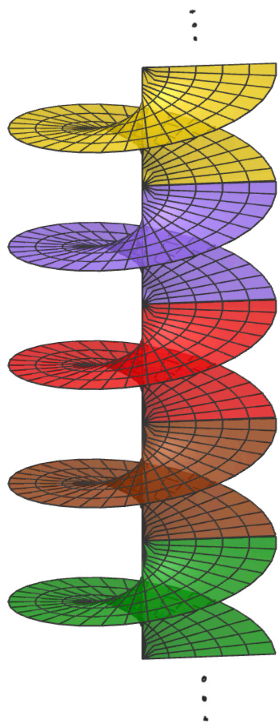


Figure 6: A portion of $\mathbb{S}$ displayed where each colour represents some restriction $\mathbb{S}_{2\pi n}$, each of which is bijectively associated with $\mathbb{C}_0$. Later, when discussing the topology of $\mathbb{S}$ we can also consider them homeomorphic to each other. Produced with https://www.geogebra.org/m/btAc29yQ#material/NYkYnjez and modified.

Remark 68. Note that $\mathbb{S}$ is also often considered as the logarithmic Riemann surface, however we will not use that fact. The name stems from the fact that the logarithm can be defined on $\mathbb{S}$.

Proposition 69. *$\mathcal{P}$ is a group homomorphism.*

Proof. We verify the required group homomorphism properties.

1. $\mathcal{P}(\boldsymbol{z}\boldsymbol{w}) = \mathcal{P}(\boldsymbol{e}^{\log \boldsymbol{z} + \log \boldsymbol{w}}) = e^{\log \boldsymbol{z} + \log \boldsymbol{w}} = e^{\log \boldsymbol{z}} e^{\log \boldsymbol{w}}$ for every $\boldsymbol{z}.\boldsymbol{w} \in \mathbb{S}$.

2. $\mathcal{P}\mathbf{1} = e^{\log \mathbf{1}} = e^0 = 1$

$\square$

Definition 70 (Lift)**.** For $z \in \mathbb{C}^*$ define its lift $\mathcal{L}z$ to be all points in $\mathbb{S}$ projecting onto it, namely $\mathcal{L}z = \{\boldsymbol{w} \in \mathbb{S} \mid \mathcal{P}\boldsymbol{w} = z\}$.

Remark 71. In the context of algebraic topology, the lift of z would be called the fiber of z.

Example 72 ($\mathcal{L}1$)**.** To find the lift of 1, we consider any $\boldsymbol{e}^z \in \mathbb{S}$ for $z \in \mathbb{C}$ and let it project onto 1. This yields the equation $e^z = 1$ which is solved by $z = 2\pi in$ for any $n \in \mathbb{Z}$. Consequently $\mathcal{L}1 = \{\boldsymbol{e}^{2\pi in} \mid n \in \mathbb{Z}\}$, which is group isomorphic to the additive group $\mathbb{Z}$.

The elements in $\mathcal{L}1$ are of particular importance and will be given their own notation.

Definition 73. For any $n \in \mathbb{Z}$ we let $\mathbf{1}_n = \boldsymbol{e}^{2\pi in}$, and we define $[1] := \{\mathbf{1}_n \mid n \in \mathbb{Z}\}$.

Proposition 74 (Characterization of $\mathcal{L}z$)**.** *Given $z \in \mathbb{C}^*$, then there exists a lift point $\boldsymbol{z} \in \mathbb{S}$ such that $\mathcal{P}\boldsymbol{z} = z$, and $\mathcal{L}z = \boldsymbol{z}[1] = \{\ \boldsymbol{z}\mathbf{1}_n \mid n \in \mathbb{Z}\}$.*

Proof. By writing $z = re^{i\theta}$ in polar form we see that $\boldsymbol{z} = r\boldsymbol{e}^{i\theta}$ is a lift point. Furthermore, any $\boldsymbol{w} \in \mathcal{L}z$ can be characterized by $\mathcal{P}\left(\frac{\boldsymbol{w}}{\boldsymbol{z}}\right) = 1$ since the projection is a group homomorphism. The characterization is equivalent to $\frac{\boldsymbol{w}}{\boldsymbol{z}} \in [1] \iff \boldsymbol{w} \in \boldsymbol{z}[1]$. It follows that $\mathcal{L}z = \boldsymbol{z}[1]$ as desired. □

We shall use the topology of $\mathbb{C}^*$ to construct the topology of $\mathbb{S}$ and then explain why $\mathbb{S}$ covers $\mathbb{C}^*$. We start by considering a basis of disks for the topology of $\mathbb{C}^*$ which will help us in our construction.

Lemma 75. *For every $z = re^{i\theta} \in \mathbb{C}^*$, consider its collection of open balls $\mathscr{B}_z = \{B_\delta(z) \mid 0 < \delta < r\}$ centered at z and of radii such that 0 is not contained. Then the union $\mathscr{B} = \bigcup_{z\in\mathbb{C}^*} \mathscr{B}_z$ forms a basis for the topology of $\mathbb{C}^*$.*

Definition 76 (Topology of $\mathbb{S}$)**.** We equip $\mathbb{S}$ with the topology generated by $\bigcup_{\boldsymbol{z}\in\mathbb{S}} \left\{\tilde{B}_\delta(\boldsymbol{z}) \mid 0 < \delta < r\right\}$, where $\tilde{B}_\delta(\boldsymbol{z}) := \mathcal{P}^{-1}_{\theta-\pi}(B_\delta(z))$ for every $\boldsymbol{z} = r\boldsymbol{e}^{i\theta} \in \mathbb{S}$ and $z = \mathcal{P}\boldsymbol{z}$.

Remark 77. We note that $\tilde{B}_\delta(\boldsymbol{z})$ is well-defined since the restricted projection $\mathcal{P}_{\theta-\pi}$ has an associated inverse $\mathcal{P}^{-1}_{\theta-\pi} : \mathbb{C}_{\theta-\pi} \to \mathbb{S}_{\theta-\pi}$ in whose domain $\mathbb{C}_{\theta-\pi} \supset B_\delta(z)$ is fully contained.

Proposition 78. *$\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$ is a covering.*

Proof. We first note that $\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$ is a continuous map. Furthermore, for any $z = re^{i\theta} \in \mathbb{C}^*$ and $0 < \delta < r$, we have that

$$\mathcal{P}^{-1}(B_\delta(z) = \coprod_{\boldsymbol{w} \in \mathcal{L}z} \tilde{B}_\delta(\boldsymbol{w}) = \coprod_{n \in \mathbb{Z}} \tilde{B}_\delta(\boldsymbol{z}\mathbf{1}_n) =: \coprod_{n \in \mathbb{Z}} B_\delta(z, n),$$

where $\boldsymbol{z}$ is a lift point of z, and $\mathcal{P}|_{B_\delta(z,n)} : B_\delta(z, n) \to B_\delta(z)$ is a homeomorphism for every $n \in \mathbb{Z}$ which verifies that $\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$ is a covering. $\square$

Remark 79. Note that in the notation $B_\delta(z, n)$ for $\tilde{B}_\delta(\boldsymbol{w})$ we assume that some fixed reference point lift $\boldsymbol{z}$ is given. For example, for $z = 1$ we shall consider the canonical reference point $\mathbf{1} = \mathbf{1}_0$.

The covering is important since it relates to the concept of function lifts and projections, which we will need later when transitioning between functions $f \in C(X, \mathbb{C}^*)$ to and their lifts $\boldsymbol{f} \in C(X, \mathbb{S})$. This will then allow us to translate operators as well, but as we shall see they will gain a multi-valued nature. We begin be defining function projections pointwise, not assuming any topological structure on X, but afterwards $X \neq \emptyset$ will always be considered a topological space.

Definition 80 (Function Projection)**.** Define, by abuse of notation, the function projection $\mathcal{P} : \mathbb{S}^X \to \mathbb{C}^{*X}$ by $(\mathcal{P}f)(x) = \mathcal{P}(f(x))$ for every $f \in \mathbb{S}^X$ and $x \in X$. We shall also use the same notation $\mathcal{P}$ for any restriction.

Remark 81. Note that $\mathcal{P}$ preserves continuity, so that we have a restriction $\mathcal{P} : C(X, \mathbb{S}) \to (X, \mathbb{C}^*)$ which will be the function projection (and its restrictions) which we are most interested in.

Remark 82. By considering componentwise multiplication in the function spaces $\mathbb{S}^X$ and $\mathbb{C}^{*X}$, we note that $\mathcal{P} : \mathbb{S}^X \to \mathbb{C}^{*X}$ is a group homomorphism sine $\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$ is so too.

Definition 83 (Function Lift)**.** Given $f \in C(X, \mathbb{C}^*)$ then $\boldsymbol{f} \in C(X, \mathbb{S})$ is a *function lift* of f if $\mathcal{P}\boldsymbol{f} = f$.

Example 84. If $f = e^g \in C(X, \mathbb{C}^*)$ for $g \in C(X, \mathbb{C})$ then a function lift is given by $\boldsymbol{f} = \boldsymbol{e}^g \in C(X, \mathbb{S})$ since $\mathcal{P}\boldsymbol{e}^{g(x)} = e^{g(x)}$ for every $x \in X$.

We are interested in spaces X from Definition 83 for which function lifts are guaranteed to exist. Such spaces will be called lift-spaces, and they are defined below.

Definition 85 (Lift-space)**.** We call X a lift-space, if for any $f \in C(X, \mathbb{C}^*)$ there exists a function lift $\boldsymbol{f} \in C(X, \mathbb{S})$ such that $\mathcal{P}\boldsymbol{f} = f$.

Example 86. The identity function on $\mathbb{C}^*$ does not have a continuous lift since a branch cut will be introduced. Therefore $\mathbb{C}^*$ is not a lift-space.

In the following definition , we consider the class of all function lifts of a function.

Definition 87 (Lift-class)**.** Let X be a lift-space and $f \in C(X, \mathbb{C}^*)$. Then the *lift-class* of f is defined as $\mathcal{L}f := \{\boldsymbol{g} \in \mathbb{C}(X, \mathbb{S}) \mid \mathcal{P}\boldsymbol{g} = f\}$.

Remark 88. We will also call $\mathcal{L}f$ the lift of f, and any $\boldsymbol{g} \in \mathcal{L}f$ a lift of f.

We may characterize $\mathcal{L}f$ similarly to the point lifts by considering the constant functions defined by $1 : x \mapsto 1$ and $\mathbf{1}_n : x \mapsto \mathbf{1}_n = \boldsymbol{e}^{2\pi in}$ for every $x \in X$ (again with abuse of notation). If we also let $[1] = \{\mathbf{1}_n \mid n \in Z\}$, then we have the following characterization of the lift.

Theorem 89 (Characterization of $\mathcal{L}f$)**.** *If X is a connected lift-space, then for any $f \in C(X, \mathbb{C}^*)$ we have that $\mathcal{L}f = \boldsymbol{f}[1]$ where $\boldsymbol{f}$ is a function lift of f.*

Proof. We start by proving the statement for the special case f = 1, where we verify that $\mathcal{L}1 = [1]$, and then we consider the general case.

1. Since $\mathcal{P}\mathbf{1}_n = 1$ we have that $[1] \subseteq \mathcal{L}1$. Too prove the reverse inclusion we consider any $\boldsymbol{g} \in \mathcal{L}1$. Then $\mathcal{P}\boldsymbol{g}(x) = 1$ for every $x \in X$, so that $\boldsymbol{g}(x) \in [1]$ holds everywhere. Suppose for contradiction that $\boldsymbol{g}$ is not constant so that $\boldsymbol{g}(x) = \mathbf{1}_m$ and $\boldsymbol{g}(y) = \mathbf{1}_n$ for some $x, y \in X$ and for $m \neq n$. We consider the disjoint open balls $\tilde{B}_{1/2}(\mathbf{1}_m)$ and $\tilde{B}_{1/2}(\mathbf{1}_n)$ from Definition 76. Then, their preimages $U_k = \boldsymbol{g}^{-1}\left(\tilde{B}_{1/2}(\mathbf{1}_k)\right)$ are disjoint and empty for $k = m, n$. Furthermore, since $U_m \ni x$ and $U_n \ni y$ are also non-empty we have a contradiction to the connectedness of X. This means the assumption that $\boldsymbol{g}$ was non-constant was faulty. As such $\boldsymbol{g}$ is constant, and since we know its range is included in the set of points $[1]$ it follows that $\boldsymbol{g} \in [1]$. This shows the reverse inclusion and we may conclude that $\mathcal{L}1 = [1]$ as desired.

2. Given some lift $\boldsymbol{f} \in \mathcal{L}f$, we shall use that $\mathcal{P}$ is group homomorphisms (Remark 82) and a chain a equivalences to prove the characterization $\mathcal{L}f = \boldsymbol{f}[1]$. We have that

$$\begin{aligned} \boldsymbol{g} \in \mathcal{L}f \iff \mathcal{P}\boldsymbol{g} = f = \mathcal{P}\boldsymbol{f} &\iff \frac{\mathcal{P}\boldsymbol{g}}{\mathcal{P}\boldsymbol{f}} = \frac{f}{f} = 1 \\ \iff \mathcal{P}\left(\frac{\boldsymbol{g}}{\boldsymbol{f}}\right) = 1 &\iff \frac{\boldsymbol{g}}{\boldsymbol{f}} \in [1] \iff \boldsymbol{g} \in \boldsymbol{f}[1], \end{aligned}$$

from which the characterization $\mathcal{L}f = \boldsymbol{f}[1]$ follows.

□

From algebraic topology we have a lifting property that we will use inductively to construct desirable lift-spaces such as $\mathbb{C}$. It is formulated for the covering $\mathcal{P} : \mathbb{S} \to \mathbb{C}^*$ below, and the proof is omitted as it can be found as Proposition 1.30 in [5].

Lemma 90. *Let $f : Y \times [0,1] \to \mathbb{C}^*$ be continuous and let $\boldsymbol{f}_0 : Y \times \{0\} \to \mathbb{S}$ be a lift of $f|_{Y\times\{0\}}$. Then there exists a unique lift $\boldsymbol{f} : Y \times [0,1] \to \mathbb{C}^*$ of f such that $\boldsymbol{f}|_{Y\times\{0\}} = \boldsymbol{f}_0$.*

Remark 91. Another way to formulate the lemma is by stating that if Y is a lift-space then so is $Y \times [0,1]$.

We shall use Lemma 90 recursively in conjunction with a gluing process to show that $\mathbb{R}^m \times [0,1]^n$ is a lift -space, which will include $\mathbb{C}$ a special case.

Theorem 92. *For every $m, n \in \mathbb{N}$, we have that $\mathbb{R}^m \times [0,1]^n$ is a lift -space.*

Proof. We will use joint induction on the statement

$$\mathbb{R}^m \times [0,1]^n \text{ is a lift-space} \tag{$P_{m.n}$}$$

over $m, n \in \mathbb{N}$.

($P_{0,0}$) This statement is trivial since it corresponds to the existence of the lift of a point.

($P_{m,n} \implies P_{m,n+1}$) Let $Y = \mathbb{R}^m \times [0,1]^n$ be a lift-space by the induction assumption, and consider any $f \in C(Y \times [0,1], \mathbb{C}^*)$. Since Y is a lift-space there exists a lift $\boldsymbol{f}_0 : Y \to \mathbb{S}$ of the section $f_0 : Y \to \mathbb{C}^*$. Then by Lemma 90 there exists a unique lift $\boldsymbol{f} : Y \times [0,1] \to \mathbb{S}$ of f satisfying $\boldsymbol{f}(y,0) = \boldsymbol{f}_0(y)$ for every $y \in Y$. We may therefore conclude that $Y \times [0,1] = \mathbb{R}^m \times [0,1]^{n+1}$ is a lift-space, which finishes this induction step.

($P_{m,n+1} \implies P_{m+1,n}$) With Y given as before we want to show that if $Y \times [0,1]$ is a lift-space, then so is $Y \times \mathbb{R}$. To do this we consider any (continuous) $f : Y \times \mathbb{R} \to \mathbb{C}^*$, for which we would like to construct a lift. This will be accomplished by constructing lifts $\boldsymbol{f}_k : Y \times I_k \to \mathbb{S}$ of the restrictions $f_k = f|_{Y\times I_k}$ where $I_k = [k, k+1]$, and by gluing these lifts together at the endpoints during the construction process using Lemma 90. Before providing further details of the gluing process we first note that I_k may be identified with $[0,1]$ by translation, and consequently we may also use the induction assumption and Lemma 90 for $Y \times I_k$. We start by fixing a lift $\boldsymbol{f}_0$ using the existence in the induction assumption. For $k \in \mathbb{N}$ we then inductively construct $\boldsymbol{f}_{k+1}$ from $\boldsymbol{f}_k$ such that they agree on their common domain, namely $\boldsymbol{f}_{k+1}|_{Y\times\{k+1\}} = \boldsymbol{f}_k|_{Y\times\{k+1\}}$. This may be accomplished using Lemma 90 for f_{k+1} with respect to the given lift $\boldsymbol{f}_{k+1,0} = \boldsymbol{f}_k|_{Y\times\{k+1\}}$ of the restriction $f_{k+1}|_{Y\times I_{k+1}}$. Using an identical process we may therefore equivalently construct $\boldsymbol{f}_k$ from $\boldsymbol{f}_{k+1}$ for negative k, such that they agree at their common domain. With all the restricted lifts $\boldsymbol{f}_k$ defined and agreeing on their common domains we may then construct the desired lift $\boldsymbol{f}$ of f by $\boldsymbol{f}|_{Y\times I_k} = \boldsymbol{f}_k$ for every $k \in \mathbb{Z}$. Note that this yields a well defined, continuous $\boldsymbol{f}$ since all $\boldsymbol{f}_k$ are continuous and agree on their common domains. Since the restrictions of $\boldsymbol{f}$ project onto the restriction s of f, then we may indeed conclude that $\mathcal{P}\boldsymbol{f} = f$. We have therefore showed that $Y \times \mathbb{R} = \mathbb{R}^{m+1} \times [0,1]^n$ is a lift-space, as we sought out to do in this induction step.

We have thus proved the base case $P_{0,0}$, the induction step $P_{m,n} \implies P_{m,n+1}$, and by combining the two induction steps, $P_{m,n} \implies P_{m,n+1} \implies P_{m,n+1}$ for $m, n \in \mathbb{N}$. By induction we therefore have that $P_{m,n}$ is true for every $m, n \in \mathbb{N}$ as desired. □

As a consequence of the theorem we therefore have that $\mathbb{C}$ is a lift-space (using the standard topology of $\mathbb{R}^2$), and by its connectivity we also use the characterization theorem on the lift.

Corollary 93. *For every continuous $f : \mathbb{C} \to \mathbb{C}^*$ there exists a (continuous) lift $\boldsymbol{f} : \mathbb{C} \to \mathbb{S}$ and $\mathcal{L}f = \boldsymbol{f}[1]$.*

3.2 The lift-projection method

With function lifts and projections defined we now have the necessary tools to transfer $\mathbb{S}$-valued multiplicative transformations to their $\mathbb{C}^*$-valued counterparts.

Definition 94 (Lift-projection method)**.** Suppose that $\mathcal{A} : V \to W$ is a multiplicative transformation acting between the subspaces $V \subseteq C(X, \mathbb{S})$ and $W \subseteq C(Y, \mathbb{S})$ for (connected) lift-spaces X, Y (such as those in Theorem 92). Then the lift-projection method defines the (potentially) multivalued transformation

$$A : f \mapsto \{\mathcal{P}\mathcal{A}\boldsymbol{g} \mid \boldsymbol{g} \in \mathcal{L}f\}. \tag{17}$$

Remark 95. The lift projection method (17) may be decomposed into three steps according to the "lift, apply, project" pattern.

1. Lift: $f \mapsto \mathcal{L}f = \boldsymbol{f}[1]$ where $\mathcal{P}\boldsymbol{f} = f$
2. Apply: $\mathcal{L}f \mapsto \{\mathcal{A}\boldsymbol{g} \mid \boldsymbol{g} \in \mathcal{L}f\} = (\mathcal{A}\boldsymbol{f})(\mathcal{A}[1])$
3. Project: we get $\{\mathcal{P}\mathcal{A}\boldsymbol{g} \mid \boldsymbol{g} \in \mathcal{L}f\} = (\mathcal{P}\mathcal{A}\boldsymbol{f})[1]_{\mathcal{A}}$ where $[1]_{\mathcal{A}} := \{\mathcal{P}\mathcal{A}\mathbf{1}_n \mid n \in \mathbb{Z}\}$

Note that $[1]_{\mathcal{A}} = A1$ is the lift-projection method applied to the constant function 1, and characterizes whether A is multivalued or not as will be seen when *derivatives and *integrals are considered. In conclusion, the lift-projection method defines that (potentially) multivalued transformation

$$A : f \mapsto (\mathcal{P}\mathcal{A}\boldsymbol{f})[1]_{\mathcal{A}}$$

where $\boldsymbol{f}$ is a lift of f.

Remark 96. We may also apply A on a set of functions F by $AF := \bigcup_{f \in F} Af$.

Example 97 (Multiplicative derivative)**.** The multiplicative derivative $D_* : H_*(U) \to H_*(U)$ from Example 61 is transformed by the lift-projection method to act on holomorphic functions $f : U \to \mathbb{C}^*$ by

$$f \mapsto \boldsymbol{f}[1] \mapsto (\boldsymbol{f}^*)(D_*[1]) = \boldsymbol{f}^* \mapsto \mathcal{P}\boldsymbol{f}^* := f^*,$$

which is single-valued. We may calculate $f^* = \mathcal{P}\boldsymbol{f}^* = \mathcal{P}\boldsymbol{e}^{(\mathbf{log}\,\boldsymbol{f})'} = e^{(\mathbf{log}\,\boldsymbol{f})'}$ using a local logarithmic branch log on which $\mathbf{log}\,\boldsymbol{f} = \log f$. Doing this gives

$$f^* = e^{(\mathbf{log}\,\boldsymbol{f})'} = e^{(\log f)'} = e^{f'/f}$$

for the multiplicative derivative.

We are also interested to see how composition works in the lift-projection method.

Proposition 98. *Suppose that $\mathcal{A} : U \to V$ and $\mathcal{B} : V \to W$ are multiplicative transformations acting between the subspaces $U \subseteq C(X, \mathbb{S})$, $V \subseteq C(Y, \mathbb{S})$ and $W \subseteq C(Z, \mathbb{S})$, where X, Y and Z are connected lift-spaces. Furthermore, let A, B and BA be the lift-projection counterparts of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}\mathcal{A}$, respectively. Then*

$$B(Af) = (BA)f[1]_{\mathcal{B}}.$$

for every $f \in U$.

Proof. By the lift-projection method we have that $Af = (\mathcal{P}\mathcal{A}\boldsymbol{f})[1]_{\mathcal{A}}$ for any $f \in U$ and where $\mathcal{P}\boldsymbol{f} = f$. We shall use the lift-projection method with respect to $\mathcal{B}$ on Af.

1. The lifts of elements in Af are of the form $(\mathcal{A}\boldsymbol{f})(\mathcal{A}\mathbf{1}_m)\mathbf{1}_n$ for $m, n \in \mathbb{Z}$.

2. After application by $\mathcal{B}$ we get elements of the form $(\mathcal{B}\mathcal{A}\boldsymbol{f})(\mathcal{B}\mathcal{A}\mathbf{1}_m)\mathcal{B}\mathbf{1}_n$ for $m, n \in \mathbb{Z}$.

3. The projection then gives elements of the form $(\mathcal{P}\mathcal{B}\mathcal{A}\boldsymbol{f})(\mathcal{P}\mathcal{B}\mathcal{A}\mathbf{1}_m)\mathcal{P}\mathcal{B}\mathbf{1}_n$ for $m, n \in \mathbb{Z}$, which corresponds to the set $(\mathcal{P}\mathcal{B}\mathcal{A}\boldsymbol{f})[1]_{\mathcal{B}\mathcal{A}}[1]_{\mathcal{B}}$.

We therefore have that $B(Af) = (\mathcal{P}\mathcal{B}\mathcal{A}\boldsymbol{f})[1]_{\mathcal{B}\mathcal{A}}[1]_{\mathcal{B}}$, and since $(BA)f = (\mathcal{P}\mathcal{B}\mathcal{A}\boldsymbol{f})[1]_{\mathcal{B}\mathcal{A}}$ the desired statement follows. □

Remark 99. In the proposition we used the notation BA to denote the lift-projection analog of $\mathcal{B}\mathcal{A}$, and it therefore represents first composing the operators and *then* applying the lift-projection method. On the other hand, when we consider $B(Af)$ we apply the lift-projected counterparts of $\mathcal{A}$ and $\mathcal{B}$ in succession to $f \in U$. We shall use the notation $B \circ A : f \mapsto B(Af)$ to denote when the lift-projection method is performed before the composition.

The statement then simplifies to $(B \circ A)f = (BA)f[1]_{\mathcal{B}}$ for every $f \in U$, which we may again simplify to

$$B \circ A = (BA)[1]_{\mathcal{B}}. \tag{18}$$

Corollary 100. *Suppose that we have multiplicative transformations $\mathcal{A}_k : V_k \to V_{k+1}$ for every $k = 1, \dots, n$, where $V_k \subseteq C(X_k, \mathbb{S})$ are subspaces and X_k are connected lift-spaces for every $k = 1, \dots, n+1$. Let $A_1, \dots, A_n$ be the lift-projection analogues of $\mathcal{A}_1, \dots, \mathcal{A}_n$ with composition $A_n \circ \cdots \circ A_1$, and let $A_n \cdots A_1$ be the lift-projection analog of $\mathcal{A}_n \cdots \mathcal{A}_1$. Then*

$$A_n \circ \cdots \circ A_1 = (A_n \cdots A_1) \prod_{k=2}^{n} [1]_{\mathcal{A}_n \cdots \mathcal{A}_k}. \tag{19}$$

Proof. We shall use induction over $n \in \mathbb{Z}_+$ to prove the statement. For $n = 1$, the product in the right hand-side is empty and consequently the singleton identity $\{1\}$, which proves (19) in this case. In the induction step, we suppose that (19) holds for some fixed $n \in \mathbb{Z}_+$ and want to show that it holds for $n + 1$, i.e. that

$$A_{n+1} \circ \cdots \circ A_1 \stackrel{?}{=} (A_{n+1} \cdots A_1) \prod_{k=2}^{n+1} [1]_{\mathcal{A}_{n+1} \cdots \mathcal{A}_k}. \tag{20}$$

We let $A = A_{n+1} \circ \cdots \circ A_1$ and $B = A_{n+1}$ in (18) so that

$$B \circ A = (BA_n \cdots A_1) \prod_{k=2}^{n} [1]_{\mathcal{B}\mathcal{A}_n \cdots \mathcal{A}_k} [1]_{\mathcal{B}} = (A_{n+1} \cdots A_1) \prod_{k=2}^{n+1} [1]_{\mathcal{A}_{n+1} \cdots \mathcal{A}_k},$$

which proves (20) as desired. □

Remark 101. Using (19) on $f \in V_1$, we find that

$$\begin{aligned}(A_n \circ \cdots \circ A_1)f &= (A_n \cdots A_1)f \prod_{k=2}^{n} [1]_{\mathcal{A}_n \cdots \mathcal{A}_k} \\ &= (\mathcal{P}\mathcal{A}_n \cdots \mathcal{A}_1 \boldsymbol{f})[1]_{\mathcal{A}_n \cdots \mathcal{A}_1} \prod_{k=2}^{n} [1]_{\mathcal{A}_n \cdots \mathcal{A}_k} \\ &= (\mathcal{P}\mathcal{A}_n \cdots \mathcal{A}_1 \boldsymbol{f}) \prod_{k=1}^{n} [1]_{\mathcal{A}_n \cdots \mathcal{A}_k},\end{aligned}$$

where $\mathcal{P}\boldsymbol{f} = f$. In particular, for $f = 1$ and letting $\boldsymbol{f} = \mathbf{1}$, we have that

$$(A_n \circ \cdots \circ A_1)1 = \underbrace{(\mathcal{P}\mathcal{A}_n \cdots \mathcal{A}_1 \mathbf{1})}_{1} \prod_{k=1}^{n} [1]_{\mathcal{A}_n \cdots \mathcal{A}_k} = \prod_{k=1}^{n} [1]_{\mathcal{A}_n \cdots \mathcal{A}_k}, \tag{21}$$

and consequently we define

$$[1]_{A_n \circ \cdots \circ A_1} := (A_n \circ \cdots \circ A_1)1 = \prod_{k=1}^{n} [1]_{\mathcal{A}_n \cdots \mathcal{A}_k}. \tag{22}$$

We see that $[1]_{A_n \circ \cdots \circ A_1}$ characterizes the multivalued behaviour with the formula

$$(A_n \circ \cdots \circ A_1)f = (\mathcal{P}\mathcal{A}_n \cdots \mathcal{A}_1 \boldsymbol{f})[1]_{A_n \circ \cdots \circ A_1} \tag{23}$$

for every $f \in V_1$ and where $\mathcal{P}\boldsymbol{f} = f$. Note that since $[1]_A = [1]_{\mathcal{A}}$, then (23) generalizes the lift-projection formula $Af = (\mathcal{P}\mathcal{A}\boldsymbol{f})[1]_{\mathcal{A}}$.

Example 102. Consider iterating the canonical multiplicative integration from Example 63, where $\mathcal{A}_k = J_* : f \mapsto \left(z \mapsto \prod_0^z f(\zeta)^{\mathrm{d}\zeta}\right)$, on the constant function 1. We get the following chain

$$1 \mapsto \{e^{2\pi i m z}\}_{m \in \mathbb{Z}} \mapsto \{e^{2\pi i m z + \pi i n z^2}\}_{m,n \in \mathbb{Z}} \mapsto \dots$$

under iterated action of J_* (and where shorthand notation "$f(z)$" has been used instead of "$z \mapsto f(z)$"). Note in particular that $[1]_{J_*} = \{z \mapsto e^{2\pi i n z} \mid n \in \mathbb{Z}\}$, which characterizes the multivalued behaviour of multiplicative integration. By combining the multiplicative integration with integration on $[0,1]$ we then get a connection to the Fourier series. We will not pursue this topic further, but will look at some other mixed problems in the next section.

Example 103 (Multiplicative expectation)**.** Recall from Example 64 the definition of the measure theoretic multiplicative integral for $\mathbb{S}$-valued functions. If we let Ω be (connected) lift-space such as those in Theorem 92 of the form $\mathbb{R}^m \times [0,1]^n$ and equip it with a probability measure P we may consider the multiplicative integral (geometric expectation) of a continuous function (random variable) $f : \Omega \to \mathbb{C}^*$. By defining $\boldsymbol{\mathcal{A}f} = \prod \boldsymbol{f}^{\,\mathrm{d}P}$ it follows that

$$[1]_{\mathcal{A}} = \left\{ \mathcal{P} \prod \mathbf{1}_n^{\mathrm{d}P} \mid n \in \mathbb{Z} \right\}$$

where

$$\prod \mathbf{1}_n^{\mathrm{d}P} = \boldsymbol{e}^{\int \mathbf{log}\,\mathbf{1}_n \,\mathrm{d}P} = \boldsymbol{e}^{\int 2\pi i n \,\mathrm{d}P} = \boldsymbol{e}^{2\pi i n} = \mathbf{1}_n$$

so that

$$[1]_{\mathcal{A}} = \left\{ \underbrace{\mathcal{P}\mathbf{1}_n}_{1} \mid n \in \mathbb{Z} \right\} = \{1\}$$

is single-valued. The multiplicative expectation of a non-zero continuous random variable may therefore be defined uniquely, and satisfies

$$Af = \mathcal{P} \prod \boldsymbol{f}^{\,\mathrm{d}P}$$

where $\mathcal{P}\boldsymbol{f} = f$.

4 Mixed additive and multiplicative Problems

So far we have mainly focused on linear and multiplicative problems in isolation, but we are also interested in the behaviour of mixed additive and multiplicative problems.

4.1 $y^* = y'$

First we provide a toy example of such a mixed problem namely the mixed differential equation $y^* = y'$ in which the left-hand side is multiplicative with respect to y and where the right-hand side is linear with respect to y. Since $y^* = e^{y'/y}$ the mixed differential equation is equivalent to the differential equation

$$y' = e^{y'/y},$$

whose solutions can be found using Maple to be

$$y(x) = -\frac{W\left(e^{c-x}\right)}{e^{\frac{1}{W\left(e^{c-x}\right)}}}, \tag{24}$$

where W is the Lambert W function (see [3] for more information) and $c \in \mathbb{R}$ is a constant. We will explain this result further by also rewriting it to a simplified form, and in Proposition 108 we will provide the derivation of (24). We begin by defining the Lambert W function together with some related functions so that their interdependent relationships can be highlighted.

Definition 104 (M, W, m and w functions)**.** Let $M(x) = xe^x$ and define a (local) inverse W to be the Lambert W function. We also define the function $m(x) = x + e^x$ and its inverse w.

Remark 105. The Lambert W function may also be defined as the inverse of the complex function $M(z) = ze^z$, and in this case will have an associated Riemann surface on which W may be defined. . This is done in [3] which also mentions that the real case contains two branches of the Lambert W function, which may be seen in figure 7.

The relationship between M and m is given by

$$e^{m(x)} = e^{x+e^x} = e^x e^{e^x} = M(e^x) \tag{25}$$

or equivalently by

$$\ln M(x) = m(\ln x), \tag{26}$$

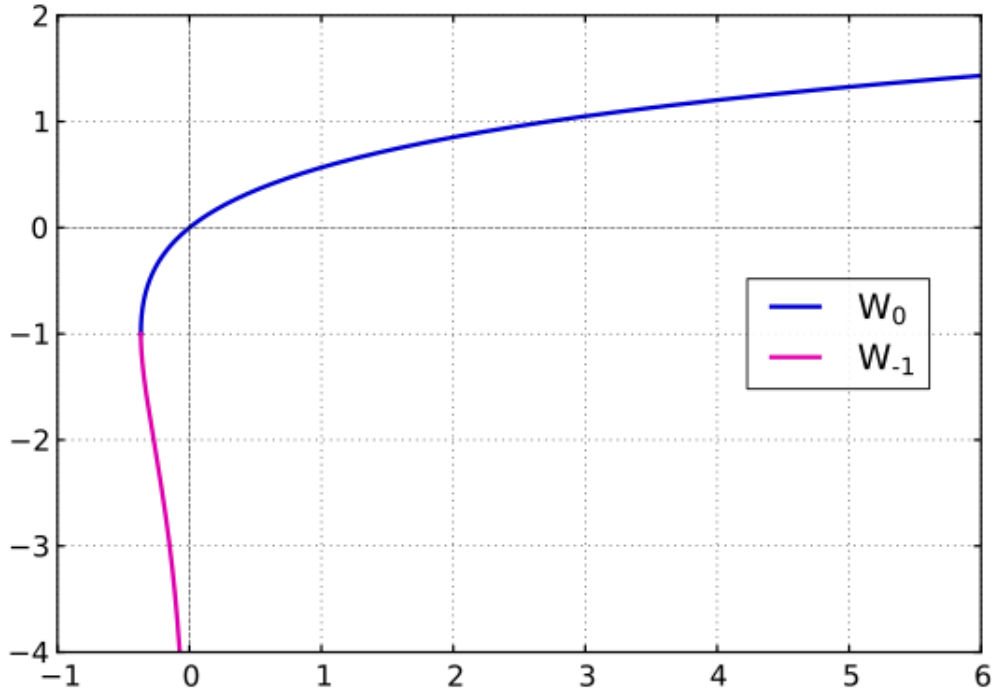


Figure 7: The two real branches of the Lambert W functions. We will only use the main branch W_0 in blue, and will use notation W for it. Image taken from https://commons.wikimedia.org/wiki/File:Mplwp_lambert_W_branches.svg

from which we may derive the corresponding relationship between their inverses W and w. If we let $v = W(e^u)$, we may first solve for u, so that

$$W(e^u) = v \iff e^u = M(v) \iff u = \ln M(v) = m(\ln v)$$

by (26). Then, we solve for v again, so that

$$u = m(\ln v) \iff w(u) = \ln v \iff e^{w(u)} = v,$$

and it follows that

$$W(e^u) = e^{w(u)}. \tag{27}$$

If we let $u = -x + c$ in the right-hand-side of (24), we may rewrite it as

$$-\frac{W(e^u)}{e^{\frac{1}{W(e^u)}}} = -\frac{e^{w(u)}}{e^{\frac{1}{e^{w(u)}}}} = -e^{w(u)-\frac{1}{e^{w(u)}}} = -e^{w(u)-e^{-w(u)}}$$

where the exponent satisfies

$$w(u) - e^{-w(u)} = -\left((-w(u)) + e^{-w(u)}\right) = -m(-w(u))$$

and consequently (24) may be rewritten as

$$y_c(x) = -e^{-m(-w(-x+c))}. \tag{28}$$

We note that $y_c(x) = y_0(x - c)$ for every $x \in \mathbb{R}$ and consequently y_c is a translation of the canonical solution y_0,

$$y_0(x) = -e^{-m(-w(-x))}. \tag{29}$$

The translation symmetry of the solutions is expected since both the derivative and *derivative are locally defined the same way everywhere.

Remark 106. As we may see in figure 8 and (29) y_0 attains negative values which might cause the reader some concern as the *derivative was defined for positively valued functions. However, as others have previously mentioned, the *derivative may also be defined for purely negatively valued functions. This can be done by $g^* = e^{g'/g} = e^{(-g)'/(-g)} = (-g)^*$ for g a negatively valued function. Another way to see why the same formula holds is by considering quotient limits

$$g^*(x) = \lim_{h \to 0} \left(\frac{g(x+h)}{g(x)} \right)^{1/h}$$

analogous to the difference limit for the ordinary derivative, and noting the the sign cancels in the quotient. We could also view it as a complex valued function where the angle is constant and hence vanishes in the *derivative.

Remark 107 (Asymptotic approximations of m, w and y_0). To understand the heuristic asymptotics of the canonical solution $y_0(x) = -e^{-m(-w(-x))}$ we first consider the asymptotics of m and w. Recall that $m(x) = x + e^x$, which yields the asymptotic approximation[1]

$$m(x) \approx \begin{cases} e^x, & x \to \infty \\ x, & x \to -\infty \end{cases}$$

and since $m : \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijection we get for the inverse w the inverse asymptotic approximations, namely

$$w(x) \approx \begin{cases} \ln x, & x \to \infty \\ x, & x \to -\infty \end{cases}$$

[1] By $a(x) \approx b(x)$, $x \to L$ we mean that $\lim_{x \to L} \frac{a(x)}{b(x)} = 1$, where $L = \pm\infty$.

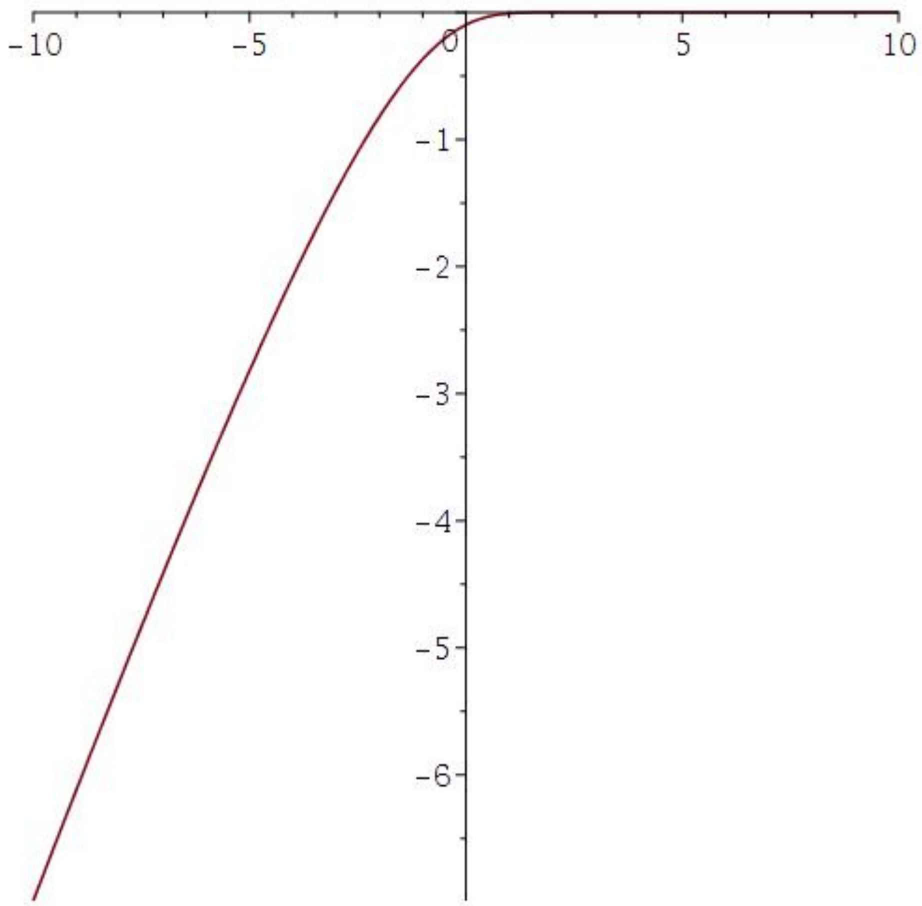


Figure 8: The graph of the canonical solution y_0 to the differential equation $y' = y^*$. The other solutions are found by horizontal displacement of the graph.

which can also be seen heuristically in figure 9.

With the asymptotics of m and w considered we may then find the asymptotics of y_0. As $x \to \infty$, then $-x \to -\infty$, which means that $-w(-x) \approx -(-x) = x$ and as such $y_0 \approx -e^{-m(x)} \approx -e^{-e^x}$ in that case. When $x \to -\infty$ then $w(-x) \approx \ln(-x)$ which tends to infinity so $y_0 \approx -e^{-m(-\ln(-x))} \approx -e^{-(-\ln(-x))} = -e^{\ln(-x)} = -(-x) = x$. Summarizing, we have the asymptotic approximation

$$y_0(x) \approx \begin{cases} -e^{-e^x}, & x \to \infty \\ x, & x \to -\infty \end{cases} \tag{30}$$

for y_0.

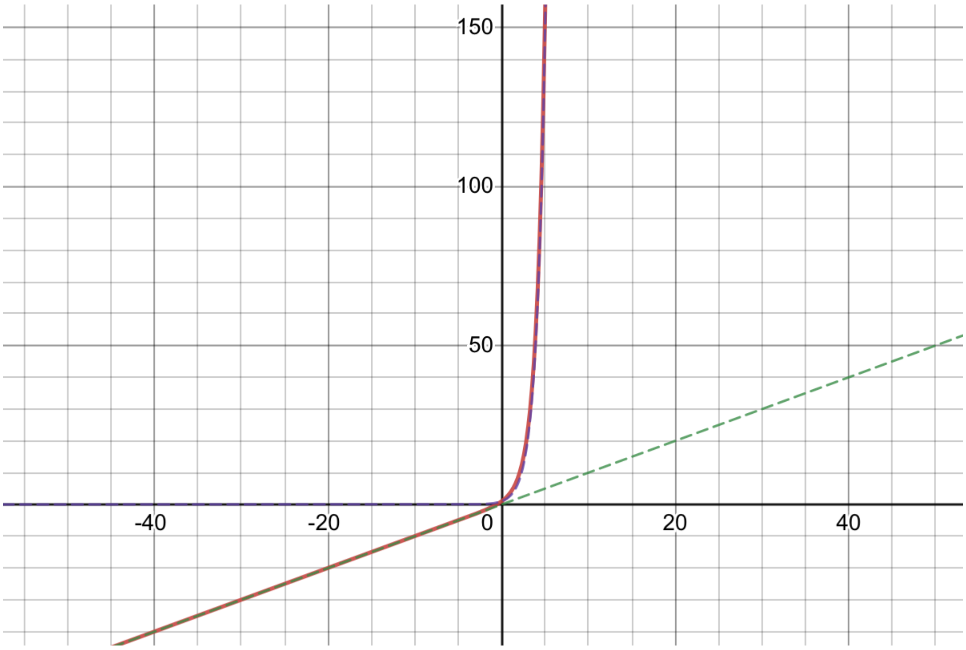


Figure 9: The graphs of $y = m(x) = x + e^x$ (red), $y = x$ (green) and $y = e^x$ (purple).

With the preliminary discussion undertaken we are now ready to state and prove the solution of $y' = y^*$.

Proposition 108 (Solution to $y' = y^*$)**.** *The smooth real solutions to $y' = y^*$ are all negative and given by $y_c(x) = -e^{-m(-w(-x+c))}$ for $x \in \mathbb{R}$, where $c \in \mathbb{R}$ a constant.*

Proof. We will show that $y' = e^{y'/y}$if and only if(24) holds, and note that the remainder of the derivation to the desired form(28) has already been presented. This will be accomplished in several steps, where first we separate the variables, then integrate both sides and finally solve the equation for y to obtain the solution.

1. To separate the variables we transform $y' = e^{y'/y}$ such that it involves the M function and we may then transform it by its inverse, the Lam-

bert W function. We have that

$$\begin{aligned} y' = e^{y'/y} &\iff y'e^{-y'/y} = 1 \\ &\iff \left(-\frac{y'}{y}\right) e^{-y'/y} = -\frac{1}{y} \\ &\iff M\left(-\frac{y'}{y'}\right) = -\frac{1}{y} \end{aligned}$$

which may be rewritten using the Lambert W function as

$$-\frac{y'}{y} = W\left(-\frac{1}{y}\right) \iff \frac{\mathrm{d}y}{yW\left(-\frac{1}{y}\right)} = -\,\mathrm{d}x$$

in separated form.

2. Next we integrate both sides

$$\int \frac{\mathrm{d}y}{yW\left(-\frac{1}{y}\right)} = \int -\,\mathrm{d}x$$

to obtain

$$\frac{1}{W\left(-\frac{1}{y}\right)} - \ln W\left(-\frac{1}{y}\right) = -x + c \tag{31}$$

for an arbitrary constant $c \in \mathbb{R}$. Indeed, we verify that the left-hand-side $f(y) = \frac{1}{W\left(-\frac{1}{y}\right)} - \ln W\left(-\frac{1}{y}\right)$ is a primitive by showing that $f'(y) = \frac{1}{yW\left(-\frac{1}{y}\right)}$. If we let $g(y) = W\left(-\frac{1}{y}\right)$ then we have that $f(y) = \frac{1}{g(y)} - \ln g(y)$ and by the chain rule it follows that

$$f'(y) = -\frac{g'(y)}{(g(y))^2} - \frac{g'(y)}{g(y)} = -\frac{g'(y)}{g(y)}\left(\frac{1}{g(y)} + 1\right). \tag{32}$$

Using the chain rule again on g, we find that $g'(y) = \frac{1}{y^2}W'\left(-\frac{1}{y}\right)$, and the derivative of the Lambert W function is $W'(z) = \frac{W(z)}{z(1+W(z))}$ (can be found by implicit differentiation of $we^w = z$, see [3]), and it follows that

$$g'(y) = \frac{1}{y^2} \cdot \frac{W\left(-\frac{1}{y}\right)}{-\frac{1}{y}\left(1 + W\left(-\frac{1}{y}\right)\right)} = \frac{g(y)}{-y(1+g(y))}. \tag{33}$$

Rewriting (33) as $-\frac{g'(y)}{g(y)} = \frac{1}{y(1+g(y))}$ and inserting into (32) we get that

$$f'(y) = \frac{1}{y(1+g(y))}\left(\frac{1}{g(y)} + 1\right) = \frac{1}{y(1+g(y))} \cdot \frac{1+g(y)}{g(y)} = \frac{1}{yg(y)}.$$

Then using $g(y) = W\left(-\frac{1}{y}\right)$ it follows that $f'(y) = \frac{1}{yW\left(-\frac{1}{y}\right)}$ as desired.

3. Next we solve (31) for y, by first transforming both sides by the exponential function

$$\frac{e^{\frac{1}{W\left(-\frac{1}{y}\right)}}}{W(-1/y)} = e^{-x+c}.$$

Since the left-hand side can be expressed in the form

$$M\left(\frac{1}{W\left(-\frac{1}{y}\right)}\right) = \frac{1}{W\left(-\frac{1}{y}\right)} e^{\frac{1}{W\left(-\frac{1}{y}\right)}},$$

we get that

$$M\left(\frac{1}{W\left(-\frac{1}{y}\right)}\right) = e^{-x+c}.$$

Next we transform both sides by the Lambert W function so that

$$\frac{1}{W\left(-\frac{1}{y}\right)} = W\left(e^{-x+c}\right) \iff 1 = W\left(-\frac{1}{y}\right) W\left(e^{-x+c}\right), \tag{34}$$

where we have the symmetric equation $1 = W(u)W(v)$ for $u = -1/y$ and $v = e^{-x=c}$. Using symmetry and the preceding calculations we then get that

$$\frac{1}{W(u)} e^{\frac{1}{W(u)}} = v \Leftrightarrow 1 = W(u)W(v) \Leftrightarrow \frac{1}{W(v)} e^{\frac{1}{W(v)}} = u, \tag{35}$$

and since $u = -1/y$ it follows that

$$y = -\frac{1}{u} = -W(v)e^{-\frac{1}{W(v)}} = -W\left(e^{-x+c}\right) e^{-\frac{1}{W\left(e^{-x+c}\right)}}. \tag{36}$$

Since (36) is of the desired form, the proof is therefore complete. □

Remark 109 (Complex $y' = y^*$)*.* Note that the complex counterpart of (24) holds locally by repeating the calcuations in Proposition 108.

4.2 Arithmetic and Geometric means

In this section we generalize the inequality of arithmetic and geometric means (AM-GM) to hold for any non-negative random variables, and then consider a generalization of the matrix AM-GM for such random variables. We begin by defining the arithmetic and geometric means.

Definition 110 (Arithmetic and geometric means)**.** Let $X : (\Omega, \mathscr{A}, P) \to [0, \infty)$ be a non-negative random variable. Then, we define its arithmetic mean by $\mathbb{A}X := \mathbb{E}X$, and its geometric mean $\mathbb{G}X$ as follows. If $X = Y$ almost surely (written $X \simeq Y$, meaning $P(X = Y) = 1$) and $Y : (\Omega, \mathscr{A}, P) \to \mathbb{R}_+$ is a positive random variable then

$$\mathbb{G}X := \prod Y^{\,\mathrm{d}P}. \tag{37}$$

Otherwise, if no such Y exists, then $P(X = 0) > 0$ and we define $\mathbb{G}X := 0$ in this case.

Remark 111. We may rewrite (37) as $\mathbb{G}X = \prod Y^{\,\mathrm{d}P} = e^{\int \ln Y \,\mathrm{d}P} = e^{\mathbb{E} \ln Y}$.

Remark 112. We could also have included the special case in the general definition by extending the logarithm to act on 0 as well by $\ln 0 := -\infty$ and conversely for its inverse – the exponential function. This would require integration of $[-\infty, \infty)$ valued functions, which is covered in measure theory. Note that the two definitions coincide, but we favored the direct method since it will be more amenable to Jensen's inequality (which will not need to be modified to include singularities, although it could).

Since we will use Jensen's inequality to prove the AM-GM inequality for non-negative random variables, we will now state it without proof as those abound in the literature.

Lemma 113 (Jensen's inequality)**.** *Let $X : \Omega \to I$ be a random variable and $\varphi : I \to \mathbb{R}$ be a concave function, where I is an interval. Then*

$$\varphi\big(\mathbb{E}X\big) \geq \mathbb{E}\big(\varphi(X)\big). \tag{38}$$

If φ is strictly concave then equality holds if and only if X is constant almost surely.

Theorem 114 (AM-GM inequality)**.** *Let $X \geq 0$ be a non-negative random variable. Then*

$$\mathbb{A}X \geq \mathbb{G}X \tag{39}$$

with equality if and only if X is constant almost everywhere.

Proof. We begin to prove the statement if $X \simeq Y > 0$ by considering Y. Note that since $\mathbb{A}X = \mathbb{A}Y$ and $\mathbb{G}Y = \mathbb{G}Y$ in this case, the AM-GM inequality would also holds for X. Let $\varphi(x) = \ln x$, which is a strictly concave function $\mathbb{R}_+ \to \mathbb{R}$ since $\varphi'(x) = -\frac{1}{x^2} < 0$ for every $x \in \mathbb{R}_+$. Therefore it follows by Jensen's inequality that

$$\ln\big(\mathbb{E}Y\big) \geq \mathbb{E}\big(\ln(Y)\big) \iff \mathbb{A}Y \geq \mathbb{G}Y = e^{\mathbb{E}\ln Y} \tag{40}$$

with equality if and only if Y is constant almost surely (and consequently X too).

Suppose now that $P(X = 0) > 0$ so that $\mathbb{G}X = 0$. Since $X \geq 0$ we know from measure theory that $\mathbb{A}X \geq 0$ with equality if and only if $X \simeq 0$. Therefore, in this case $\mathbb{A}X \geq 0 = \mathbb{G}X$ with equality if and only if $X \simeq 0$ is almost surely 0. We have thus proved the AM-GM inequality for the two cases which concludes the proof. □

Next we extend the matrix form of the AM-GM inequality (Theorem10.9 in [6]) to a probabilistic setting with an analogous proof.

Theorem 115 ($A_G \geq G_A$)**.** *Let $(\mathcal{X}, \mathcal{F}_X, P_X)$ and $(\mathcal{Y}, \mathcal{F}_Y, P_Y)$ be probability spaces (with the identity random variables*[2] *X and Y), and let $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ be integrable. Define the functions*

$$\begin{cases} A(y) = \mathbb{A}f(X, y) \\ G(x) = \mathbb{G}f(x, Y) \end{cases} \tag{41}$$

and let

$$\begin{cases} A_G = \mathbb{G}A(Y) \\ G_A = \mathbb{A}G(X) \end{cases} \tag{42}$$

so that the inequality states

$$A_G \geq G_A. \tag{43}$$

If $P_Y(A(Y) = 0) = 0$ then equality holds in (43) if and only if there exists a function $a : \mathcal{Y} \to [0, \infty)$ such that $f(x, y) = a(y)$ for almost every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and in the trivial case when $P_Y(A(Y) = 0) > 0$ where $A_G = 0 = G_A$.

Proof. Suppose first that $A(y) > 0$ for every $y \in \mathcal{Y}$. Then by the AM-GM inequality with respect to Y it follows that

$$\int \frac{f(x, y)}{A(y)} \,\mathrm{d}P_Y(y) = \mathbb{A}\left(\frac{f(x, Y)}{A(Y)}\right) \geq \mathbb{G}\left(\frac{f(x, Y)}{A(Y)}\right) = \frac{\mathbb{G}f(x, Y)}{\mathbb{G}A(Y)} = \frac{G(x)}{A_G} \tag{44}$$

[2] $X : \mathcal{X} \to \mathcal{X}$ is the identity random variable of the probability space $(\mathcal{X}, \mathcal{F}_X, P_X)$ when $X(x) = x$ for every $x \in \mathcal{X}$

or equivalently that

$$A_G \int \frac{f(x,y)}{A(y)} \, \mathrm{d}P_Y(y) \geq G(x) \tag{45}$$

for every $x \in \mathcal{X}$. By integrating (45) with respect to x and using linearity of the integral (so scalar A_G can be moved outside) and Tonelli's theorem we get that

$$\begin{gathered} A_G \iint \frac{f(x,y)}{A(y)} \, \mathrm{d}P_Y(y) \, \mathrm{d}P_X(x) \geq \int G(x) \, \mathrm{d}P_X(x) \\ \Leftrightarrow \\ A_G \iint \frac{f(x,y)}{A(y)} \, \mathrm{d}P_X(x) \, \mathrm{d}P_Y(y) \geq G_A. \end{gathered} \tag{46}$$

we simplify the double integral in the left-hand side to get that

$$\begin{aligned} &\iint \frac{f(x,y)}{A(y)} \, \mathrm{d}P_X(x) \, \mathrm{d}P_Y(y) = \int \frac{1}{A(y)} \left(\int f(x,y) \, \mathrm{d}P_X(x) \right) \, \mathrm{d}P_Y(y) \\ &= \int \frac{1}{A(y)} A(y) \, \mathrm{d}P_Y(y) = \int 1 \, \mathrm{d}P_Y = 1 \end{aligned} \tag{47}$$

since P_Y is a probability, and it follows that $A_G \geq G_A$ in this case. Note that by the AM-GM inequality equality in (45) holds if and only if $y \mapsto \frac{f(x,y)}{A(y)}$ is constant almost surely with respect y, and if the equality in (45) should extend to an equality in (46) we also need equality in (45) for almost every x. Therefore we have in this case that $A_G = G_A$if and only if $(x,y) \mapsto \frac{f(x,y)}{A(y)} = c$ is constant for almost every x and y which means that $f(x,y) = cA(y) =: a(y)$ almost everywhere.

Now we consider the case where A is not necessarily a strictly positive function, and consider its null set $N = \{A(Y) = 0\} = A^{-1}(\{0\})$ where the previous case was when $N = \emptyset$. First we consider the case where $P_Y(N) = 0$ where N is almost surely empty, in which case we replace $\frac{f(x,y)}{A(y)}$ by 0 for $y \in N$. Then by replacing $\mathcal{Y}$ by $\tilde{\mathcal{Y}} = \mathcal{Y} \setminus N$ in previous calculations those conclusions hold and since their difference has zero measure the integrals over$\tilde{\mathcal{Y}}$ may be replaced by integrals over $\mathcal{Y}$ and the conclusions of the theorem holds in this case too.

Finally we consider the case where $P_Y(N) = P_Y(A(Y) =)) > 0$, where by definition of the geometric mean we have that $A_G = \mathbb{G}A(Y) = 0$. Furthermore, since $\int f(x,y) \, \mathrm{d}P_X = A(y) = 0$ for for every $y \in N$, Tonelli's theorem yields that

$$\iint_{\mathcal{X}\times N} f \, \mathrm{d}P_X \times P_Y = \int_{\mathcal{X}} \int_N f(x,y) \, \mathrm{d}P_X(x) \, \mathrm{d}P_Y(y) = \int_N A(y) \, \mathrm{d}P_Y(y) = 0 \tag{48}$$

so that $f = 0$ almost everywhere on $\mathcal{X} \times N$. Then for almost every $x \in \mathcal{X}$, then the restricted section $f_x : N \to [0, \infty)$ with $f_x(y) = f(x,y)$ for $y \in N$ satisfies $f_x \simeq 0$ and consequently $G(x) = 0$ for those x. Since $G(x) = 0$ for almost every x, we have that $G_A = 0 = A_G$ which concludes the proof of the last case. □

Remark 116. We find the value of c by integrating $\frac{f(x,y)}{A(y)} = c$ with respect to x so that $1 = \frac{A(y)}{A(y)} = \int \left(\frac{f(x,y)}{A(y)}\right) \mathrm{d}P_X = \int c \, \mathrm{d}P_X = cP_X(\mathcal{X}) = c \cdot 1 = c$, which means $A \simeq a$. This is unique to the probabilistic setting of $\mathcal{X}$.

Note that we only use the probabilistic assumption of $\mathcal{X}$ and $\mathcal{Y}$ in the proof of Theorem 115 when applying the AM-GM inequality with respect to y in (44), in (47) when calculating $\int 1 \, \mathrm{d}P_Y = 1$ and in the use of Tonelli's theorem where only σ-finiteness of the measures is required. Consequently by iterating the proof of Theorem 115 we may instead assume that $\mathcal{X}$ is equipped with a σ-finite measure μ. We state it below without proof, adopting the measure theoretic notation for $\mathcal{X}$ and omit the equality conditions for brevity.

Theorem 117. *Let $(\mathcal{X}, \mathscr{A}, \mu)$ be a σ-finite measure space and $(\mathcal{Y}, \mathcal{F}, P)$ be a probability space with the identity random variable Y, and let $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ be integrable. Then*

$$\mathbb{G}\left(\int f(x,y) \, \mathrm{d}\mu(x)\right) \geq \int \mathbb{G} f(x,y) \, \mathrm{d}\mu(x). \tag{49}$$

Theorem 118 (Generalized Hölder's inequality)**.** *Let $(\mathcal{X}, \mathscr{A}, \mu)$ and $(\mathcal{Y}, \mathscr{B}, \nu)$ be a σ-finite measure spaces and $(\Omega, \mathcal{F}, P)$ be a probability space. Furthermore, suppose that the random variable $Y : \Omega \to \mathcal{Y}$ has distribution function F_Y with a corresponding non-negative density $f_Y = \frac{\mathrm{d}F_Y}{\mathrm{d}\nu} > 0$, and let $p(y) = \frac{1}{f_Y(y)}$. Then given a class of functions $\{g_y : \mathcal{X} \to [0, \infty)\}_{y \in \mathcal{Y}}$ such that $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ given by $f(x,y) = g_y(x)^{p(y)}$ is integrable, we have that*

$$\prod \|g_y\|_{p(y)}^{\mathrm{d}\nu(y)} \geq \left\| \prod g_y^{\mathrm{d}\nu(y)} \right\|_1 \tag{50}$$

Remark 119. Note that if $f_Y > 0$ does not hold everywhere then we may restrict ourselves to the subspace of $\mathcal{Y}$ where it holds, namely its support.

Proof. We expand both sides in (49);

$$\mathbb{G}\left(\int f(x,y)\,\mathrm{d}\mu(x)\right) = \prod\left(\int g_y(x)^{p(y)}\,\mathrm{d}\mu(x)\right)^{\mathrm{d}F_Y(y)}$$
$$= \prod\left(\int g_y^{p(y)}\,\mathrm{d}\mu\right)^{f_Y(y)\,\mathrm{d}\nu(y)}$$
$$= \prod\left(\int g_y^{p(y)}\,\mathrm{d}\mu\right)^{\frac{1}{p(y)}\,\mathrm{d}\nu(y)}$$
$$= \prod \|g_y\|_{p(y)}^{\mathrm{d}\nu(y)}$$

for the left-hand-side and similarly

$$\int \mathbb{G}f(x,y)\,\mathrm{d}\mu(x) = \int \prod g_y(x)^{p(y)\,\mathrm{d}F_Y(y)}\,\mathrm{d}\mu(x) = \int \prod g_y^{\mathrm{d}\nu(y)}\,\mathrm{d}\mu = \left\|\prod g_y^{\mathrm{d}\nu(y)}\right\|$$

for the right-hand-side.

□

Corollary 120 (Discrete Hölder's inequality)**.** *If $\mathcal{Y} = K$ is a discrete space with counting measure and $p_k = p(k)$ for $k \in K$, then*

$$\prod_{k\in K} \|g_k\|_{p_k} \geq \left\|\prod_{k\in K} g_k\right\|_1$$

under the assumptions of Theorem 118.

Remark 121 (Hölder's inequality). If $K = \{0,1\}$, $p_0 = p$, $p_1 = q$, $g = g_0$ and $h = g_1$ we get Hölder's inequality $\|g\|_p\|h\|_q \geq \|gh\|_1$.

Corollary 122 (Continuous Hölder's inequality)**.** *If $\mathcal{Y} = I$ is an interval with Lebesgue measure, we get the continuous Hölder inequality*

$$\prod \|g_y\|_{p(y)}^{\mathrm{d}y} \geq \left\|\prod g_y^{\mathrm{d}y}\right\|_1$$

under the conditions of Theorem 118.

Example 123 (Exponential and Gaussian Hölder inequalities)**.** If Y is an exponential random variable in the Continuous Hölder inequality we get the exponential Hölder inequality

$$\prod_0^\infty \|g_y\|_{e^y}^{\mathrm{d}y} \geq \left\|\prod_0^\infty g_y^{\mathrm{d}y}\right\|_1$$

and if Y is a Gaussian random variable we get the Gaussian Hölder inequality

$$\prod_{-\infty}^{\infty} \|g_y\|_{\sqrt{2\pi e^{y^2}}}^{\mathrm{d}y} \geq \left\| \prod_{-\infty}^{\infty} g_y^{\mathrm{d}y} \right\|_1 .$$

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